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Research Article

Generating Functions for Some Hypergeometric Functions of Four Variables via Laplace Integral Representations

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Generating functions plays an essential role in the investigation of several useful properties of the sequences which they generate. In this paper, we establish certain generating relations, involving some quadruple hypergeometric functions introduced by Bin-Saad and Younis. Some interesting special cases of our main results are also considered.

1. Introduction

The hypergeometric series is the most useful and important special function, and it has been studied to solve various problems in many areas of mathematics, physics, statistics, and engineering [1–5]. Hypergeometric series in several variables appear in numerous fields of applied mathematics, mathematical physics, and chemistry. Very recently, Bin-Saad and Younis [6] introduced thirty new hypergeometric functions of four variables $X_i^{(4)}(i=1,2,\cdots,30)$, eight of them are defined below

$$X_{11}^{(4)}(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_1,c_2,c_3;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+n+p}(a_2)_{n+q}(a_3)_{p+q}}{(c_1)_{m+n}(c_2)_p(c_3)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}; \tag{1}$$

$$X_{12}^{(4)}(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_2,c_1,c_1,c_3;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+n+p}(a_2)_{n+q}(a_3)_{p+q}}{(c_1)_{n+p}(c_2)_m(c_3)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}; \tag{2}$$

$$X_{16}^{(4)}(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_2;c_1,c_1,c_2,c_3;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+n+p+q}(a_2)_{n+q}(a_3)_{p}}{(c_1)_{m+n}(c_2)_p(c_3)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}; \tag{3}$$

$$X_{17}^{(4)}(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_2;c_2,c_1,c_1,c_3;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+n+p+q}(a_2)_{n+q}(a_3)_{p}}{(c_1)_{n+p}(c_2)_m(c_3)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}; \tag{4}$$

$$X_{21}^{(4)}(a_1,a_1,a_2,a_1,a_1,a_2,a_3,a_2;c_1,c_1,c_2,c_3;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+n+q}(a_2)_{n+p+q}(a_3)_{p}}{(c_1)_{m+n}(c_2)_p(c_3)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}; \tag{5}$$

$$X_{22}^{(4)}(a_1,a_1,a_2,a_1,a_1,a_2,a_3,a_2;c_2,c_1,c_1,c_3;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+n+q}(a_2)_{n+p+q}(a_3)_{p}}{(c_1)_{n+p}(c_2)_m(c_3)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}; \tag{6}$$

$$X_{27}^{(4)}(a_1,a_1,a_3,a_1,a_1,a_2,a_4,a_4;c_2,c_1,c_1,c_3;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+n+q}(a_2)_{n}(a_3)_{p}(a_4)_{p+q}}{(c_1)_{n+p}(c_2)_m(c_3)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}; \tag{7}$$

$$X_{28}^{(4)}(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_4;c_1,c_1,c_2,c_3;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+n+p+q}(a_2)_{n}(a_3)_{p}(a_4)_{q}}{(c_1)_{m+n}(c_2)_p(c_3)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}; \tag{8}$$

for

$$\left(|x| < \frac{1}{4}, |y| < 1, |z| < 1, |u| < 1\right). \quad (9)$$

Here, $(a)_n$ corresponds to the Pochhammer symbol given as

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma a} = a(a+1)(a+2)\cdots(a+n-1), n \in \mathbb{N}, \quad (10)$$

and $(a)_0 = 1$. Given the following integral representations:

$$\begin{aligned} &X_{11}^{(4)}(a_1, a_1, a_1, a_2, a_1, a_2, a_3, a_3; c_1, c_1, c_2, c_3; x, y, z, u) \\ &\quad = \frac{1}{\Gamma(a_1)}\frac{1}{\Gamma(a_2)}\int_0^\infty\int_0^\infty e^{-(s+t)} s^{a_1-1} t^{a_2-1} \\ &\quad \times {}_0F_1(-;c_1; s^2x + sty)\Psi_2(a_3; c_2, c_3; sz, tu)\,ds\,dt, \\ &\quad \cdot(\Re(a_1) > 0, \Re(a_2) > 0); \end{aligned} \quad (11)$$

$$\begin{aligned} &X_{12}^{(4)}(a_1, a_1, a_1, a_2, a_1, a_2, a_3, a_3; c_2, c_1, c_1, c_3; x, y, z, u) \\ &\quad = \frac{1}{\Gamma(a_2)}\frac{1}{\Gamma(a_3)}\int_0^\infty\int_0^\infty e^{-(s+t)} s^{a_2-1} t^{a_3-1} \\ &\quad \times H_7(a_1; c_2, c_1; x, sy+tz)\,{}_0F_1(-;c_3; stu)\,ds\,dt, \\ &\quad \cdot(\Re(a_2) > 0, \Re(a_3) > 0); \end{aligned} \quad (12)$$

$$\begin{aligned} &X_{16}^{(4)}(a_1, a_1, a_1, a_1, a_1, a_2, a_3, a_2; c_1, c_1, c_2, c_3; x, y, z, u) \\ &\quad = \frac{1}{\Gamma(a_1)}\frac{1}{\Gamma(a_2)}\int_0^\infty\int_0^\infty e^{-(s+t)} s^{a_1-1} t^{a_2-1} \\ &\quad \times {}_0F_1(-;c_1; s^2x + sty)\,{}_1F_1(a_3; c_2; sz)\,{}_0F_1(-;c_3; stu)\,ds\,dt, \\ &\quad \cdot(\Re(a_1) > 0, \Re(a_2) > 0); \end{aligned} \quad (13)$$

$$\begin{aligned} &X_{17}^{(4)}(a_1, a_1, a_1, a_1, a_1, a_2, a_3, a_2; c_2, c_1, c_1, c_3; x, y, z, u) \\ &\quad = \frac{1}{\Gamma(a_1)}\frac{1}{\Gamma(a_2)}\int_0^\infty\int_0^\infty e^{-(s+t)} s^{a_1-1} t^{a_2-1} \\ &\quad \times \Phi_3(a_3; c_1; sz, sty)\,{}_0F_1(-;c_2; s^2x)\,{}_0F_1(-;c_3; stu)\,ds\,dt, \\ &\quad \cdot(\Re(a_1) > 0, \Re(a_2) > 0); \end{aligned} \quad (14)$$

$$\begin{aligned} &X_{21}^{(4)}(a_1, a_1, a_2, a_1, a_1, a_2, a_3, a_2; c_1, c_1, c_2, c_3; x, y, z, u) \\ &\quad = \frac{1}{\Gamma(a_1)}\frac{1}{\Gamma(a_2)}\int_0^\infty\int_0^\infty e^{-(s+t)} s^{a_1-1} t^{a_2-1} \\ &\quad \times {}_0F_1(-;c_1; s^2x + sty)\,{}_1F_1(a_3; c_2; tz)\,{}_0F_1(-;c_3; stu)\,ds\,dt, \\ &\quad \cdot(\Re(a_1) > 0, \Re(a_2) > 0); \end{aligned} \quad (15)$$

$$\begin{aligned} &X_{22}^{(4)}(a_1, a_1, a_2, a_1, a_1, a_2, a_3, a_2; c_2, c_1, c_1, c_3; x, y, z, u) \\ &\quad = \frac{1}{\Gamma(a_1)}\frac{1}{\Gamma(a_3)}\int_0^\infty\int_0^\infty e^{-(s+t)} s^{a_1-1} t^{a_3-1} \\ &\quad \times \Psi_2(a_2; c_1, c_3; sy + tz, su)\,{}_0F_1(-;c_2; s^2x)\,ds\,dt, \\ &\quad \cdot(\Re(a_1) > 0, \Re(a_3) > 0); \end{aligned} \quad (16)$$

$$\begin{aligned} &X_{27}^{(4)}(a_1, a_1, a_3, a_1, a_1, a_2, a_4, a_4; c_2, c_1, c_1, c_3; x, y, z, u) \\ &\quad = \frac{1}{\Gamma(a_1)}\frac{1}{\Gamma(a_4)}\int_0^\infty\int_0^\infty e^{-(s+t)} s^{a_1-1} t^{a_4-1} \\ &\quad \times \Phi_2(a_2, a_3; c_1; sy, tz)\,{}_0F_1(-;c_2; s^2x)\,{}_0F_1(-;c_3; stu)\,ds\,dt, \\ &\quad \cdot(\Re(a_1) > 0, \Re(a_4) > 0); \end{aligned} \quad (17)$$

$$\begin{aligned} &X_{28}^{(4)}(a_1, a_1, a_1, a_1, a_1, a_2, a_3, a_4; c_1, c_1, c_2, c_3; x, y, z, u) \\ &\quad = \frac{1}{\Gamma(a_1)}\int_0^\infty e^{-s} s^{a_1-1} \times \Phi_3(a_2; c_1; sy, s^2x)\,{}_1F_1 \\ &\quad \cdot(a_3; c_2; sz)\,{}_1F_1(a_4; c_3; su)\,ds, (\Re(a_1) > 0), \end{aligned} \quad (18)$$

where ${}_0F_1$, ${}_1F_1$ are Kummer's functions and $\Phi_2, \Phi_3, \Psi_2, H_7$ are Humbert functions defined, respectively, by (see [7])

$${}_0F_1(-;b;x) = \sum_{n=0}^{\infty} \frac{1}{(b)_n}\frac{x^n}{n!}; \quad (19)$$

$${}_1F_1(a;b;x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n}\frac{x^n}{n!}; \quad (20)$$

$$\Phi_2(a, b; c; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_m (b)_n}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}; \quad (21)$$

$$\Phi_3(b; c; x, y) = \sum_{m,n=0}^{\infty} \frac{(b)_m}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}; \quad (22)$$

$$\Psi_2(a; b, c; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}}{(b)_m (c)_n}\frac{x^m}{m!}\frac{y^n}{n!}; \quad (23)$$

$$H_7(a; b, c; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{2m+n}}{(b)_m (c)_n}\frac{x^m}{m!}\frac{y^n}{n!}. \quad (24)$$

Several families of generating functions have been established in diverse ways. These are playing important roles in the theory of special functions of applied mathematics and mathematical physics. One can refer to the extensive work of Srivastava and Manocha [8] for a systematic introduction and to several interesting and useful applications of the various methods of obtaining linear, bilinear, bilateral, or mixed multilateral generating functions for a fairly wide variety of sequences of hypergeometric functions and polynomials in one, two, or more variables, among much abundant literature. Many authors have been presented various generating functions in many different ways (see, for details, [9–11] and the references cited therein). In this paper, we aim at establishing some generating functions for the quadruple functions $X_i^{(4)}(i = 11,12,16,17,21,22,27,28)$.

2. Generating Relations

Here, by using the integral representations in (11)–(18), we give certain generating relations involving hypergeometric functions of three and four variables as follows:

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_{11}^{(4)}(a_1+k,a_1+k,a_1+k,a_2+k,a_1+k,a_2+k,c_2,c_2;c_1,c_1,c_2,c_2;x,y,z,u)=(1-z)^{-a_1}(1-u)^{-a_2}\cdot\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{w}{(1-z)(1-u)}\right)^k\times X_3\left(a_1+k,a_2+k;c_1,c_2;\frac{x}{(1-z)^2},\cdot\frac{y}{(1-z)(1-u)},\frac{zu}{(1-z)(1-u)}\right); \tag{25}$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_{12}^{(4)}(a_1,a_1,a_1,a_2+k,a_1,a_2+k,a_3+k,a_3+k;c_2,c_1,c_1,c_3;x,y,z,u\tau)=(1+u)^{-a_2}(1+\tau)^{-a_3}\cdot\sum_{k,m=0}^{\infty}\frac{(a_1)_{2m}}{(c_2)_m k!m!}\left(\frac{w}{(1+u)(1+\tau)}\right)^k x^m\times F_{28}^{(4)}\cdot(a_1+2m,a_1+2m,c_3,c_3,a_2+k,a_3+k,a_2+k,a_3+k;c_1,c_1,c_3,c_3;\lambda_1 y,\lambda_2 z,\lambda_1 u,\lambda_2\tau),\cdot\left(\lambda_1=\frac{1}{1+u},\lambda_2=\frac{1}{1+\tau}\right); \tag{26}$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_{16}^{(4)}(a_1+k,a_1+k,a_1+k,a_1+k,a_1+k,a_2+k,a_3,a_2+k;c_1,c_1,c_2,c_3;x,y,z,u)=(1-z)^{-a_1}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{w}{1-z}\right)^k\times X_{16}^{(4)}(a_1+k,a_1+k,a_1+k,a_1+k,a_1+k,a_2+k,c_2-a_3,a_2+k;c_1,c_1,c_2,c_3;\lambda^2x,\lambda y-\lambda z,\lambda u),\left(\lambda=\frac{1}{1-z}\right); \tag{27}$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_{17}^{(4)}(a_1+k,a_1+k,a_1+k,a_1+k,a_1+k,a_2+k,a_3,a_2+k;c_2,c_1,c_1,c_3;x^2,y,z,u)=(1+2x)^{-a_1}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{w}{1+2x}\right)^k\times K_8\left(a_1+k,a_1+k,a_1+k,a_1+k,a_2+k,a_2+k,a_3,c_2-\frac{1}{2};c_1,c_3,c_1,2c_2-1;\lambda y,\lambda u,\lambda z,4\lambda x\right),\left(\lambda=\frac{1}{1+2x}\right); \tag{28}$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_{21}^{(4)}(a_1+k,a_1+k,a_2+k,a_1+k,a_1+k,a_2+k,a_3,a_2+k;c_1,c_1,c_2,c_3;x,y,z,u\tau)=(1+u)^{-a_1}(1+\tau-z)^{-a_2}\cdot\sum_{k,p=0}^{\infty}\frac{(a_2+k)_p(c_2-a_3)_p}{(c_2)_p k!p!}\left(\frac{w}{(1+u)(1+\tau-z)}\right)^k\left(\frac{z}{z-\tau-1}\right)^p\times X_{11}^{(4)}(a_1+,a_1+k,a_1+k,a_2+k+p,a_1+k,a_2+k+p,c_3,c_3;c_1,c_1,c_3,c_3;\lambda_1^2x,\lambda_2 y,\lambda_1 u,\lambda_3\tau),\cdot\left(\lambda_1=\frac{1}{1+u},\lambda_2=\frac{1}{(1+u)(1+\tau-z)},\lambda_3=\frac{1}{1+\tau-z}\right); \tag{29}$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_{22}^{(4)}(a_1+k,a_1+k,a_2,a_1+k,a_1+k,a_2,a_3+k,a_2;c_2,c_1,c_1,c_3;x^2,y,z,u)=(1+2x)^{-a_1}\cdot\sum_{k,q=0}^{\infty}\frac{(a_1+k)_q(a_2)_q}{(c_3)_q k!q!}\left(\frac{w}{1+2x}\right)^k\left(\frac{u}{1+2x}\right)^q\times F_M\left(c_2-\frac{1}{2},a_2+q,a_2+q,a_1+k+q,a_3+k,a_1+k+q;2c_2-1,c_1,c_1;\frac{4x}{1+2x},z,\frac{y}{1+2x}\right); \tag{30}$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_{27}^{(4)}(a_1+k,a_1+k,a_3,a_1+k,a_1+k,a_2,a_4+k,a_4+k;c_2,c_1,c_1,c_3;x^2,y,z,u)=(1+2x)^{-a_1}\cdot\sum_{k,q=0}^{\infty}\frac{(a_1+k)_q(a_4+k)_q}{(c_3)_q k!q!}\left(\frac{w}{1+2x}\right)^k\left(\frac{u}{1+2x}\right)^q\times F_N\left(c_2-\frac{1}{2},a_3,a_2,a_1+k+q,a_4+k+q,a_1+k+q;2c_2-1,c_1,c_1;\frac{4x}{1+2x},z,\frac{y}{1+2x}\right); \tag{31}$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_{28}^{(4)}(a_1+k,a_1+k,a_1+k,a_1+k,a_1+k,a_2,a_3,a_4;c_1,c_1,c_2,c_3;x,y,z,u)=(1-z-u)^{-a_1}\cdot\sum_{k,n=0}^{\infty}\frac{(a_1+k)_n(a_2)_n}{(c_1)_n k!n!}\left(\frac{w}{1-z-u}\right)^k\left(\frac{y}{1-z-u}\right)^n\times X_8\left(a_1+k+n,c_2-a_3,c_3-a_4;c_1+n,c_2,c_3;\cdot\frac{x}{(1-z-u)^2},\frac{z}{z+u-1},\frac{u}{z+u-1}\right); \tag{32}$$

where X_3, X_8 are the Exton functions of three variables [12], Lauricella functions F_M, F_N [13], Exton function of four variables K_8 [14], and Sharma and Parihar function $F_{28}^{(4)}$ [15] are defined, respectively, by

$$X_3(a_1,a_2;c_1,c_2;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+n+p}(a_2)_{n+p}}{(c_1)_{m+n}(c_2)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}; \tag{33}$$

$$X_8(a_1,a_2,a_3;c_1,c_2,c_3;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+n+p}(a_2)_n(a_3)_p}{(c_1)_m(c_2)_n(c_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}; \tag{34}$$

$$F_M(a_1,a_2,a_2,b_1,b_2,b_1;c_1,c_2,c_2;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_m(a_2)_{n+p}(b_1)_{m+p}(b_2)_n}{(c_1)_m(c_2)_{n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}; \tag{35}$$

$$F_N(a_1,a_2,a_3,b_1,b_2,b_1;c_1,c_2,c_2;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_m(a_2)_n(a_3)_p(b_1)_{m+p}(b_2)_n}{(c_1)_m(c_2)_{n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}; \tag{36}$$

$$K_8(a_1,a_1,a_1,a_1,b_1,b_1,b_2,b_3;c_1,c_2,c_1,c_3;x,y,z)=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{m+n+p+q}(b_1)_{m+n}(b_2)_p(b_3)_q}{(c_1)_{m+p}(c_2)_n(c_3)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}; \tag{37}$$

$$F_{28}^{(4)}(a_1,a_1,a_2,a_2,b_1,b_2,b_1,b_2;c_1,c_1,c_2,c_3;x,y,z)=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{m+n}(a_2)_{p+q}(b_1)_{m+p}(b_2)_{n+q}}{(c_1)_{m+n}(c_2)_p(c_3)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}. \tag{38}$$

Proof. To prove the above equations, we require the following results (see, e.g., [7, 16, 17]):

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}f(m+n)\frac{x^m}{m!}\frac{y^n}{n!}=\sum_{N=0}^{\infty}f(N)\frac{(x+y)^N}{N!}; \tag{39}$$

$$\Gamma(z)=s^z\int_0^{\infty}e^{-st}t^{z-1}dt,\ \Re(z)>0; \tag{40}$$

$$(a)_n=\frac{\Gamma(a+n)}{\Gamma a},\ a\neq 0,-1,-2,\cdots; \tag{41}$$

$$(a)_{m+n}=(a)_m(a+m)_n; \tag{42}$$

$${}_0F_1\left(-;a;x^2\right)=e^{-2x}{}_1F_1\left(a-\frac{1}{2};2a-1;4x\right); \tag{43}$$

$${}_1F_1(a;b;x)=e^x{}_1F_1(b-a;b;-x); \tag{44}$$

$$\Psi_2(c,c;c,x,y)=\exp\,(x+y){}_0F_1(-;c;xy). \tag{45}$$

For convenience and simplicity, by denoting the left-hand side of (25) by δ, and using (11), one gets

$$\delta=\sum_{k=0}^{\infty}\frac{w^k}{k!\Gamma(a_1+k)\Gamma(a_2+k)}\int_0^{\infty}\int_0^{\infty}s^{a_1+k-1}t^{a_2+k-1}\times{}_0F_1\left(-;c_1;s^2x+sty\right)\Psi_2(c_2;c_2,c_2;sz,tu)dsdt. \tag{46}$$

In view of (40) and (46), we have

$$\delta=\sum_{k,m,n=0}^{\infty}\frac{w^kx^my^n}{(c_1)_{m+n}k!m!n!\Gamma(a_1+k)\Gamma(a_2+k)}\int_0^{\infty}\int_0^{\infty}e^{-s(1-z)}e^{-t(1-u)}\times s^{a_1+k+2m+n-1}t^{a_2+k+n-1}{}_0F_1(-;c_2;stzu)dsdt. \tag{47}$$

The function ${}_0F_1$ which appears in the above equation can be replaced by its series form and then interchanging the order of the summation and integral sign which is permissible here, we get

$$\delta=\sum_{k,m,n,p=0}^{\infty}\frac{w^kx^my^n(zu)^p}{(c_1)_{m+n}(c_2)_pk!m!n!p!\Gamma(a_1+k)\Gamma(a_2+k)}\times\int_0^{\infty}\int_0^{\infty}e^{-s(1-z)}e^{-t(1-u)}s^{a_1+k+2m+n+p-1}t^{a_2+k+n+p-1}dsdt. \tag{48}$$

Now, the use of (41) and (42) in above equation and the simplification with series manipulation completes the proof of relation (25).

The proof of all remaining relations runs in the same way, considering the appropriate integral representation and Laplace transform during the proof. □

3. Special Cases

Here, we shall consider several interesting special cases of our main results stated in the previous section.

Putting $k=0$ in equations (25) to (28), we obtain the following relations:

$$X_{11}^{(4)}(a_1,a_1,a_1,a_2,a_1,a_2,c_2,c_2;c_1,c_1,c_2,c_2;x,y,z,u)=(1-z)^{-a_1}(1-u)^{-a_2}\times X_3\left(a_1,a_2;c_1,c_2;\frac{x}{(1-z)^2};\frac{y}{(1-z)(1-u)},\frac{zu}{(1-z)(1-u)}\right); \tag{49}$$

$$X_{12}^{(4)}(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_2,c_1,c_1,c_3;x,y,z,u\tau)=(1+u)^{-a_2}(1+\tau)^{-a_2}\sum_{m=0}^{\infty}\frac{(a_1)_{2m}}{(c_2)_mm!}x^m\times F_{28}^{(4)}(a_1+2m,a_1+2m,c_3,c_3,a_2,a_3,a_2,a_3;c_1,c_1,c_3,c_3;\frac{y}{1+u},\frac{z}{1+\tau},\frac{u}{1+u},\frac{\tau}{1+\tau}); \tag{50}$$

$$X_{16}^{(4)}(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_2;c_1,c_1,c_2,c_3;x,y,z,u)=(1-z)^{-a_1}X_{16}^{(4)}\cdot\left(a_1,a_1,a_1,a_1,a_1,a_2,c_2,-a_3,a_2;c_1,c_1,c_2,c_3;\frac{x}{(1-z)^2},\frac{y}{1-z},\frac{z}{z-1},\frac{u}{1-z}\right); \tag{51}$$

$$X_{17}^{(4)}(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_2;c_2,c_1,c_1,c_3;x^2,y,z,u)=(1+2x)^{-a_1}\times K_8\left(a_1,a_1,a_1,a_1,a_2,a_2,a_3,c_2-\frac{1}{2};c_1,c_3,c_1,2c_2-1;\lambda y,\lambda u,\lambda z,4\lambda x\right),\cdot\left(\lambda=\frac{1}{1+2x}\right). \tag{52}$$

Equations (49), (51), and (52) with $u=0$ yield the Exton's results (see [12]).

Now, if we take $x=0$ in (25) to (27) and (29), and simplification, we shall obtain the following generating relations:

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}H_B(a_1+k,a_2+k,c_2;c_1,c_3,c_2;y,u,z) = (1-z)^{-a_1}(1-u)^{-a_2}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{w}{(1-z)(1-u)}\right)^k \times F_4\left(a_1+k,a_2+k;c_1,c_2;\frac{y}{(1-z)(1-u)},\frac{zu}{(1-z)(1-u)}\right); \quad (53)$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}H_A(a_2+k,a_3+k,a_1;c_3,c_1;u\tau,y,z) = (1+u)^{-a_2}(1+\tau)^{-a_3}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{w}{(1+\tau)(1+u)}\right)^k \times F_{28}^{(4)}\Big(a_1,a_1,c_3,c_3,a_2+k,a_3+k,a_2+k,a_3+k;c_1,c_1,c_3,c_3;\frac{y}{1+u},\frac{z}{1+\tau},\frac{u}{1+u},\frac{\tau}{1+\tau}\Big); \quad (54)$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}F_E(a_1+k,a_1+k,a_1+k,a_3,a_2+k,a_2+k;c_2,c_1,c_3;z,y,u) = (1-z)^{-a_1}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{w}{1-z}\right)^k \times F_E\Big(a_1+k,a_1+k,a_1+k,c_2-a_3,a_2+k,a_2+k;c_2,c_1,c_3;\frac{z}{z-1},\frac{y}{1-z},\frac{u}{1-z}\Big); \quad (55)$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}F_E(a_2+k,a_2+k,a_2+k,a_3,a_1+k,a_1+k;c_2,c_1,c_3;z,y,u\tau) = (1+u)^{-a_1}(1+\tau-z)^{-a_2}\sum_{k,p=0}^{\infty}\frac{(a_2+k)_p(c_2-a_3)_p}{(c_2)_p\,k!p!} \cdot\left(\frac{w}{(1+u)(1+\tau-z)}\right)^k\left(\frac{z}{z-\tau-1}\right)^p \times H_B\left(a_1+k,a_2+k+p,c_3;c_1,c_3,c_3;\lambda_1 y,\lambda_2\tau,\frac{u}{u+1}\right), \cdot\left(\lambda_1=\frac{1}{(1+u)(1+\tau-z)},\lambda_2=\frac{1}{1+\tau-z}\right). \quad (56)$$

Note that the special cases of each of the above generating relations can be easily derived by assigning the value zero to k. For example,

$$H_B(a_1,a_2,c_2;c_1,c_2;y,u,z) = (1-z)^{-a_1}(1-u)^{-a_2}F_4 \cdot\left(a_1,a_2;c_1,c_2;\frac{y}{(1-z)(1-u)},\frac{zu}{(1-z)(1-u)}\right), \quad (57)$$

which, when $a_2=c_3$, yields [[7], pp. 309 (125)].

Another interesting special case of (56) occurs when we set $u=0$. We thus find that

$$F_2(a_1,a_2,c_2;c_1,c_2;y,z) = (1-z)^{-a_1}\,{}_2F_1\left(a_1,a_2;c_1;\frac{y}{(1-z)}\right), \quad (58)$$

which is due to Srivastava and Karlsson [7].

Also,

$$H_A(a_2,a_3,a_1;c_3,c_1;u\tau,y,z) = (1+u)^{-a_2}(1+\tau)^{-a_3}F_{28}^{(4)} \cdot\left(a_1,a_1,c_3,c_3,a_2,a_3,a_2,a_3;c_1,c_1,c_3,c_3;\frac{y}{1+u},\frac{z}{1+\tau},\frac{u}{1+u},\frac{\tau}{1+\tau}\right), \quad (59)$$

which, for $y=z=0$, $x=u/1+u$ and $y=\tau/1+\tau$, yields the well-known result [7].

Furthermore, by setting $k=0$ and $y=0$ in (25) and (32), we obtain the Exton's results [12].

Finally, if in (29), we let $z=0$, we shall obtain generating relation between Exton's series X_3 and the quadruple hypergeometric series $X_{11}^{(4)}$:

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_3(a_1+k,a_2+k;c_1,c_3;x,y,u\tau) = (1+u)^{-a_1}(1+\tau)^{-a_2} \cdot\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{w}{(1+u)(1+\tau)}\right)^k \times X_{11}^{(4)}(a_1+k,a_1+k,a_1+k,a_2+k,a_1+k,a_2+k,c_3,c_3;c_1,c_1,c_3,c_3;\lambda_1^2x,\lambda_1\lambda_2y,\lambda_1u,\lambda_2\tau), \cdot\left(\lambda_1=\frac{1}{1+u},\lambda_2=\frac{1}{1+\tau}\right). \quad (60)$$

Formula (61), with $x=0$, yields the generating relation

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}F_4(a_1+k,a_2+k;c_1,c_3;y,u\tau) = (1+u)^{-a_1}(1+\tau)^{-a_2} \cdot\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{w}{(1+u)(1+\tau)}\right)^k \times H_B\left(a_1+k,a_2+k,c_3;c_1,c_3,c_3; \cdot\frac{y}{(1+u)(1+\tau)},\frac{\tau}{1+\tau},\frac{u}{1+u}\right). \quad (61)$$

For $k=0$, we have the elegant transformation

$$F_4(a_1,a_2;c_1,c_3;y,u\tau) = (1+u)^{-a_1}(1+\tau)^{-a_2}H_B \cdot\left(a_1,a_2,c_3;c_1,c_3,c_3;\frac{y}{(1+u)(1+\tau)},\frac{\tau}{1+\tau},\frac{u}{1+u}\right); \quad (62)$$

where ${}_2F_1$, F_2, F_4, H_A, H_B, and F_E are the Gaussian hypergeometric function, Appell's functions, Srivastava's functions, and Lauricella function defined, respectively, by (see [7])

$$ {}_2F_1(a, b ; c ; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} ; \tag{63} $$

$$ F_2(a, b, c ; d, e ; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} (b)_m (c)_n}{(d)_m (e)_n} \frac{x^m}{m!} \frac{y^n}{n!} ; \tag{64} $$

$$ F_4(a, b ; c, d ; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} (b)_{m+n}}{(c)_m (d)_n} \frac{x^m}{m!} \frac{y^n}{n!} ; \tag{65} $$

$$ H_A(a_1, b_1, b_2 ; c_1, c_2 ; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+p} (b_1)_{m+n} (b_2)_{n+p}}{(c_1)_m (c_2)_{n+p}} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} ; \tag{66} $$

$$ H_B(a_1, b_1, b_2 ; c_1, c_2, c_3 ; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+p} (b_1)_{m+n} (b_2)_{n+p}}{(c_1)_m (c_2)_n (c_3)_p} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} ; \tag{67} $$

$$ \begin{aligned} & F_E(a_1, a_1, a_1, b_1, b_2, b_2 ; c_1, c_2, c_3 ; x, y, z) \\ & \quad = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n+p} (b_1)_m (b_2)_{n+p}}{(c_1)_m (c_2)_n (c_3)_p} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} . \end{aligned} \tag{68} $$

4. Conclusion

Based on the integral representations for quadruple hypergeometric functions (1)–(8), we obtained certain generating functions for these functions. Some particular cases and the consequences of our main results are also considered. We concluded this investigation by remarking that the scheme suggested in the derivation of the results can be applied to find other new generating functions for other quadruple hypergeometric functions and study their special cases.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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