Research Article

Some Compatible and Weakly-Compatible Four Self-Mapping Results Approach to Nonlinear Integral Equations in Fuzzy Cone Metric Spaces

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This paper is aimed at proving some unique common fixed point theorems by using the compatible and weakly-compatible four self-mappings in fuzzy cone metric (FCM) space. We prove the results under the generalized rational contraction conditions in FCM spaces with the help of one self-map are continuous. Furthermore, we prove some rational contraction results with the weaker condition of the self-mapping continuity. Ultimately, our theoretical work has been utilized to prove the existence solution of the two nonlinear integral equations. This is an illustrative application of how FCM spaces can be used in other integral type operators.

1. Introduction

The theory of fixed-point theory was introduced by Banach [1]. He proved a “Banach contraction principle,” which is stated as follows: “A self-mapping on a complete metric space verifying the contraction condition has a unique fixed point (FP).” Later on, many researchers have been generalized this principle in many directions and proved different contractive type FP and common fixed point (CFP) for single-valued and multivalued mappings in the context of metric spaces. Chatterjea [2], Chatterjea [3], and Kannan [4] proved some single-valued contractive type FP theorems. While Ali et al. [5], Covitz and Nadler [6], Daker and Kaneko [7, 8], Khan [9], and Patle et al. [10] proved multivalued contractive type FP and CFP results by using different types of spaces.

Zadeh [11], in 1965, introduced the concept of fuzzy sets. Later on, this concept was used in topology and functional

analysis by many researchers. Kramosil and Michalek [12] introduced the notion of fuzzy metric FM space, and they established some basic properties. After that, George and Veeramani [13] presented the stronger form of the FM. Grabiec [14] proved two FP theorems by using the concept of complete and compact FM spaces. Gregori and Sapena [15] established some FP contraction results in the sense of [13, 15]. Hadzic and Pap [16] proved a FP theorem for multivalued mappings in probabilistic metric spaces and presented applications in FM spaces. Imdad and Ali [17] and Rehman et al. [18] proved some FP theorems in complete FM spaces. Pant and Chauhan [19] established some CFP theorems by using weakly-compatible mappings in menger spaces and FM spaces. Kiyani et al. [20] and Sadeghi et al. [21] proved some results for set-valued contractive type mappings in FM spaces.

The concept of cone metric space (CMS) was proposed by many researchers but it became popular after being redis-

covered by Huang and Zhang [22]. They proved the convergence properties and FP theorems for nonlinear contractive type mappings. By using the concept of Huang and Zhang [22], many authors have contributed their work to the problems on CMSs. Some of such works can be found in ([23–28]).

In 2015, the notion of fuzzy cone metric space (FCM space) was introduced by Oner et al. [29]. They proved the key attributes of FCM space and a “fuzzy cone Banach contraction theorem for FP” in FCM space. In [30], Rehman and Li extended and improved a “fuzzy cone Banach contraction theorem” and established some generalized-contraction results for FP in FCM spaces. Rehman et al. [31, 32] proved different contractive type CFP-theorems in FCM spaces. Recently, the concept of weakly compatible self-mappings in FCM spaces was given by Jabeen et al. [33].

This paper is aimed at proving some unique CFP-theorems under the generalized rational contraction conditions in FCM spaces by using compatibility and weak-compatibility of four self-mappings. We prove our results by using the one self-map are continuous. Furthermore, we prove some results without the continuity of self-mappings with supportive examples. In addition, we present an application of two nonlinear integral equations (NIEs) for the existence of a common solution to support our main work. This paper is managed as follows: in Section 2, we present the basic preliminary concept. While in Section 3, we prove our main results for unique CFP-theorems under the generalized rational contraction conditions in FCM spaces by using compatibility and weak-compatibility of four self-mappings. In Section 4, we present NIEs as an application to support our main work.

2. Preliminaries

In this section, we recall some basic definitions and lemmas.

Definition 1 (see [34]). An operation $*$: $[0, 1]^2 \rightarrow [0, 1]$ is called a continuous t -norm if:

- (i) $*$ is associative, commutative, and continuous
- (ii) $1 * q_1 = q_1$ and $q_1 * q_2 \leq q_3 * q_4$, whenever $q_1 \leq q_3$ and $q_2 \leq q_4$, for all $q_1, q_2, q_3, q_4 \in [0, 1]$

Schweizer and Sklar [34] define the following basic continuous t -norms are

- (i) The minimum; $q_1 * q_2 = \min \{q_1, q_2\}$
- (ii) The product; $q_1 * q_2 = q_1 q_2$
- (iii) The Lukasiewicz; $q_1 * q_2 = \max \{q_1 + q_2 - 1, 0\}$

For detail study (see [34]).

Definition 2 (see [29]). A 3-tuple $(U, M_o, *)$ is called a FCM space if C is a cone of $\mathbb{E}$, U is an arbitrary set, $*$ is a contin-

uous t -norm and M_o is a fuzzy set on $U^2 \times \text{int}(P)$ satisfying the following conditions:

- (1) $M_o(\lambda_1, \lambda_2, t) > 0$ and $M_o(\lambda_1, \lambda_2, t) = 1 \Leftrightarrow \lambda_1 = \lambda_2$
 - (2) $M_o(\lambda_1, \lambda_2, t) = M_o(\lambda_2, \lambda_1, t)$
 - (3) $M_o(\lambda_1, \lambda_2, t) * M_o(\lambda_2, \lambda_3, s) \leq M_o(\lambda_1, \lambda_3, t + s)$
 - (4) $M_o(\lambda_1, \lambda_2, \cdot) : \text{int}(P) \rightarrow [0, 1]$ is continuous
- $\forall \lambda_1, \lambda_2, \lambda_3 \in U$ and $t, s \in \text{int}(P)$.

Definition 3 (see [29]). Let $(U, M_o, *)$ be a FCM space, $\exists \lambda_1 \in U$ and $\{\lambda_j\}$ be any sequence in U .

- (i) $\{\lambda_j\}$ converges to λ_1 if for any $c \in (0, 1)$, $t \gg \theta$, and $\exists j_1 \in \mathbb{N}$ such that $M_o(\lambda_j, \lambda_1, t) > 1 - c$, for $j \geq j_1$. This can be written as $\lim_{j \rightarrow \infty} \lambda_j = \lambda_1$, or $\lambda_j \rightarrow \lambda_1$ as $j \rightarrow \infty$
- (ii) (λ_j) is Cauchy if for any $c \in (0, 1)$, $t \gg \theta$, and $\exists j_1 \in \mathbb{N}$ such that $M_o(\lambda_j, \lambda_k, t) > 1 - c$, for $j, k \geq j_1$
- (iii) $(U, M_o, *)$ is complete if every Cauchy sequence is convergent in U
- (iv) $\{\lambda_j\}$ is FC contractive if $\exists a \in (0, 1)$ so that

$$\frac{1}{M_o(\lambda_j, \lambda_{j+1}, t)} - 1 \leq a \left(\frac{1}{M_o(\lambda_{j-1}, \lambda_j, t)} - 1 \right), \text{ for } t \gg \theta, j \geq 1. \quad (1)$$

Lemma 4 (see [29]). “Let $(U, M_o, *)$ be a FCM space and a sequence $\lambda_j \rightarrow \lambda_1 \in U$ iff $M_o(\lambda_j, \lambda_1, t) \rightarrow 1$ as $j \rightarrow \infty$ for each $t \gg \theta$ ”.

Definition 5 (see [30]). Let $(U, M_o, *)$ be a FCM space. The FCM M_o is triangular if

$$\frac{1}{M_o(\lambda_1, \lambda_3, t)} - 1 \leq \left(\frac{1}{M_o(\lambda_1, \lambda_2, t)} - 1 \right) + \left(\frac{1}{M_o(\lambda_2, \lambda_3, t)} - 1 \right), \forall \lambda_1, \lambda_2, \lambda_3 \in U, t \gg \theta. \quad (2)$$

Definition 6 (see [29]). Let $(U, M_o, *)$ be a FCM space and $A : U \rightarrow U$. Then, A is said to be FC contractive if there is $a \in (0, 1)$ so that

$$\frac{1}{M_o(A\lambda_1, A\lambda_2, t)} - 1 \leq a \left(\frac{1}{M_o(\lambda_1, \lambda_2, t)} - 1 \right), \quad (3)$$

$\forall \lambda_1, \lambda_2 \in U$, and $t \gg \theta$.

Definition 7 (see [23]). Let $U \neq \emptyset$ set and let $B, h : U \rightarrow U$ be the self-mappings on U . If there exists $\xi \in U$ such that $B\rho = h\rho = \xi$ for some $\rho \in U$. Then, ρ is called a coincidence

point of B and h , and ξ is known as a point of coincidence of the mappings B, h . A pair of self-mappings (B, h) is known to be weakly-compatible if the self-mappings commute at their coincidence point, i.e., $hB(\rho) = Bh(\rho)$ for $\rho \in U$.

Proposition 8 (see [23]). *Let B, h be weakly-compatible self-mappings on U . If B and h have a unique point of coincidence, that is, $B\rho = h\rho = \xi$, then, ξ is a unique CFP of the mappings B and h .*

Definition 9 (see [32]). A self-mapping pair (h, B) is said to be compatible on a FCM space $(U, M_o, *)$, if, $\lim_{j \rightarrow \infty} M_o(Bh\lambda_j, hB\lambda_j, t) = 1$ for $t \gg \theta$, whenever $\{\lambda_j\}$ is a sequence in U so that $\lim_{j \rightarrow \infty} B\lambda_j = \lim_{j \rightarrow \infty} h\lambda_j = \xi$ for some $\xi \in U$.

3. Main Results

Now, we are in the position to present our main results.

Theorem 10. *Let $A, B, g, h : U \rightarrow U$ be the four self-mappings on a complete FCM space $(U, M_o, *)$ in which a FCM M_o is triangular and satisfies*

$$\begin{aligned} \frac{1}{M_o(A\lambda, B\mu, t)} - 1 &\leq a \left(\frac{1}{M_o(h\lambda, g\mu, t)} - 1 \right) \\ &+ b \left(\frac{M_o(h\lambda, g\mu, t)}{M_o(h\lambda, B\mu, 2t) * M_o(g\mu, A\lambda, 2t)} - 1 \right) \\ &+ c \left(\frac{1}{M_r(h\lambda, A\lambda, t)} - 1 + \frac{1}{M_r(g\mu, B\mu, t)} - 1 \right) \\ &+ d \left(\frac{1}{M_r(g\mu, A\lambda, t)} - 1 + \frac{1}{M_r(h\lambda, B\mu, t)} - 1 \right), \end{aligned} \quad (4)$$

$\forall \lambda, \mu \in U, t \gg \theta$, and $0 \leq a, b, c, d < 1$ with $(a + b + 2c + 2d) < 1$. If $A(U) \subseteq g(U)$, $B(U) \subseteq h(U)$ and consider that [(1)]

- (1) h is a continuous self-mapping
- (2) A pair (A, h) is compatible, and
- (3) A pair (B, g) is weakly-compatible

Then, the mappings A, B, g , and h have a unique CFP in U .

Proof. Fix $\lambda_0 \in U$ and by the hypothesis $A(U) \subseteq g(U)$, $B(U) \subseteq h(U)$, we define the iterative sequences in U so that

$$\xi_{2j+1} = g\lambda_{2j+1} = A\lambda_{2j} \text{ and } \xi_{2j+2} = h\lambda_{2j+2} = B\lambda_{2j+1}, j \geq 0. \quad (5)$$

Then, by (4),

$$\begin{aligned} \frac{1}{M_o(g\lambda_{2j+1}, h\lambda_{2j+2}, t)} - 1 &= \frac{1}{M_o(A\lambda_{2j}, B\lambda_{2j+1}, t)} - 1 \\ &\leq a \left(\frac{1}{M_o(h\lambda_{2j}, g\lambda_{2j+1}, t)} - 1 \right) \\ &+ b \left(\frac{M_o(h\lambda_{2j}, g\lambda_{2j+1}, t)}{M_o(h\lambda_{2j}, B\lambda_{2j+1}, 2t) * M_o(g\lambda_{2j+1}, A\lambda_{2j}, 2t)} - 1 \right) \\ &+ c \left(\frac{1}{M_o(h\lambda_{2j}, A\lambda_{2j}, t)} - 1 + \frac{1}{M_o(g\lambda_{2j+1}, B\lambda_{2j+1}, t)} - 1 \right) \\ &+ d \left(\frac{1}{M_o(g\lambda_{2j+1}, A\lambda_{2j}, t)} - 1 + \frac{1}{M_o(h\lambda_{2j}, B\lambda_{2j+1}, t)} - 1 \right) \\ &= a \left(\frac{1}{M_o(h\lambda_{2j}, g\lambda_{2j+1}, t)} - 1 \right) + b \left(\frac{M_o(h\lambda_{2j}, g\lambda_{2j+1}, t)}{M_o(h\lambda_{2j}, h\lambda_{2j+2}, 2t)} - 1 \right) \\ &+ c \left(\frac{1}{M_o(h\lambda_{2j}, g\lambda_{2j+1}, t)} - 1 + \frac{1}{M_o(g\lambda_{2j+1}, h\lambda_{2j+2}, t)} - 1 \right) \\ &+ d \left(\frac{1}{M_o(h\lambda_{2j}, h\lambda_{2j+2}, t)} - 1 \right). \end{aligned} \quad (6)$$

By Definition 2 (3), $M_o(h\lambda_{2j}, h\lambda_{2j+2}, 2t) \geq M_o(h\lambda_{2j}, g\lambda_{2j+1}, t) * M_o(g\lambda_{2j+1}, h\lambda_{2j+2}, t)$, for $t \gg \theta$. One writes

$$\begin{aligned} \frac{1}{M_o(g\lambda_{2j+1}, h\lambda_{2j+2}, t)} - 1 &\leq a \left(\frac{1}{M_o(h\lambda_{2j}, g\lambda_{2j+1}, t)} - 1 \right) \\ &+ b \left(\frac{M_o(h\lambda_{2j}, g\lambda_{2j+1}, t)}{M_o(h\lambda_{2j}, g\lambda_{2j+1}, t) * M_r(g\lambda_{2j+1}, h\lambda_{2j+2}, t)} - 1 \right) \\ &+ c \left(\frac{1}{M_o(h\lambda_{2j}, g\lambda_{2j+1}, t)} - 1 + \frac{1}{M_o(g\lambda_{2j+1}, h\lambda_{2j+2}, t)} - 1 \right) \\ &+ d \left(\frac{1}{M_o(h\lambda_{2j}, g\lambda_{2j+1}, t)} - 1 + \frac{1}{M_o(g\lambda_{2j+1}, h\lambda_{2j+2}, t)} - 1 \right). \end{aligned} \quad (7)$$

This implies that

$$\frac{1}{M_o(g\lambda_{2j+1}, h\lambda_{2j+2}, t)} - 1 \leq \beta \left(\frac{1}{M_o(h\lambda_{2j}, g\lambda_{2j+1}, t)} - 1 \right), \text{ for } t \gg \theta, \quad (8)$$

where $\gamma = a + c + d/1 - b - c - d < 1$ since $(a + b + 2c + 2d) < 1$

$d) < 1$. Similarly,

$$\begin{aligned}
& \frac{1}{M_o(h\lambda_{2j+2}, g\lambda_{2j+3}, t)} - 1 = \frac{1}{M_o(A\lambda_{2j+2}, B\lambda_{2j+1}, t)} - 1 \\
& \leq a \left(\frac{1}{M_o(g\lambda_{2j+1}, h\lambda_{2j+2}, t)} - 1 \right) \\
& \quad + b \left(\frac{M_o(g\lambda_{2j+1}, h\lambda_{2j+2}, t)}{M_o(h\lambda_{2j+2}, B\lambda_{2j+1}, 2t) * M(g\lambda_{2j+1}, A\lambda_{2j+2}, 2t)} - 1 \right) \\
& \quad + c \left(\frac{1}{M_o(h\lambda_{2j+2}, A\lambda_{2j+2}, t)} - 1 + \frac{1}{M_o(g\lambda_{2j+1}, B\lambda_{2j+1}, t)} - 1 \right) \\
& \quad + d \left(\frac{1}{M_o(g\lambda_{2j+1}, A\lambda_{2j+2}, t)} - 1 + \frac{1}{M_o(h\lambda_{2j+2}, B\lambda_{2j+1}, t)} - 1 \right) \\
& = a \left(\frac{1}{M_o(g\lambda_{2j+1}, h\lambda_{2j+2}, t)} - 1 \right) + b \left(\frac{M_o(g\lambda_{2j+1}, h\lambda_{2j+2}, t)}{M(g\lambda_{2j+1}, g\lambda_{2j+3}, 2t)} - 1 \right) \\
& \quad + c \left(\frac{1}{M_o(h\lambda_{2j+2}, g\lambda_{2j+3}, t)} - 1 + \frac{1}{M_o(g\lambda_{2j+1}, h\lambda_{2j+2}, t)} - 1 \right) \\
& \quad + d \left(\frac{1}{M_o(g\lambda_{2j+1}, g\lambda_{2j+3}, t)} - 1 \right). \tag{9}
\end{aligned}$$

Again, by Definition 2 (3), $M_o(g\lambda_{2j+1}, g\lambda_{2j+3}, 2t) \geq M_o(g\lambda_{2j+1}, h\lambda_{2j+2}, t) * M_o(h\lambda_{2j+2}, g\lambda_{2j+3}, t)$, for $t \gg \theta$. We have

$$\begin{aligned}
& \frac{1}{M_o(h\lambda_{2j+2}, g\lambda_{2j+3}, t)} - 1 \leq a \left(\frac{1}{M_o(g\lambda_{2j+1}, h\lambda_{2j+2}, t)} - 1 \right) \\
& \quad + b \left(\frac{M_o(g\lambda_{2j+1}, h\lambda_{2j+2}, t)}{M_o(g\lambda_{2j+1}, h\lambda_{2j+2}, t) * M_r(h\lambda_{2j+2}, g\lambda_{2j+3}, t)} - 1 \right) \\
& \quad + c \left(\frac{1}{M_o(g\lambda_{2j+1}, h\lambda_{2j+2}, t)} - 1 + \frac{1}{M_o(h\lambda_{2j+2}, g\lambda_{2j+3}, t)} - 1 \right) \\
& \quad + d \left(\frac{1}{M_o(g\lambda_{2j+1}, h\lambda_{2j+2}, t)} - 1 + \frac{1}{M_o(h\lambda_{2j+2}, g\lambda_{2j+3}, t)} - 1 \right). \tag{10}
\end{aligned}$$

This implies that

$$\frac{1}{M_o(h\lambda_{2j+2}, g\lambda_{2j+3}, t)} - 1 \leq \beta \left(\frac{1}{M_o(g\lambda_{2j+1}, h\lambda_{2j+2}, t)} - 1 \right), \text{ for } t \gg \theta, \tag{11}$$

where the value of β is the same as in (8). Now, from (3), (8), (11), and by induction, we have

$$\begin{aligned}
& \frac{1}{M_o(\xi_{2j+2}, \xi_{2j+3}, t)} - 1 \leq \beta \left(\frac{1}{M_o(\xi_{2j+1}, \xi_{2j+2}, t)} - 1 \right) \\
& \leq \beta^2 \left(\frac{1}{M_o(\xi_{2j+1}, \xi_{2j+2}, t)} - 1 \right) \\
& \leq \dots \leq \beta^{2j+2} \left(\frac{1}{M_o(\xi_0, \xi_1, t)} - 1 \right) \longrightarrow 0, \text{ as } j \longrightarrow \infty. \tag{12}
\end{aligned}$$

Its prove that a sequence $\{\xi_j\}_{j \geq 0}$ is a FC contractive, and we get that

$$\lim_{j \longrightarrow \infty} M_o(\xi_j, \xi_{j+1}, t) = 1, \text{ for } t \gg \theta. \tag{13}$$

Since M_o is triangular, then $\forall k > j \geq j_0$,

$$\begin{aligned}
& \frac{1}{M_o(\xi_j, \xi_k, t)} - 1 \leq \left(\frac{1}{M_o(\xi_j, \xi_{j+1}, t)} - 1 \right) + \left(\frac{1}{M_o(\xi_{j+1}, \xi_{j+2}, t)} - 1 \right) \\
& \quad + \dots + \left(\frac{1}{M_o(\xi_{k-1}, \xi_k, t)} - 1 \right) \\
& \leq (\beta^j + \beta^{j+1} + \dots + \beta^{k-1}) \left(\frac{1}{M_o(\xi_0, \xi_1, t)} - 1 \right) \\
& \leq \frac{\beta^j}{1 - \beta} \left(\frac{1}{M_o(\xi_0, \xi_1, t)} - 1 \right) \longrightarrow 0, \text{ as } j \longrightarrow \infty. \tag{14}
\end{aligned}$$

Hence, proved that $\{\xi_j\}$ is a Cauchy sequence. Now by the completeness of FCM space $(U, M_o, *)$, $\exists \xi \in U$ so that $\xi_j \longrightarrow \xi$ as $j \longrightarrow \infty$. Now for its subsequences, we have that

$$g\lambda_{2j+1} \longrightarrow \xi, h\lambda_{2j+2} \longrightarrow \xi, A\lambda_{2j} \longrightarrow \xi, \text{ and } B\lambda_{2j+1} \longrightarrow \xi \text{ as } j \longrightarrow \infty. \tag{15}$$

Since, a self-mapping $h : U \longrightarrow U$ is continuous, therefore

$$h(g\lambda_{2j+1}) \longrightarrow h\xi, h(h\lambda_{2j+2}) \longrightarrow h\xi, h(A\lambda_{2j}) \longrightarrow h\xi, \text{ and } h(B\lambda_{2j+1}) \longrightarrow h\xi \text{ as } j \longrightarrow \infty. \tag{16}$$

By hypothesis (2), a (A, h) is compatible, therefore,

$$\begin{aligned}
& \lim_{j \longrightarrow \infty} M_o(A(h\lambda_{2j}), h(A\lambda_{2j}), t) = \lim_{j \longrightarrow \infty} M_r(A(h\lambda_{2j}), h\xi, t) \\
& = 1, \Rightarrow \lim_{j \longrightarrow \infty} M_r(h(A\lambda_{2j}), h\xi, t) = 1, \text{ for } t \gg \theta. \tag{17}
\end{aligned}$$

Next, we have to prove that $h\xi = \xi$, then, by Definition 2 (3),

$$M_o(h\xi, \xi, 2t) \geq M_r(h\xi, A(h\lambda_{2j}, t) * M_r(A(h\lambda_{2j}, \xi, t) \text{ for } t \gg \theta. \tag{18}$$

Since, a pair (A, h) is compatible, by using limit $j \longrightarrow \infty$, and by the view of (15), (17), and (18), we have

$$\begin{aligned}
& M_o(h\xi, \xi, 2t) \geq \lim_{j \longrightarrow \infty} M_r(h\xi, A(h\lambda_{2j}, t) * \lim_{j \longrightarrow \infty} M_r(A(h\lambda_{2j}, \xi, t) \\
& = 1 * 1 = 1 \text{ for } t \gg \theta. \tag{19}
\end{aligned}$$

Hence, $M_o(h\xi, \xi, 2t) = 1 \Rightarrow h\xi = \xi$, for $t \gg \theta$. Now, we

prove that $A\xi = \xi$, then again by Definition 2 (3),

$$M_o(A\xi, \xi, 2t) \geq M_r(A\xi, h(A\lambda_{2j}, t) * M_r(h(A\lambda_{2j}, \xi, t) \text{ for } t \gg \theta. \quad (20)$$

Again by using limit $j \rightarrow \infty$, and by the view of (15), (17), and (20), we have

$$\begin{aligned} M_o(A\xi, \xi, 2t) &\geq \lim_{j \rightarrow \infty} M_r(A\xi, h(A\lambda_{2j}), t) * \lim_{j \rightarrow \infty} M_r(h(A\lambda_{2j}), \xi, t) \\ &= 1 * 1 = 1 \text{ for } t \gg \theta. \end{aligned} \quad (21)$$

Hence, $M_o(A\xi, \xi, 2t) = 1 \Rightarrow A\xi = \xi$, for $t \gg \theta$. Thus, we get that $A\xi = h\xi = \xi$. Next, we have to prove that $B\xi = g\xi$. Now by hypothesis (1), i.e., $A(U) \subseteq g(U)$, and there exists $\rho \in U$ such that $\xi = A\xi = g\rho$. Then, by view of (4), for $t \gg \theta$,

$$\begin{aligned} \frac{1}{M_o(B\rho, g\rho, t)} - 1 &= \frac{1}{M_o(A\xi, B\rho, t)} - 1 \leq a \left(\frac{1}{M_o(h\xi, g\rho, t)} - 1 \right) \\ &\quad + b \left(\frac{M_o(h\xi, g\rho, t)}{M_o(h\xi, B\rho, 2t) * M_o(g\rho, A\xi, 2t)} - 1 \right) \\ &\quad + c \left(\frac{1}{M_o(h\xi, A\xi, t)} - 1 + \frac{1}{M_o(g\rho, B\rho, t)} - 1 \right) \\ &\quad + d \left(\frac{1}{M_o(g\rho, A\xi, t)} - 1 + \frac{1}{M_o(h\xi, B\rho, t)} - 1 \right) \\ &= a \left(\frac{1}{M_o(h\xi, \xi, t)} - 1 \right) + b \left(\frac{M_o(h\xi, g\rho, t)}{M_o(h\xi, B\rho, 2t) * M_o(\xi, A\xi, 2t)} - 1 \right) \\ &\quad + c \left(\frac{1}{M_o(h\xi, \xi, t)} - 1 + \frac{1}{M_o(g\rho, B\rho, t)} - 1 \right) \\ &\quad + d \left(\frac{1}{M_o(\xi, A\xi, t)} - 1 + \frac{1}{M_o(h\xi, B\rho, t)} - 1 \right) \\ &= b \left(\frac{M_o(h\xi, g\rho, t)}{M_o(h\xi, B\rho, 2t)} - 1 \right) + c \left(\frac{1}{M_o(g\rho, B\rho, t)} - 1 \right) \\ &\quad + d \left(\frac{1}{M_o(g\rho, B\rho, t)} - 1 \right). \end{aligned} \quad (22)$$

Again, by Definition 2 (3), $M_o(h\xi, B\rho, 2t) \geq M_o(h\xi, g\rho, t) * M_o(g\rho, B\rho, t)$, for $t \gg \theta$. It follows that

$$\begin{aligned} \frac{1}{M_o(B\rho, g\rho, t)} - 1 &\leq b \left(\frac{M_o(h\xi, g\rho, t)}{M_o(h\xi, g\rho, t) * M_o(g\rho, B\rho, t)} - 1 \right) \\ &\quad + (c + d) \left(\frac{1}{M_o(g\rho, B\rho, t)} - 1 \right) \\ &= (b + c + d) \left(\frac{1}{M_o(g\rho, B\rho, t)} - 1 \right) \text{ for } t \gg \theta. \end{aligned} \quad (23)$$

Noticing that $(b + c + d) < 1$, therefore, $M_o(B\rho, g\rho, t) = 1 \Rightarrow B\rho = g\rho$ for $t \gg \theta$, hence, $B\rho = g\rho = \xi$. Now by hypothesis (3), a pair (B, g) is weakly compatible, therefore,

$$g\xi = g(B\rho) = B(g\rho) = B\xi. \quad (24)$$

Next, we have to prove that $B\xi = \xi$, then again by view of

(4) and by using Definition 2 (3), for $t \gg \theta$,

$$\begin{aligned} \frac{1}{M_o(B\xi, \xi, t)} - 1 &= \frac{1}{M_o(B\xi, A\xi, t)} - 1 \leq a \left(\frac{1}{M_o(h\xi, g\xi, t)} - 1 \right) \\ &\quad + b \left(\frac{M_o(h\xi, g\xi, t)}{M_o(h\xi, B\xi, 2t) * M_o(g\xi, A\xi, 2t)} - 1 \right) \\ &\quad + c \left(\frac{1}{M_r(h\xi, A\xi, t)} - 1 + \frac{1}{M_r(g\xi, B\xi, t)} - 1 \right) \\ &\quad + d \left(\frac{1}{M_r(g\xi, A\xi, t)} - 1 + \frac{1}{M_r(h\xi, B\xi, t)} - 1 \right) \\ &\leq a \left(\frac{1}{M_o(\xi, B\xi, t)} - 1 \right) \\ &\quad + b \left(\frac{M_o(h\xi, g\xi, t)}{M_o(h\xi, g\xi, t) * M_o(g\xi, B\xi, t) * M_o(g\xi, \xi, t) * M_o(\xi, A\xi, t)} - 1 \right) \\ &\quad + c \left(\frac{1}{M_r(h\xi, \xi, t)} - 1 + \frac{1}{M_r(g\xi, B\xi, t)} - 1 \right) \\ &\quad + d \left(\frac{1}{M_r(B\xi, \xi, t)} - 1 + \frac{1}{M_r(\xi, B\xi, t)} - 1 \right). \end{aligned} \quad (25)$$

After simplification, we obtain

$$\begin{aligned} \frac{1}{M_r(B\xi, \xi, t)} - 1 &\leq (a + b + 2d) \left(\frac{1}{M_r(B\xi, \xi, t)} - 1 \right), \\ &\Rightarrow (1 - a - b - 2d) \left(\frac{1}{M_r(B\xi, \xi, t)} - 1 \right) \leq 0, \text{ for } t \gg \theta. \end{aligned} \quad (26)$$

Since $(1 - a - b - 2d) \neq 0$, therefore, $M_r(B\xi, \xi, t) = 1 \Rightarrow B\xi = \xi$, for $t \gg \theta$, which further implies that $g\xi = \xi$. Hence, proved that $h\xi = g\xi = A\xi = B\xi = \xi$, that is, ξ is the CFP of the mappings A, B, g , and h . $\square$

Uniqueness: let $\eta \in U$ be the other CFP of the mappings A, B, g , and h in U such that $h\eta = g\eta = A\eta = B\eta = \eta$. Then by view of (4) and by using Definition 2 (3), for $t \gg \theta$,

$$\begin{aligned} \frac{1}{M_r(\xi, \eta, t)} - 1 &= \frac{1}{M_r(A\xi, B\eta, t)} - 1 \leq a \left(\frac{1}{M_o(h\xi, g\eta, t)} - 1 \right) \\ &\quad + b \left(\frac{M_o(h\xi, g\eta, t)}{M_o(h\xi, B\eta, 2t) * M_o(g\eta, A\xi, 2t)} - 1 \right) \\ &\quad + c \left(\frac{1}{M_r(h\xi, A\xi, t)} - 1 + \frac{1}{M_r(g\eta, B\eta, t)} - 1 \right) \\ &\quad + d \left(\frac{1}{M_r(g\eta, A\xi, t)} - 1 + \frac{1}{M_r(h\xi, B\eta, t)} - 1 \right) \\ &\leq a \left(\frac{1}{M_o(\xi, \eta, t)} - 1 \right) \\ &\quad + b \left(\frac{M_o(\xi, \eta, t)}{M_o(\xi, \xi, t) * M_o(\xi, \eta, t) * M_o(\eta, \eta, t) * M_o(\eta, \xi, t)} - 1 \right) \\ &\quad + 2d \left(\frac{1}{M_r(\eta, \xi, t)} - 1 \right). \end{aligned} \quad (27)$$

After simplification, we obtain

$$\begin{aligned} \frac{1}{M_r(\xi, \eta, t)} - 1 &\leq (a + b + 2d) \left(\frac{1}{M_r(\xi, \eta, t)} - 1 \right), \\ \Rightarrow (1 - a - b - 2d) \left(\frac{1}{M_r(\xi, \eta, t)} - 1 \right) &\leq 0, \text{ for } t \gg \theta. \end{aligned} \quad (28)$$

Since $(1 - a - b - 2d) \neq 0$, therefore, $M_r(\xi, \eta, t) = 1 \Rightarrow \xi = \eta$, for $t \gg \theta$. This completes the proof.

Corollary 11. Let $A, B, g, h : U \longrightarrow U$ be the four self-mappings on a complete FCM space $(U, M_o, *)$ in which a FCM M_o is triangular and satisfies

$$\begin{aligned} \frac{1}{M_o(A\lambda, B\mu, t)} - 1 &\leq a \left(\frac{1}{M_o(h\lambda, g\mu, t)} - 1 \right) \\ &+ b \left(\frac{M_o(h\lambda, g\mu, t)}{M_o(h\lambda, B\mu, 2t) * M_o(g\mu, A\lambda, 2t)} - 1 \right) \\ &+ c \left(\frac{1}{M_r(h\lambda, A\lambda, t)} - 1 + \frac{1}{M_r(g\mu, B\mu, t)} - 1 \right), \end{aligned} \quad (29)$$

$\forall \lambda, \mu \in U, t \gg \theta$, and $0 \leq a, b, c < 1$ with $(a + b + 2c) < 1$. If $A(U) \subseteq g(U)$, $B(U) \subseteq h(U)$ and consider that

- (1) h is a continuous self-mapping
- (2) A pair (A, h) is compatible, and
- (3) A pair (B, g) is weakly-compatible

Then, the mappings A, B, g , and h have a unique CFP in U .

Corollary 12. Let $A, B, g, h : U \longrightarrow U$ be the four self-mappings on a complete FCM space $(U, M_o, *)$ in which a FCM M_o is triangular and satisfies

$$\begin{aligned} \frac{1}{M_o(A\lambda, B\mu, t)} - 1 &\leq a \left(\frac{1}{M_o(h\lambda, g\mu, t)} - 1 \right) \\ &+ b \left(\frac{M_o(h\lambda, g\mu, t)}{M_o(h\lambda, B\mu, 2t) * M_o(g\mu, A\lambda, 2t)} - 1 \right) \\ &+ d \left(\frac{1}{M_r(g\mu, A\lambda, t)} - 1 + \frac{1}{M_r(h\lambda, B\mu, t)} - 1 \right), \end{aligned} \quad (30)$$

$\forall \lambda, \mu \in U, t \gg \theta$, and $0 \leq a, b, d < 1$ with $(a + b + 2d) < 1$. If $A(U) \subseteq g(U)$, $B(U) \subseteq h(U)$ and consider that

- (1) h is a continuous self-mapping
- (2) A pair (A, h) is compatible, and
- (3) A pair (B, g) is weakly-compatible

Then, the mappings A, B, g , and h have a unique CFP in U .

Corollary 13. Let $A, B, g, h : U \longrightarrow U$ be the four self-mappings on a complete FCM space $(U, M_o, *)$ in which a FCM M_o is triangular and satisfies

$$\begin{aligned} \frac{1}{M_o(A\lambda, B\mu, t)} - 1 &\leq a \left(\frac{1}{M_o(h\lambda, g\mu, t)} - 1 \right) \\ &+ c \left(\frac{1}{M_r(h\lambda, A\lambda, t)} - 1 + \frac{1}{M_r(g\mu, B\mu, t)} - 1 \right) \\ &+ d \left(\frac{1}{M_r(g\mu, A\lambda, t)} - 1 + \frac{1}{M_r(h\lambda, B\mu, t)} - 1 \right), \end{aligned} \quad (31)$$

$\forall \lambda, \mu \in U, t \gg \theta$, and $0 \leq a, c, d < 1$ with $(a + 2c + 2d) < 1$. If $A(U) \subseteq g(U)$, $B(U) \subseteq h(U)$ and consider that

- (1) h is a continuous self-mapping
- (2) A pair (A, h) is compatible, and
- (3) A pair (B, g) is weakly-compatible

Then, the mappings A, B, g , and h have a CFP in U .

Example 14. Assume that $U = [0, \infty)$, $*$ is a continuous t -norm and $M_o : U \times U \times (0, \infty) \longrightarrow [0, 1]$ be written as

$$M_o(\lambda, \mu, t) = \frac{t}{t + |\lambda - \mu|}, \forall \lambda, \mu \in U, t \gg \theta. \quad (32)$$

Then, it is easy to verify that FCM M_o is triangular and $(U, M_o, *)$ is a complete FCM space. Now, the mappings, $A, g, h, B : U \longrightarrow U$, be defined by, (for all $\lambda \in U$);

$$A(\lambda) = B(\lambda) = \begin{cases} \frac{1}{3} \left(\frac{3\lambda}{4} + \frac{1}{8} \right), & \text{if } \lambda \neq 0, \\ 0, & \text{if } \lambda = 0, \end{cases} \quad (33)$$

and

$$h(\lambda) = g(\lambda) = \begin{cases} \left(\frac{3\lambda}{4} + \frac{1}{8} \right), & \text{if } \lambda \neq 0, \\ 0, & \text{if } \lambda = 0. \end{cases} \quad (34)$$

Since, from the above equation, $A(U) = B(U)$ and $g(U) = h(U)$, so that we conclude that $A(U) \subseteq g(U)$ or $B(U) \subseteq h(U)$. Then,

$$\frac{1}{M_o(h\lambda, g\mu, t)} - 1 = \frac{|h\lambda - g\mu|}{t} = \frac{3|\lambda - \mu|}{4t}, \text{ for } t \gg \theta, \quad (35)$$

and

$$\frac{1}{M_o(A\lambda, B\mu, t)} - 1 = \frac{|A(\lambda) - B(\mu)|}{t} = \frac{1}{3} \left(\frac{1}{M_o(h\lambda, g\mu, t)} - 1 \right) \text{ for } t \gg \theta. \quad (36)$$

Hence, the mappings A, B, g , and h on U satisfying the

fuzzy cone-contraction condition in FCM space. Now, from Definition 2 (3), $M_r(h\lambda, B\mu, 2t) \geq M_r(h\lambda, g\mu, t) * M_r(g\mu, B\mu, t)$ and $M_r(g\mu, A\lambda, 2t) \geq M_r(g\mu, h\lambda, t) * M_r(h\lambda, A\lambda, t)$, for $t \gg \theta$. Now, we calculate the following terms of (4), for $t \gg \theta$,

$$\begin{aligned}
& \frac{M_o(h\lambda, g\mu, t)}{M_o(h\lambda, B\mu, 2t) * M_r(g\mu, A\lambda, 2t)} - 1 \\
& \leq \frac{1}{M_o(h\lambda, g\mu, t) * M_r(h\lambda, A\lambda, t) * M_r(g\mu, B\mu, t)} - 1 \\
& = \frac{1}{(t/t + |h\lambda - g\mu|) \cdot (t/t + |h\lambda - A\lambda|) \cdot (t/t + |g\mu - B\mu|)} - 1 \\
& = \frac{(t + |h\lambda - g\mu|) \cdot (t + |h\lambda - A\lambda|) \cdot (t + |g\mu - B\mu|)}{t^3} - 1 \\
& = \frac{(t + 3/4|\lambda - \mu|) \cdot [(t + 1/12(6\lambda + 1)) \cdot (t + 1/12(6\mu + 1))] }{t^3} - 1 \\
& = \frac{(t + 3/4|\lambda - \mu|) \cdot [1/144(6\lambda + 1) \cdot (6\mu + 1) + (t/6)(3\lambda + 3\mu + 1) + (3t^2/4)|\lambda - \mu|] }{t^3} \\
& = \frac{(t + 3/4|\lambda - \mu|) \cdot [1/144(6\lambda + 1) \cdot (6\mu + 1) + (t/6)(3\lambda + 3\mu + 1)] }{t^3} + \frac{3}{4t}|\lambda - \mu|.
\end{aligned} \tag{37}$$

Next, we calculate

$$\begin{aligned}
& \left(\frac{1}{M_r(h\lambda, A\lambda, t)} - 1 + \frac{1}{M_r(g\mu, B\mu, t)} - 1 \right) \\
& = \frac{1}{t/t + |h\lambda - A\lambda|} - 1 + \frac{1}{t/t + |g\mu - B\mu|} - 1 \\
& = \frac{|h\lambda - A\lambda|}{t} + \frac{|g\mu - B\mu|}{t} = \frac{1}{6t}(3\lambda + 3\mu + 1) \text{ for } t \gg \theta.
\end{aligned} \tag{38}$$

Similarly,

$$\begin{aligned}
& \left(\frac{1}{M_r(g\mu, A\lambda, t)} - 1 + \frac{1}{M_r(h\lambda, B\mu, t)} - 1 \right) \\
& = \frac{1}{t/t + |g\mu - A\lambda|} - 1 + \frac{1}{t/t + |h\lambda - B\mu|} - 1 \\
& = \frac{|g\mu - A\lambda|}{t} + \frac{|h\lambda - B\mu|}{t} = \frac{1}{6t}(3\lambda + 3\mu + 1) \text{ for } t \gg \theta.
\end{aligned} \tag{39}$$

Thus, after routine calculation, all the conditions of Theorem 10 are satisfied with $a = 1/3$, $b = 2/7$, and $c = d = 1/14$, and the mappings A, B, g , and h on U have a unique CFP, i.e., $0 \in U$.

If we choose the self-mappings $B = A$ and $g = h$ in Theorem 10, we obtain the following corollary.

Corollary 15. Let $B, h : U \longrightarrow U$ be two self-mappings on a complete FCM space $(U, M_o, *)$ in which a FCM M_o is triangular and satisfies

$$\begin{aligned}
\frac{1}{M_o(B\lambda, B\mu, t)} - 1 & \leq a \left(\frac{1}{M_o(h\lambda, h\mu, t)} - 1 \right) \\
& + b \left(\frac{M_o(h\lambda, h\mu, t)}{M_o(h\lambda, B\mu, 2t) * M_o(h\mu, B\lambda, 2t)} - 1 \right) \\
& + c \left(\frac{1}{M_r(h\lambda, B\lambda, t)} - 1 + \frac{1}{M_r(h\mu, B\mu, t)} - 1 \right) \\
& + d \left(\frac{1}{M_r(h\mu, B\lambda, t)} - 1 + \frac{1}{M_r(h\lambda, B\mu, t)} - 1 \right),
\end{aligned} \tag{40}$$

$\forall \lambda, \mu \in U$, $t \gg \theta$, and $0 \leq a, b, c, d < 1$ with $(a + b + 2c + 2d) < 1$. If $B(U) \subseteq h(U)$, h is a continuous self-mapping, and a pair (B, h) is weakly-compatible. Then, the mappings B and h have a unique CFP in U .

Next result, we shall prove without the continuity of self-mapping, i.e., h and we replaced the completeness of U by the completeness of $B(U)$ or $h(U)$.

Theorem 16. Let $B, h : U \longrightarrow U$ be two self-mappings on a complete FCM space $(U, M_o, *)$ in which a FCM M_o is triangular and satisfies

$$\begin{aligned}
\frac{1}{M_o(B\lambda, B\mu, t)} - 1 & \leq a \left(\frac{1}{M_o(h\lambda, h\mu, t)} - 1 \right) \\
& + b \left(\frac{M_o(h\lambda, h\mu, t)}{M_o(h\lambda, B\mu, 2t) * M_o(h\mu, B\lambda, 2t)} - 1 \right) \\
& + c \left(\frac{1}{M_r(h\lambda, B\lambda, t)} - 1 + \frac{1}{M_r(h\mu, B\mu, t)} - 1 \right) \\
& + d \left(\frac{1}{M_r(h\mu, B\lambda, t)} - 1 + \frac{1}{M_r(h\lambda, B\mu, t)} - 1 \right),
\end{aligned} \tag{41}$$

$\forall \lambda, \mu \in U$, $t \gg \theta$, and $0 \leq a, b, c, d < 1$ with $(a + b + 2c + 2d) < 1$. If $B(U) \subseteq h(U)$, $B(U)$ or $h(U)$ is complete and a pair (B, h) is weakly-compatible. Then, the mappings B and h have a unique CFP in U .

Proof. From the proof of Theorem 10, we assume that $\{\xi_j\}_{j \geq 0}$ is a Cauchy sequence in $h(U)$, and the iterative sequences are earlier defined in the proof of Theorem 10, that are,

$$\xi_{2j+1} = h\lambda_{2j+1} = B\lambda_{2j} \text{ and } \xi_{2j+2} = h\lambda_{2j+2} = B\lambda_{2j+1}, j \geq 0. \tag{42}$$

We know that $h(U)$ is complete, and $\exists \xi, \rho \in U$, so that $\xi_{2j+1} \longrightarrow \xi = h\rho$, as $j \longrightarrow \infty$. Therefore,

$$\lim_{j \longrightarrow \infty} M_o(\xi_{2j+1}, \xi, t) = \lim_{j \longrightarrow \infty} M_o(h\lambda_{2j+1}, \xi, t) = 1 \text{ for } t \gg \theta. \tag{43}$$

Since by M_o triangular property,

$$\frac{1}{M_o(h\rho, B\rho, t)} - 1 \leq \left(\frac{1}{M_o(h\rho, \xi_{2j+2}, t)} - 1 \right) + \left(\frac{1}{M_o(\xi_{2j+2}, B\rho, t)} - 1 \right) \text{ for } t \gg \theta. \quad (44)$$

Now by the view of (41), (43), and by using Definition 2 (3), for $t \gg \theta$, we have that

$$\begin{aligned} \frac{1}{M_o(\xi_{2j+2}, B\rho, t)} - 1 &= \frac{1}{M_o(B\lambda_{2j+1}, B\rho, t)} - 1 \leq a \left(\frac{1}{M_o(h\lambda_{2j+1}, h\rho, t)} - 1 \right) \\ &+ b \left(\frac{M_o(h\lambda_{2j+1}, h\rho, t)}{M_o(h\lambda_{2j+1}, B\rho, 2t) * M_o(h\rho, B\lambda_{2j+1}, 2t)} - 1 \right) \\ &+ c \left(\frac{1}{M_r(h\lambda_{2j+1}, B\lambda_{2j+1}, t)} - 1 + \frac{1}{M_r(h\rho, B\rho, t)} - 1 \right) \\ &+ d \left(\frac{1}{M_r(h\rho, B\lambda_{2j+1}, t)} - 1 + \frac{1}{M_r(h\lambda_{2j+1}, B\rho, t)} - 1 \right) \\ &= a \left(\frac{1}{M_o(h\lambda_{2j+1}, h\rho, t)} - 1 \right) + b \left(\frac{M_o(h\lambda_{2j+1}, h\rho, t)}{M_o(h\lambda_{2j+1}, B\rho, 2t) * M_o(h\rho, h\lambda_{2j+2}, 2t)} - 1 \right) \\ &+ c \left(\frac{1}{M_r(h\lambda_{2j+1}, h\lambda_{2j+2}, t)} - 1 + \frac{1}{M_r(h\rho, B\rho, t)} - 1 \right) \\ &+ d \left(\frac{1}{M_r(h\rho, h\lambda_{2j+2}, t)} - 1 + \frac{1}{M_r(h\lambda_{2j+1}, B\rho, t)} - 1 \right) \\ &\leq a \left(\frac{1}{M_o(h\lambda_{2j+1}, h\rho, t)} - 1 \right) \\ &+ b \left(\frac{M_o(h\lambda_{2j+1}, h\rho, t)}{M_o(h\lambda_{2j+1}, h\rho, t) * M_o(h\rho, B\rho, t) * M_o(h\rho, h\lambda_{2j+2}, 2t)} - 1 \right) \\ &+ c \left(\frac{1}{M_r(h\lambda_{2j+1}, h\lambda_{2j+2}, t)} - 1 + \frac{1}{M_r(h\rho, B\rho, t)} - 1 \right) \\ &+ d \left(\frac{1}{M_r(h\rho, h\lambda_{2j+2}, t)} - 1 + \frac{1}{M_r(h\lambda_{2j+1}, B\rho, t)} - 1 \right) \\ &\longrightarrow (b + c + d) \left(\frac{1}{M_r(h\rho, B\rho, t)} - 1 \right), \text{ as } j \longrightarrow \infty. \end{aligned} \quad (45)$$

Hence,

$$\limsup_{j \longrightarrow \infty} \left(\frac{1}{M_o(\xi_{2j+2}, B\rho, t)} - 1 \right) \leq (b + c + d) \left(\frac{1}{M_r(h\rho, B\rho, t)} - 1 \right), \text{ for } t \gg \theta. \quad (46)$$

Now from (44) and (46),

$$\begin{aligned} \frac{1}{M_o(h\rho, B\rho, t)} - 1 &\leq (b + c + d) \left(\frac{1}{M_r(h\rho, B\rho, t)} - 1 \right), \\ &\Rightarrow (1 - b - c - d) \left(\frac{1}{M_r(h\rho, B\rho, t)} - 1 \right) \leq 0, \text{ for } t \gg \theta. \end{aligned} \quad (47)$$

Since, $(1 - b - c - d) \neq 0$, therefore, $M_r(h\rho, B\rho, t) = 1 \Rightarrow h\rho = B\rho = \xi$, for $t \gg \theta$. Now by the weak-compatibility of (B, h) ,

$$B\xi = B(h\rho) = h(B\rho) = h\xi. \quad (48)$$

Next, we have to prove that $B\xi = \xi$. Then, again by view

of (41) and by using Definition 2 (3), for $t \gg \theta$, we have

$$\begin{aligned} \frac{1}{M_o(B\xi, \xi, t)} - 1 &= \frac{1}{M_o(B\xi, B\rho, t)} - 1 \leq a \left(\frac{1}{M_o(h\xi, h\rho, t)} - 1 \right) \\ &+ b \left(\frac{M_o(h\xi, h\rho, t)}{M_o(h\xi, B\rho, 2t) * M_o(h\rho, B\xi, 2t)} - 1 \right) \\ &+ c \left(\frac{1}{M_r(h\xi, B\xi, t)} - 1 + \frac{1}{M_r(h\rho, B\rho, t)} - 1 \right) \\ &+ d \left(\frac{1}{M_r(h\rho, B\xi, t)} - 1 + \frac{1}{M_r(h\xi, B\rho, t)} - 1 \right) \\ &\leq a \left(\frac{1}{M_o(B\xi, \xi, t)} - 1 \right) \\ &+ b \left(\frac{M_o(h\xi, h\rho, t)}{M_o(h\xi, h\rho, t) * M_o(h\rho, B\rho, t) * M_o(h\rho, \xi, t) * M_o(\xi, B\xi, t)} - 1 \right) \\ &+ c \left(\frac{1}{M_r(h\xi, B\xi, t)} - 1 + \frac{1}{M_r(h\rho, B\rho, t)} - 1 \right) \\ &+ d \left(\frac{1}{M_r(\xi, B\xi, t)} - 1 + \frac{1}{M_r(B\xi, \xi, t)} - 1 \right) \\ &= (a + b + 2d) \left(\frac{1}{M_r(\xi, B\xi, t)} - 1 \right). \end{aligned} \quad (49)$$

Hence,

$$\begin{aligned} \frac{1}{M_o(B\xi, \xi, t)} - 1 &\leq (a + b + 2d) \left(\frac{1}{M_r(B\xi, \xi, t)} - 1 \right), \\ &\Rightarrow (1 - a - b - 2d) \left(\frac{1}{M_r(B\xi, \xi, t)} - 1 \right) \\ &\leq 0, \text{ for } t \gg \theta. \end{aligned} \quad (50)$$

As $(1 - a - b - 2d) \neq 0$, therefore, $M_r(B\xi, \xi, t) = 1 \Rightarrow B\xi = \xi$, for $t \gg \theta$. Hence, proved that $B\xi = h\xi = \xi$. $\square$

Uniqueness: let $\exists \eta \in U$ be the other CFP of the mapping B and h in U such that $B\eta = h\eta = \eta$. Then by the view of (41) and by using Definition 2 (3),

$$\begin{aligned} \frac{1}{M_o(\xi, \eta, t)} - 1 &= \frac{1}{M_o(B\xi, B\eta, t)} - 1 \leq a \left(\frac{1}{M_o(h\xi, h\eta, t)} - 1 \right) \\ &+ b \left(\frac{M_o(h\xi, h\eta, t)}{M_o(h\xi, B\eta, 2t) * M_o(h\eta, B\xi, 2t)} - 1 \right) \\ &+ c \left(\frac{1}{M_r(h\xi, B\xi, t)} - 1 + \frac{1}{M_r(h\eta, B\eta, t)} - 1 \right) \\ &+ d \left(\frac{1}{M_r(h\eta, B\xi, t)} - 1 + \frac{1}{M_r(h\xi, B\eta, t)} - 1 \right) \\ &\leq a \left(\frac{1}{M_o(\xi, \eta, t)} - 1 \right) \\ &+ b \left(\frac{M_o(\xi, \eta, t)}{M_o(\xi, \eta, t) * M_o(\eta, \eta, t) * M_o(\eta, \xi, t) * M_o(\xi, \xi, t)} - 1 \right) \\ &+ c \left(\frac{1}{M_r(\xi, \xi, t)} - 1 + \frac{1}{M_r(\eta, \eta, t)} - 1 \right) \\ &+ d \left(\frac{1}{M_r(\eta, \xi, t)} - 1 + \frac{1}{M_r(\xi, \eta, t)} - 1 \right) = (a + b + 2d) \left(\frac{1}{M_r(\eta, \xi, t)} - 1 \right). \end{aligned} \quad (51)$$

Hence,

$$\begin{aligned} \frac{1}{M_r(\eta, \xi, t)} - 1 &\leq (a + b + 2d) \left(\frac{1}{M_r(\eta, \xi, t)} - 1 \right), \\ \Rightarrow (1 - a - b - 2d) \left(\frac{1}{M_r(\eta, \xi, t)} - 1 \right) &\leq 0, \text{ for } t \gg \theta. \end{aligned} \quad (52)$$

As $(1 - a - b - 2d) \neq 0$, therefore, $M_r(\xi, \eta, t) = 1 \Rightarrow \xi = \eta$, for $t \gg \theta$. Hence, proved that the mappings B, h have a unique CFP in U .

Corollary 17. Let $B, h : U \longrightarrow U$ be two self-mappings on a complete FCM space $(U, M_o, *)$ in which a FCM M_o is triangular and satisfies

$$\begin{aligned} \frac{1}{M_o(B\lambda, B\mu, t)} - 1 &\leq a \left(\frac{1}{M_o(h\lambda, h\mu, t)} - 1 \right) \\ &+ b \left(\frac{M_o(h\lambda, h\mu, t)}{M_o(h\lambda, B\mu, 2t) * M_o(h\mu, B\lambda, 2t)} - 1 \right) \\ &+ c \left(\frac{1}{M_r(h\lambda, B\lambda, t)} - 1 + \frac{1}{M_r(h\mu, B\mu, t)} - 1 \right), \end{aligned} \quad (53)$$

$\forall \lambda, \mu \in U, t \gg \theta$, and $0 \leq a, b, c < 1$ with $(a + b + 2c) < 1$. If $B(U) \subseteq h(U)$, $B(U)$ or $h(U)$ is complete and a pair (B, h) is weakly-compatible. Then, the mappings B, h have a unique CFP in U .

Corollary 18. Let $B, h : U \longrightarrow U$ be two self-mappings on a complete FCM space $(U, M_o, *)$ in which a FCM M_o is triangular and satisfies

$$\begin{aligned} \frac{1}{M_o(B\lambda, B\mu, t)} - 1 &\leq a \left(\frac{1}{M_o(h\lambda, h\mu, t)} - 1 \right) \\ &+ b \left(\frac{M_o(h\lambda, h\mu, t)}{M_o(h\lambda, B\mu, 2t) * M_o(h\mu, B\lambda, 2t)} - 1 \right) \\ &+ d \left(\frac{1}{M_r(h\mu, B\lambda, t)} - 1 + \frac{1}{M_r(h\lambda, B\mu, t)} - 1 \right), \end{aligned} \quad (54)$$

$\forall \lambda, \mu \in U, t \gg \theta$, and $0 \leq a, b, d < 1$ with $(a + b + 2d) < 1$. If $B(U) \subseteq h(U)$, $B(U)$ or $h(U)$ is complete and a pair (B, h) is weakly-compatible. Then, the mappings B, h have a unique CFP in U .

Corollary 19. Let $B, h : U \longrightarrow U$ be two self-mappings on a complete FCM space $(U, M_o, *)$ in which a FCM M_o is triangular and satisfies

$$\begin{aligned} \frac{1}{M_o(B\lambda, B\mu, t)} - 1 &\leq a \left(\frac{1}{M_o(h\lambda, h\mu, t)} - 1 \right) \\ &+ b \left(\frac{M_o(h\lambda, h\mu, t)}{M_o(h\lambda, B\mu, 2t) * M_o(h\mu, B\lambda, 2t)} - 1 \right), \end{aligned} \quad (55)$$

for all $\lambda, \mu \in U, t \gg \theta$, and $0 \leq a, b < 1$ with $(a + b) < 1$. If $B(U) \subseteq h(U)$, $B(U)$ or $h(U)$ is complete and a pair (B, h) is

weakly-compatible. Then, the mappings B, h have a unique CFP in U .

Corollary 20. Let $B, h : U \longrightarrow U$ be two self-mappings on a complete FCM space $(U, M_o, *)$ in which a FCM M_o is triangular and satisfies

$$\begin{aligned} \frac{1}{M_o(B\lambda, B\mu, t)} - 1 &\leq a \left(\frac{1}{M_o(h\lambda, h\mu, t)} - 1 \right) \\ &+ c \left(\frac{1}{M_r(h\lambda, B\lambda, t)} - 1 + \frac{1}{M_r(h\mu, B\mu, t)} - 1 \right) \\ &+ d \left(\frac{1}{M_r(h\mu, B\lambda, t)} - 1 + \frac{1}{M_r(h\lambda, B\mu, t)} - 1 \right), \end{aligned} \quad (56)$$

$\forall \lambda, \mu \in U, t \gg \theta$, and $0 \leq a, c, d < 1$ with $(a + 2c + 2d) < 1$. If $B(U) \subseteq h(U)$, $B(U)$ or $h(U)$ is complete and a pair (B, h) is weakly-compatible. Then, the mappings B, h have a unique CFP in U .

Example 21. From Example 14, let us define $B, h : U \longrightarrow U$, for all $\lambda \in U$ such that

$$B(\lambda) = \begin{cases} \frac{1}{6} \left(\frac{3\lambda}{4} + \frac{1}{4} \right), & \text{if } \lambda \neq 0, \\ 0, & \text{if } \lambda = 0, \end{cases} \quad (57)$$

and

$$h(\lambda) = \begin{cases} \left(\frac{3\lambda}{4} + \frac{1}{4} \right), & \text{if } \lambda \neq 0, \\ 0, & \text{if } \lambda = 0. \end{cases} \quad (58)$$

Hence, from the above equations, we have that $B(U) \subseteq h(U)$. Then, for $t \gg \theta$, we have that

$$\begin{aligned} \frac{1}{M_o(B\lambda, B\mu, t)} - 1 &= \frac{|B\lambda - B\mu|}{t} = \frac{|\lambda - \mu|}{8t} \leq \frac{12\lambda + 3\mu + 5}{36t} \\ &\leq \frac{1}{8t} |\lambda - \mu| + \frac{5}{72t} (3\lambda + 3\mu + 2) \\ &= \frac{1}{6t} \left| \frac{3\lambda}{4} - \frac{3\mu}{4} \right| + \frac{5}{144t} (3\lambda + 3\mu + 2) + \frac{5}{144t} (3\lambda + 3\mu + 2) \\ &= \frac{1}{6t} \left| \frac{3\lambda}{4} - \frac{3\mu}{4} \right| + \frac{1}{6t} \left(\frac{5}{24} (3\lambda + 3\mu + 2) \right) + \frac{1}{6t} \left(\frac{5}{24} (3\lambda + 3\mu + 2) \right) \\ &\leq \frac{1}{6} \left(\frac{1}{M_o(h\lambda, h\mu, t)} - 1 \right) + \frac{1}{6} \left(\frac{5}{24t} (3\lambda + 1) + \frac{5}{24t} (3\mu + 1) \right) \\ &\quad + \frac{1}{6} \left(\frac{1}{24t} |18\mu - 3\lambda + 5| + \frac{1}{24t} |18\lambda - 3\mu + 5| \right) \\ &= \frac{1}{6} \left(\frac{1}{M_o(h\lambda, h\mu, t)} - 1 \right) + \frac{1}{6} \left(\left| \frac{h\lambda - B\lambda}{t} \right| + \left| \frac{h\mu - B\mu}{t} \right| \right) \\ &\quad + \frac{1}{6} \left(\left| \frac{h\mu - B\lambda}{t} \right| + \left| \frac{h\lambda - B\mu}{t} \right| \right) = \frac{1}{6} \left(\frac{1}{M_o(h\lambda, h\mu, t)} - 1 \right) \\ &\quad + c \left(\frac{1}{M_r(h\lambda, B\lambda, t)} - 1 + \frac{1}{M_r(h\mu, B\mu, t)} - 1 \right) \\ &\quad + \frac{1}{6} \left(\frac{1}{M_r(h\mu, B\lambda, t)} - 1 + \frac{1}{M_r(h\lambda, B\mu, t)} - 1 \right). \end{aligned} \quad (59)$$

Thus, after simplification, we conclude that all the

conditions of Corollary 20 are satisfied with $a = c = d = 1/6$ and the mappings B, h have a unique CFP, i.e., $0 \in U$.

4. Supportive Application

This section will deal with the nonlinear integral equations (NIEs) to support our main work. Let $U = C([0, q], \mathbb{R})$ be the space of all $\mathbb{R}$ -valued continuous functions on $[0, q]$, where $0 < q \in \mathbb{R}$. The two NIEs are

$$\lambda(\tau) = \int_0^p Q_1(\tau, v, \lambda(v))dv, \text{ and } \mu(\tau) = \int_0^p Q_2(\tau, v, \mu(v))dv, \forall \lambda, \mu \in U, \quad (60)$$

where $\tau \in [0, q]$ and $Q_1, Q_2 : [0, q] \times [0, q] \times \mathbb{R} \rightarrow \mathbb{R}$, and a metric $d : U \times U \rightarrow \mathbb{R}$ is defined by

$$d(\lambda, \mu) = \sup_{\tau \in [0, q]} |\lambda(\tau) - \mu(\tau)| = \|\lambda - \mu\|, \text{ where } \lambda, \mu \in C([0, q], \mathbb{R}) = U. \quad (61)$$

The operation $*$ is defined by $\rho_1 * \rho_2 = \rho_1 \rho_2$ for all $\rho_1, \rho_2 \in [0, q]$. A FM $M_o : U \times U \times (0, \infty) \rightarrow [0, 1]$ is defined by

$$M_r(\lambda, \mu, t) = \frac{t}{t + d(\lambda, \mu)}, \text{ for } t > 0, \forall \lambda, \mu \in U. \quad (62)$$

Then, FM M_o is triangular and $(U, M_o, *)$ is a complete FCM space.

Theorem 22. *The two NIEs are*

$$\lambda(\tau) = \int_0^q Q_1(\tau, v, x(v))dv, \text{ and } \mu(\tau) = \int_0^q Q_2(\tau, v, \mu(v))dv, \quad (63)$$

where $\tau, v \in [0, q]$ and $\lambda, \mu \in U$. Assume that $Q_1, Q_2 : [0, q] \times [0, q] \times \mathbb{R} \rightarrow \mathbb{R}$ are so that $A_\lambda, B_\mu \in U$ for all $\lambda, \mu \in U$, where

$$A_\lambda(\tau) = \int_0^q Q_1(\tau, v, \lambda(v))dv, \text{ and } B_\mu(\tau) = \int_0^q Q_2(\tau, v, \mu(v))dv. \quad (64)$$

If $\exists 0 < k^* < 1$ such that $\forall \lambda, \mu \in U$,

$$\|A_\lambda - B_\mu\| \leq k^* N_{(A, B, g, h, \lambda, \mu)}, \quad (65)$$

where

$$N_{(A, B, g, h, \lambda, \mu)} = \max \left\{ \begin{aligned} &\|\lambda - \mu\|, \|\lambda - A_\lambda\| + \|\mu - B_\mu\|, \|\mu - A_\lambda\| + \|\lambda - B_\mu\|, \\ &\frac{1}{t^2} (t + \|\lambda - \mu\|) (t \|\lambda - A_\lambda\| + t \|\mu - B_\mu\| + \|\lambda - A_\lambda\| \cdot \|\mu - B_\mu\|) \end{aligned} \right\}, \quad (66)$$

then, the two nonlinear integral equations defined in (60) have a unique common solution in U .

Proof. Define the integral operators $A, B, g, h : U \rightarrow U$ by

$$A(\lambda) = A_\lambda, B(\mu) = B_\mu, g(\mu) = \mu \text{ and } h(\lambda) = \lambda. \quad (67)$$

The NIEs in (60) have a unique common solution iff A, B, g , and h have a unique CFP in U . Now, we prove that Theorem 10 applies to the integral operators A, B, g , and h . Then, $\forall \lambda, \mu \in U$, we may have the following four cases arises:

(a) If $N_{(A, B, g, h, \lambda, \mu)} = \|\lambda - \mu\|$ in (66). Then, by using (62), (65), and (67), we have

$$\begin{aligned} \frac{1}{M_r(A\lambda, B\mu, t)} - 1 &= \frac{d(A\lambda, B\mu)}{t} \leq k^* \frac{N_{(A, B, g, h, \lambda, \mu)}}{t} = k^* \frac{\|\lambda - \mu\|}{t} \\ &= k^* \left(\frac{1}{M_r(h\lambda, g\mu, t)} - 1 \right). \end{aligned} \quad (68)$$

This implies that

$$\frac{1}{M_r(A\lambda, B\mu, t)} - 1 \leq k^* \left(\frac{1}{M_r(h\lambda, g\mu, t)} - 1 \right), \text{ for } t \gg \theta, \quad (69)$$

$\forall \lambda, \mu \in U$. Hence, the integral operators A, B, g , and h satisfy the conditions of Theorem 10 with $k^* = a$ and $b = c = d = 0$ in (4). The operators A, B, g , and h have a unique CFP, i.e., the unique common solution of the two NIEs (60) in U .

(b) If $N_{(A, B, g, h, \lambda, \mu)} = 1/t^2 (t + \|\lambda - \mu\|) (t \|\lambda - A_\lambda\| + t \|\mu - B_\mu\| + \|\lambda - A_\lambda\| \cdot \|\mu - B_\mu\|)$ in (66). Then, by using (62) and (65), we have

$$\begin{aligned} \frac{1}{M_r(A\lambda, B\mu, t)} - 1 &= \frac{d(A\lambda, B\mu)}{t} \leq k^* \frac{N_{(A, B, g, h, \lambda, \mu)}}{t} \\ &= k^* \frac{1}{t^3} (t + \|\lambda - \mu\|) (t \|\lambda - A_\lambda\| + t \|\mu - B_\mu\| \\ &\quad + \|\lambda - A_\lambda\| \cdot \|\mu - B_\mu\|). \end{aligned} \quad (70)$$

It yields that

$$\begin{aligned} \frac{1}{M_r(A\lambda, B\mu, t)} - 1 &\leq k^* \frac{1}{t^3} (t + \|\lambda - \mu\|) (t \|\lambda - A_\lambda\| + t \|\mu - B_\mu\| \\ &\quad + \|\lambda - A_\lambda\| \cdot \|\mu - B_\mu\|), \end{aligned} \quad (71)$$

$\forall \lambda, \mu \in U$, and $t \gg \theta$. Now first, we have to simplify the term $M_o(h\lambda, g\mu, t)/M_o(h\lambda, B\mu, 2t) * M_o(g\mu, A\lambda, 2t) - 1$ by

using Definition 2 (3) and (62), for $t \gg \theta$,

$$\begin{aligned}
 & \frac{M_o(h\lambda, g\mu, t)}{M_o(h\lambda, B\mu, 2t) * M_o(g\mu, A\lambda, 2t)} - 1 \\
 & \leq \frac{M_o(h\lambda, g\mu, t)}{M_o(h\lambda, g\mu, t) * M_o(g\mu, B\mu, t) * M_r(g\mu, h\lambda, t) * M_o(h\lambda, A\lambda, t)} - 1 \\
 & = \frac{1}{M_r(h\lambda, g\mu, t) * M_r(g\mu, B\mu, t) * M_r(h\lambda, A\lambda, t)} - 1 \\
 & = \frac{1}{(t + \|h\lambda - g\mu\|)(t + \|g\mu - B\mu\|)(t + \|h\lambda - A\lambda\|)} - 1 \\
 & = \frac{1}{t^3} \left(t^2 \|h\lambda - g\mu\| + t^2 (\|g\mu - B\mu\| + \|h\lambda - A\lambda\|) + t \|h\lambda - g\mu\| (\|g\mu - B\mu\| + \|h\lambda - A\lambda\|) \right) \\
 & = \frac{1}{t^3} \left(t^2 \|h\lambda - g\mu\| + t^2 (\|g\mu - B\mu\| + \|h\lambda - A\lambda\|) + t \|h\lambda - g\mu\| \cdot \|g\mu - B\mu\| + t \|h\lambda - g\mu\| \cdot \|h\lambda - A\lambda\| + t \|g\mu - B\mu\| \cdot \|h\lambda - A\lambda\| \right) \\
 & = \frac{1}{t^3} (t^2 \|h\lambda - g\mu\| + (t + \|h\lambda - g\mu\|)(t \|g\mu - B\mu\| + t \|h\lambda - A\lambda\| + \|h\lambda - A\lambda\| \cdot \|g\mu - B\mu\|)).
 \end{aligned} \tag{72}$$

This implies that

$$\begin{aligned}
 & \frac{M_o(h\lambda, g\mu, t)}{M_o(h\lambda, B\mu, 2t) * M_o(g\mu, A\lambda, 2t)} - 1 \\
 & \leq \frac{1}{t^3} (t^2 \|h\lambda - g\mu\| + (t + \|h\lambda - g\mu\|)(t \|g\mu - B\mu\| \\
 & \quad + t \|h\lambda - A\lambda\| + \|h\lambda - A\lambda\| \cdot \|g\mu - B\mu\|)),
 \end{aligned} \tag{73}$$

$\forall \lambda, \mu \in U$, and $t \gg \theta$. Now, from (71) and (73), we have

$$\frac{1}{M_r(A\lambda, B\mu, t)} - 1 \leq k^* \left(\frac{M_o(h\lambda, g\mu, t)}{M_o(h\lambda, B\mu, 2t) * M_o(g\mu, A\lambda, 2t)} - 1 \right), \text{ for } t \gg \theta, \tag{74}$$

$\forall \lambda, \mu \in U$. Hence, the integral operators A, B, g , and h satisfy the conditions of Theorem 10 with $k^* = b$ and $a = c = d = 0$ in (4). The operators A, B, g , and h have a unique CFP, i.e., the unique common solution of the two NIEs (60) in U .

(c) If $\mathbf{N}_{(A,B,g,h,\lambda,\mu)} = \|\lambda - A\lambda\| + \|\mu - B\mu\|$ in (66). Then by using (62), (65), and (67), we have

$$\begin{aligned}
 \frac{1}{M_r(A\lambda, B\mu, t)} - 1 &= \frac{d(A\lambda, B\mu)}{t} \leq k^* \frac{\mathbf{N}_{(A,B,g,h,\lambda,\mu)}}{t} \\
 &= k^* \frac{\|\lambda - A\lambda\| + \|\mu - B\mu\|}{t} \\
 &= k^* \left(\frac{1}{M_r(h\lambda, A\lambda, t)} - 1 + \frac{1}{M_r(g\mu, B\mu, t)} - 1 \right).
 \end{aligned} \tag{75}$$

This implies that

$$\frac{1}{M_r(A\lambda, B\mu, t)} - 1 \leq k^* \left(\frac{1}{M_r(h\lambda, A\lambda, t)} - 1 + \frac{1}{M_r(g\mu, B\mu, t)} - 1 \right), \text{ for } t \gg \theta, \tag{76}$$

for all $\lambda, \mu \in U$. Hence, the integral operators A, B, g , and h satisfy the conditions of Theorem 10 with $k^* = c$ and $a = b = d = 0$ in (4). The operators A, B, g , and h have a unique CFP, i.e., the unique common solution of the two NIEs (60) in U .

(d) If $\mathbf{N}_{(A,B,g,h,\lambda,\mu)} = \|\mu - A\lambda\| + \|\lambda - B\mu\|$ in (66). Then, by using (62), (65), and (67), we have

$$\begin{aligned}
 \frac{1}{M_r(A\lambda, B\mu, t)} - 1 &= \frac{d(A\lambda, B\mu)}{t} \leq k^* \frac{\mathbf{N}_{(A,B,g,h,\lambda,\mu)}}{t} \\
 &= k^* \frac{\|\mu - A\lambda\| + \|\lambda - B\mu\|}{t} \\
 &= k^* \left(\frac{1}{M_r(g\mu, A\lambda, t)} - 1 + \frac{1}{M_r(h\lambda, B\mu, t)} - 1 \right).
 \end{aligned} \tag{77}$$

This implies that

$$\frac{1}{M_r(A\lambda, B\mu, t)} - 1 \leq k^* \left(\frac{1}{M_r(g\mu, A\lambda, t)} - 1 + \frac{1}{M_r(h\lambda, B\mu, t)} - 1 \right), \text{ for } t \gg \theta, \tag{78}$$

for all $\lambda, \mu \in U$. Hence, the integral operators A, B, g , and h satisfy the conditions of Theorem 10 with $k^* = d$ and $a = b = c = 0$ in (4). The operators A, B, g , and h have a unique CFP, i.e., the unique common solution of the two NIEs (60) in U . $\square$

5. Conclusion

In this paper, we proved some unique CFP theorems by using the compatible and weakly-compatible four self-mappings in FCM space. We proved the results under the generalized rational contraction conditions in FCM spaces with help of one self-map are continuous. Moreover, we proved some rational contraction results with the weaker condition of self-map continuity. Ultimately, we provide an application of the two NIEs for our theoretical results that have been utilized to prove the existence common solution of the two NIEs to support our main work. This is an illustrative application of how FCM spaces can be used in other integral type operators.

Data Availability

Data sharing does not apply to this article as no data set were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Authors' Contributions

The authors have equally contributed to the final manuscript.

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