Research Article
Kannan-Type Contractions on New Extended $b$-Metric Spaces

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Abstract
This article is focused on the generalization of some fixed point theorems with Kannan-type contractions in the setting of new extended $b$-metric spaces. An idea of asymptotic regularity has been incorporated to achieve the new results. As an application, the existence of a solution of the Fredholm-type integral equation is presented.

1. Introduction and Preliminaries
The existence of fixed points for some operators has a noteworthy contribution in many branches of applied and pure mathematics. The theory of a fixed point provides very valuable and effective tools in mathematics. It has a wide range of implications in nonlinear analysis and has been established in two directions. One is to change the space under consideration (see the works of Bakhtin [1], Jleli and Samet [2], Karapinar [3], Kamran et al. [4], etc.), and the other is to change the contraction conditions (see the works of Cirić [5], Popescu [6], Rakotch [7], etc.).

In 1922, the Polish mathematician Banach [8] established a remarkable result relevant to a metric fixed point theory, that is known as the Banach contraction principle (BCP). The work of Banach is well regarded and an adaptable consequence in the theory of fixed points. BCP laid a foundation of research in this field, which is further investigated by many researchers from 1922 till now. One of the prominent generalizations of the BCP was presented by Kannan [9]. Additional works on the existence of (common) fixed points can be seen in [10–14].

In 1989, Bakhtin [1] introduced the notion of $b$-metric spaces. Later on, the concept of $b$-metric spaces was further used by Czerwik [15] to establish different fixed point results in $b$-metric spaces. The study of $b$-metric spaces endowed an imperative place in the fixed point theory with multiple aspects. Many mathematicians (Abdeljawad et al. [16, 17], Ali et al. [18], Akkouchi [19], Chifu and Karapinar [20], Kadelburg and Radenović [21], Parvaneh et al. [11], Gupta et al. [12], Mlaiki et al. [22], etc.) led the foundation to improve the fixed point theory in $b$-metric spaces. Another innovative task has been achieved by Kamran et al. [4] in 2017 by introducing the notion of an extended $b$-metric space, which generalizes the notion of a $b$-metric space. Some fixed point results are proved in this new setting; see for instance the work presented in [23–25].

The work of Kannan [9] refined the concept of the Banach contraction mapping by introducing a new contraction, now known as Kannan contraction. The Kannan fixed point result has been further extended and generalized in the setup of $b$-metric spaces [15] and for generalized metric spaces [26].

In 2019, the notion of a new extended $b$-metric space has been initiated by Aydi et al. [27], where the control...
function depends on three variables. This fact is new since all precedent control functions depend on two variables. The objective of this work is at investigating Kannan-type contractions in the context of new extended b-metric spaces by extending the main results of Gornicki [28]. For this purpose, some basic concepts are needed in the sequel.

**Definition 1** (see [15]). Let $X$ be a nonempty set and $s \geq 1$ be a real number. A function $d_b : X \times X \rightarrow \mathbb{R}$ is called a $b$-metric, if it satisfies the following for all $j, \kappa, \ell \in X$:

1. $d_b(j, \kappa) \geq 0$
2. $d_b(j, \kappa) = 0$, if and only if $j = \kappa$
3. $d_b(j, \kappa) = d_b(\kappa, j)$
4. $d_b(j, \kappa) \leq s[d_b(j, \ell) + d_b(\ell, \kappa)]$

The pair $(X, d_b)$ is called a $b$-metric space. If $s = 1$, then, a $b$-metric space becomes a metric space. In 2017, Kamran et al. [4] generalized the $b$-metric space setting to an extended $b$-metric space (in the same direction, see also [29, 30]).

**Definition 2** (see [4]). Let $X$ be a nonempty set and $\theta : X \times X \rightarrow [1, \infty)$ be a function. The map $d_\theta : X \times X \rightarrow \mathbb{R}$ is called an extended $b$-metric if for all $j, \kappa, \ell \in X$, it satisfies the following axioms:

1. $d_\theta(j, \kappa) \geq 0$
2. $d_\theta(j, \kappa) = 0$, if and only if $j = \kappa$
3. $d_\theta(j, \kappa) = d_\theta(\kappa, j)$
4. $d_\theta(j, \kappa) \leq \theta(j, \ell)[d_\theta(j, \ell) + d_\theta(\ell, \kappa)]$

In 2019, Aydi et al. [27] introduced the notion of new extended $b$-metric spaces. Here, the control function depends on 3 variables.

**Definition 3** (see [27]). Let $X$ be a nonempty set and $\theta : X \times X \times X \rightarrow [1, \infty)$ be a function. The map $d_\theta : X \times X \rightarrow \mathbb{R}$ is called a new extended $b$-metric if for all $j, \kappa, \ell \in X$, it satisfies the following axioms:

1. $d_\theta(j, \kappa) \geq 0$
2. $d_\theta(j, \kappa) = 0$, if and only if $j = \kappa$
3. $d_\theta(j, \kappa) = d_\theta(\kappa, j)$
4. $d_\theta(j, \kappa) \leq \theta(j, \ell)[d_\theta(j, \ell) + d_\theta(\ell, \kappa)]$

The pair $(X, d_\theta)$ is named to be a new extended $b$-metric space. If $\theta(j, \kappa, \ell) = s$ (for $s \geq 1$), we get Definition 1.

**Example 4** (see [27]). Let $X = \mathbb{N}$. Define $d_\theta : X \times X \rightarrow \mathbb{R}$ by

\[
d_\theta(j, \kappa) = \begin{cases} 
0, & \text{if } j = \kappa, \\
1, & \text{if } j \text{ is even and } \kappa \text{ is odd}, \\
\frac{1}{j}, & \text{if } j \text{ is odd and } \kappa \text{ is even}, \\
1, & \text{otherwise},
\end{cases}
\]

where

\[
\theta(j, \kappa, \ell) = \begin{cases} 
1, & \text{if } j = \ell \text{ and } \kappa \text{ is even or odd}, \\
\frac{1}{j}, & \text{if } j \neq \ell, j \text{ and } \ell \text{ are even and } \kappa \text{ odd}, \\
\frac{1}{2}, & \text{if } j \neq \ell, j \text{ and } \ell \text{ are odd and } \kappa \text{ even}, \\
\frac{1}{3}, & \text{if } j \neq \ell, j \ell \text{ and } \kappa \text{ are all even or all odd}, \\
\frac{j + \kappa(1 + j)}{j(1 + \kappa)}, & \text{if } j \neq \ell, j \text{ is even and } \ell \text{ is odd and } \kappa \text{ is even}, \\
\frac{\ell + \kappa(\ell + 1)}{\ell(\kappa + 1)}, & \text{if } j \neq \ell, j \text{ is odd and } \ell \text{ is even and } \kappa \text{ is even}, \\
2 + \ell, & \text{if } j \neq \ell, j \text{ is odd and } \ell \text{ and } \kappa \text{ are all even or all odd}, \\
\frac{j + 1}{j}, & \text{if } j \neq \ell, j \text{ is even and } \ell \text{ is odd and } \kappa \text{ is odd}.
\end{cases}
\]

Here, $(X, d_\theta)$ is a new extended $b$-metric space.

On the other hand, by taking $j = 2p + 1$, $\ell = 4p + 1$, and $\kappa = 2p$, we have

\[
d_\theta(j, \ell) = \frac{d_\theta(2p + 1, 4p + 1)}{d_\theta(2p + 1, 2p) + d_\theta(2p, 4p + 1)} = p.
\]

It is not possible to find $s \geq 1$ so that (b4) holds. Thus, $d_\theta$ is not a $b$-metric on $X$.

**Example 5.** Consider $X = \{1, 2, 3\}$. Take $d_\theta : X \times X \rightarrow [0, \infty)$ as

\[
d_\theta(j, \kappa) = (j - \kappa)^2,
\]

where $\theta : X \times X \times X \rightarrow [1, \infty)$ as $\theta(j, \kappa, \ell) = j + \kappa + 2\ell$. Then, $(X, d_\theta)$ is a new extended $b$-metric space.
Proof. The first three conditions are trivially verified. To check the triangular inequality, we proceed as follows:

\[ d_\theta(1, 2) \leq \theta(1, 3, 2)[d_\theta(1, 3) + d_\theta(3, 2)] \leq (1 + 3 + 4)[4 + 1]. \] (5)

so \( 1 < 40 \). Similarly, we can check the other two pairs.

Therefore, for all \( j, \kappa, \ell \in X \), \( d_\theta(j, \ell) \leq \theta(j, \kappa, \ell)[d_\theta(j, \kappa) + d_\theta(\kappa, \ell)] \). \( \square \)

Definition 6. Let \( (X, d_\theta) \) be a new extended \( b \)-metric space.

(i) A sequence \( \{a_n\} \subset X \) is called convergent to \( a \in X \) if for \( \varepsilon > 0 \), there is \( N(\varepsilon) \in \mathbb{N} \) such that \( \lim_{n \to \infty} d_\theta(a_n, a) < \varepsilon \) for all \( n \geq N(\varepsilon) \)

(ii) A sequence \( \{a_n\} \subset X \) is called Cauchy if for \( \varepsilon > 0 \), there is \( N(\varepsilon) \in \mathbb{N} \) such that \( \lim_{n \to \infty} d_\theta(a_m, a_n) < \varepsilon \) for all \( n, m \geq N(\varepsilon) \)

(iii) The space is called complete if every Cauchy sequence in \( X \) is convergent in \( X \).

Definition 7. Let \( (X, d_\theta) \) be a new extended \( b \)-metric space. Denote by \( \mathcal{B}(a, r) = \{ b \in X : d_\theta(b, a) < r \} \) and \( \mathcal{B}[a, r] = \{ b \in X : d_\theta(b, a) \leq r \} \) the open and closed balls in \( X \), respectively.

(i) A subset \( U \) of \( X \) is called open if for any \( u \in U \), there exists an \( \varepsilon > 0 \) such that \( \mathcal{B}(u, \varepsilon) \subset U \)

(ii) A subset \( V \) of \( X \) is called closed if for any \( \{v_n\} \subset V \) such that \( \lim_{n \to \infty} v_n = v \), then \( v \in X \)

In this paper, we are going to prove some Kannan-type fixed point theorems in the setting of new extended \( b \)-metric spaces. Some examples are also provided to make effective the obtained results.

2. Main Results

We define a Kannan-type fixed point contraction on new extended \( b \)-metric spaces.

Definition 8. Let \( (X, d_\theta) \) be a new extended \( b \)-metric space. A mapping \( T : X \to X \) is a Kannan-type contraction if there are \( K \in (0, 1/2) \) and \( 0 \leq L < 1 \) such that

\[ d_\theta(T(a), T(b)) \leq K[d_\theta(a, T(a)) + d_\theta(b, T(b))] + Ld_\theta(a, T(a)), \quad \text{for all } a, b \in X. \] (6)

Our first main result is as follows:

Theorem 9. Consider a complete new extended \( b \)-metric space \( (X, d_\theta) \) such that \( d_\theta \) is a continuous functional. Let \( T : X \to X \) be a mapping such that there are \( \xi \in (0, 1/2) \) and \( L \in (0, 1) \) so that

\[ d_\theta(T(a), T(b)) \leq \xi[d_\theta(a, T(a)) + d_\theta(b, T(b))] + Ld_\theta(b, T(a)), \quad \forall a, b \in X. \] (7)

Assume that

\[ \sup_{n \geq 1} \xi(a_n, a_{n+1}, a_m) < \frac{L}{1 - \xi}, \] (8)

for each \( a_0 \in X \), such that \( a_n = T^n(a_0), n = 1, 2, \ldots \).

Then, \( T \) has only one fixed point \( v \in X \). Further, for any \( a \in X \), the iterative sequence \( \{T^n(a)\} \) converges to \( v \), and

\[ d_\theta(T(a_n), v) \leq \frac{\xi}{1 - L} \left( \frac{\xi}{1 - L} \right)^n d_\theta(T(a_0), a_0), \quad n = 0, 1, 2, \ldots. \] (9)

Proof. For an arbitrary point \( a_0 \in X \), construct the iterative sequence

\[ a_{n+1} = T(a_n) = T^{n+1}(a_0). \] (10)

If for some \( n \), \( a_n = a_{n+1} \), so \( a_n \) is a fixed point of \( T \). Otherwise, assume that \( a_n \neq a_{n+1} \) for all \( n \geq 0 \). Since

\[ d_\theta(a_n, a_{n+1}) = d_\theta(T(a_{n-1}), T(a_n)), \] (11)

one writes

\[ d_\theta(T(a_{n-1}), T(a_n)) \leq \xi[d_\theta(a_{n-1}, T(a_{n-1})) + d_\theta(a_n, T(a_n))] + Ld_\theta(a_n, T(a_{n-1})). \] (12)

Then,

\[ d_\theta(a_n, a_{n+1}) \leq \xi[d_\theta(a_{n-1}, a_n) + d_\theta(a_n, a_{n+1})] + Ld_\theta(a_n, a_n). \] (13)

That is,

\[ d_\theta(a_n, a_{n+1}) \leq \left( \frac{\xi}{1 - \xi} \right) d_\theta(a_{n-1}, a_n). \] (14)

Continuing in this way, we have

\[ d_\theta(a_n, a_{n+1}) \leq \left( \frac{\xi}{1 - \xi} \right)^n d_\theta(a_0, a_1), \] (15)

\[ d_\theta(T(a_{n-1}), T(a_n)) \leq \left( \frac{\xi}{1 - \xi} \right)^n d_\theta(a_0, T(a_0)). \] (16)

Let \( m, n \in \mathbb{N} \) be such that \( m > n \). Applying triangular inequality, we get
\[
\begin{align*}
d_\phi(a_n, a_m) &\leq \theta(a_n, a_{n+1}, a_m) [d_\phi(a_n, a_{n+1}) + d_\phi(a_{n+1}, a_m)] \\
&= \theta(a_n, a_{n+1}, a_m) [d_\phi(a_n, a_{n+1}) + \theta(a_{n+1}, a_m)] \\
&\leq \theta(a_n, a_{n+1}, a_m) [d_\phi(a_n, a_{n+1}) + \theta(a_{n+1}, a_m)] \\
&\quad \cdot (a_{n+1}, a_{n+2}, a_m) [d_\phi(a_{n+1}, a_{n+2}) + \theta(a_{n+2}, a_m)] \\
&\leq \theta(a_n, a_{n+1}, a_m) [d_\phi(a_n, a_{n+1}) + \theta(a_{n+1}, a_m)] \\
&\quad \cdot (a_{n+1}, a_{n+2}, a_m) [d_\phi(a_{n+1}, a_{n+2}) + \theta(a_{n+2}, a_m)] \\
&\quad \cdot (a_{n+2}, a_{n+3}, a_m) [\theta(a_{n+3}, a_{n+1}, a_m)] \\
&\quad \cdot (a_{n+3}, a_{n+4}, a_m) [\theta(a_{n+4}, a_{n+2}, a_m)] \\
&\quad \cdots \\
&\quad \cdot (a_m, a_{m-1}, a_m) [\theta(a_m, a_{m-1}, a_m)]. \\
\end{align*}
\]

That is, \[\lim_{n \to \infty} d_\phi(a_n, a_m) = 0.\] (23)

Hence, the sequence \(\{a_n\}\) is a Cauchy sequence. By the completeness of \(X\), there is \(v \in X\) such that \(\lim_{n \to \infty} a_n = v\).

We claim that \(v\) is a fixed point of \(T\). We have

\[
d_\phi(T(a_n), T(v)) \leq \xi d_\phi(a_n, T(a_n)) + d_\phi(v, T(v)) + Ld_\phi(v, T(a_n)).
\]

That is,

\[
d_\phi(a_{n+1}, T(v)) \leq \xi d_\phi(a_n, a_{n+1}) + d_\phi(v, T(v)) + Ld_\phi(v, a_{n+1}).
\]

As \(n \to \infty\), we have in view of the assumption that \(d_\phi\) is continuous,

\[
d_\phi(v, T(v)) \leq \xi d_\phi(v, T(v)),
\]

which holds unless \(d_\phi(v, T(v)) = 0\), and so, \(T(v) = v\).

The uniqueness is as follows:

Let \(\tau\) be another fixed point of \(T\). We have

\[
0 \leq d_\phi(v, \tau) = d_\phi(T(v), T(\tau)) \leq \xi [d_\phi(v, T(v)) + d_\phi(\tau, T(\tau))] + Ld_\phi(\tau, v).
\]

That is,

\[
d_\phi(v, \tau) \leq Ld_\phi(v, \tau).
\]

It is only possible if \(d_\phi(v, \tau) = 0\). Thus, \(v \in X\) is the unique fixed point of \(T\). Further, we have

\[
d_\phi(T(a_{n-1}), T(a_n)) \leq \xi [d_\phi(T(a_{n-2}), T(a_{n-1})) + d_\phi(T(a_{n-1}), T(a_n))] + Ld_\phi(a_{n-1}, a_n).
\]

Then,

\[
d_\phi(T(a_n), T(a_n)) \leq \left(\frac{\xi}{1 - \xi}\right) d_\phi(T(a_{n-2}), T(a_{n-1})).
\]

Also,

\[
d_\phi(T(a_n), v) \leq \xi [d_\phi(T(a_{n-1}), T(a_n)) + d_\phi(v, T(v))] + Ld_\phi(v, T(a_n)) \leq \xi d_\phi(T(a_{n-1}), T(a_n)) + Ld_\phi(v, T(a_n)).
\]
Using (15),

\[ d_\theta(T(a_n), v) \leq \xi \left( \frac{\xi}{1 - \xi} \right)^n d_\theta(T(a_0), a_0) + Ld_\theta(v, T(a_n)). \]

(32)

That is,

\[ d_\theta(T(a_n), v) \leq \frac{\xi}{1 - L} \left( \frac{\xi}{1 - \xi} \right)^n d_\theta(T(a_0), a_0), \quad n = 0, 1, 2, \ldots. \]

(33)

The following examples illustrate Theorem 9. We deal with noncompact sets.

**Example 10.** Let \( X = l_\infty \) be the space of all bounded sequences of real numbers, that is,

\[ l_\infty = \{ \eta = \{ \eta_n \} : |\eta_n| \leq C_n, \forall n \in \mathbb{N} \}, \]

where \( C_n \in \mathbb{R} \) may depend on the sequence \( \eta \) but does not depend on \( n \). Take that

\[ d_\theta(\eta, \zeta) = \sup_{n \in \mathbb{N}} |\eta_n - \zeta_n|^2, \]

\[ \eta = \{ \eta_n \}, \]

\[ \zeta = \{ \zeta_n \}, \]

are in \( X \).

Then, \( X \) is a complete new extended \( b \)-metric space with \( \theta : X \times X \times X \rightarrow [1, \infty) \) being defined by

\[ \theta(\eta, \zeta, \varphi) = \sup_{n \in \mathbb{N}} \frac{|\eta_n - \zeta_n + \varphi_n|^2}{|\eta_n + \zeta_n + \varphi_n|^2} + 3. \]

(36)

Consider \( T : X \rightarrow X \) given as \( T(\eta) = \{(\eta_n - 1)/5\}, \forall n = 1, 2, 3, \ldots \). For each \( \eta, \zeta \in X \), we have

\[ d_\theta(T\eta, T\zeta) = \sup_{n \in \mathbb{N}} \frac{|\eta_n - 1 - \zeta_n - 1|^2}{\left| |\eta_n + \zeta_n + 1\right|^2} \]

\[ \leq \sup_{n \in \mathbb{N}} \frac{2|\eta_n - 1|^2}{5} + \frac{2|\zeta_n - 1|^2}{5} \]

\[ + \frac{1}{8} \left( \sup_{n \in \mathbb{N}} \frac{4|\eta_n|^2}{5} + \frac{4|\zeta_n|^2}{5} \right) \]

\[ \leq \frac{1}{8} \left[ d_\theta(\eta, T\eta) + d_\theta(\zeta, T\zeta) \right]. \]

(37)

Thus, (7) holds with \( \xi = 1/8 \) and \( L \in [0, 1) \). Also, \( 1 - \xi \)

\( \left\langle \xi, 7 \right\rangle = 5 \) and \( \theta(\eta, \zeta, \varphi) < 7 \) for all \( \eta, \zeta, \varphi \in X \). Hence, (8) holds, and by Theorem 9, \( T \) has a fixed point.

**Example 11.** Let \( X = C[a, b] \) be the set of all real-valued continuous functions defined on \([a, b]\). Define

\[ d_\theta(x, y) = \sup_{t \in [a, b]} |x(t) - y(t)|^2, \quad \forall x, y \in C[a, b], \]

and \( \theta : X \times X \times X \rightarrow [1, \infty) \) as

\[ \theta(x, y, z) = \sup_{t \in [a, b]} \frac{|x(t)| + |y(t)| + |z(t)|}{|x(t)| + |y(t)| + |z(t)| + 1} + 2. \]

Then, \( X \) is a complete new extended \( b \)-metric space, consider a mapping \( T : X \rightarrow X \) given as

\[ T(\eta(t)) = \frac{\eta(t) - 1}{7}, \quad \forall \eta \in C[a, b]. \]

(40)

For all \( \eta, \zeta \in X \), we have

\[ d_\theta(T\eta, T\zeta) = \sup_{t \in [a, b]} \frac{|\eta(t) - 1 - \zeta(t) - 1|^2}{7} + \sup_{t \in [a, b]} \frac{|\eta(t) - 1|^2}{7} \]

\[ \leq \sup_{t \in [a, b]} \frac{2|\eta(t)|^2}{7} + \sup_{t \in [a, b]} \frac{|\zeta(t) - 1|^2}{7} \]

\[ + \frac{1}{18} \sup_{t \in [a, b]} \frac{|\eta(t)|^2}{7} \]

\[ + \frac{1}{18} \sup_{t \in [a, b]} \frac{|\zeta(t)|^2}{7} \]

\[ \leq \frac{1}{18} \sup_{t \in [a, b]} \frac{|\eta(t) + 1|^2}{5} + \frac{1}{18} \sup_{t \in [a, b]} \frac{|\zeta(t) + 1|^2}{5} \]

\[ < \frac{1}{8} [d_\theta(\eta, T\eta) + d_\theta(\zeta, T\zeta)]. \]

(41)

Thus, (7) holds with \( \xi = 1/6 \) and \( L \in [0, 1) \). Also, \( 1 - \xi \)

\( \left\langle \xi, 5 \right\rangle = 5 \) and \( \theta(\eta, \zeta, \varphi) < 5 \) for all \( \eta, \zeta, \varphi \in X \). That is, (8) holds. Since all the conditions of Theorem 9 are satisfied, \( T \) has a fixed point.

**Example 12.** Choose \( X = \{1/4, 1/16, \ldots, 1/4^n, \ldots\} \cup \{0, 1\} \). Define \( \theta : X \times X \times X \rightarrow [1, \infty) \) and \( d_\theta : X \times X \rightarrow [0, \infty) \) by

\[ \theta(x, y, z) = x + y + z + 2, \quad d_\theta(x, y) = (x - y)^2. \]

(42)

Let \( T : X \rightarrow X \) be given as

\[ Tu = \begin{cases} \frac{1}{4^n}, & \text{if } u = \frac{1}{4^n}, n = 0, 1, 2, 3, \ldots, \\ u, & \text{if } u = 0. \end{cases} \]

(43)

Then, for all \( x, y \in X \) with neither \( x = 0 \) nor \( y = 0 \), we have
\[
d_0(Tx, Ty) = \left( 1 + \frac{1}{4^{m+1}} \right) \leq \left( 1 + \frac{1}{4^{m+1}} \right)^{2} + 2 \left( \frac{1}{4^{m+1}} \right)
= \frac{2}{9} \left( \frac{3}{4^{m+1}} \right)^{2} \leq 2 \left( \frac{1}{9} \frac{3}{4^{m+1}} \right)^{2} = 2 \left( \frac{1}{9} \frac{3}{4^{m+1}} \right)^{2} = \frac{1}{9} \frac{3}{4^{m+1}}
\]

If \( x = 0 \) and \( y \neq 0 \), then
\[
d_0(Tx, Ty) = 0 = \left| - \frac{1}{4^{m+1}} \right| \leq \left( \frac{1}{4^{m+1}} \right)^{2} \left( \frac{3}{4^{m+1}} \right)^{2} \leq 2 \left( \frac{1}{9} \frac{3}{4^{m+1}} \right)^{2} = \frac{1}{9} \frac{3}{4^{m+1}}
\]

Thus, (7) is satisfied for \( \xi = 2/9 \) and for each \( L \in [0, 1] \).

Also, \( 1 - \xi = \xi/7 \). If \( a_0 = 0 \), then, for the iterative sequence \( a_n = T^n a_0 = 0 \) for each \( n \in N \), we have \( \lim_{n \to \infty} \theta(a_n, a_{n+1}, a_m) = \lim_{n \to \infty} \theta(a_n, a_{n+1}, a_m) + a_m + 2 < 7/2 \). If \( a_0 \neq 0 \) (say \( a_0 = 1/4 \) for some \( k \in \{0, 1, 2, \ldots\} \)); then, for the iterative sequence \( a_n = T^n a_0 = 1/4^{k+n} \) for each \( n \in N \), we have \( \lim_{n \to \infty} \theta(a_n, a_{n+1}, a_m) = \lim_{n \to \infty} \theta(a_n, a_{n+1}, a_m) + a_m + 2 < 7/2 \). Hence, Theorem 9 ensures the existence of a fixed point of \( T \).

Remark 13. In the following, we ensure the completeness of the spaces given in precedent examples.

(a) Completeness of \( \ell_{\infty} \)

Let \( X = \ell_{\infty} \) and let \( \{x_m\} = \{\xi_1^{(m)}, \ldots, \xi_{n}^{(m)} \} \) be a Cauchy sequence in \( X \). Define a metric \( d_0 \) on \( X \) as \( d_0(x, y) = \sup_{j} \left| \xi_j - \eta_j \right| \) for each \( x = (\xi_j) \) and \( y = (\eta_j) \). Since \( \{x_m\} \) is a Cauchy sequence, for \( \varepsilon > 0 \), there is \( N \in N \) such that for all \( m, n \geq N \),
\[
d_0(x_m, x_n) = \sup_{j} \left| \xi_j^{(m)} - \xi_j^{(n)} \right| < \varepsilon \forall m, n \geq N
\]

That is,
\[
\left| \xi_j^{(m)} - \xi_j^{(n)} \right| < \varepsilon, \quad \forall j \text{ and } m, n \geq N.
\]

where \( \varepsilon = \sqrt{\varepsilon} \) is arbitrary. Hence, for every fixed \( j \), \( \{\xi_j^{(1)}, \xi_j^{(2)}, \ldots\} \) is a Cauchy sequence of complex numbers and it converges, so \( \xi_j^{(m)} \to \xi_j \) as \( m \to \infty \). Construct a sequence \( x = (\xi_1, \xi_2, \ldots) \) by using these infinitely many limits to show that \( x \in \ell_{\infty} \) and \( x_m \to x \). From (48) with \( n \to \infty \), we have
\[
\left| \xi_j^{(m)} - \xi_j \right| < \varepsilon, \quad m \geq N.
\]

Since \( x_m \in \ell_{\infty} \), there is a real number \( c_m \) so that \( |\xi_j^{(m)}| \leq c_m, \forall j \). Hence,
\[
\left| \xi_j - \xi_j^{(m)} \right| \leq \xi_j - \xi_j^{(m)} \leq \left( \xi_j - \xi_j^{(m)} \right) \leq \left( \xi_j - \xi_j^{(m)} \right) \leq \left( \xi_j - \xi_j^{(m)} \right) \leq \left( \xi_j - \xi_j^{(m)} \right)^2
\]

That is,
\[
\left| \xi_j \right| \leq \sqrt{2} \xi_j + \xi_j^{(m)} \leq \left( \xi_j - \xi_j^{(m)} \right)^2.
\]

So, (51) holds for each \( j \). It implies that \( \{\xi_j\} \) is a bounded sequence of complex numbers. This leads to \( \{\xi_j\} \in \ell_{\infty} \). Also, from (48), we have
\[
d_0(x_m, x) = \sup_{j} \left| \xi_j^{(m)} - \xi_j \right|^2 \leq \varepsilon, \quad m > N.
\]

This implies that \( x_m \to x \in \ell_{\infty} \). \( \ell_{\infty} \) is endowed with the new extended metric \( d_0 \).

(b) Completeness of \( C[a, b] \)

Let \( X \) be the function space \( C[a, b] \), where \( [a, b] \) is any closed interval in \( \mathbb{R} \). Define
\[
d_0(x, y) = \sup_{t \in [a, b]} |x(t) - y(t)|^2, \quad \forall x, y \in C[a, b].
\]

Let \( \{x_m\} \) be a Cauchy sequence in \( C[a, b] \). Then, for each \( \varepsilon > 0 \), there exists an \( N \in \mathbb{N} \) such that
\[
d_0(x_m, x_n) = \sup_{t \in [a, b]} |x_m(t) - x_n(t)|^2 < \varepsilon, \quad \forall m, n \geq N.
\]

Hence, for each fixed \( t_0 \in [a, b] \), we have
\[
|x_m(t_0) - x_n(t_0)|^2 < \varepsilon, \quad \forall m, n \geq N.
\]

That is,
\[
|x_m(t_0) - x_n(t_0)| \leq \varepsilon, \quad \forall m, n \geq N.
\]

This shows that \( \{x_1(t_0), x_2(t_0), \ldots\} \) is a Cauchy sequence of real numbers; hence, it converges. That is, \( x_m(t_0) \to x(t_0) \) as \( m \to \infty \). In this way, we can associate to each \( t \in [a, b] \) a unique real number \( x(t) \). This defines a function \( x \) (pointwise) in \([a, b] \). Further, we need to show that \( x_m \to x \in C[a, b] \). From (53), we have as \( n \to \infty \)
\[
sup_{t \in [a, b]} |x_m(t) - x(t)|^2 < \varepsilon, \quad m \geq N.
\]

Hence, for each \( t \in [a, b] \) and \( m > N \), we have
\[
|x_m(t) - x(t)|^2 < \varepsilon.
\]
Theorem 14. Let \((X, d_0)\) be a complete new extended \(b\)-metric space where \(d_0\) is a continuous functional and \(M\) is a nonempty closed subset of \(X\). Let \(T : M \rightarrow M\) satisfy
\[
d_0(T(a), T(b)) \leq \xi [d_0(a, T(a)) + d_0(b, T(b))] + Ld_0(b, T(a)), \quad \forall a, b \in M \text{ and } L, \xi \in [0, 1).
\]
(58)
Assume that there exist \(\alpha, \beta \in \mathbb{R}\) with \(0 < \alpha < 1\) and \(\beta > 0\) such that for an arbitrary \(a \in M\), there is \(\mu^* \in M\) verifying
\[
d_0(\mu^*, T(\mu^*)) \leq \alpha d_0(a, T(a)), \quad d_0(\mu^*, a) \leq \beta d_0(a, T(a)).
\]
(59)
Also, for an arbitrary \(a_0 \in M\), assume that the sequence \(\{a_n = T^n(a_0)\}\) verifies
\[
\sup_{m \geq 1} \lim_{n \to \infty} \theta(a_n, a_{n+1}, a_m) < \frac{1}{\alpha}.
\]
(60)
Then, \(T\) has only one fixed point.

Proof. Let \(a_0\) be an arbitrary element of \(M\). Consider the sequence \(\{a_n = T^n(a_0)\}\) in \(M\). We have
\[
d_0(T(a_{n+1}), a_{n+1}) \leq \alpha d_0(T(a_n), a_n), \quad d_0(T(a_n), a_n) \leq \beta d_0(T(a_{n-1}), a_{n-1}), \quad n = 0, 1, 2, \ldots.
\]
(61)
Since
\[
d_0(a_{n+1}, a_n) = d_0(T(a_n), a_n) \leq \beta d_0(T(a_{n-1}), a_{n-1}), \quad n = 0, 1, 2, \ldots,
\]
\[
\beta d_0(T(a_{n-1}), a_{n-2}) \leq \cdots \leq \beta^2 d_0(T(a_0), a_0),
\]
(62)
we have
\[
d_0(a_{n+1}, a_n) \leq \beta^2 d_0(T(a_0), a_0).
\]
(63)
Let \(m, n \in \mathbb{N}\) such that \(m > n\). Applying triangular inequality, we get
\[
d_0(a_m, a_n) \leq \theta(a_m, a_{m+1}, a_{m+1}) d_0(a_{m+1}, a_n) + \sum_{k=n}^{m-1} \theta(a_k, a_{k+1}, a_{m+1}) d_0(a_{k+1}, a_n)
\]
\[
= \theta(a_m, a_{m+1}, a_n) [d_0(a_{m+1}, a_n) + \sum_{k=n}^{m-2} \theta(a_k, a_{k+1}, a_{m+1}) d_0(a_{k+1}, a_n)]
\]
\[
\leq \theta(a_m, a_{m+1}, a_n) [d_0(a_{m+1}, a_n) + \sum_{k=n}^{m-2} \theta(a_k, a_{k+1}, a_{m+1}) d_0(a_{k+1}, a_n)]
\]
\[
\leq \theta(a_m, a_{m+1}, a_n) [d_0(a_{m+1}, a_n) + \sum_{k=n}^{m-2} \theta(a_k, a_{k+1}, a_{m+1}) d_0(a_{k+1}, a_n)]
\]
\[
\leq \theta(a_m, a_{m+1}, a_n) [d_0(a_{m+1}, a_n) + \sum_{k=n}^{m-2} \theta(a_k, a_{k+1}, a_{m+1}) d_0(a_{k+1}, a_n)]
\]
(64)
Using (63), we get
\[
d_0(a_m, a_n) \leq \theta(a_m, a_{m+1}, a_n) d_0(T(a_0), a_n) + \theta(a_{m+1}, a_{m+2}, a_{m+1}) \theta(a_{m+2}, a_{m+3}, a_{m+2}) \cdots \theta(a_{m-1}, a_m) \beta d_0(T(a_0), a_n)
\]
\[
\leq \theta(a_m, a_{m+1}, a_n) \theta(a_{m+1}, a_{m+2}, a_{m+1}) \cdots \theta(a_{m-1}, a_m) \theta(a_m, a_{m+1}, a_{m+1}) \beta d_0(T(a_0), a_n)
\]
\[
\leq \theta(a_m, a_{m+1}, a_n) \theta(a_{m+1}, a_{m+2}, a_{m+1}) \cdots \theta(a_{m-1}, a_m) \theta(a_m, a_{m+1}, a_{m+1}) \beta d_0(T(a_0), a_n).
\]
(65)
Since \(\sup_{n \geq 1} \lim_{n \to \infty} \theta(a_n, a_{n+1}, a_m) \alpha < 1\), the series \(\sum_{n=1}^{\infty} \alpha^n \prod_{i=1}^{n} \theta(a_i, a_{i+1}, a_m)\) is convergent for each \(m \in \mathbb{N}\) by ratio test.

Let
\[
S_m = \sum_{n=1}^{\infty} \alpha^n \prod_{i=1}^{n} \theta(a_i, a_{i+1}, a_m),
\]
(66)
\[
S_n = \sum_{j=1}^{n} \alpha^j \prod_{i=1}^{j} \theta(a_i, a_{i+1}, a_m).
\]
(67)

So, for \(m > n\), the above inequality implies that
\[
d_0(a_m, a_n) \leq d_0(a_m, a_1) (S_{n-1} - S_{m-1}).
\]
(68)

Letting \(n \to \infty\), the sequence \(\{a_n\}\) is a Cauchy sequence. By the completeness of \(M\), there is \(v \in M\) such that \(\lim_{n \to \infty} a_n = v\).

We will prove that \(v\) is a fixed point of \(T\). By using (58), we get
\[
d_0(T(a_n), T(v)) \leq \xi [d_0(a_n, T(a_n)) + d_0(v, T(v)) + Ld_0(v, T(a_n))]
\]
\[
\Rightarrow d_0(a_{n+1}, T(v)) \leq \xi [d_0(a_n, a_{n+1}) + d_0(v, T(v))]
\]
\[
+ Ld_0(v, T(v)).
\]
(69)

As \(n \to \infty\), we have
\[
d_0(v, T(v)) \leq \xi d_0(v, T(v)).
\]

Hence, \(T(v) = v\).

The uniqueness is as follows:

Assume on contrary that there is \(\tau(\neq v) \in M\) so that \(T(\tau) = \tau\); then,
\[
0 < d_0(v, \tau) = d_0(T(v), T(\tau)) \leq \xi [d_0(v, T(v)) + d_0(\tau, T(\tau))] + Ld_0(\tau, T(v)) = Ld_0(v, \tau).
\]
(70)

That is, \(d_0(v, \tau) < Ld_0(v, \tau)\), which is a contradiction. Thus, \(v \in X\) is the unique fixed point of \(T\).
Remark 15. To prove Theorem 9 in new extended $b$-metric spaces, by using the following conditions:

$$
d_b(μ^*, T(μ^*)) ≤ α d_b(a, T(a)),
$$

$$
d_b(μ^*, a) ≤ β d_b(a, T(a)),
$$

we proceed as follows:

For any $a ∈ X$, take $μ^* = T(a)$. Then,

$$
d_b(μ^*, T(μ^*)) = d_b(T(a), T(μ^*)) ≤ [d_b(a, T(a)) + d_b(μ^*, T(μ^*))]
+ L d_b(μ^*, T(a)) (1 − ξ)d_b(μ^*, T(μ^*))
≤ ξ d_b(a, T(a)) + d_b(μ^*, T(μ^*)) ≤ \left( \frac{ξ}{1 − ξ} \right) d_b(a, T(a)),
$$

where by assumption $(ξ/1 − ξ) < 1$ and $d_b(μ^*, a) = d_b(T(a), a)$. Now, for arbitrary $a_0 ∈ X$, we can inductively define a sequence $\{a_{n+1} = T(a_n)\}$. By Theorem 14, this sequence is convergent. So, $\lim_{n→∞} a_n = v$. Thus, $T(v) = v$.

Also, for each $a ∈ X$,

$$
d_b(T(a_{n+1}), T(a_n)) ≤ ξ d_b(T(a_{n+2}), T(a_{n+1})) + d_b(T(a_{n+1}), T(a_n))
+ L d_b(μ^*, T(a_{n+1})) = d_b(T(a_{n+1}), T(a_n))
≤ \left( \frac{ξ}{1 − ξ} \right) d_b(T(a_{n+2}), T(a_{n+1})),
$$

$$
d_b(T(a_n), v) ≤ ξ d_b(T(a_{n+1}), T(a_n)) + d_b(v, T(v)) + L d_b(μ^*, T(a_n))
≤ ξ d_b(T(a_{n+1}), T(a_n)) + L d_b(μ^*, T(a_n)) = d_b(T(a_n), v)
≤ \frac{ξ}{1 − L} \left( \frac{ξ}{1 − ξ} \right)^n d_b(T(a), a), \quad n = 0, 1, 2, \ldots.
$$

Theorem 16. Let $(X, d_b)$ be a complete new extended $b$-metric space such that $d_b$ is a continuous functional. Let $T : X → X$ satisfy

$$
d_b(T(a), T(b)) ≤ α d_b(a, T(a)) + β d_b(b, T(b)) + γ d_b(a, b)
+ L d_b(a, T(b)), \quad ∀a, b ∈ X,
$$

where $α, β, γ, L$ are nonnegative real numbers such that $α + β + γ + L < 1$ and $β + γ > 0$. Assume that for an arbitrary $a_0 ∈ M$, we have

$$
\sup_{m≥2} \lim_{n→∞} θ(a_n, a_{n+1}, a_m) < \frac{1}{p},
$$

where $p = ((β + γ)/(1 − α))$ and $a_n = T^m(a_0)$.

Then, $T$ has only one fixed point.

Proof. For an arbitrary $a_0 ∈ X$, take the sequence $\{T^m(a_0)\}$. Substituting $a = T^{m−1}(a_0) = T(a_{n−1}) = a_n$ and $b = T^{m−2}(a_0) = T(a_{n−2}) = a_{n−1}$ in (74), we obtain

$$
d_b(T(a_n), T(a_{n+1})) ≤ α d_b(T(a_{n+1}), T(a_n)) + β d_b(T(a_{n−2}), T(a_{n−1}))
+ γ d_b(T(a_{n−1}), T(a_n)) + L d_b(T(a_{n−1}), T(a_{n−2})).
$$

(76)

That is,

$$
(1 − α)d_b(T(a_n), T(a_{n+1})) ≤ (β + γ)d_b(T(a_{n−1}), T(a_n)).
$$

(77)

Thus,

$$
d_b(T(a_n), T(a_{n+1})) ≤ \left( \frac{β + γ}{1 − α} \right) d_b(T(a_{n−1}), T(a_n)).
$$

(78)

Moreover,

$$
d_b(T(a_n), T(a_{n+1})) ≤ p d_b(T(a_{n−1}), T(a_n)) \leq p^n d_b(T(a_1), T(a_0)), \quad ∀n > 1.
$$

(79)

Thus, we reach

$$
d_b(T(a_n), T(a_{n+1})) ≤ p^n d_b(a_0, a_1), \quad ∀n ∈ N.
$$

(80)

By assumption on the parameters $α, β, γ$, and $L$, one has

$$
p = ((β + γ)/(1 − α)) < 1.
$$

Following the same steps as given in Theorem 14, one can show that $\{a_n\}$ is a Cauchy sequence. By the completeness of $X$, there is $v ∈ X$ such that $T^m(a_0) → v$. To prove $T(v) = v$, replace $a = T^m(a_0)$ and $b = v$ in (74). We have

$$
d_b(T^{m+1}(a_0), T(v)) ≤ α d_b(T(v), T(a_0)) + β d_b(v, T(v))
+ γ d_b(a_0, v) + L d_b(T(a_0), T(v)).
$$

(81)

Then,

$$
d_b(a_{n+1}, T(v)) ≤ α d_b(T(v), T(a_0)) + β d_b(v, T(v))
+ γ d_b(a_n, v) + L d_b(T(v), a_{n+1}).
$$

(82)

That is,

$$
\lim_{n→∞} d_b(a_{n+1}, T(v)) ≤ \lim_{n→∞} (α d_b(T(v), T(a_0)) + β d_b(v, T(v))
+ γ d_b(a_n, v) + L d_b(T(v), a_{n+1})).
$$

(83)
We have $d_{\theta}(v, T(v)) \leq (\beta + L)d_{\theta}(v, T(v))$, which holds unless $T(v) = v$.

The uniqueness is as follows:

Let $\tau$ and $v$ be two fixed points, such that $\tau \neq v$. Then, using inequality (74), we get

$$d_{\theta}(T(\tau), T(v)) \leq [d_{\theta}(\tau, T(\tau)) + \beta d_{\theta}(v, T(v)) + \gamma d_{\theta}(\tau, v)] + L d_{\theta}(T(v), \tau) d_{\theta}(\tau, v) \leq [(L + \gamma) d_{\theta}(\tau, v)] 1 \leq L + \gamma.$$  

which is a contradiction. Hence, $T$ has only one fixed point. \qed

Now, we use the concept of an asymptotically regular mapping [31, 32] in new extended $b$-metric spaces.

**Definition 17.** Let $(X, d_{\theta})$ be a new extended $b$-metric space. A mapping $T : X \to X$ satisfying the condition

$$\lim_{n \to \infty} d_{\theta}(T^{n+1} a, T^n a) = 0, \quad \text{for all } a \in X,$$

is called asymptotically regular.

**Example 18.** Let $X = \{0\} \cup [1, 3]$. Define $T : X \to X$ by $T 0 = 1$ and $T x = 0$, for $0 < x \leq 3$. Consider $d_{\theta}(x, y) = (x - y)^2$ and $\theta(x, y, z) = (x + y + z + 1)/(x + y + z + 1)$. We claim that $T$ satisfies condition (7). Indeed,

**Case 1.** If $x = y = 0$, then, (7) gives $0 \leq 2 \xi + L$, which is true for all $\xi \in [0, 1/2]$ and $L \in (0, 1)$.

**Case 2.** If $x \in [1, 3]$, $y = 0$, then, (7) gives $1 \leq \xi(1 + x^2)$, which is true for all $\xi \in [0, 1/2]$ and $L \in (0, 1)$.

**Case 3.** If $x, y \in [1, 3]$, then, (7) implies that $0 \leq \xi(x^2 + y^2) + L y^2$, which is true for all $\xi \in [0, 1/2]$ and $L \in (0, 1)$. Notice that $T$ is fixed point free. The iterative sequence $\{x_n\}$ is not convergent, so $T$ is not asymptotically regular.

**Theorem 19.** Let $(X, d_{\theta})$ be a complete new extended $b$-metric space such that $d_{\theta}$ is a continuous functional. Let $T : X \to X$ be an asymptotically regular self mapping such that there is $\xi < 1$ so that

$$d_{\theta}(T(a), T(b)) \leq \xi [d_{\theta}(a, T(a)) + d_{\theta}(b, T(b))], \quad \forall a, b \in X.$$  

Then, $T$ has only one fixed point $v \in X$.

**Proof.** Let $a \in X$ and take $a_n = T^n(a)$ be defined inductively. Let $m, n \in \mathbb{N}$ such that $m > n$; then, according to asymptotic regularity,

$$d_{\theta}(T^{n+1}(a), T^{n+1}(a)) \leq \xi [d_{\theta}(T^n(a), T^{n+1}(a)) + d_{\theta}(T^n(a), T^{n+1}(a))]$$

$$\to 0, \quad \text{as } n \to \infty.$$  

Thus, the sequence $\{T^n(a)\}$ is a Cauchy sequence. By the completeness of $X$, there is $v \in X$ such that

$$\lim_{n \to \infty} T^n(a) = v.$$  

To prove that $v$ is a fixed point of $T$, we proceed as follows:

$$d_{\theta}(T(a_n), T(v)) \leq \xi (d_{\theta}(a_n, T(a_n)) + d_{\theta}(v, T(v))).$$

That is,

$$d_{\theta}(T(a_n), T(v)) \leq \xi (d_{\theta}(a_n, a_{n+1}) + d_{\theta}(v, T(v))).$$

Now, we use the concept of an asymptotically regular mapping. We have

$$\lim_{n \to \infty} d_{\theta}(T^{n+1} a, T^n a) = 0, \quad \text{for all } a \in X,$$  

which implies that $T(v) = v$.

To prove uniqueness, let $\tau$ be another fixed point of $T$. We have

$$d_{\theta}(v, \tau) = d_{\theta}(T(v), T(\tau)) \leq \xi (d_{\theta}(v, T(v)) + d_{\theta}(\tau, T(\tau))).$$

This is true unless $d_{\theta}(v, \tau) = 0$, and so, $v = \tau$. Hence, $v$ is the only fixed point of $T$. Further, for each $a \in X$, the iterative sequence $\{T^n(a)\}$ converges to $v$. \qed

**Remark 20.** It is noteworthy that if the mapping is asymptotically regular, then, the condition on $\theta(a_n, a_{n+1}, a_m)$ can be relaxed.

**Theorem 21.** Let $(X, d_{\theta})$ be a complete new extended $b$-metric space such that $d_{\theta}$ is a continuous functional. Let $T : X \to X$ be an asymptotically regular mapping such that there is $0 < S < 1$ so that

$$d_{\theta}(T(a), T(b)) \leq S [d_{\theta}(a, T(a)) + d_{\theta}(b, T(b)) + d_{\theta}(a, b)], \quad \forall a, b \in X.$$  

Then, $T$ has only one fixed point $v \in X$ provided that

$$\lim_{n \to \infty} S + S\theta(a_n, a_{n+1}, a_m)\theta(a_{n+1}, a_{m+1}, a_m)$$

$$- S \theta(a_n, a_{n+1}, a_m)\theta(a_{n+1}, a_{m+1}, a_m)$$

exists for $a_n = T^n(a), m > n$ and $a$ is arbitrary in $X$. 


Proof. Let \( a \in X \) and take \( a_n = T^n(a) \) defined inductively. Let \( m, n \in \mathbb{N} \) such that \( m > n \); then, using (93), we have

\[
d_0(T^{m+1}(a), T^{n+1}(a)) \leq S[d_0(T^n(a), T^{m+1}(a)) + d_0(T^m(a), T^{m+1}(a)) + d_0(T^n(a), T^m(a))] \\
= S[d_0(T^n(a), T^{m+1}(a)) + d_0(T^m(a), T^{m+1}(a))] + Sd_0(T^n(a), T^m(a)) \\
\leq S[d_0(T^n(a), T^{m+1}(a)) + d_0(T^m(a), T^{m+1}(a))] + S \theta(a_n, a_{n+1}, a_m) \\
\cdot \left[ d_0(T^n(a), T^{m+1}(a)) + d_0(T^m(a), T^{m+1}(a)) \right] + S \theta(a_n, a_{n+1}, a_m)d_0(T^n(a), T^m(a)) \\
+ S \theta(a_n, a_{n+1}, a_m)d_0(T^n(a), T^{m+1}(a)) \\
\leq S[d_0(T^n(a), T^{m+1}(a)) + d_0(T^m(a), T^{m+1}(a))] + S \theta(a_n, a_{n+1}, a_m)d_0(T^n(a), T^m(a)) \\
+ S \theta(a_n, a_{n+1}, a_m)d_0(T^n(a), T^{m+1}(a)) \\
\leq \left( \frac{S + S \theta(a_n, a_{n+1}, a_m)}{1 - S \theta(a_n, a_{n+1}, a_m)} \right) d_0(T^n(a), T^{m+1}(a)) \\
\leq \left( \frac{S + \theta(a_n, a_{n+1}, a_m)}{1 - \theta(a_n, a_{n+1}, a_m)} \theta(a_n, a_{n+1}, a_m) \right) d_0(T^{m+1}(a), T^m(a)) \rightarrow 0, \quad \text{as } n \rightarrow \infty.
\]

(95)

Thus, the sequence \( \{T^n(a)\} \) is a Cauchy sequence. By the completeness of \( X \), there is \( v \in X \) such that

\[
\lim_{n \to \infty} T^n(a) = v.
\]

(96)

Now, by using triangular inequality and (93), we get

\[
d_0(T(a_n), T(v)) \leq S[d_0(a_n, T(a_n)) + d_0(v, T(v)) + d_0(a_n, v)],
\]

(97)

so

\[
d_0(a_{n+1}, T(v)) \leq S[d_0(a_n, a_{n+1}) + d_0(v, T(v)) + d_0(a_n, v)].
\]

(98)

At the limit,

\[
\lim_{n \to \infty} a_{n+1}, T(v) \leq \lim_{n \to \infty} S[d_0(a_n, a_{n+1}) + d_0(v, T(v)) + d_0(a_n, v)].
\]

(99)

Thus, \( d_0(v, T(v)) \leq Sd_0(v, T(v)) \), which is possible only if \( T(v) = v \).

The uniqueness is as follows:

Suppose that there is \( \tau (\neq v) \in M \) so that \( T(\tau) = \tau \), then

\[
d_0(T(\tau), T(v)) \leq S[d_0(\tau, T(\tau)) + d_0(T(\tau), T(v)) + d_0(\tau, T(v))] \leq S,
\]

(100)

which a contradiction. Hence, \( T \) has only one fixed point. Thus, for each \( a \in X \), \( \{T^n(a)\} \) converges to \( v \). \( \square \)

3. Application

Let \( X = C[a, b] \) be the set of all real-valued continuous functions on \( [a, b] \), and let \( d_0 : X \times X \to [0, \infty) \) be defined as

\[
d_0(x, y) = \sup_{t \in [a, b]} |x(t) - y(t)|^2, \quad \text{with } \theta(x, y, z)
\]

\[
= \sup_{t \in [a, b]} \frac{|x(t)| + |y(t)| + |z(t)|}{1 + 2}.
\]

(101)

One can easily verify that \( X \) is a complete new extended \( b \)-metric space. Consider the Fredholm integral equation

\[
x(t) = \int_a^b K(t, \tau, x(\tau))d\tau + f(t), \quad \text{for all } t \in [a, b].
\]

(102)

where \( f : [a, b] \to \mathbb{R} \) and \( K : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R} \) are continuous.

Theorem 22. Let \( X = C[a, b] \) and let the operator \( I : X \to X \) be defined by

\[
I(x(t)) = \int_a^b K(t, \tau, x(\tau))d\tau + f(t), \quad \text{for all } t \in [a, b].
\]

(103)

Assume that the following condition holds for each \( x, y \in X \)

\[
|K(t, \tau, x(\tau)) - K(t, \tau, y(\tau))|^2 \leq \frac{\xi}{2(b - a)} |x(\tau) - I(x(\tau)) + y(\tau) - I(y(\tau))|^2,
\]

(104)

for all \( t, \tau \in [a, b] \), where \( \xi \in [0, 1/2] \). Then, the integral equation (102) has a solution, provided that for every iterative sequence \( \{x_n = I^n x_0\} \), for each \( x_0 \in X \), we have
\[ \sup_{m \geq 1} \lim_{n \to \infty} \theta(x_n, x_{n+1}, x_m) < \frac{1 \mp \xi}{\xi}. \quad (105) \]

**Proof.** It is required to prove that the operator \( I \) satisfies the conditions of Theorem 9. For this, we will use the following inequality for \( \beta > 1 \):

\[ \left( \frac{a + b}{2} \right)^\beta \leq \frac{a^\beta + b^\beta}{2}. \quad (106) \]

For \( x, y \in X \), consider

\[ \sup_{n \in [a,b]} |l(x(n)) - l(y(n))|^2 = \left| \int_a^b f(t) - f(t) \right|^2 \]

\[ \leq \left| \int_a^b (K(t, \tau, x(t)) - K(t, \tau, y(t))) \right|^2 dt \]

\[ \leq \left( \int_a^b \left| K(t, \tau, x(t)) - K(t, \tau, y(t)) \right|^2 dt \right)^{\frac{1}{2}} \left( \int_a^b \left| y(t) - x(t) \right|^2 dt \right)^{\frac{1}{2}} \]

\[ \leq \xi \left( \int_a^b \left| y(t) - x(t) \right|^2 dt \right)^{\frac{1}{2}} \quad (107) \]

That is,

\[ d_\theta(Ix, y) \leq \xi [d_\theta(x, Tx) + d_\theta(y, Ty)], \quad \forall x, y \in X. \quad (108) \]

This implies that (7) holds for \( L = 0 \). Hence, by Theorem 9, the operator \( I \) has a fixed point, provided that for every iterative sequence \( x_n = Ix_n \), for each \( x_n \in X \), we have \( \sup_{m \geq 1} \lim_{n \to \infty} \theta(x_n, x_{n+1}, x_m) < (1 - \xi)\xi \), that is, the Fredholm integral equation (102) has a solution. \( \Box \)

**4. Conclusion**

(i) The idea of new extended \( b \)-metric spaces was elaborated with examples

(ii) Some results involving Kannan-type contractions on new extended \( b \)-metric spaces are provided

(iii) Results presented by Gornicki [28] are generalized and modified

**Data Availability**

The data used to support the findings of this study are available from the corresponding author upon request.

**Conflicts of Interest**

The authors declare that they have no competing interests.

**Authors’ Contributions**

All authors contributed equally to the writing of this paper.

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**References**


