Hindawi
Journal of Function Spaces
Volume 2021, Article ID 7613684, 12 pages
https://doi.org/10.1155/2021/7613684

Research Article

Kannan-Type Contractions on New Extended b-Metric Spaces

Hassen Aydi,[1,2,3] Muhammad Aslam,[4] Dur-e-Shehwar Sagheer,[5] Samina Batul,[5] Rashid Ali,[5] and Eskandar Ameer[6]

[1]*Université de Sousse, Institut Supérieur d'Informatique et des Techniques de Communication, H. Sousse 4000, Tunisia*
[2]*Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Ga-Rankuwa, South Africa*
[3]*China Medical University Hospital, China Medical University, Taichung 40402, Taiwan*
[4]*Department of Mathematics, College of Sciences, King Khalid University, Abha 61413, Saudi Arabia*
[5]*Department of Mathematics, Capital University of Science and Technology, Islamabad, Pakistan*
[6]*Department of Mathematics, Taiz University, Taiz 6803, Yemen*

Correspondence should be addressed to Eskandar Ameer; eskandarameer@gmail.com

Received 16 May 2021; Accepted 20 August 2021; Published 13 September 2021

Academic Editor: Liliana Guran

This article is focused on the generalization of some fixed point theorems with Kannan-type contractions in the setting of new extended b-metric spaces. An idea of asymptotic regularity has been incorporated to achieve the new results. As an application, the existence of a solution of the Fredholm-type integral equation is presented.

1. Introduction and Preliminaries

The existence of fixed points for some operators has a noteworthy contribution in many branches of applied and pure mathematics. The theory of a fixed point provides very valuable and effective tools in mathematics. It has a wide range of implications in nonlinear analysis and has been established in two directions. One is to change the space under consideration (see the works of Bakhtin [1], Jleli and Samet [2], Karapinar [3], Kamran et al. [4], etc.), and the other is to change the contraction conditions (see the works of Ćirić [5], Popescu [6], Rakotch [7], etc.).

In 1922, the Polish mathematician Banach [8] established a remarkable result relevant to a metric fixed point theory, that is known as the Banach contraction principle (BCP). The work of Banach is well regarded and an adaptable consequence in the theory of fixed points. BCP laid a foundation of research in this field, which is further investigated by many researchers from 1922 till now. One of the prominent generalizations of the BCP was presented by Kannan [9]. Additional works on the existence of (common) fixed points can be seen in [10–14].

In 1989, Bakhtin [1] introduced the notion of b-metric spaces. Later on, the concept of b-metric spaces was further used by Czerwik [15] to establish different fixed point results in b-metric spaces. The study of b-metric spaces endowed an imperative place in the fixed point theory with multiple aspects. Many mathematicians (Abdeljawad et al. [16, 17], Ali et al. [18], Akkouchi [19], Chifu and Karapinar [20], Kadelburg and Radenović [21], Parvaneh et al. [11], Gupta et al. [12], Mlaiki et al. [22], etc.) led the foundation to improve the fixed point theory in b-metric spaces. Another innovative task has been achieved by Kamran et al. [4] in 2017 by introducing the notion of an extended b-metric space, which generalizes the notion of a b-metric space. Some fixed point results are proved in this new setting; see for instance the work presented in [23–25].

The work of Kannan [9] refined the concept of the Banach contraction mapping by introducing a new contraction, now known as Kannan contraction. The Kannan fixed point result has been further extended and generalized in the setup of b-metric spaces [15] and for generalized metric spaces [26].

In 2019, the notion of a new extended b-metric space has been initiated by Aydi et al. [27], where the control

function depends on three variables. This fact is new since all precedent control functions depend on two variables. The objective of this work is at investigating Kannan-type contractions in the context of new extended b- metric spaces by extending the main results of Gornicki [28]. For this purpose, some basic concepts are needed in the sequel.

Definition 1 (see [15]). Let X be a nonempty set and $s \geq 1$ be a real number. A function $d_b : X \times X \longrightarrow \mathbb{R}$ is called a b -metric, if it satisfies the following for all $j, \kappa, \ell \in X$:

(b1) $d_b(j, \kappa) \geq 0$

(b2) $d_b(j, \kappa) = 0$, if and only if $j = \kappa$

(b3) $d_b(j, \kappa) = d_b(\kappa, j)$

(b4) $d_b(j, \kappa) \leq s[d_b(j, \ell) + d_b(\ell, \kappa)]$

The pair (X, d_b) is called a b-metric space. If $s = 1$, then, a b-metric space becomes a metric space. In 2017, Kamran et al. [4] generalized the b-metric space setting to an extended b-metric space (in the same direction, see also [29, 30]).

Definition 2 (see [4]). Let X be a nonempty set and $\theta : X \times X \longrightarrow [1,\infty)$ be a function. The map $d_\theta : X \times X \longrightarrow \mathbb{R}$ is called an extended b-metric if for all $j, \kappa, \ell \in X$, it satisfies the following axioms:

(i) $d_\theta(j, \kappa) \geq 0$

(ii) $d_\theta(j, \kappa) = 0$, if and only if $j = \kappa$

(iii) $d_\theta(j, \kappa) = d_\theta(\kappa, j)$

(iv) $d_\theta(j, \ell) \leq \theta(j, \ell)[d_\theta(j, \kappa) + d_\theta(\kappa, \ell)]$

In 2019, Aydi et al. [27] introduced the notion of new extended b-metric spaces. Here, the control function depends on 3 variables.

Definition 3 (see [27]). Let X be a nonempty set and $\theta : X \times X \times X \longrightarrow [1,\infty)$ be a function. The map $d_\theta : X \times X \longrightarrow \mathbb{R}$ is called a new extended b-metric if for all $j, \kappa, \ell \in X$, it satisfies the following axioms:

(i) $d_\theta(\mathrm{j}, \kappa) \geq 0$

(ii) $d_\theta(j, \kappa) = 0$, if and only if $j = \kappa$

(iii) $d_\theta(j, \kappa) = d_\theta(\kappa, j)$

(iv) $d_\theta(j, \ell) \leq \theta(j, \kappa, \ell)[d_\theta(j, \kappa) + d_\theta(\kappa, \ell)]$

The pair (X, d_θ) is named to be a new extended b-metric space. If $\theta(j, \kappa, \ell) = s$ (for $s \geq 1$), we get Definition 1.

Example 4 (see [27]). Let $X = \mathbb{N}$. Define $d_\theta : X \times X \longrightarrow \mathbb{R}$ by

$$d_\theta(j, \kappa) = \begin{cases} 0, & \Longleftrightarrow j = \kappa, \\ \dfrac{1}{j}, & \text{if } j \text{ is even and } \kappa \text{ is odd}, \\ \dfrac{1}{\kappa}, & \text{if } j \text{ is odd and } \kappa \text{ is even}, \\ 1, & \text{otherwise}, \end{cases} \tag{1}$$

where

$$\theta(j, \kappa, \ell) = \begin{cases} 1, & \text{if } j = \ell \text{ and } \kappa \text{ is even or odd}, \\ \dfrac{j\ell}{j + \ell}, & \text{if } j \neq \ell, j \text{ and } \ell \text{ are even and } \kappa \text{ odd}, \\ \dfrac{\kappa}{2}, & \text{if } j \neq \ell, j \text{ and } \ell \text{ are odd and } \kappa \text{ even}, \\ \dfrac{3}{2}, & \text{if } j \neq \ell, j, \ell \text{ and } \kappa \text{ are all even or all odd}, \\ \dfrac{j + \kappa(1 + j)}{j(1 + \kappa)}, & \text{if } j \neq \ell, j \text{ is even}, \ell \text{ is odd and } \kappa \text{ is even}, \\ \dfrac{\ell + \kappa(\ell + 1)}{\ell(\kappa + 1)}, & \text{if } j \neq \ell, j \text{ is odd}, \ell \text{ is even and } \kappa \text{ is even}, \\ \dfrac{2 + \ell}{1 + \ell}, & \text{if } j \neq \ell, j \text{ is odd}, \ell \text{ is even and } \kappa \text{ is odd}, \\ \dfrac{j + 1}{j}, & \text{if } j \neq \ell, j \text{ is even}, \ell \text{ is odd and } \kappa \text{ is odd}. \end{cases} \tag{2}$$

Here, (X, d_θ) is a new extended b-metric space.

On the other hand, by taking $j = 2p + 1$, $\ell = 4p + 1$, and $\kappa = 2p$, we have

$$\frac{d_\theta(j, \ell)}{d_\theta(j, \kappa) + d_\theta(\kappa, \ell)} = \frac{d_\theta(2p + 1, 4p + 1)}{d_\theta(2p + 1, 2p) + d_\theta(2p, 4p + 1)} = p. \tag{3}$$

It is not possible to find $s \geq 1$ so that $(b4)$ holds. Thus, d_θ is not a b-metric on X.

Example 5. Consider $X = \{1, 2, 3\}$. Take $d_\theta : X \times X \longrightarrow [0, \infty)$ as

$$d_\theta(j, \kappa) = (j - \kappa)^2, \tag{4}$$

where $\theta : X \times X \times X \longrightarrow [1,\infty)$ as $\theta(j, \kappa, \ell) = j + \kappa + 2\ell$. Then, (X, d_θ) is a new extended b-metric space.

Proof. The first three conditions are trivially verified. To check the triangular inequality, we proceed as follows:

$$d_\theta(1,2) \le \theta(1,3,2)[d_\theta(1,3) + d_\theta(3,2)] \le (1+3+4)[4+1], \tag{5}$$

so $1 < 40$. Similarly, we can check the other two pairs.

Therefore, for all j, κ, ℓ in X, $d_\theta(j,\ell) \le \theta(j,\kappa,\ell)[d_\theta(j,\kappa) + d_\theta(\kappa,\ell)]$. □

Definition 6. Let (X, d_θ) be a new extended b -metric space.

(i) A sequence $\{a_n\} \subset X$ is called convergent to $a \in X$ if for $\varepsilon > 0$, there is $N(\varepsilon) \in \mathbb{N}$ such that $\lim_{n\longrightarrow\infty} d_\theta(a_n, a) < \varepsilon$ for all $n \ge N(\varepsilon)$

(ii) A sequence $\{a_n\} \subset X$ is called Cauchy if for $\varepsilon > 0$, there is $N(\varepsilon) \in \mathbb{N}$ such that $\lim_{n\longrightarrow\infty} d_\theta(a_n, a_m) < \varepsilon$ for all $n, m \ge N(\varepsilon)$

(iii) The space is called complete if every Cauchy sequence in X is convergent in X

Definition 7. Let (X, d_θ) be a new extended b -metric space. Denote by $\mathscr{B}(a,r) = \{b \in X : d_\theta(b,a) < r\}$ and $\mathscr{B}[a,r] = \{b \in X : d_\theta(b,a) \le r\}$ the open and closed balls in X, respectively.

(i) A subset U of X is called open if for any $u \in U$, there exists an $\varepsilon > 0$ such that $\mathscr{B}(u,\varepsilon) \subset U$

(ii) A subset V of X is called closed if for any $\{v_n\} \subset V$ such that $\lim_{n\longrightarrow\infty} v_n = v$, then $v \in X$

In this paper, we are going to prove some Kannan-type fixed point theorems in the setting of new extended b-metric spaces. Some examples are also provided to make effective the obtained results.

2. Main Results

We define a Kannan-type fixed point contraction on new extended b-metric spaces.

Definition 8. Let (X, d_θ) be a new extended b-metric space. A mapping $T : X \longrightarrow X$ is a Kannan-type contraction if there are $K \in [0, 1/2)$ and $0 \le L < 1$ such that

$$\begin{aligned} d_\theta(T(a), T(b)) \le{}& K[d_\theta(a, T(a)) + d_\theta(b, T(b))] \\ &+ Ld_\theta(b, T(a)), \quad \text{for all } a, b \in X. \end{aligned} \tag{6}$$

Our first main result is as follows:

Theorem 9. *Consider a complete new extended b-metric space (X, d_θ) such that d_θ is a continuous functional. Let $T : X \longrightarrow X$ be a mapping such that there are $\xi \in (0, 1/2)$ and $L \in [0, 1)$ so that*

$$\begin{aligned} d_\theta(T(a), T(b)) \le{}& \xi[d_\theta(a, T(a)) + d_\theta(b, T(b))] \\ &+ Ld(b, T(a)), \quad \forall a, b \in X. \end{aligned} \tag{7}$$

Assume that

$$\sup_{m\ge 1} \lim_{n\longrightarrow\infty} \theta(a_n, a_{n+1}, a_m) < \frac{1-\xi}{\xi}, \tag{8}$$

for each $a_0 \in X$, such that $a_n = T^n(a_0)$, $n = 1, 2, \cdots$.

Then, T has only one fixed point $v \in X$. Further, for any $a \in X$, the iterative sequence $\{T^n(a)\}$ converges to v, and

$$d_\theta(T(a_n), v) \le \frac{\xi}{1-L}\left(\frac{\xi}{1-\xi}\right)^n d_\theta(T(a_0), a_0), \quad n = 0, 1, 2, \cdots. \tag{9}$$

Proof. For an arbitrary point $a_0 \in X$, construct the iterative sequence

$$a_{n+1} = T(a_n) = T^{n+1}(a_0). \tag{10}$$

If for some n, $a_n = a_{n+1}$, so a_n is a fixed point of T. Otherwise, assume that $a_n \ne a_{n+1}$ for all $n \ge 0$. Since

$$d_\theta(a_n, a_{n+1}) = d_\theta(T(a_{n-1}), T(a_n)), \tag{11}$$

one writes

$$\begin{aligned} d_\theta(T(a_{n-1}), T(a_n)) \le{}& \xi[d_\theta(a_{n-1}, T(a_{n-1})) + d_\theta(a_n, T(a_n))] \\ &+ Ld_\theta(a_n, Ta_{n-1}). \end{aligned} \tag{12}$$

Then,

$$d_\theta(a_n, a_{n+1}) \le \xi[d_\theta(a_{n-1}, a_n) + d_\theta(a_n, a_{n+1})] + Ld_\theta(a_n, a_n). \tag{13}$$

That is,

$$d_\theta(a_n, a_{n+1}) \le \left(\frac{\xi}{1-\xi}\right) d_\theta(a_{n-1}, a_n). \tag{14}$$

Continuing in this way, we have

$$d_\theta(a_n, a_{n+1}) \le \left(\frac{\xi}{1-\xi}\right)^n d_\theta(a_0, a_1), \tag{15}$$

$$d_\theta(T(a_{n-1}), T(a_n)) \le \left(\frac{\xi}{1-\xi}\right)^n d_\theta(a_0, T(a_0)). \tag{16}$$

Let $m, n \in \mathbb{N}$ be such that $m > n$. Applying triangular inequality, we get

$$\begin{aligned}
d_\theta(a_n, a_m) &\le \theta(a_n, a_{n+1}, a_m)[d_\theta(a_n, a_{n+1}) + d_\theta(a_{n+1}, a_m)] \\
&= \theta(a_n, a_{n+1}, a_m) d_\theta(a_n, a_{n+1}) + \theta(a_n, a_{n+1}, a_m) d_\theta(a_{n+1}, a_m) \\
&\le \theta(a_n, a_{n+1}, a_m) d_\theta(a_n, a_{n+1}) + \theta(a_n, a_{n+1}, a_m)\theta \\
&\quad \cdot (a_{n+1}, a_{n+2}, a_m)\,[d_\theta(a_{n+1}, a_{n+2}) + d_\theta(a_{n+2}, a_m)] \\
&\le \theta(a_n, a_{n+1}, \mathrm{a}_m) d_\theta(a_n, a_{n+1}) + \theta(a_n, a_{n+1}, a_m)\theta \\
&\quad \cdot (a_{n+1}, a_{n+2}, a_m) d_\theta(a_{n+1}, a_{n+2}) + \cdots + \theta(a_n, a_{n+1}, a_m)\theta \\
&\quad \cdot (a_{n+1}, a_{n+2}, a_m)\theta(a_{n+2}, a_{n+3}, a_m) \cdots \theta \\
&\quad \cdot (a_{m-2}, a_{m-1}, a_m) d_\theta(a_{m-1}, a_m).
\end{aligned} \tag{17}$$

Since

$$d_\theta(a_n, a_{n+1}) \le \left(\frac{\xi}{1-\xi}\right)^n d_\theta(a_0, a_1)\ n = 0, 1, 2, \cdots, \tag{18}$$

one writes

$$\begin{aligned}
d_\theta(a_n, a_m) &\le \theta(a_n, a_{n+1}, a_m)\left(\frac{\xi}{1-\xi}\right)^n d_\theta(a_0, a_1) + \theta \\
&\quad \cdot (a_n, a_{n+1}, a_m)\theta(a_{n+1}, a_{n+2}, a_m)\left(\frac{\xi}{1-\xi}\right)^{n+1} d_\theta \\
&\quad \cdot (a_0, a_1) + \cdots + \theta(a_n, a_{n+1}, a_m)\theta(a_{n+1}, a_{n+2}, a_m)\theta \\
&\quad \cdot (a_{n+2}, a_{n+3}, a_m) \cdots \theta(a_{m-2}, a_{m-1}, a_m)\left(\frac{\xi}{1-\xi}\right)^{m-1} d_\theta(a_0, a_1) \\
&\le \Bigg\{\theta(a_1, a_2, a_m)\theta(a_2, a_3, a_m) \cdots \theta(a_{n-1}, a_n, a_m)\theta \\
&\quad \cdot (a_n, a_{n+1}, a_m)\left(\frac{\xi}{1-\xi}\right)^n + \theta(a_1, a_2, a_m)\theta(a_2, a_3, a_m) \cdots \theta \\
&\quad \cdot (a_n, a_{n+1}, a_m)\theta(a_{n+1}, a_{n+2}, a_m)\left(\frac{\xi}{1-\xi}\right)^{n+1} + \cdots + \theta \\
&\quad \cdot (a_1, a_2, a_m)\theta(a_2, a_3, a_m) \cdots \theta(a_n, a_{n+1}, a_m)\theta \\
&\quad \cdot (a_{n+1}, a_{n+2}, a_m) \cdots \theta(a_{m-3}, a_{m-2}, a_m)\theta(a_{m-2}, a_{m-1}, a_m) \\
&\quad \cdot \left(\frac{\xi}{1-\xi}\right)^{m-1}\Bigg\} d_\theta(a_0, a_1).
\end{aligned} \tag{19}$$

Since

$$\sup_{m \ge 1} \lim_{n,m \longrightarrow \infty} \theta(a_n, a_{n+1}, a_m)\left(\frac{\xi}{1-\xi}\right) < 1, \tag{20}$$

in view of (8), the series $\sum_{n=1}^{\infty}(\xi/(1-\xi))^n \prod_{i=1}^{n}\theta(a_i, a_{i+1}, a_m)$ is convergent for each $m \in \mathbb{N}$ by ratio test.

Let $S = \sum_{n=1}^{\infty}(\xi/(1-\xi))^n \prod_{i=1}^{n}\theta(a_i, a_{i+1}, a_m)$ and

$$S_n = \sum_{j=1}^{n}\left(\frac{\xi}{1-\xi}\right)^j \prod_{i=1}^{j}\theta(a_i, a_{i+1}, a_m). \tag{21}$$

So, for $m > n$, the above inequality implies that

$$d_\theta(a_n, a_m) \le d_\theta(a_0, a_1)(S_{m-1} - S_{n-1}). \tag{22}$$

That is,

$$\lim_{n \longrightarrow \infty} d_\theta(a_n, a_m) = 0. \tag{23}$$

Hence, the sequence $\{a_n\}$ is a Cauchy sequence. By the completeness of X, there is $v \in X$ such that $\lim_{n \longrightarrow \infty} a_n = v$.

We claim that v is a fixed point of T. We have

$$d_\theta(T(a_n), T(v)) \le \xi(d_\theta(a_n, T(a_n)) + d_\theta(v, T(v)) + L d_\theta(v, T(a_n)). \tag{24}$$

That is,

$$d_\theta(a_{n+1}, T(v)) \le \xi(d_\theta(a_n, a_{n+1}) + d_\theta(v, T(v))) + L d_\theta(v, a_{n+1}). \tag{25}$$

As $n \longrightarrow \infty$, we have in view of the assumption that d_θ is continuous,

$$d_\theta(v, T(v)) \le \xi d_\theta(v, Tv), \tag{26}$$

which holds unless $d_\theta(v, T(v)) = 0$, and so, $T(v) = v$.

The uniqueness is as follows:

Let τ be another fixed point of T. We have

$$\begin{aligned}
0 \le d_\theta(v, \tau) = d_\theta(T(v), T(\tau)) &\le \xi[d_\theta(v, T(v)) + d_\theta(\tau, T(\tau))] \\
&\quad + L d_\theta(\tau, T(v)).
\end{aligned} \tag{27}$$

That is,

$$d_\theta(v, \tau) \le L d_\theta(v, \tau). \tag{28}$$

It is only possible if $d_\theta(v, \tau) = 0$. Thus, $v \in X$ is the unique fixed point of T. Further, we have

$$\begin{aligned}
d_\theta(T(a_{n-1}), T(a_n)) &\le \xi[d_\theta(T(a_{n-2}), T(a_{n-1})) + d_\theta(T(a_{n-1}), T(a_n))] \\
&\quad + L d_\theta(a_n, a_n).
\end{aligned} \tag{29}$$

Then,

$$d_\theta(T(a_{n-1}), T(a_n)) \le \left(\frac{\xi}{1-\xi}\right) d_\theta(T(a_{n-2}), T(a_{n-1})). \tag{30}$$

Also,

$$\begin{aligned}
d_\theta(T(a_n), v) &\le \xi[d_\theta(T(a_{n-1}), T(a_n)) + d_\theta(v, T(v))] + L d_\theta(v, T(a_n)) \\
&\le \xi d_\theta(T(a_{n-1}), T(a_n)) + L d_\theta(v, T(a_n)).
\end{aligned} \tag{31}$$

Using (15),

$$d_\theta(T(a_n), v) \le \xi\left(\frac{\xi}{1-\xi}\right)^n d_\theta(T(a_0), a_0) + Ld_\theta(v, T(a_n)). \tag{32}$$

That is,

$$d_\theta(T(a_n), v) \le \frac{\xi}{1-L}\left(\frac{\xi}{1-\xi}\right)^n d_\theta(T(a_0), a_0), \quad n = 0, 1, 2, \cdots. \tag{33}$$

□

The following examples illustrate Theorem 9. We deal with noncompact sets.

Example 10. Let $X = \ell_\infty$ be the space of all bounded sequences of real numbers, that is,

$$\ell_\infty = \{\eta = \{\eta_n\}: |\eta_n| \le C_\eta, \forall n \in \mathbb{N}\}, \tag{34}$$

where $C_\eta \in \mathbb{R}$ may depend on the sequence η but does not depend on n. Take that

$$\begin{aligned} d_\theta(\eta, \zeta) &= \sup_{n\in\mathbb{N}} |\eta_n - \zeta_n|^2, \\ \eta &= \{\eta_n\}, \\ \zeta &= \{\zeta_n\}, \end{aligned} \tag{35}$$

are in X.

Then, X is a complete new extended b-metric space with $\theta : X \times X \times X \longrightarrow [1,\infty)$ being defined by

$$\theta(\eta, \zeta, \varphi) = \sup_{n\in\mathbb{N}} \frac{|\eta_n + \zeta_n + \varphi_n|}{|\eta_n + \zeta_n + \varphi_n| + 1} + 3. \tag{36}$$

Consider $T : X \longrightarrow X$ given as $T(\eta) = \{(\eta_n - 1)/5\}, \forall n = 1, 2, 3, \cdots$. For each $\eta, \zeta \in X$, we have

$$\begin{aligned} d_\theta(T\eta, T\zeta) &= \sup_{n\in\mathbb{N}}\left|\frac{\eta_n - 1}{5} - \frac{\zeta_n - 1}{5}\right|^2 = \sup_{n\in\mathbb{N}}\left|\frac{\eta_n}{5} - \frac{\zeta_n}{5}\right|^2 \\ &\le \sup_{n\in\mathbb{N}} 2\left|\frac{\eta_n}{5}\right|^2 + \sup_{n\in\mathbb{N}} 2\left|\frac{\zeta_n}{5}\right|^2 = \frac{1}{8}\sup_{n\in\mathbb{N}} 16\left|\frac{\eta_n}{5}\right|^2 \\ &\quad + \frac{1}{8}\sup_{n\in\mathbb{N}} 16\left|\frac{\zeta_n}{5}\right|^2 = \frac{1}{8}\sup_{n\in\mathbb{N}}\left|\frac{4\eta_n}{5}\right|^2 + \frac{1}{8}\sup_{n\in\mathbb{N}}\left|\frac{4\zeta_n}{5}\right|^2 \\ &\le \frac{1}{8}\sup_{n\in\mathbb{N}}\left|\frac{4\eta_n + 1}{5}\right|^2 + \frac{1}{8}\sup_{n\in\mathbb{N}}\left|\frac{4\zeta_n + 1}{5}\right|^2 \\ &= \frac{1}{8}[d_\theta(\eta, T\eta) + d_\theta(\zeta, T\zeta)]. \end{aligned} \tag{37}$$

Thus, (7) holds with $\xi = 1/8$ and $L \in [0, 1)$. Also, $(1 - \xi)/\xi = 7$ and $\theta(\eta, \zeta, \varphi) < 7$ for all $\eta, \zeta, \varphi \in X$. Hence, (8) holds, and by Theorem 9, T has a fixed point.

Example 11. Let $X = C[a, b]$ be the set of all real-valued continuous functions defined on [a, b]. Define

$$d_\theta(x, y) = \sup_{t\in[a,b]} |x(t) - y(t)|^2, \quad \forall x, y \in C[a, b], \tag{38}$$

and $\theta : X \times X \times X \longrightarrow [1,\infty)$ as

$$\theta(x, y, z) = \sup_{t\in[a,b]} \frac{|x(t)| + |y(t)| + |z(t)|}{|x(t)| + |y(t)| + |z(t)| + 1} + 2. \tag{39}$$

Then, X is a complete new extended b-metric space, consider a mapping $T : X \longrightarrow X$ given as

$$T(\eta(t)) = \frac{\eta(t) - 1}{7}, \quad \forall \eta \in C[a, b]. \tag{40}$$

For all $\eta, \zeta \in X$, we have

$$\begin{aligned} d_\theta(T\eta, T\zeta) &= \sup_{t\in[a,b]}\left|\frac{\eta(t) - 1}{7} - \frac{\zeta(t) - 1}{7}\right|^2 = \sup_{t\in[a,b]}\left|\frac{\eta(t)}{7} - \frac{\zeta(t)}{7}\right|^2 \\ &\le \sup_{t\in[a,b]} 2\left|\frac{\eta(t)}{7}\right|^2 + \sup_{t\in[a,b]} 2\left|\frac{\zeta(t)}{7}\right|^2 = \frac{1}{18}\sup_{t\in[a,b]} 36\left|\frac{\eta(t)}{7}\right|^2 \\ &\quad + \frac{1}{18}\sup_{t\in[a,b]} 36\left|\frac{\zeta(t)}{7}\right|^2 = \frac{1}{18}\sup_{t\in[a,b]}\left|\frac{6\eta(t)}{7}\right|^2 + \frac{1}{18}\sup_{n\in\mathbb{N}}\left|\frac{6\zeta(t)}{7}\right|^2 \\ &\le \frac{1}{18}\sup_{t\in[a,b]}\left|\frac{6\eta(t) + 1}{5}\right|^2 + \frac{1}{18}\sup_{t\in[a,b]}\left|\frac{6\zeta(t) + 1}{5}\right|^2 \\ &< \frac{1}{6}[d_\theta(\eta, T\eta) + d_\theta(\zeta, T\zeta)]. \end{aligned} \tag{41}$$

Thus, (7) holds with $\xi = 1/6$ and $L \in [0, 1)$. Also, $(1 - \xi)/\xi = 5$ and $\theta(\eta, \zeta, \varphi) < 5$ for all $\eta, \zeta, \varphi \in X$. That is, (8) holds. Since all the conditions of Theorem 9 are satisfied, T has a fixed point.

Example 12. Choose $X = \{1/4, 1/16, \cdots, 1/4^n, \cdots\} \cup \{0, 1\}$. Define $\theta : X \times X \times X \longrightarrow [1,\infty)$ and $d_\theta : X \times X \longrightarrow [0,\infty)$ by

$$\theta(x, y, z) = x + y + z + 2, d_\theta(x, y) = (x - y)^2. \tag{42}$$

Let $T : X \longrightarrow X$ be given as

$$Tu = \begin{cases} \dfrac{1}{4^{n+1}}, & \text{if } u = \dfrac{1}{4^n}, n = 0, 1, 2, 3, \cdots, \\ u, & \text{if } u = 0. \end{cases} \tag{43}$$

Then, for all $x, y \in X$ with neither $x = 0$ nor $y = 0$, we have

$$d_\theta(Tx, Ty) = \left|\frac{1}{4^{n+1}} - \frac{1}{4^{m+1}}\right|^2 \le 2\left|\frac{1}{4^{n+1}}\right|^2 + 2\left|\frac{1}{4^{m+1}}\right|^2 = \frac{2}{9}\left|\frac{3}{4^{n+1}}\right|^2 + \frac{2}{9}\left|\frac{3}{4^{m+1}}\right|^2 = \frac{2}{9}[d_\theta(x, Tx) + d_\theta(y, Ty)]. \tag{44}$$

If $x = 0$ and $y \neq 0$, then

$$d_\theta(Tx, Ty) = \left|0 - \frac{1}{4^{m+1}}\right|^2 = \left|\frac{1}{4^{m+1}}\right|^2 = \frac{2}{9}\left|\frac{3}{4^{m+1}}\right|^2 = \frac{2}{9}[d_\theta(x, Tx) + d_\theta(y, Ty)]. \tag{45}$$

Thus, (7) is satisfied for $\xi = 2/9$ and for each $L \in [0, 1)$. Also, $(1 - \xi)/\xi = 7/2$. If $a_0 = 0$, then, for the iterative sequence $a_n = T^n a_0 = 0$ for each $n \in \mathbb{N}$, we have $\lim_{n,m\longrightarrow\infty} \theta(a_n, a_{n+1}, a_m) = \lim_{n,m\longrightarrow\infty} a_n + a_{n+1} + a_m + 2 < 7/2$. If $a_0 \neq 0$ (say $a_0 = 1/4^k$ for some $k \in \{0, 1, 2, \cdots\}$); then, for the iterative sequence $a_n = T^n a_0 = 1/4^{k+n}$ for each $n \in \mathbb{N}$, we have $\lim_{n,m\longrightarrow\infty} \theta(a_n, a_{n+1}, a_m) = \lim_{n,m\longrightarrow\infty} a_n + a_{n+1} + a_m + 2 < 7/2$. Hence, Theorem 9 ensures the existence of a fixed point of T.

Remark 13. In the following, we ensure the completeness of the spaces given in precedent examples.

(a) Completeness of ℓ_∞

Let $X = \ell_\infty$ and let $\{x_m\} = \{\xi_1^{(m)}, \xi_2^{(m)}, \xi_3^{(m)}, \cdots\}$ be a Cauchy sequence in X. Define a metric on X as $d_\theta(x, y) = \sup_j |\xi_j - \eta_j|^2$, where $x = (\xi_j)$ and $y = (\eta_j)$. Since $\{x_m\}$ is a Cauchy sequence, for $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for all $m, n \ge N$,

$$d_\theta(x_n, x_m) = \sup_j \left|\xi_j^{(m)} - \xi_j^{(n)}\right|^2 < \varepsilon \forall m, n \ge N \Longrightarrow \left|\xi_j^{(m)} - \xi_j^{(n)}\right|^2 < \varepsilon, \quad \forall j \text{ and } m, n \ge N. \tag{46}$$

That is,

$$\left|\xi_j^{(m)} - \xi_j^{(n)}\right| < \varepsilon_1, \quad \forall j \text{ and } m, n \ge N, \tag{47}$$

where $\varepsilon_1 = \sqrt{\varepsilon}$ is arbitrary. Hence, for every fixed j, $(\xi_j^{(1)}, \xi_j^{(2)}, \cdots)$ is a Cauchy sequence of complex numbers and it converges, so $\xi_j^{(m)} \longrightarrow \xi_j$ as $m \longrightarrow \infty$. Construct a sequence $x = (\xi_1, \xi_2, \cdots)$ by using these infinitely many limits to show that $x \in \ell_\infty$ and $x_m \longrightarrow x$. From (48) with $n \longrightarrow \infty$, we have

$$\left|\xi_j^{(m)} - \xi_j\right| < \varepsilon_1, \quad m \ge N. \tag{48}$$

Since $x_m = \xi_j^{(m)} \in \ell_\infty$, there is a real number c_m so that $|\xi_j^{(m)}| \le c_m$, $\forall j$. Hence,

$$|\xi_j|^2 = \left|\xi_j - \xi_j^{(m)} + \xi_j^{(m)}\right|^2 \le \left(\left|\xi_j - \xi_j^{(m)}\right| + \left|\xi_j^{(m)}\right|\right)^2 \le 2\left(\left|\xi_j - \xi_j^{(m)}\right|^2 + \left|\xi_j^{(m)}\right|^2\right). \tag{49}$$

That is,

$$|\xi_j| \le \sqrt{2\varepsilon + 2(c_m)^2}. \tag{50}$$

So, (51) holds for each j. It implies that $\{\xi_j\}$ is a bounded sequence of complex numbers. This leads to $\{\xi_j\} \in \ell_\infty$. Also, from (48), we have

$$d_\theta(x_m, x) = \sup_j \left|\xi_j^{(m)} - \xi_j\right|^2 \le \varepsilon, \quad m > N. \tag{51}$$

This implies that $x_m \longrightarrow x \in \ell_\infty$ (ℓ_∞ is endowed with the new extended metric d_θ).

(b) Completeness of $C[a, b]$

Let X be the function space $C[a, b]$, where $[a, b]$ is any closed interval in $\mathbb{R}$. Define

$$d_\theta(x, y) = \sup_{t\in[a,b]} |x(t) - y(t)|^2, \quad \forall x, y \in C[a, b]. \tag{52}$$

Let $\{x_m\}$ be a Cauchy sequence in $C[a, b]$. Then, for each $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$d_\theta(x_m, x_n) = \sup_{t\in[a,b]} |x_m(t) - x_n(t)|^2 < \varepsilon, \quad \forall m, n \ge N. \tag{53}$$

Hence, for each fixed $t_0 \in [a, b]$, we have

$$|x_m(t_0) - x_n(t_0)|^2 < \varepsilon, \quad \forall m, n \ge N. \tag{54}$$

That is,

$$|x_m(t_0) - x_n(t_0)| < \sqrt{\varepsilon} = \varepsilon_1, \quad (\text{say}) \forall m, n \ge N. \tag{55}$$

This shows that $(x_1(t_0), x_2(t_0), \cdots)$ is a Cauchy sequence of real numbers; hence, it converges. That is, $x_m(t_0) \longrightarrow x(t_0)$ as $m \longrightarrow \infty$. In this way, we can associate to each $t \in [a, b]$ a unique real number $x(t)$. This defines a function x (pointwise) in $[a, b]$. Further, we need to show that $x_m \longrightarrow x \in C[a, b]$. From (53), we have as $n \longrightarrow \infty$

$$\sup_{t\in[a,b]} |x_m(t) - x(t)|^2 < \varepsilon, \quad m \ge N. \tag{56}$$

Hence, for each $t \in [a, b]$ and $m > N$, we have

$$|x_m(t) - x(t)|^2 < \varepsilon, \tag{57}$$

which implies that $|x_m(t) - x(t)| < \varepsilon_1$. Thus, $\{x_m(t)\}$ converges uniformly to $x(t)$ on the interval $[a, b]$. Since $t \longrightarrow x_m(t)$ is continuous and the convergence is uniform; hence,

$x(t)$ is continuous. This leads to the completeness of $C[a, b]$ with the new extended metric d_θ.

Theorem 14. *Let (X, d_θ) be a complete new extended b -metric space where d_θ is a continuous functional and M is a nonempty closed subset of X. Let $T : M \longrightarrow M$ satisfy*

$$d_\theta(T(a), T(b)) \le \xi[d_\theta(a, T(a)) + d_\theta(b, T(b))] + Ld_\theta(b, T(a)), \quad \forall a, b \in M \text{ and } L, \xi \in [0, 1). \tag{58}$$

Assume that there exist $\alpha, \beta \in \mathbb{R}$ with $0 < \alpha < 1$ and $\beta > 0$ such that for an arbitrary $a \in M$, there is $\mu^* \in M$ verifying

$$d_\theta(\mu^*, T(\mu^*)) \le \alpha d_\theta(a, T(a)), d_\theta(\mu^*, a) \le \beta d_\theta(a, T(a)). \tag{59}$$

Also, for an arbitrary $a_0 \in M$, assume that the sequence $\{a_n = T^n(a_0)\}$ verifies

$$\sup_{m\ge1} \lim_{n\longrightarrow\infty} \theta(a_n, a_{n+1}, a_m) < \frac{1}{\alpha}. \tag{60}$$

Then, T has only one fixed point.

Proof. Let a_0 be an arbitrary element of M. Consider the sequence $\{a_n = T^n(a_0)\}$ in M. We have

$$\begin{aligned} d_\theta(T(a_{n+1}), a_{n+1}) &\le \alpha d_\theta(T(a_n), a_n), d_\theta(a_{n+1}, a_n) \\ &\le \beta d_\theta(T(a_n), a_n), \quad n = 0, 1, 2, \cdots. \end{aligned} \tag{61}$$

Since

$$d_\theta(a_{n+1}, a_n) = d_\theta(T(a_n), a_n) \le \beta d_\theta(T(a_n), a_n), \quad n = 0, 1, 2, \cdots,$$

$$\begin{aligned} \beta d_\theta(T(a_n), a_n) &\le \beta\alpha d_\theta(T(a_{n-1}), a_{n-1}) \le \beta\alpha^2 d_\theta(T(a_{n-2}), a_{n-2}) \\ &\le \cdots \le \beta\alpha^n d_\theta(T(a_0), a_0), \end{aligned} \tag{62}$$

we have

$$d_\theta(a_{n+1}, a_n) \le \beta\alpha^n d_\theta(T(a_0), a_0). \tag{63}$$

Let $m, n \in \mathbb{N}$ such that $m > n$. Applying triangular inequality, we get

$$\begin{aligned} d_\theta(a_n, a_m) &\le \theta(a_n, a_{n+1}, a_m)[d_\theta(a_n, a_{n+1}) + d_\theta(a_{n+1}, a_m)] \\ &= \theta(a_n, a_{n+1}, a_m) d_\theta(a_n, a_{n+1}) + \theta(a_n, a_{n+1}, a_m) d_\theta \\ &\quad \cdot (a_{n+1}, a_m) \le \theta(a_n, a_{n+1}, a_m) d_\theta(a_n, a_{n+1}) + \theta \\ &\quad \cdot (a_n, a_{n+1}, a_m)\theta(a_{n+1}, a_{n+2}, a_m)(d_\theta(a_{n+1}, a_{n+2}) + d_\theta(a_{n+2}, a_m)) \\ &\le \theta(a_n, a_{n+1}, a_m) d_\theta(a_n, a_{n+1}) + \theta(a_n, a_{n+1}, a_m)\theta(a_{n+1}, a_{n+2}, a_m) d_\theta \\ &\quad \cdot (a_{n+1}, a_{n+2}) + \cdots + \theta(a_n, a_{n+1}, a_m)\theta(a_{n+1}, a_{n+2}, a_m)\theta \\ &\quad \cdot (a_{n+2}, a_{n+3}, a_m) \cdots \theta(a_{m-2}, a_{m-1}, a_m) d_\theta(a_{m-1}, a_m). \end{aligned} \tag{64}$$

Using (63), we get

$$\begin{aligned} d_\theta(a_n, a_m) &\le \theta(a_n, a_{n+1}, a_m)\beta\alpha^n d_\theta(T(a_0), a_0) + \theta(a_n, a_{n+1}, a_m)\theta \\ &\quad \cdot (a_{n+1}, a_{n+2}, a_m)\beta\alpha^{n+1} d_\theta(T(a_0), a_0) + \cdots + \theta \\ &\quad \cdot (a_n, a_{n+1}, a_m)\theta(a_{n+1}, a_{n+2}, a_m)\theta(a_{n+2}, a_{n+3}, a_m) \cdots \theta \\ &\quad \cdot (a_{m-2}, a_{m-1}, a_m)\, \beta\alpha^{m-1} d_\theta(T(a_0), a_0) \\ &\le [\theta(a_1, a_2, a_m)\theta(a_2, a_3, a_m) \cdots \theta(a_{n-1}, a_n, a_m)\theta \\ &\quad \cdot (a_n, a_{n+1} a_m)\alpha^n + \theta(a_1, a_2, a_m)\theta(a_2, a_3, a_m) \cdots \theta \\ &\quad \cdot (a_n, a_{n+1}, a_m)\theta(a_{n+1}, a_{n+2}, a_m)\alpha^{n+1} + \cdots + \theta(a_1, a_2, a_m)\theta \\ &\quad \cdot (a_2, a_3, a_m) \cdots \theta(a_n, a_{n+1}, a_m)\theta(a_{n+1}, a_{n+2} a_m) \cdots \theta \\ &\quad \cdot (a_{m-2}, a_{m-1}, a_m)\alpha^{m-1}]\, \beta d_\theta(T(a_0), a_0). \end{aligned} \tag{65}$$

Since $\sup_{m\ge1}\lim_{n,m\longrightarrow\infty}\theta(a_n, a_{n+1}, a_m)\alpha < 1$, the series $\sum_{n=1}^{\infty}\alpha^n\prod_{i=1}^{n}\theta(a_i, a_{i+1}, a_m)$ is convergent for each $m \in \mathbb{N}$ by ratio test.

Let

$$\begin{aligned} S &= \sum_{n=1}^{\infty} \alpha^n \prod_{i=1}^{n} \theta(a_i, a_{i+1}, a_m), \\ S_n &= \sum_{j=1}^{n} \alpha^j \prod_{i=1}^{j} \theta(a_i, a_{i+1}, a_m). \end{aligned} \tag{66}$$

So, for $m > n$, the above inequality implies that

$$d_\theta(a_n, a_m) \le d_\theta(a_0, a_1)(S_{m-1} - S_{n-1}). \tag{67}$$

Letting $n \longrightarrow \infty$, the sequence $\{a_n\}$ is a Cauchy sequence. By the completeness of M, there is $v \in M$ such that $\lim_{n\longrightarrow\infty} a_n = v$.

We will prove that v is a fixed point of T. By using (58), we get

$$\begin{aligned} d_\theta(T(a_n), T(v)) &\le \xi(d_\theta(a_n, T(a_n)) + d_\theta(v, Tv)) + Ld_\theta(v, T(a_n)) \\ &\Longrightarrow d_\theta(a_{n+1}, T(v)) \le \xi(d_\theta(a_n, a_{n+1}) + d_\theta(v, Tv)) \\ &\quad + Ld_\theta(v, a_{n+1}). \end{aligned} \tag{68}$$

As $n \longrightarrow \infty$, we have

$$d_\theta(v, T(v)) \le \xi d_\theta(v, T(v)). \tag{69}$$

Hence, $T(v) = v$.

The uniqueness is as follows:

Assume on contrary that there is $\tau(\neq v) \in M$ so that $T(\tau) = \tau$; then,

$$\begin{aligned} 0 < d_\theta(v, \tau) = d_\theta(T(v), T(\tau)) &\le \xi[d_\theta(v, T(v)) + d_\theta(\tau, T(\tau))] \\ &\quad + Ld_\theta(\tau, T(v)) = Ld_\theta(\tau, v). \end{aligned} \tag{70}$$

That is, $d_\theta(v, \tau) < Ld_\theta(v, \tau)$, which is a contradiction. Thus, $v \in X$ is the unique fixed point of T. □

Remark 15. To prove Theorem 9 in new extended b-metric spaces, by using the following conditions:

$$\begin{aligned} d_\theta(\mu^*, T(\mu^*)) &\le \alpha d_\theta(a, T(a)), \\ d_\theta(\mu^*, a) &\le \beta d_\theta(a, T(a)), \end{aligned} \tag{71}$$

we proceed as follows:

For any $a \in X$, take $\mu^* = T(a)$. Then,

$$\begin{aligned} d_\theta(\mu^*, T(\mu^*)) = d_\theta(T(a), T(\mu^*)) &\le \xi[d_\theta(a, T(a)) + d_\theta(\mu^*, T(\mu^*))] \\ &\quad + Ld_\theta(\mu^*, T(a))\,(1-\xi)d_\theta(\mu^*, T(\mu^*)) \\ &\le \xi d_\theta(a, T(a))d_\theta(\mu^*, T(\mu^*)) \le \left(\frac{\xi}{1-\xi}\right)d_\theta(a, T(a)), \end{aligned} \tag{72}$$

where by assumption $(\xi/1-\xi) < 1$ and $d_\theta(\mu^*, a) = d_\theta(T(a), a)$. Now, for arbitrary $a_0 \in X$, we can inductively define a sequence $\{a_{n+1} = T(a_n)\}$. By Theorem 14, this sequence is convergent. So, $\lim_{n\longrightarrow\infty} a_n = v$. Thus, $T(v) = v$.

Also, for each $a \in X$,

$$\begin{aligned} d_\theta(T(a_{n-1}), T(a_n)) &\le \xi[d_\theta(T(a_{n-2}), T(a_{n-1})) + d_\theta(T(a_{n-1}), T(a_n))] \\ &\quad + Ld_\theta(a_n, T(a_{n-1})) \Longrightarrow d_\theta(T(a_{n-1}), T(a_n)) \\ &\le \left(\frac{\xi}{1-\xi}\right)d_\theta(T(a_{n-2}), T(a_{n-1})), \end{aligned}$$

$$\begin{aligned} d_\theta(\mathrm{T}(a_n), v) &\le \xi[d_\theta(T(a_{n-1}), T(a_n)) + d_\theta(v, T(v))] + Ld_\theta(v, T(a_n)) \\ &\le \xi d_\theta(T(a_{n-1}), T(a_n)) + Ld_\theta(v, T(a_n)) \Longrightarrow d_\theta(T(a_n), v) \\ &\le \frac{\xi}{1-L}\left(\frac{\xi}{1-\xi}\right)^n d_\theta(T(a), a), \quad n = 0, 1, 2, \cdots. \end{aligned} \tag{73}$$

Theorem 16. *Let (X, d_θ) be a complete new extended b-metric space such that d_θ is a continuous functional. Let $T : X \longrightarrow X$ satisfy*

$$\begin{aligned} d_\theta(T(a), T(b)) &\le \alpha d_\theta(a, T(a)) + \beta d_\theta(b, T(b)) + \gamma d_\theta(a, b) \\ &\quad + Ld_\theta(a, T(b)), \quad \forall a, b \in X, \end{aligned} \tag{74}$$

where α, β, γ, L are nonnegative real numbers such that $\alpha + \beta + \gamma + L < 1$ and $\beta + \gamma > 0$. Assume that for an arbitrary $a_0 \in M$, we have

$$\sup_{m\ge 1} \lim_{n\longrightarrow\infty} \theta(a_n, a_{n+1}, a_m) < \frac{1}{p}, \tag{75}$$

where $p = ((\beta+\gamma)/(1-\alpha))$ and $a_n = T^n(a_0)$.

Then, T has only one fixed point.

Proof. For an arbitrary $a_0 \in X$, take the sequence $\{T^n(a_0)\}$. Substituting $a = T^{n-1}(a_0) = T(a_{n-1}) = a_n$ and $b = T^{n-2}(a_0) = T(a_{n-2}) = a_{n-1}$ in (74), we obtain

$$\begin{aligned} d_\theta(T(a_n), T(a_{n-1})) &\le \alpha d_\theta(T(a_{n-1}), T(a_n)) + \beta d_\theta(T(a_{n-2}), T(a_{n-1})) \\ &\quad + \gamma d_\theta(T(a_{n-1}), T(a_{n-2})) + Ld_\theta(T(a_{n-1}), T(a_{n-1})). \end{aligned} \tag{76}$$

That is,

$$(1-\alpha)d_\theta(T(a_n), T(a_{n-1})) \le (\beta+\gamma)d_\theta(T(a_{n-1}), T(a_{n-2})). \tag{77}$$

Thus,

$$d_\theta(T(a_n), T(a_{n-1})) \le \left(\frac{\beta+\gamma}{1-\alpha}\right)d_\theta(T(a_{n-1}), T(a_{n-2})). \tag{78}$$

Moreover,

$$\begin{aligned} d_\theta(T(a_n), T(a_{n-1})) &\le pd_\theta(T(a_{n-1}), T(a_{n-2})) \\ &\le p^2 d_\theta(T(a_{n-2}), T(a_{n-3})) \vdots \\ &\le p^{n-1} d_\theta(T(a_1), T(a_0)), \quad \forall n > 1. \end{aligned} \tag{79}$$

Thus, we reach

$$d_\theta(T(a_n), T(a_{n-1})) \le p^n d_\theta(a_0, a_1), \quad \forall n \in \mathbb{N}. \tag{80}$$

By assumption on the parameters α, β, and γ, one has $p = ((\beta+\gamma)/(1-\alpha)) < 1$.

Following the same steps as given in Theorem 14, one can show that $\{\mathrm{a}_n\}$ is a Cauchy sequence. By the completeness of X, there is $v \in X$ such that $T^n(a_0) \longrightarrow v$. To prove $T(v) = v$, replace $a = T^n(a_0)$ and $b = v$ in (74). We have

$$\begin{aligned} d_\theta\left(T^{n+1}(a_0), T(v)\right) &\le \alpha d_\theta\left(T^n(a_0), T^{n+1}(a_0)\right) + \beta d_\theta(v, T(v)) \\ &\quad + \gamma d_\theta(T^n(a_0), \mathrm{v}) + Ld_\theta(T^n(a_0), T(v)). \end{aligned} \tag{81}$$

Then,

$$\begin{aligned} d_\theta(a_{n+1}, T(v)) &\le \alpha d_\theta\left(T^n(a_0), T^{n+1}(a_0)\right) + \beta d_\theta(v, T(v)) \\ &\quad + \gamma d_\theta(a_n, v) + Ld_\theta(Tv, a_{n+1}). \end{aligned} \tag{82}$$

That is,

$$\begin{aligned} \lim_{n\longrightarrow\infty} d_\theta(a_{n+1}, T(v)) &\le \lim_{n\longrightarrow\infty} \left(\alpha d_\theta\left(T^n(a_0), T^{n+1}(a_0)\right) + \beta d_\theta(v, T(v))\right. \\ &\quad \left. + \gamma d_\theta(a_n, v) + Ld_\theta(T(v), a_{n+1})\right). \end{aligned} \tag{83}$$

We have $d_\theta(v, T(v)) \le (\beta + L)d_\theta(v, T(v))$, which holds unless $T(v) = v$.

The uniqueness is as follows:

Let τ and v be two fixed points, such that $\tau \neq v$. Then, using inequality (74), we get

$$d_\theta(T(\tau), T(v)) \le [\alpha d_\theta(\tau, T(\tau)) + \beta d_\theta(v, T(v)) + \gamma d_\theta(\tau, v)] + L d_\theta(T(v), \tau) d_\theta(\tau, v) \le [(L+\gamma) d_\theta(\tau, v)] 1 \le L + \gamma, \tag{84}$$

which is a contradiction. Hence, T has only one fixed point. □

Now, we use the concept of an asymptotically regular mapping [31, 32] in new extended b-metric spaces.

Definition 17. Let (X, d_θ) be a new extended b-metric space. A mapping $T : X \longrightarrow X$ satisfying the condition

$$\lim_{n\longrightarrow\infty} d_\theta\left(T^{n+1}a, T^n a\right) = 0, \quad \text{for all } a \in X, \tag{85}$$

is called asymptotically regular.

Example 18. Let $X = \{0\} \cup [1, 3]$. Define $T : X \longrightarrow X$ by $T0 = 1$ and $Tx = 0$, for $0 < x \le 3$. Consider $d_\theta(x, y) = (x - y)^2$ and $\theta(x, y, z) = (x + y + z + 11)/(x + y + z + 1)$. We claim that T satisfies condition (7). Indeed,

Case 1. If $x = y = 0$, then, (7) gives $0 \le 2\xi + L$, which is true for all $\xi \in [0, 1/2)$ and $L \in (0, 1)$.

Case 2. If $x \in [1, 3], y = 0$, then, (7) gives $1 \le \xi(1 + x^2)$, which is true for all $\xi \in [0, 1/2)$ and $L \in (0, 1)$.

Case 3. If $x, y \in [1, 3]$, then, (7) implies that $0 \le \xi(x^2 + y^2) + Ly^2$, which is true for all $\xi \in [0, 1/2)$ and $L \in (0, 1)$. Notice that T is fixed point free. The iterative sequence $\{x_n = T^n 0\}$ is not convergent, so T is not asymptotically regular.

Theorem 19. *Let (X, d_θ) be a complete new extended b-metric space such that d_θ is a continuous functional. Let $T : X \longrightarrow X$ be an asymptotically regular self mapping such that there is $\xi < 1$ so that*

$$d_\theta(T(a), T(b)) \le \xi[d_\theta(a, T(a)) + d_\theta(b, T(b))], \quad \forall a, b \in X. \tag{86}$$

Then, T has only one fixed point $v \in X$.

Proof. Let $a \in X$ and take $a_n = T^n(a)$ be defined inductively. Let $m, n \in \mathbb{N}$ such that $m > n$; then, according to asymptotic regularity,

$$d_\theta\left(T^{n+1}(a), T^{m+1}(a)\right) \le \xi\left[d_\theta\left(T^n(a), T^{n+1}(a)\right) + d_\theta\left(T^m(a), T^{m+1}(a)\right)\right] \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{87}$$

Thus, the sequence $\{T^n(a)\}$ is a Cauchy sequence. By the completeness of X, there is $v \in X$ such that

$$\lim_{n\longrightarrow\infty} T^n(a) = v. \tag{88}$$

To prove that v is a fixed point of T, we proceed as follows:

$$d_\theta(T(a_n), T(v)) \le \xi(d_\theta(a_n, T(a_n)) + d_\theta(v, T(v))). \tag{89}$$

That is,

$$d_\theta(T(a_n), T(v)) \le \xi(d_\theta(a_n, a_{n+1}) + d_\theta(v, T(v))). \tag{90}$$

Taking limit $n \longrightarrow \infty$ and using the asymptotic regularity of T, one obtains

$$d_\theta(v, T(v)) \le \xi d_\theta(v, T(v)), \tag{91}$$

which implies that $T(v) = v$.

To prove uniqueness, let τ be another fixed point of T. We have

$$d_\theta(v, \tau) = d_\theta(T(v), T(\tau)) \le \xi(d_\theta(v, T(v)) + d_\theta(\tau, T(\tau))). \tag{92}$$

This is true unless $d_\theta(v, \tau) = 0$, and so, $v = \tau$. Hence, v is the only fixed point of T. Further, for each $a \in X$, the iterative sequence $\{T^n(a)\}$ converges to v. □

Remark 20. It is noteworthy that if the mapping is asymptotically regular, then, the condition on $\theta(a_n, a_{n+1}, a_m)$ can be relaxed.

Theorem 21. *Let (X, d_θ) be a complete new extended b-metric space such that d_θ is a continuous functional. Let $T : X \longrightarrow X$ be an asymptotically regular mapping such that there is $0 < S < 1$ so that*

$$d_\theta(T(a), T(b)) \le S[d_\theta(a, T(a)) + d_\theta(b, T(b)) + d_\theta(a, b)], \quad \forall a, b \in X. \tag{93}$$

Then, T has only one fixed point $v \in X$ provided that

$$\lim_{n\longrightarrow\infty} \frac{S + S\theta(a_n, a_{n+1}, a_m)\theta(a_{n+1}, a_{m+1}, a_m)}{1 - S\theta(a_n, a_{n+1}, a_m)\theta(a_{n+1}, a_{m+1}, a_m)} \tag{94}$$

exists for $a_n = T^n(a), m > n$ and a is arbitrary in X.

Proof. Let $a \in X$ and take $a_n = T^n(a)$ defined inductively. Let $m, n \in \mathbb{N}$ such that $m > n$; then, using (93), we have

$$\begin{aligned}
d_\theta\left(T^{n+1}(a), T^{m+1}(a)\right) &\le S\left[d_\theta\left(T^n(a), T^{n+1}(a)\right) + d_\theta\left(T^m(a), T^{m+1}(a)\right) + d_\theta(T^n(a), T^m(a))\right] \\
&= S\left[d_\theta\left(T^n(a), T^{n+1}(a)\right) + d_\theta\left(T^m(a), T^{m+1}(a)\right)\right] + Sd_\theta(T^n(a), T^m(a)) \\
&\le S\left[d_\theta\left(T^n(a), T^{n+1}(a)\right) + d_\theta\left(T^m(a), T^{m+1}(a)\right)\right] + S\theta(a_n, a_{n+1}, a_m) \\
&\quad \cdot \left[d_\theta\left(T^n(a), T^{n+1}(a)\right) + d_\theta\left(T^{n+1}(a), T^m(a)\right)\right] = S\left[d_\theta\left(T^n(a), T^{n+1}(a)\right) + d_\theta\left(T^m(a), T^{m+1}(a)\right)\right] \\
&\quad + S\theta(a_n, a_{n+1}, a_m) d_\theta\left(T^n(a), T^{n+1}(a)\right) + S\theta(a_n, a_{n+1}, a_m) d_\theta\left(T^{n+1}(a), T^m(a)\right) \\
&\le S\left[d_\theta\left(T^n(a), T^{n+1}(a)\right) + d_\theta\left(T^m(a), T^{m+1}(a)\right)\right] + S\theta(a_n, a_{n+1}, a_m) d_\theta\left(T^n(a), T^{n+1}(a)\right) \\
&\quad + S\theta(a_n, a_{n+1}, a_m)\theta(a_{n+1}, a_{m+1}, a_m)\left[d_\theta\left(T^{n+1}(a), T^{m+1}(a)\right) + d_\theta\left(T^{m+1}(a), T^m(a)\right)\right] d_\theta \\
&\quad \cdot \left(T^{n+1}(a), T^{m+1}(a)\right) \le \left(\frac{S + S\theta(a_n, a_{n+1}, a_m)}{1 - S\theta(a_n, a_{n+1}, a_m)\theta(a_{n+1}, a_{m+1}, a_m))}\right) d_\theta\left(T^n(a), T^{n+1}(a)\right) \\
&\quad + \left(\frac{S + S\theta(a_n, a_{n+1}, a_m)\theta(a_{n+1}, a_{m+1}, a_m))}{1 - S\theta(a_n, a_{n+1}, a_m)\theta(a_{n+1}, a_{m+1}, a_m))}\right) d_\theta\left(T^{m+1}(a), T^m(a)\right) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{95}$$

Thus, the sequence $\{T^n(a)\}$ is a Cauchy sequence. By the completeness of X, there is $v \in X$ such that

$$\lim_{n \longrightarrow \infty} T^n(a) = v. \tag{96}$$

Now, by using triangular inequality and (93), we get

$$d_\theta(T(a_n), T(v)) \le S[d_\theta(a_n, T(a_n)) + d_\theta(v, T(v)) + d_\theta(a_n, v)], \tag{97}$$

so

$$d_\theta(a_{n+1}, T(v)) \le S[d_\theta(a_n, a_{n+1}) + d_\theta(v, T(v)) + d_\theta(a_n, v)]. \tag{98}$$

At the limit,

$$\lim_{n \longrightarrow \infty} d_\theta(a_{n+1}, T(v)) \le \lim_{n \longrightarrow \infty} S[d_\theta(a_n, a_{n+1}) + d_\theta(v, T(v)) + d_\theta(a_n, v)]. \tag{99}$$

Thus, $d_\theta(v, T(v)) \le Sd_\theta(v, T(v))$, which is possible only if $T(v) = v$.

The uniqueness is as follows:

Suppose that there is $\tau(\neq v) \in M$ so that $T(\tau) = \tau$, then

$$d_\theta(T(\tau), T(v)) \le S[d_\theta(\tau, T(\tau)) + d_\theta(v, T(v)) + d_\theta(\tau, v)]1 \le S, \tag{100}$$

which a contradiction. Hence, T has only one fixed point. Thus, for each $a \in X$, $\{T^n(a)\}$ converges to v. □

3. Application

Let $X = C[a, b]$ be the set of all real-valued continuous functions on $[a, b]$, and let $d_\theta : X \times X \longrightarrow [0,\infty)$ be defined as

$$\begin{aligned}
d_\theta(x, y) &= \sup_{t \in [a,b]} |x(t) - y(t)|^2, \quad \text{with } \theta(x, y, z) \\
&= \sup_{t \in [a,b]} \frac{|x(t)| + |y(t)| + |z(t)|}{|x(t)| + |y(t)| + |z(t)| + 1} + 2.
\end{aligned} \tag{101}$$

One can easily verify that X is a complete new extended b-metric space. Consider the Fredholm integral equation

$$x(t) = \int_a^b K(t, \tau, x(\tau))d\tau + f(t), \quad \text{for all } t \in [a, b], \tag{102}$$

where $f : [a, b] \longrightarrow \mathbb{R}$ and $K : [a, b] \times [a, b] \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous.

Theorem 22. *Let $X = C[a, b]$ and let the operator $I : X \longrightarrow X$ be defined by*

$$I(x(t)) = \int_a^b K(t, \tau, x(\tau))d\tau + f(t), \quad \textit{for all } t \in [a, b]. \tag{103}$$

Assume that the following condition holds for each $x, y \in X$

$$\begin{aligned}
&|K(t, \tau, x(\tau)) - K(t, \tau, y(\tau))|^2 \\
&\quad \le \frac{\xi}{2(b - a)} |x(\tau) - I(x(\tau)) + y(\tau) - I(y(\tau))|^2,
\end{aligned} \tag{104}$$

for all $t, \tau \in [a, b]$, where $\xi \in [0, 1/2)$. Then, the integral equation (102) has a solution, provided that for every iterative sequence $\{x_n = I^n x_0\}$, for each $x_0 \in X$, we have

$$\sup_{m\geq 1}\lim_{n\longrightarrow\infty}\theta(x_n, x_{n+1}, x_m) < \frac{1-\xi}{\xi}. \tag{105}$$

Proof. It is required to prove that the operator I satisfies the conditions of Theorem 9. For this, we will use the following inequality for $\beta > 1$:

$$\left(\frac{a+b}{2}\right)^{\beta} \leq \frac{a^{\beta}+b^{\beta}}{2}. \tag{106}$$

For $x, y \in X$, consider

$$\begin{aligned}
\sup_{t\in[a,b]}|I(x(t)) - I(y(t))|^2 &= \left|\int_a^b K(t,\tau,x(\tau))d\tau + f(t) - \int_a^b K(t,\tau,y(\tau))d\tau - f(t)\right|^2 \\
&= \left|\int_a^b (K(t,\tau,x(\tau)) - K(t,\tau,y(\tau)))d\tau\right|^2 \\
&\leq \int_a^b |K(t,\tau,x(\tau)) - K(t,\tau,y(\tau))|^2 d\tau \\
&\leq \int_a^b \frac{\xi}{2(b-a)}|x(\tau) - I(x(\tau)) + y(\tau)) - I(y(\tau))|^2 d\tau \\
&\leq \frac{\xi}{2(b-a)}\int_a^b 2\left[|x(\tau) - I(x(\tau))|^2 + |y(\tau) - I(y(\tau))|^2\right]d\tau \\
&\leq \xi \sup_{t\in[a,b]}\left[|x(t) - I(x(t))|^2 + \sup_{t\in[a,b]}|y(t) - I(y(t))|^2\right].
\end{aligned} \tag{107}$$

That is,

$$d_\theta(Ix, Iy) \leq \xi[d_\theta(x, Tx) + d_\theta(y, Ty)], \quad \forall x, y \in X. \tag{108}$$

This implies that (7) holds for $L = 0$. Hence, by Theorem 9, the operator I has a fixed point, provided that for every iterative sequence $x_n = I^n x_0$, for each $x_0 \in X$, we have $\sup_{m\geq 1}\lim_{n\longrightarrow\infty}\theta(x_n, x_{n+1}, x_m) < (1-\xi)/\xi$, that is, the Fredholm integral equation (102) has a solution. □

4. Conclusion

(i) The idea of new extended b-metric spaces was elaborated with examples

(ii) Some results involving Kannan-type contractions on new extended b-metric spaces are provided

(iii) Results presented by Gornicki [28] are generalized and modified

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper.

Acknowledgments

M. Aslam extends his appreciation to the deanship of scientific research at King Khalid University, Abha 61413, Saudi Arabia, for funding this work through the research group program under grant number R.G. P-1/23/42.

References

[1] I. A. Bakhtin, "The contraction mapping principle in quasi metric spaces," *Funct. Anal. Unianowsk Gos. Ped. Inst.*, vol. 30, pp. 26–37, 1989.

[2] M. Jleli and B. Samet, "A generalized metric space and related fixed point theorems," *Fixed Point Theory and Applications*, vol. 2015, no. 1, Article ID 61, 2015.

[3] E. Karapinar, "A short survey on the recent fixed point results on b-metric spaces," *Constructive Mathematical Analysis*, vol. 1, no. 1, pp. 15–44, 2018.

[4] T. Kamran, M. Samreen, and Q. UL Ain, "A generalization of b-metric space and some fixed point theorems," *Mathematics*, vol. 5, no. 2, p. 19, 2017.

[5] L. B. Ćirić, "A generalization of Banach's contraction principle," *Proceedings of the American Mathematical Society*, vol. 45, no. 2, pp. 267–273, 1974.

[6] O. Popescu, "Some new fixed point theorems for α-Geraghty contraction type maps in metric spaces," *Fixed Point Theory and Applications*, vol. 2014, no. 1, Article ID 190, 2014.

[7] E. Rakotch, "A note on contractive mappings," *Proceedings of the American Mathematical Society*, vol. 13, no. 3, pp. 459–465, 1962.

[8] S. Banach, "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales," *Fundamenta Mathematicae*, vol. 3, pp. 133–181, 1922.

[9] R. Kannan, "Some results on fixed points," *Bulletin of the Calcutta Mathematical Society*, vol. 60, pp. 71–76, 1968.

[10] H. Aydi, E. Karapinar, and W. Shatanawi, "Coupled fixed point results for (ψ, φ)-weakly contractive condition in ordered partial metric spaces," *Computers and Mathematics with Applications*, vol. 62, no. 12, pp. 4449–4460, 2011.

[11] V. Parvaneh, M. R. Haddadi, and H. Aydi, "On best proximity point results for some type of mappings," *Journal of Function Spaces*, vol. 2020, Article ID 6298138, 6 pages, 2020.

[12] V. Gupta, E. Ozgur, S. Rajani, and M. De la Sen, "Various fixed point results in complete G_b-metric spaces," *Dynamic System and Applicatios*, vol. 30, no. 2, pp. 277–293, 2021.

[13] Y. C. Kwun, A. A. Shahid, W. Nazeer, S. I. Butt, M. Abbas, and S. M. Kang, "Tricorns and multicorns in noor orbit with s-convexity," *IEEE Access*, vol. 7, pp. 95297–95304, 2019.

[14] L. Luo, R. Ullah, G. Rahmat, S. I. Butt, and M. Numan, "Approximating common fixed points of an evolution family on a metric space via Mann iteration," *Journal of Mathematics*, vol. 2021, Article ID 6764280, 7 pages, 2021.

[15] S. Czerwik, "Contraction mappings in b-metric spaces," *Acta Mathematica et Informatica Universitatis Ostraviensis*, vol. 1, pp. 5–11, 1993.

[16] T. Abdeljawad, R. P. Agarwal, E. Karapınar, and P. S. Kumari, "Solutions of the nonlinear integral equation and fractional

differential equation using the technique of a fixed point with a numerical experiment in extended b-metric space," *Symmetry*, vol. 11, no. 5, p. 686, 2019.

[17] T. Abdeljawad, E. Karapınar, S. K. Panda, and N. Mlaiki, "Solutions of boundary value problems on extended-Branciari b-distance," *Journal of Inequalities and Applications*, vol. 2020, no. 1, Article ID 103, 2020.

[18] M. U. Ali, H. Aydi, and M. Alansari, "New generalizations of set valued interpolative Hardy-Rogers type contractions in b-metric spaces," *Journal of Function Spaces*, vol. 2021, Article ID 6641342, 8 pages, 2021.

[19] M. Akkouchi, "A common fixed point theorem for expansive mappings under strict implicit conditions on b-metric spaces," *Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica*, vol. 50, pp. 5–15, 2011.

[20] C. Chifu and E. Karapinar, "On contractions via simulation functions on extended b-metric spaces," *Miskolc Mathematical Notes*, vol. 21, no. 1, pp. 127–141, 2020.

[21] Z. Kadelburg and S. Radenović, "Notes on some recent papers concerning F-contractions in b-metric spaces," *Constructive Mathematical Analysis*, vol. 1, no. 2, pp. 108–112, 2018.

[22] N. Mlaiki, N. Souayah, T. Abdeljawad, and H. Aydi, "A new extension to the controlled metric type spaces endowed with a graph," *Advances in Difference Equations*, vol. 2021, no. 1, Article ID 94, 2021.

[23] M. Anwar, D. Shehwar, and R. Ali, "Fixed point theorems on (α, F)-contractive mapping in extended b-metric spaces," *Journal of Mathematical Analysis*, vol. 11, pp. 43–51, 2020.

[24] H. A. Hammad, H. Aydi, and N. Mlaiki, "Contributions of the fixed point technique to solve the 2D Volterra integral equations, Riemann–Liouville fractional integrals, and Atangana–Baleanu integral operators," *Advances in Difference Equations*, vol. 2021, no. 1, 2021.

[25] M. Samreen, T. Kamran, and M. Postolache, "Extended b-metric space, extended b-comparison function and nonlinear contractions," *UPB Scientific Bulletin, Series A*, vol. 80, no. 4, 2018.

[26] A. Azam and M. Arshad, "Kannan fixed point theorem on generalized metric spaces," *Journal of Nonlinear Sciences and Applications*, vol. 1, no. 1, pp. 45–48, 2008.

[27] H. Aydi, A. Felhi, T. Kamran, E. Karapınar, and M. U. Ali, "On nonlinear contractions in new extended b-metric spaces," *Applications & Applied Mathematics*, vol. 14, no. 1, 2019.

[28] J. Gornicki, "Fixed point theorems for Kannan type mappings," *Journal of Fixed Point Theory and Applications*, vol. 19, no. 3, pp. 2145–2152, 2017.

[29] T. Abdeljawad, N. Mlaiki, H. Aydi, and N. Souayah, "Double controlled metric type spaces and some fixed point results," *Mathematics*, vol. 6, no. 12, p. 320, 2018.

[30] N. Mlaiki, H. Aydi, N. Souayah, and T. Abdeljawad, "Controlled metric type spaces and the related contraction principle," *Mathematics*, vol. 6, no. 10, p. 194, 2018.

[31] F. E. Browder and W. V. Petryshyn, "The solution by iteration of nonlinear functional equations in Banach spaces," *Bulletin of the American Mathematical Society*, vol. 72, no. 3, pp. 571–576, 1966.

[32] R. E. Bruck and S. Reich, "Nonexpansive projections and resolvents of accretive operators in Banach spaces," *Houston Journal of Mathematics*, vol. 3, pp. 459–470, 1977.

[33] A. Branciari, "A fixed point theorem of Banach-Caccippoli type on a class of generalized metric spaces," *Universitatis Debreceniensis*, vol. 57, pp. 31–37, 2000.

[34] S. Kakutani, "A generalization of Tychonoff's fixed point theorem," *Duke Math*, vol. 8, pp. 457–459, 1968.

[35] R. Kannan, "Some results on fixed points-II," *The American Mathematical Monthly*, vol. 76, no. 4, pp. 405–408, 1969.