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Research Article

Common Fixed-Point Theorems in the Partial *b*-Metric Spaces and an Application to the System of Boundary Value Problems

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In this paper, we investigate the conditions for the existence of the common fixed points of generalized contractions in the partial b-metric spaces endowed with an arbitrary binary relation. We establish some unique common fixed-point theorems. The obtained results may generalize and improve earlier fixed-point results. We provide examples to illustrate our findings. As an application, we discuss the common solution to the system of boundary value problems.

1. Introduction, Preliminaries, and Motivations

The b-metric space was introduced by Czerwik [1]. It is obtained by modifying the triangle property of the metric space. Every metric is a b-metric, but the converse is not true. Almost all the fixed-point theorems in the metric spaces have been proved true in the b-metric spaces; for example, see [2–10] and references therein.

Matthews [11] introduced the notion of the partial metric space as a part of the study of denotational semantics of the dataflow network. In this space, the usual metric is replaced by a partial metric having a property that the self-distance of any point of the space may not be zero. Every metric is a partial metric, but the converse is not true. Matthews [11] also initiated the fixed-point theory in the partial metric space. He proved the Banach contraction principle in this space to be applied in program verification. We can find so many fixed-point theorems in the metric spaces which have been proved in the partial metric spaces by many fixed-point theorists ([12, 26] and references therein).

Shukla [13] introduced the concept of partial b-metric by modifying the triangle property of the partial metric and investigated fixed points of Banach contraction and Kannan contraction in the partial b-metric spaces. Mustafa et al. [14] modified the triangle property of partial b-metric and established a convergence criterion and some working rules in partial b-metric spaces. Moreover, Mustafa et al. [14] investigated common fixed-point results for (ϕ, ψ)-weakly contractive mappings. Dolicanin-Đekic [15] obtained the fixed-point theorems for Ciric-type contractions in the partial b-metric spaces. Singh et al. [16] investigated some conditions to show the existence of the common fixed points of power graphic (F, ψ)-contractions defined on the partial b-metric space endowed with directed graphs. More results on F-contractions can be seen in [8, 17, 18].

Let X be a nonempty set, then the nonempty binary relation $\mathfrak{R}$ is a subset of X^2. The set X^2 itself is known as universal relation, and the empty set is known as an empty relation; both are trivial relations. If any two elements $\alpha, \beta \in X$ are related with respect to $\mathfrak{R}$, then we shall write $(\alpha, \beta) \in \mathfrak{R}$. We shall use the notation $[\alpha, \beta] \in \mathfrak{R}$ if either $(\alpha, \beta) \in \mathfrak{R}$ or $(\beta, \alpha) \in \mathfrak{R}$. $\mathfrak{R}$ is reflexive if $(\alpha, \alpha) \in \mathfrak{R}$, for all $\alpha \in X$. $\mathfrak{R}$ is symmetric if $(\alpha, \beta) \in \mathfrak{R}$ implies $(\beta, \alpha) \in \mathfrak{R}$, for all $\alpha, \beta \in X$. $\mathfrak{R}$ is antisymmetric if $(\alpha, \beta) \in \mathfrak{R}$ and $(\beta, \alpha) \in \mathfrak{R}$ implies $\alpha = \beta$, for all $\alpha, \beta \in X$. $\mathfrak{R}$ is transitive if $(\alpha, \beta) \in \mathfrak{R}$ and $(\beta, \gamma) \in \mathfrak{R}$ implies $(\alpha, \gamma) \in \mathfrak{R}$, for all $\alpha, \beta \in X$. The inverse, transpose, or dual of binary relation $\mathfrak{R}$ is denoted by $\mathfrak{R}^{-1}$ and defined as follows: $\mathfrak{R}^{-1} = \{(\alpha, \beta) \in X \mid (\beta, \alpha) \in \mathfrak{R}\}$. Let $\mathfrak{R}^s = \mathfrak{R} \cup \mathfrak{R}^{-1}$, then it is easy to prove that $(\alpha, \beta) \in \mathfrak{R}^s$ if and only if $[\alpha, \beta] \in \mathfrak{R}$.

Definition 1 (see [19]). Let T be a self-mapping on a nonempty set X. A binary relation $\mathfrak{R}$ on X is said to be T-closed if for all $\alpha, \beta \in X$,

$$(\alpha, \beta) \in \mathfrak{R} \Rightarrow (T(\alpha), T(\beta)) \in \mathfrak{R}. \tag{1}$$

Definition 2 (see [19]). Let $\mathfrak{R}$ be a binary relation on X. A path in $\mathfrak{R}$ from α to β is a sequence $\{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n\} \subseteq X$ such that

(1) $\alpha_0 = \alpha$ and $\alpha_n = \beta$

(2) $(\alpha_j, \alpha_{j+1}) \in \mathfrak{R}$ for all $j \in \{0, 1, 2, \cdots \cdots \cdots, n-1\}$

The set of all paths from α to β in $\mathfrak{R}$ is denoted by $\Gamma(\alpha, \beta, \mathfrak{R})$. The path of length n involves $n+1$ element of X.

Definition 3 (see [19]). A metric space (X, d) equipped with the binary relation $\mathfrak{R}$ is called $\mathfrak{R}$-regular (or d-self-closed) if for each sequence $\{\alpha_n\}$in$X$, whenever $(\alpha_n, \alpha_{n+1}) \in \mathfrak{R}$ and $\alpha_n \xrightarrow{d} \alpha$, we have $(\alpha_n, \alpha) \in \mathfrak{R}$, for all $n \in \mathbb{N} \cup \{0\}$.

Alam and Imdad [19] used nonempty binary relation on the nonempty set X to obtain the following relation-theoretic contraction principle.

Theorem 4 (see [19]). *Let (X, d) be a complete metric space and $\mathfrak{R}$ be a binary relation on X. Let T be a self-mapping defined on (X, d) satisfying the following conditions:*

(a) There exists $\alpha_0 \in X$ such that $(\alpha_0, T(\alpha_0)) \in \mathfrak{R}$ and $\mathfrak{R}$ is T-closed

(b) Either T is continuous or (X, d) is $\mathfrak{R}$ − regular

(c) There exists $k \in [0, 1)$ such that $d(T(\alpha), T(\beta))) \leq kd(\alpha, \beta)$ for $\alpha, \beta \in X$ with $(\alpha, \beta) \in \mathfrak{R}$

Then T admits a fixed point in X. Moreover, if $\Gamma(\alpha, \beta, \mathfrak{R}^s)$ is a nonempty set for all $\alpha, \beta \in X$, then the fixed point is unique.

al-Sulami et al. [20] generalized Theorem 4 by replacing Banach contraction with θ-contraction as follows.

Theorem 5 (see [20]). *Let (X, d) be a complete metric space and $\mathfrak{R}$ be a binary relation on X. Let T be a self-mapping defined on (X, d) satisfying the following conditions:*

(a) There exists $\alpha_0 \in X$ such that $(\alpha_0, T(\alpha_0)) \in \mathfrak{R}$ and $\mathfrak{R}$ is T-closed

(b) Either T is continuous or (X, d) is $\mathfrak{R}$ − regular

(c) There exists $k \in [0, 1)$ such that $\theta(d(T(\alpha), T(\beta))) \leq [\theta(d(\alpha, \beta))]^k$ for $\alpha, \beta \in X$ with $(\alpha, \beta) \in \mathfrak{R}$

Then T admits a fixed point in X. Moreover, if $\Gamma(\alpha, \beta, \mathfrak{R}^s)$ is a nonempty set for all $\alpha, \beta \in X$, then the fixed point is unique.

Definition 6 (see [21]). Let T and S be two self-mappings on a nonempty set X. A binary relation $\mathfrak{R}$ on X is said to be (T, S)-closed if for all $\alpha, \beta \in X$,

$$(\alpha, \beta) \in \mathfrak{R} \Rightarrow (T(\alpha), S(\beta)) \in \mathfrak{R} \text{ or } (S(\alpha), T(\beta)) \in \mathfrak{R}. \tag{2}$$

Zada and Sarwar [21] generalized Theorem 4 by replacing Banach contraction with F-contraction as follows.

Theorem 7 (see [21]). *Let (X, d) be a complete metric space and $\mathfrak{R}$ be a binary relation on X. If the self-mappings T and S defined on (X, d) satisfy the following conditions:*

(a) There exists $\alpha_0 \in X$ such that $(\alpha_0, T(\alpha_0)) \in \mathfrak{R}$ and $\mathfrak{R}$ is (T, S)-closed

(b) Either T, S are continuous or (X, d) is $\mathfrak{R}$ − regular

(c) There exists $\tau > 0$, such that for all $(\alpha, \beta) \in \mathfrak{R}$ with $d(T(\alpha), S(\beta)) > 0$,

$$\tau + F(d(T(\alpha), S(\beta))) \leq F\left(d(\alpha, \beta) + \frac{d(\alpha, S(\beta))d(\beta, T(\alpha))}{1 + d(\alpha, \beta)}\right) \tag{3}$$

Then T and S have a unique common fixed point in X. Moreover, if $\Gamma(\alpha, \beta, \mathfrak{R}^s)$ is nonempty for all $\alpha, \beta \in X$, then the common fixed point is unique.

Liu et al. [22] introduced the $(D, \mathscr{C})$-contractions where the mapping D maps positive real numbers to positive real numbers and satisfies the conditions $(D_1) - (D_3)$:

(D_1) D is nondecreasing

(D_2) $\lim_{n\longrightarrow\infty} D(t_n) = 0 \Longleftrightarrow \lim_{n\longrightarrow\infty} t_n = 0$, for each positive sequence $\{t_n\}$

(D_3) D is continuous

$\mathscr{C} : (0,\infty) \longrightarrow (0,\infty)$ is a comparison function; that is, it satisfies the following conditions:

(i) $\mathscr{C}$ is monotone increasing, that is,

$$\alpha < \beta \Longrightarrow \mathscr{C}(\alpha) < \mathscr{C}(\beta) \tag{4}$$

(ii) $\lim_{n\longrightarrow\infty} \mathscr{C}^n(t) = 0$ for all $t > 0$, where $\mathscr{C}^n$ stands for the n^{th} iterate of $\mathscr{C}$

Let $\mathscr{D} = \{D : (0,\infty) \longrightarrow (0,\infty) | D \text{ satisfies } (D_1) - (D_3)\}$. If D is defined by $D(t) = t; D(t) = \ln t$, then D belongs to $\mathscr{D}$.

Note that if $\mathscr{C}$ is a comparison function, then $\mathscr{C}(t) < t$, for every $t > 0$. The mappings $\mathscr{C}(t) = \alpha t, 0 < \alpha < 1, t > 0$, and $\mathscr{C}(t) = t/(1+t), t > 0$, are comparison functions.

Definition 8 ([22], $(D, \mathscr{C})$-contraction). Let T be a self-mapping defined on the metric space (X, d). Let

$$\mathfrak{I} = \{ (\alpha, \beta) \in X^2 : d(T(\alpha), T(\beta)) > 0 \}. \quad (5)$$

The mapping T is called $(D, \mathscr{C})$-contraction if it satisfies the following condition:

$$D(d(T(\alpha), T(\beta))) \leq \mathscr{C}(D(d(\alpha, \beta))), \text{for all } \alpha, \beta \in \mathfrak{I}. \quad (6)$$

Definition 9 ([22], generalized $(D, \mathscr{C})$-contraction). Let T be a self-mapping defined on the metric space (X, d). If the mapping T satisfies the condition $D(d(T(\alpha), T(\beta))) \leq \mathscr{C}(D (M(\alpha, \beta)))$, for all $(\alpha, \beta) \in \mathfrak{I}$, where $M(\alpha, \beta)$ is defined by $M(\alpha, \beta) = \max \{d(\alpha, \beta), d(\alpha, T\alpha), d(\beta, T\beta), (d(\alpha, T\beta) + d(\beta, T\alpha))/2 \}$. Then it is called generalized $(D, \mathscr{C})$-contraction.

Liu et al. established the following theorem for $(D, \mathscr{C})$ -contractions.

Theorem 10 (see [22]). *Every generalized $(D, \mathscr{C})$-contraction has a unique fixed point in a complete metric space (X, d).*

In this paper, in Section 3, we investigate common fixed-point results for generalized contractions in the partial b -metric spaces endowed with binary relation $\mathfrak{R}$. The obtained results generalize Theorems 4, 5, 7, 10. We support the results with a nontrivial example and counter the remarks given in [23].

2. Basic Notions in the Partial b-Metric Spaces

Let X be a nonempty set, and the mapping $P : X \times X \longrightarrow$ $\longrightarrow [0,\infty)$ satisfies the following axioms:

(1) $x = y \Leftrightarrow P(x, x) = P(x, y) = P(y, y), \forall x, y \in X$

(2) $P(x, x) \leq P(x, y) \forall x, y \in X$

(3) $P(x, y) = P(y, x) \forall x, y \in X$

(4) $P(x, z) \leq P(x, y) + P(y, z) - P(y, y) \forall x, y, z \in X$

(5) There exists a real number $s \geq 1$ such that

$$P(x, z) \leq s[P(x, y) + P(y, z)] - P(y, y) \forall x, y, z \in X \quad (7)$$

According to Matthews [11], if the mapping P satisfies axioms (1-4), we say that it is a *partial metric* on the set X and (X, P) is called *partial metric space*. According to Shukla [13], if P satisfies axioms (1, 2, 3, and 5), then it is a *partial b -metric* on the set X and (X, P_b) is called *partial b -metric space*. For convenience, we denote partial b-metric by P_b.

Every partial b-metric P_b induces a b-metric $d_{P_b} : X \times X \longrightarrow [0,\infty)$ defined by

$$d_{P_b}(x, y) = 2P_b(x, y) - P_b(x, x) - P_b(y, y) \forall x, y \in X. \quad (8)$$

It is called *induced b -metric* on X.

Let $B_{P_b}(x, \epsilon) = \{ y \in X : P_b(x, y) < \epsilon + P_b(x, x)\}$, then $\{B_{P_b}(x, \epsilon): x \in X, \epsilon > 0\}$ is a collection of P_b *-balls* which forms a base for *partial b -metric topology*.

The following relation can be observed.

Remark 11.

(1) In (X, P_b), $P_b(x, y) = 0, \Rightarrow x = y, \forall x, y \in X$, but the converse is not true (in this case, (X, P_b) reduces to a b-metric space) Figure 1.

Example 1 (see [13]). Let $X = [0,\infty), l > 1$, be a constant and $P_b : X \times X \longrightarrow [0,\infty)$ be defined by

$$P_b(x, y) = (\max \{x, y\})^l + |x - y|^l \text{ for all } x, y \in X. \quad (9)$$

Then (X, P_b) is a partial b-metric space with coefficient $s = 2^l > 1$, but it is neither a b-metric space nor a partial metric space.

Example 2 (see [13]). Let $P : X \times X \longrightarrow [0,\infty)$ and $d^* : X \times X \longrightarrow [0,\infty)$ be the partial metric and b-metric on X, respectively. Then the mapping $P_b : X \times X \longrightarrow [0,\infty)$ defined by $P_b(x, y) = P(x, y) + d^*(x, y)$ for all $x, y \in X$ defines a partial b-metric on X.

Example 3 (see [13]). Let $P : X \times X \longrightarrow [0,\infty)$ be a partial metric. Then the mapping $P_b : X \times X \longrightarrow [0,\infty)$ defined by $P_b(x, y) = ((P(x, y))^l$ for all $x, y \in X$ and $l \geq 1$ is a partial b-metric on X provided $s = 2^{l-1}$.

Definition 12 (see [13]). A sequence $\{x_n\}_{n \in N}$ in the partial b-metric space (X, P_b, s) is called a convergent sequence if there exists $x \in X$ such that

$$\lim_{n \longrightarrow \infty} P_b (x_n, x) = P_b(x, x). \quad (10)$$

The uniqueness of the limit of a convergent sequence may not be guaranteed in the partial b-metric spaces (see [23]).

Definition 13 (see [13]). A sequence $\{x_n\}_{n \in N}$ in a partial b -metric space (X, P_b, s) is called the Cauchy sequence if

$$\lim P_b(x_n,_{m,n \longrightarrow \infty} x_m) = P_b(x, x). \quad (11)$$

The partial b-metric space (X, P_b, s) is said to be complete if every Cauchy sequence $\{x_n\}_{n \in N}$ in X converges to a point $x \in X$.

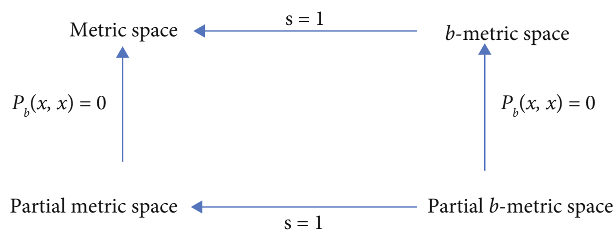

FIGURE 1

Lemma 14 (see [14]).

(1) *Every Cauchy sequence in the b -metric space is also Cauchy in the partial b-metric space and vice versa*

(2) *The partial b-metric space is complete if and only if b -metric space (induced b-metric space) is complete*

(3) *For any sequence* $\{x_n\}_{n\in N}$ *in X,* $\lim_{n\longrightarrow\infty} d_{P_b}(x^*, x_n) = 0$ *if and only if*

$$\lim_{n\longrightarrow\infty} P_b(x^*, x_n) = P_b(x^*, x^*) = \lim P_{b\, n,m\longrightarrow\infty}(x_n, x_m) \tag{12}$$

3. Common Fixed-Point Theorems in the Partial b-Metric Spaces

This section is the main part of this paper. It contains some new common fixed-point theorems in the partial b-metric spaces. The existence theorems given in [12, 15, 19–22, 24, 27] can be seen as a special case of the results proved in this section.

The results in this paper are based on the following contractive condition.

Definition 15. Let T and S be two self-mappings on the partial b-metric space (X, P_b, s) and $\mathfrak{R}$ be a binary relation on X. Let

$$\mathfrak{J} = \{ (x, y) \in \mathfrak{R} : P_b(T(x), S(y)) > 0 \}. \tag{13}$$

The mappings T and S form a *$D\mathscr{C}$ -contraction* if there exists a continuous comparison function $\mathscr{C}$ and $D \in \mathscr{D}$ such that

$$D\left(s^2 P_b(T(x), S(y))\right) \le \mathscr{C}(D(P_b(x, y))), \text{for all } x, y \in \mathfrak{J}. \tag{14}$$

In [23], it was remarked that some contraction conditions on partial b-metric spaces imply contraction conditions on b -metric spaces (see Theorem 2.6 in [23]). In the following example, we show that the contraction condition (14) is independent of these remarks.

Example 4. Let $X = [0,\infty)$ and $\mathfrak{R} = X^2$. Let $P_b : X \times X \longrightarrow [0,\infty)$ be defined by

$$P_b(x, y) = (\max \{x, y\})^2 + |x - y|^2 \text{ for all } x, y \in X. \tag{15}$$

Then (X, P_b) is a partial b-metric space with coefficient $s = 4$. The associated b-metric is given by

$$d_{P_b}(x, y) = 2\left((\max \{x, y\})^2 + |x - y|^2\right) - x^2 - y^2. \tag{16}$$

Define $T \equiv S : [0, 1] \longrightarrow [0, 1]$ by $T(x) = x/5$ (if $x \in [0, 1)$) and $T(1) = 0$. Consider

$$D\left(s^2 d_{P_b}\left(T(1), T\left(\frac{5}{6}\right)\right)\right) \le \mathscr{C}\left(D\left(d_{P_b}\left(1, \frac{5}{6}\right)\right)\right). \tag{17}$$

This implies,

$$D\left(\frac{16}{36}\right) \le \mathscr{C}\left(D\left(\frac{13}{36}\right)\right) < D\left(\frac{13}{36}\right), \tag{18}$$

a contradiction to the definition of mapping $D \in \mathscr{D}$. On the other hand, for partial b-metric, we have

$$D\left(s^2 P_b\left(T(1), T\left(\frac{5}{6}\right)\right)\right) = D\left(\frac{32}{36}\right) \le \mathscr{C}\left(D\left(P_b\left(1, \frac{5}{6}\right)\right)\right) < D\left(\frac{37}{36}\right). \tag{19}$$

Note that we have taken $(1, 5/6) \in \mathfrak{R}$. Similarly, it can be shown that the above conclusion holds for all other cases.

Since, in general, b-metric is discontinuous mapping (see [5]), so by Example 2, the partial b-metric is not continuous in general. The following lemma is necessary for the upcoming results.

Lemma 16 (see [14]). *Let (X, P_b, s) be a partial b-metric space. If there exists a $\{x_n\}$ in (X, P_b, s) and x^*, y^* such that $\lim_{n\longrightarrow\infty} x_n = x^*$. Then*

$$\frac{1}{s} P_b(x^*, y^*) \le \lim_{n\longrightarrow\infty} \inf P_b(x_n, y^*) \le \lim_{n\longrightarrow\infty} \sup P_b(x_n, y^*) \le s P_b(x^*, y^*). \tag{20}$$

3.1. Main Results. We state our main results which describe the conditions for the existence of the common fixed points of $D\mathscr{C}$-contraction in the partial b-metric spaces.

Theorem 17. *Let (X, P_b) be a complete partial b-metric space and $\mathfrak{R}$ be a transitive binary relation on X. Let T and S form a $D\mathscr{C}$-contraction. Then T and S have a common fixed point in X, if the following conditions are satisfied.*

(a) *There exists $\alpha_0 \in X$ such that $(\alpha_0, T(\alpha_0)) \in \mathfrak{R}$*

(b) *$\mathfrak{R}$ is (T, S)-closed*

(c) *T and S are continuous*

Proof. By assumption (a), there exists $\alpha_0 \in X$ such that $(\alpha_0, T(\alpha_0)) \in \mathfrak{R}$. Taking $\alpha_0 \in X$ as the initial point, we define the sequence $\{\alpha_n\}$ in X by

$$\begin{cases} \alpha_1 = T(\alpha_0), \alpha_2 = S(\alpha_1), \text{continuing with the same pattern, we have} \\ \alpha_{2n+1} = T(\alpha_{2n}), \alpha_{2n+2} = S(\alpha_{2n+1}), \text{where } n \in \mathbb{N} \cup \{0\}. \end{cases} \quad (21)$$

□ □

Moreover, by assumptions (a) and (b), we have

$$\begin{aligned} (\alpha_1, \alpha_2) &= (T(\alpha_0), S(\alpha_1)) \in \mathfrak{R}, \\ (\alpha_2, \alpha_3) &= (S(\alpha_1), T(\alpha_2)) \in \mathfrak{R}, \\ (\alpha_3, \alpha_4) &= (T(\alpha_2), S(\alpha_3)) \in \mathfrak{R}, \\ (\alpha_4, \alpha_5) &= (S(\alpha_3), T(\alpha_4)) \in \mathfrak{R}. \end{aligned} \quad (22)$$

In general, we have $(\alpha_{2n}, \alpha_{2n+1}) = (S(\alpha_{2n-1}), T(\alpha_{2n})) \in \mathfrak{R}$ and $(\alpha_{2n+1}, \alpha_{2n+2}) = (T(\alpha_{2n}), S(\alpha_{2n+1})) \in \mathfrak{R}$.

Case 1. If $\alpha_{2n^*} = \alpha_{2n^*+1}$, for some n^*, then

$$\alpha_{2n^*+1} = \alpha_{2n^*+2}. \quad (23)$$

Indeed, on the contrary, if $\alpha_{2n^*+1} \neq \alpha_{2n^*+2}$, then $(\alpha_{2n^*+1}, \alpha_{2n^*+2}) \in \mathfrak{I}$, and by contractive condition (14), we have

$$\begin{aligned} D(P_b(\alpha_{2n^*+1}, \alpha_{2n^*+2}) &\leq D\left(s^2 P_b\left(T\left(\alpha_{2n^*}\right), S\left(\alpha_{2n^*+1}\right)\right)\right) \\ &\leq \mathscr{C}(D(P_b(\alpha_{2n^*}, \alpha_{2n^*+1})). \end{aligned} \quad (24)$$

Since $\mathscr{C}(t) < \mathrm{t}$, for every $t > 0$, we obtain

$$D(P_b(\alpha_{2n^*+1}, \alpha_{2n^*+2}) < D(P_b(\alpha_{2n^*}, \alpha_{2n^*+1}). \quad (25)$$

Since the function D is nondecreasing, so $P_b(\alpha_{2n^*+1}, \alpha_{2n^*+2}) < P_b(\alpha_{2n^*}, \alpha_{2n^*+1})$. This contradicts the second condition of partial b-metric spaces ($P_b(x, x) \leq P_b(x, y) \forall x, y \in X$). Hence, $\alpha_{2n^*} = \alpha_{2n^*+1}$ implies $\alpha_{2n^*+1} = \alpha_{2n^*+2}$. Consequently, α_{2n^*} is a common fixed point of T, and that is $\alpha_{2n^*} = T(\alpha_{2n^*}) = S(\alpha_{2n^*+1}) = S(\alpha_{2n^*})$.

Case 2. If $\alpha_{2n} \neq \alpha_{2n+1}$ for all $n \in \mathbb{N}$. We have $P_b(T(\alpha_{2n}), S(\alpha_{2n-1})) > 0$ for all $n \in \mathbb{N}$. Since $(\alpha_{2n}, \alpha_{2n-1}) \in \mathfrak{R}$, so $(\alpha_{2n}, \alpha_{2n-1}) \in \mathfrak{I}$. Setting $\alpha = \alpha_{2n}$ and $\beta = \alpha_{2n-1}$ in (14), we get

$$\begin{aligned} D(P_b(\alpha_{2n+1}, \alpha_{2n})) &\leq D\left(s^2 P_b(\alpha_{2n+1}, \alpha_{2n})\right) \\ &= D\left(s^2 P_b(T(\alpha_{2n}), S(\alpha_{2n-1}))\right) \\ &\leq \mathscr{C}(D(P_b(\alpha_{2n}, \alpha_{2n-1}), \end{aligned} \quad (26)$$

for all $n \in \mathbb{N}$.

Similarly, setting $\alpha = \alpha_{2n}$ and $\beta = \alpha_{2n+1}$ in (14), we get

$$\begin{aligned} D(P_b(\alpha_{2n+1}, \alpha_{2n+2}) &\leq D\left(s^2 P_b(T(\alpha_{2n}), S(\alpha_{2n+1}))\right) \\ &\leq \mathscr{C}(D(P_b(\alpha_{2n}, \alpha_{2n+1})). \end{aligned} \quad (27)$$

In general, for all $h(n) \in \mathbb{N}$, either even or odd, we have

$$\begin{aligned} D\left(P_b\left(\alpha_{h(n)}, \alpha_{h(n)+1}\right)\right) &\leq \mathscr{C}\left(D\left(P_b\left(\alpha_{h(n)-1}, \alpha_{h(n)}\right)\right)\right) \\ &\leq \mathscr{C}^2\left(D\left(P_b\left(\alpha_{h(n)-2}, \alpha_{h(n)-1}\right)\right)\right) \vdots \\ &\leq \mathscr{C}^{h(n)}(D(P_b(\alpha_0, \alpha_1))). \end{aligned} \quad (28)$$

Taking limit $n \longrightarrow \infty$ in the above inequality, we get

$$\begin{aligned} 0 &\leq \lim_{n\longrightarrow\infty} D\left(P_b\left(\alpha_{h(n)}, \alpha_{h(n)+1}\right)\right) \\ &\leq \lim_{n\longrightarrow\infty} \mathscr{C}^{h(n)}(D(P_b(\alpha_0, \alpha_1))) = 0. \end{aligned} \quad (29)$$

This implies $\lim_{n\longrightarrow\infty} D(P_b(\alpha_{h(n)}, \alpha_{h(n)+1}) = 0$, and by (D_2), we have
$\lim_{n\longrightarrow\infty} P_b(\alpha_{h(n)}, \alpha_{h(n)+1}) = 0$. This implies (by (8)) that

$$\lim_{n\longrightarrow\infty} d_{P_b}\left(\alpha_{h(n)}, \alpha_{h(n)+1}\right) = 0. \quad (30)$$

By axiom (2), we have $\lim_{n\longrightarrow\infty} P_b(\alpha_{h(n)}, \alpha_{h(n)}) \leq \lim_{n\longrightarrow\infty} P_b(\alpha_{h(n)}, \alpha_{h(n)+1}) = 0$. Thus, for all $n, m \geq 1$, we have

$$\lim_{n,m\longrightarrow\infty} d_{P_b}\left(\alpha_{h(m)}, \alpha_{h(n)}\right) = 2 \lim_{n,m\longrightarrow\infty} P_b\left(\alpha_{h(m)}, \alpha_{h(n)}\right). \quad (31)$$

We claim that $\{\alpha_n\}$ is a Cauchy sequence in (X, d_{P_b}), for this is sufficient to prove that $\{\alpha_{2n}\}$ is Cauchy sequence. On the contrary, if $\{\alpha_{2n}\}$ is not Cauchy, then for some subsequences $\{\alpha_{2n_k}\}_{k=1}^{\infty}$ and $\{\alpha_{2m_k}\}_{k=1}^{\infty}$, there exist ϵ>0, and a positive integer $k(\epsilon)$, such that for all $n_k, m_k > k$, we have $d_{P_b}(\alpha_{2m_k}, \alpha_{2n_k}) \geq \varepsilon$ and $d_{P_b}(\alpha_{2m_k}, \alpha_{2n_k-2}) < \varepsilon$; thus,

$$\varepsilon \leq d_{P_b}\left(\alpha_{2m_k}, \alpha_{2n_k}\right) \leq s d_{P_b}\left(\alpha_{2m_k}, \alpha_{2m_k+1}\right) + s d_{P_b}\left(\alpha_{2m_k+1}, \alpha_{2n_k}\right). \quad (32)$$

As $k \longrightarrow \infty$ in the above inequality, we have

$$\frac{\varepsilon}{s} \leq \lim_{k\longrightarrow\infty} \sup d_{P_b}\left(\alpha_{2m_k+1}, \alpha_{2n_k}\right). \quad (33)$$

By using triangular inequality (axiom (5)), we get

$$d_{P_b}\left(\alpha_{2m_k}, \alpha_{2n_k-1}\right) \leq s d_{P_b}\left(\alpha_{2m_k}, \alpha_{2n_k-2}\right) + s d_{P_b}\left(\alpha_{2n_k-2}, \alpha_{2n_k-1}\right). \quad (34)$$

Taking limit $k \longrightarrow \infty$, we have

$$\lim_{k\longrightarrow\infty} \sup d_{P_b}\left(\alpha_{2m_k}, \alpha_{2n_k-1}\right) \leq s\varepsilon. \quad (35)$$

Also, we have the following information:

$$\begin{aligned} d_{P_b}\left(\alpha_{2m_k}, \alpha_{2n_k}\right) &\leq s d_{P_b}\left(\alpha_{2m_k}, \alpha_{2n_k-2}\right) + s d_{P_b}\left(\alpha_{2n_k-2}, \alpha_{2n_k}\right) \\ &\leq s d_{P_b}\left(\alpha_{2m_k}, \alpha_{2n_k-2}\right) + s^2 d_{P_b}\left(\alpha_{2n_k-2}, \alpha_{2n_k-1}\right) \\ &\quad + s^2 d_{P_b}\left(\alpha_{2n_k-1}, \alpha_{2n_k}\right). \end{aligned} \quad (36)$$

Taking limit $k \longrightarrow \infty$, we have

$$\lim_{k\longrightarrow\infty} \sup d_{P_b}\left(\alpha_{2m_k}, \alpha_{2n_k}\right) \leq s\varepsilon. \quad (37)$$

By axiom (5), we have

$$d_{P_b}\left(\alpha_{2m_k+1}, \alpha_{2n_k-1}\right) \leq s d_{P_b}\left(\alpha_{2m_k+1}, \alpha_{2m_k}\right) + s d_{P_b}\left(\alpha_{2m_k}, \alpha_{2n_k-1}\right). \quad (38)$$

Taking limit $k \longrightarrow \infty$ and using (35), we have

$$\lim_{k\longrightarrow\infty} \sup d_{P_b}\left(\alpha_{2m_k+1}, \alpha_{2n_k-1}\right) \leq s^2\varepsilon. \quad (39)$$

By using (31), we have the following information from (33), (35), (37), and (39):

$$\frac{\varepsilon}{2s} \leq \lim_{k\longrightarrow\infty} \sup P_b\left(\alpha_{2m_k+1}, \alpha_{2n_k}\right), \quad (40)$$

$$\lim_{k\longrightarrow\infty} \sup P_b\left(\alpha_{2m_k}, \alpha_{2n_k-1}\right) \leq \frac{s\varepsilon}{2}, \quad (41)$$

$$\lim_{k\longrightarrow\infty} \sup P_b\left(\alpha_{2m_k}, \alpha_{2n_k}\right) \leq \frac{s\varepsilon}{2}, \quad (42)$$

$$\lim_{k\longrightarrow\infty} \sup P_b\left(\alpha_{2m_{k+1}}, \alpha_{2n_k-1}\right) \leq \frac{s^2\varepsilon}{2}, \quad (43)$$

Since $(\alpha_{2m_k}, \alpha_{2n_k-1}) \in \mathfrak{I}$, by (14), we have

$$\begin{aligned} D\left(\frac{s\in}{2}\right) &= D\left(s^2\cdot\frac{s\in}{2}\right) \leq D\left(s^2 \lim_{k\longrightarrow\infty} \sup P_b\left(\alpha_{2m_k+1}, \alpha_{2n_k}\right)\right) \\ &= \lim_{k\longrightarrow\infty} \sup D\left(s^2 P_b\left(T\left(\alpha_{2m_k}\right), S\left(\alpha_{2n_{k-1}}\right)\right)\right) \\ &\leq \lim_{k\longrightarrow\infty} \sup \mathscr{C}\left(D\left(P_b\left(\alpha_{2m_k}, \alpha_{2n_{k-1}}\right)\right)\right) \\ &= \mathscr{C}\left(D\left(\lim_{k\longrightarrow\infty} \sup P_b\left(\alpha_{2m_k}, \alpha_{2n_{k-1}}\right)\right)\right) \\ &\leq \mathscr{C}\left(D\left(\frac{s\in}{2}\right)\right) < D\left(\frac{s\in}{2}\right). \end{aligned} \quad (44)$$

This is a contradiction to the definition of function D. Thus, $\{\alpha_n\}$ is a Cauchy sequence in (X, d_{P_b}). By Lemma 14 (1), $\{\alpha_n\}$ is a Cauchy sequence in (X, P_b). Since (X, P_b) is a complete Partial b-metric space, so by Lemma 14 (2), (X, d_{P_b}) is also a complete metric space. Thus, there exists $\alpha^* \in X$ such that $\alpha_n \longrightarrow \alpha^*$, that is, $\lim_{n\longrightarrow\infty} d_{P_b}(\alpha_n, \alpha^*) = 0$. By Lemma 14 (3), we get

$$\lim_{n\longrightarrow\infty} P_b(\alpha_n, \alpha^*) = P_b(\alpha^*, \alpha^*) = \lim_{n,m\longrightarrow\infty} P_b(\alpha_n, \alpha_m). \quad (45)$$

Since $\lim_{n,m\longrightarrow\infty} P_b(\alpha_n, \alpha_m) = 0$, so that $P_b(\alpha^*, \alpha^*) = 0$. Thus, $\{\alpha_n\}$ converges to α^* in (X, P_b).

Now, we claim that $T(\alpha^*) = S(\alpha^*) = \alpha^*$. By (40), we have

$$\begin{aligned} &\lim_{n\longrightarrow\infty} P_b(\alpha_{2n+1}, \alpha^*) = 0, \\ &\lim_{n\longrightarrow\infty} P_b(\alpha_{2n+2}, \alpha^*) = 0. \end{aligned} \quad (46)$$

Since T and S are continuous, we have

$$\begin{aligned} &\lim_{n\longrightarrow\infty} P_b(T(\alpha_{2n}), T(\alpha^*)) = 0, \\ &\lim_{n\longrightarrow\infty} P_b(S(\alpha_{2n+1}), S(\alpha^*)) = 0. \end{aligned} \quad (47)$$

By Lemma 16, we have

$$\begin{aligned} \frac{1}{s} P_b(\alpha^*, T(\alpha^*)) &\leq \lim_{n\longrightarrow\infty} \inf P_b(\alpha_{2n+1}, T(\alpha^*)) \\ &= \lim_{n\longrightarrow\infty} \inf P_b(T(\alpha_{2n}), T(\alpha^*)) = 0. \end{aligned} \quad (48)$$

Thus, $P_b(T(\alpha^*), \alpha^*) = P_b(\alpha^*, \alpha^*) = P_b(T(\alpha^*), T(\alpha^*))$.

This implies $T(\alpha^*) = \alpha^*$. Similar arguments lead us to have $S(\alpha^*) = \alpha^*$.Hence,

$T(\alpha^*) = S(\alpha^*) = \alpha^*$; that is, T and S have a common fixed point $\alpha^* \in X$.

If $\Gamma(\alpha, \beta, \mathfrak{R}) \neq \emptyset$, then we have the following theorem.

Theorem 18. *Let (X, P_b) be a complete partial b-metric space and $\mathfrak{R}$ be a transitive binary relation on X. Let T and S form a $D\mathscr{C}$-contraction. Suppose that $\Gamma(\alpha, \beta, \mathfrak{R}) \neq \emptyset$ and statement of Theorem 17 holds, then the mappings T and S admit a unique common fixed point in X.*

Proof. We have proved the existence of a common fixed point in Theorem 17 . On the contrary, suppose that v and v^* are two distinct common fixed points of T and S in X. Then the class of paths of finite length ℓ in $\mathfrak{R}$ from v to v^* is $\Gamma(v, v^*, \mathfrak{R})$. Let one of the paths be $\{A_0, A_1, A_2, \cdots, A_\ell\}$ in X from v to v^* with

$$A_0 = v, A_\ell = v^*, \left(A_{\jmath}, A_{\jmath+1}\right) \in \mathfrak{R}\,;\ \jmath = 0, 1, 2, 3 \cdots\cdots\cdots\cdots (\ell - 1). \quad (49)$$

By transitivity of $\mathfrak{R}$, we have

$$(\nu, A_1) \in \mathfrak{R}, (A_1, A_2) \in \mathfrak{R}, \cdots, (A_{\ell-1}, \nu^*) \in \mathfrak{R} \Rightarrow (\nu, \nu^*) \in \mathfrak{R}. \tag{50}$$

It is given that T and S form a $D\mathscr{C}$-contraction, that is,

$$D\left(s^2 P_b(T(\nu), S(\nu^*))\right) \leq \mathscr{C}(D(P_b(\nu, \nu^*). \tag{51}$$

This implies $D(s^2 P_b(\nu, \nu^*)) \leq \mathscr{C}(D(P_b(\nu, \nu^*))) < D(P_b(\nu, \nu^*))$. This is a contradiction to the definition of D. Hence, $\nu = \nu^*$. This shows that ν is a unique common fixed point of T and S.□ □

Remark 19. If the mappings T and S are discontinuous, then we have the following theorem.

Theorem 20. *Let (X, P_b) be an $\mathfrak{R}$ – regular complete partial b-metric space. Let T and S form a $D\mathscr{C}$-contraction. Suppose that $\mathfrak{R}$ is an antisymmetric relation, then T and S admit a common fixed point in X if they meet the conditions (a) and (b):*

(a) There exists $\alpha_0 \in X$ such that $(\alpha_0, T(\alpha_0)) \in \mathfrak{R}$

(b) $\mathfrak{R}$ is (T, S)-closed

Proof. By Theorem 17, we know that $(\alpha_n, \alpha_{n+1}) \in \mathfrak{R}$ and $\alpha_n \longrightarrow \alpha^*$ as $n \longrightarrow \infty$. It is given that (X, P_b) is $\mathfrak{R}$ – regular, so $(\alpha_n, \alpha *) \in \mathfrak{R}$, for all $n \in \mathbb{N}$. There are two possible cases.□ □

Case 1. If the sequence $\{\alpha_n\}$ is constant. Let $\alpha_n = \alpha^*$ for each $n \in \mathbb{N}$ so that $\alpha_{2n} = \alpha^*$ and $T(\alpha^*) = T(\alpha_{2n}) = \alpha_{2n+1.}$ Since (X, P_b) is $\mathfrak{R}$ – regular, so $(\alpha_{2n+1}, \alpha^*) = (T(\alpha^*), \alpha^*) \in \mathfrak{R}$. We know that $(\alpha_{2n}, \alpha_{2n+1}) \in \mathfrak{R}$; thus, $(\alpha^*, T(\alpha^*)) \in \mathfrak{R}$. As $\mathfrak{R}$ is an antisymmetric relation, so $\alpha^* = T(\alpha^*)$, by the same arguments we have $\alpha^* = S(\alpha^*)$ as required.

Case 2. If $\{\alpha_n\}$ is not constant and arbitrary, we claim that $P_b(\alpha^*, S(\alpha^*)) = 0$. Let $P_b(\alpha^*, S(\alpha^*)) > 0$. It is proved in Theorem 17 that $\lim_{i \longrightarrow \infty} \alpha_{2i+1} = \alpha^*$, so there must be an integer $n_0 > 0$, such that

$$P_b(\alpha_{2i+1}, S(\alpha^*)) > 0, P_b(\alpha_{2i}, \alpha^*) < \frac{P_b(\alpha^*, S(\alpha^*))}{2}, \text{for all } i \geq n_0. \tag{52}$$

It is assumed that (X, P_b) is $\mathfrak{R}$ – regular, and by Theorem 17, we know that $\alpha_{2i} \longrightarrow \alpha^*$ as $i \longrightarrow \infty$;thus, $(\alpha_{2i}, \alpha^*) \in \mathfrak{R}$. By contractive condition (2.1), monotonicity of D, and Lemma 16, we have

$$\begin{aligned} D(p_b(\alpha^*, S(\alpha^*))) &\leq D\left(s \lim_{i \longrightarrow \infty} \inf\ P_b(\alpha_{2i+1}, S(\alpha^*))\right) \\ &\leq D\left(s^2 \lim_{i \longrightarrow \infty} \inf\ P_b(\alpha_{2i+1}, S(\alpha^*))\right) \\ &= \lim_{i \longrightarrow \infty} \inf\ D\left(s^2 P_b(T(\alpha_{2i}), S(\alpha^*))\right) \\ &\leq \lim_{i \longrightarrow \infty} \inf\ \mathscr{C}(D(P_b(\alpha_{2i}, \alpha^*))) \\ &< \lim_{i \longrightarrow \infty} \inf\ \mathscr{C}\left(D\left(\frac{P_b(\alpha^*, S(\alpha^*))}{2} \right)\right) \\ &< D\left(\frac{P_b(\alpha^*, S(\alpha^*))}{2} \right). \end{aligned} \tag{53}$$

This is a contradiction to the definition of mapping D. Thus, $P_b(\alpha^*, S(\alpha^*)) = 0$. Also, we have the following information:

$$P_b(S(\alpha^*), S(\alpha^*)) = 0 = P_b(\alpha^*, \alpha^*). \tag{54}$$

Thus, $\alpha^* = S(\alpha^*)$. By interchanging roles of S and T, we have $\alpha^* = T(\alpha^*)$.

Hence, $T(\alpha^*) = S(\alpha^*) = \alpha^*$; that is, α^* is a common fixed point of T and S in X.

The following is the most general theorem of this section.

Theorem 21. *Let (X, P_b) be an $\mathfrak{R}$ – regular complete partial b-metric space and $\mathfrak{R}$ be a transitive and antisymmetric binary relation on X. Let T and S form a $D\mathscr{C}$-contraction. Suppose that $\Gamma(\alpha, \beta, \mathfrak{R}) \neq \varnothing$, and assumptions (a) and (b) in Theorem 17 hold. Then the mappings T and S admit a unique common fixed point in X.*

Proof. See the proofs of Theorems 17, 18, and 20, respectively.□ □

Remark 22.

(1) The results in this section are independent of the observation made in [23], and hence, our results are a real generalization of the related results in literature (see [12, 19–22])

(2) Theorem 21 remains true if $P_b(\alpha, \beta)$ is replaced by $M(\alpha, \beta)$

The following example explains the main results.

Example 5. Let $X = \{ \mathscr{a}_n = \mathscr{n}(\mathscr{n} + 1)/2 : \mathscr{n} \in \mathbb{N}\}$.Define the partial b-metric function $P_b : X \times X \longrightarrow [0,\infty)$ by

$$P_b(\alpha, \beta) = (\max \{\alpha, \beta\})^2, \text{for all } \alpha, \beta \in X. \tag{55}$$

Then $(X,\ P_b, 2)$ is a complete partial b-metric space. Define $D{:}(0,\infty) \longrightarrow (0,\infty)$ by $D(\mathscr{a}) = \mathscr{a} e^{\mathscr{a}}$ for each $\mathscr{a} > 0$, then $D \in \mathscr{D}$. Let the function $\mathscr{C} : (0,\infty) \longrightarrow (0,\infty)$ be defined by $\mathscr{C}(r) = r/2$ for all $r \in (0,\infty)$. Then $\mathscr{C}$ is continuous comparison. Define the binary relation $\mathfrak{R}$ on X by

$$\mathfrak{R} = \{(a_n, a_m): a_n + a_m \geq 2 \text{ for each } m \geq n\}. \tag{56}$$

Define the mappings $T, S: X \longrightarrow X$ by

$$T(a_n) = \begin{cases} a_1, & n = 1, \\ \dfrac{n(n-1)}{2}, & n \geq 2, \end{cases}$$
$$S(a_m) = \begin{cases} a_1, & m \in \{1, 2\}, \\ \dfrac{(m-1)(m-2)}{2}, & m \geq 3, m \in \mathbb{N}. \end{cases} \tag{57}$$

We observe that there exists $a_1 \in X$ such that $(a_1, T(a_1)) \in \mathfrak{R}$ (since $(a_1 + T(a_1) = 2)$ by definition of $\mathfrak{R}$, so assumption (a) is satisfied in Theorem 17. Let $(a_n, a_m) \in \mathfrak{R}$, then we have $T(a_n) + S(a_m) \geq 2$ for each $m \geq n$, so $(T(a_n), S(a_m)) \in \mathfrak{R}$. Thus, $\mathfrak{R}$ is (T, S)-closed (this verifies assumption (b) of Theorem 17. Also, T, S are continuous (assumption (c) is satisfied). Now, we show that T and S form $D\mathscr{C}$-contraction. It is noted that the mappings T, S do not form Banach contraction in the partial b-metric sense. Indeed,

$$\lim_{n\longrightarrow\infty} \frac{P_b(T(a_n), S(a_1))}{P_b a_n, a_1)} = \lim_{n\longrightarrow\infty} \frac{|n^2 - n|^2}{|n^2 + n|^2} = 1. \tag{58}$$

We noticed that $P_b(T(a_n), S(a_m)) > 0$ for each $m \geq n$. Thus, $(a_n, a_m) \in \mathfrak{I}$.Consider

$$4P_b(T(a_n), S(a_m))e^{4P_b(T(a_n),S(a_m))} \leq \frac{1}{2}P_b(a_n, a_m)e^{P_b(a_n,a_m)}. \tag{59}$$

This implies

$$\frac{8P_b(T(a_n), S(a_m))}{P_b(a_n, a_m)} \leq e^{P_b(a_n,a_m)-4P_b(T(a_n),S(a_m))}. \tag{60}$$

For $n = 1$ and $m = 2$, the inequality (41) reduces to $e^5 \geq 8/9$. Thus, (41) holds for this case. For $n = 2$ and $m = 3$, the inequality (41) gets the form $e^{32} \geq 2/9$. Similarly, for each $m \geq n$, (41) holds true. Thus, we have

$$D\left(s^2 P_b(T(\alpha), S(\beta))\right) \leq \mathscr{C}(D(P_b(\alpha, \beta))), \text{for all } \alpha, \beta \in X. \tag{61}$$

We note that $a_1 = T(a_1) = S(a_1)$.

3.2. Discussions. In this part of the current section, we state some corollaries which are themselves prominent fixed-point theorems in the literature.

The following corollary generalizes the results presented by Jleli and Samet [6] and al-Sulami et al. [20].

Corollary 23. *Let (X, P_b) be a complete partial b-metric space and $\mathfrak{R}$ be a transitive and antisymmetric binary relation on X. If the self-mappings T and S defined on (X, P_b) satisfy the following conditions:*

(a) *$\Gamma(\alpha, \beta, \mathfrak{R})$ is nonempty for all $\alpha, \beta \in X$*
(b) *There exists $\alpha_0 \in X$ such that $(\alpha_0, T(\alpha_0)) \in \mathfrak{R}$ and $\mathfrak{R}$ is (T, S)-closed*
(c) *Either T, S are continuous or (X, P_b) is $\mathfrak{R}$ – regular*
(d) *There exists a function $\theta \in \Theta$ and $k \in (0, 1)$, such that for all $\alpha, \beta \in \mathfrak{I}$,*

$$\theta\left(s^2 P_b(T(\alpha), S(\beta))\right) \leq [\theta(P_b(\alpha, \beta))]^k \tag{62}$$

Then the mappings T and S admit a unique common fixed point.

Proof. Setting $\mathscr{C}(t) = (\ln k)t$ and $D(t) = \theta(s^2 t)$ in Theorem 17 and following the proofs of Theorems 17, 18, and 20 respectively, we obtain the required result.□ □

The following corollary generalizes and improves the results presented by Zada and Sarwar [21] and Wardowski [25].

Corollary 24. *Suppose that the self-mappings T and S defined on the complete partial b-metric space (X, P_b) satisfy the following conditions:*

(a) *$\Gamma(\alpha, \beta, \mathfrak{R})$ is nonempty for all $\alpha, \beta \in X$*
(b) *There exists $\alpha_0 \in X$ such that $(\alpha_0, T(\alpha_0)) \in \mathfrak{R}$ and $\mathfrak{R}$ is (T, S)-closed*
(c) *Either T, S are continuous or (X, P_b) is $\mathfrak{R}$ – regular*
(d) *There exists $F \in \mathscr{F}$ and $\tau > 0$,such that for all $\alpha, \beta \in \mathfrak{I}$,*

$$\tau + F\left(s^2 P_b(T(\alpha), S(\beta))\right) \leq F(P_b(\alpha, \beta)) \tag{63}$$

If $\mathfrak{R}$ is a transitive and antisymmetric binary relation on X, then the mappings T, S admit a unique common fixed point.

Proof. Setting $\mathscr{C}(t) = e^{-\tau}t$ and $D(t) = e^{s^2 F(t)}$ in Theorem 17 and following the proofs of Theorems 17, 18, and 20, respectively, we obtain the required result.□ □

Corollary 25 (see [21]). *Let (X, P_b) be a complete partial b -metric space and $\mathfrak{R}$ be a transitive and antisymmetric binary relation on X. If the self-mappings T and S defined on (X, P_b) satisfy the following conditions:*

(a) *$\Gamma(\alpha, \beta, \mathfrak{R})$ is nonempty for all $\alpha, \beta \in X$*
(b) *There exists $\alpha_0 \in X$ such that $(\alpha_0, T(\alpha_0)) \in \mathfrak{R}$ and $\mathfrak{R}$ is (T, S)-closed*

(c) *Either T, S are continuous or (X, P_b) is $\mathfrak{R}$ − regular*

(d) *There exists $F \in \mathscr{F}$ and $\tau > 0$, such that for all $\alpha, \beta \in \mathfrak{I}$*
,

$$\tau + F(sP_b(T(\alpha), S(\beta))) \leq F\left(P_b(\alpha, \beta) + \frac{P_b(\alpha, S(\beta))P_b(\beta, T(\alpha))}{1 + P_b(\alpha, \beta)}\right) \tag{64}$$

Then T and S have a unique common fixed point in X.

Proof. This proof follows the proof of Corollary 24.

The following corollary improves the fixed-point results presented by Geraghty [24].□ □

Corollary 26. *Let (X, P_b) be a complete partial b-metric space and $\mathfrak{R}$ be a transitive and antisymmetric binary relation on X. If the self-mappings T and S defined on (X, P_b) satisfy the following conditions:*

(a) *$\Gamma(\alpha, \beta, \mathfrak{R})$ is nonempty for all $\alpha, \beta \in X$*

(b) *There exists $\alpha_0 \in X$ such that $(\alpha_0, T(\alpha_0)) \in \mathfrak{R}$ and $\mathfrak{R}$ is (T, S)-closed*

(c) *Either T, S are continuous or (X, P_b) is $\mathfrak{R}$ − regular*

(d) *For all $\alpha, \beta \in X$ and $(\alpha, \beta) \in \mathfrak{R}$*

$$s^2 P_b(T(\alpha), S(\beta)) \leq \gamma(P_b(\alpha, \beta))P_b(\alpha, \beta), \tag{65}$$

where $\gamma : [0,\infty) \longrightarrow [0,\infty)$ such that $\lim_{r \longrightarrow t^+} \gamma(r) < 1/s$, for each $t \in (0,\infty)$

Proof. By defining $\mathscr{C}(t) = t\gamma(t)$ and $D(t) = s^2 t$ in Theorem 17 and following the proofs of Theorems 17, 18, and 20, respectively, we obtain the required result.□ □

Remark 27.

(1) For $s = 1$, Theorems 17, 18, and 20 establish criteria for the existence of common fixed points of J_c-contractions in the partial metric spaces [12] and correspondingly for Corollaries 23, 24, 25, and 26

(2) For the zero self-distance ($P_b(\alpha, \beta) = 0$ for all α, β) and for the zero self-distance with $s = 1$, the results stated in Remark 27 (1) hold in the b-metric spaces and metric spaces, respectively

4. Application to the System of Boundary Value Problems

We will apply Theorem 17 to achieve the existence of a common solution to the following system of boundary value problems:

$$-\frac{d^2 v}{dt^2} = \mathscr{H}(t, v(t))\,;\, t \in \mathscr{I}, v(0) = v(1) = 0, \tag{66}$$

$$-\frac{d^2 w}{dt^2} = \mathscr{K}(t, w(t))\,;\, t \in \mathscr{I}, w(0) = w(1) = 0, \tag{67}$$

where $\mathscr{I} = [0, 1]$, $C(\mathscr{I})$ represents the set of continuous functions defined on $\mathscr{I}$. The functions $\mathscr{H}, \mathscr{K} : [0, 1] \times C(\mathscr{I}) \longrightarrow \mathbb{R}$ are continuous and nondecreasing according to ordinates. We define the binary relation $\mathfrak{R}$ on $C(\mathscr{I})$ as follows:

$$\mathfrak{R} = \{(v, w) \in C(\mathscr{I}) \times C(\mathscr{I}) : v(t) \leq w(t) \forall t \in \mathscr{I}\}. \tag{68}$$

The associated Green function $\mathscr{g} : \mathscr{I} \times \mathscr{I} \longrightarrow \mathscr{I}$ to (66) and (67) can be defined as follows:

$$\mathscr{g}(t, \ell) = \begin{cases} t(1 - \ell) \text{ if } 0 \leq t \leq \ell \leq 1, \\ \ell(1 - t) \text{ if } 0 \leq \ell \leq t \leq 1. \end{cases} \tag{69}$$

Let the mapping $d_* : C(\mathscr{I}) \times C(\mathscr{I}) \longrightarrow [0,\infty)$ be defined by

$$\begin{aligned} d_*(v, w) &= \left\| (v - w)^2 \right\|_\infty \\ &= \sup | v(t) - w(t)|^2, \forall v, w \in C(\mathscr{I}), t \in \mathscr{I}. \end{aligned} \tag{70}$$

It is claimed that $(C(\mathscr{I}), d_*, 2)$ is a complete b-metric space. By integration, we see that (66) and (67) can be written as $v = S(v)$ and $w = T(w)$, where $S, T : C(\mathscr{I}) \longrightarrow C(\mathscr{I})$ are defined by

$$\begin{aligned} S(v)(t) &= \int_0^1 \mathscr{g}(t, \ell)\mathscr{H}(\ell, v(\ell))d\ell, \\ T(w)(t) &= \int_0^1 \mathscr{g}(t, \ell)\mathscr{K}(\ell, w(\ell))d\ell. \end{aligned} \tag{71}$$

It is remarked that the common solution to (66) and (67) is the common fixed point of the operators S, T. Suppose the following conditions:

(a) $\exists \hslash > 0$ such that for $v(t) \neq w(t) (\forall t)$, we have

$$|\mathscr{H}(t, v(t)) - \mathscr{K}(t, w(t))|^2 \leq 16 e^{-\hslash} |v(t) - w(t)|^2 \forall t \in \mathscr{I} \tag{72}$$

(b) $\exists v_0, w_0 \in C(\mathscr{I})$ such that

$$\begin{aligned} v_0(t) &\leq \int_0^1 g(t, \mathscr{b})\, \mathscr{H}(\mathscr{b}, v_0(\mathscr{b}))d\mathscr{b}, \\ w_0(t) &\leq \int_0^1 g(t, \mathscr{b})\, \mathscr{K}(\mathscr{b}, w_0(\mathscr{b}))d\mathscr{b} \end{aligned} \tag{73}$$

The following theorem states the conditions under which equations (66) and (67) have a common solution.

Theorem 28. *Let the functions $\mathscr{H}, \mathscr{K} : [0, 1] \times C(\mathscr{I}) \longrightarrow \mathbb{R}$ satisfy conditions (a) and (b). Then equations (66) and (67) have a common solution.*

Proof. We will apply Theorem 17 to show the existence of the common solution to (66) and (67). By condition (b), there exists v_0 such that $(v_0, S(v_0)) \in \mathfrak{R}$. Since the functions $\mathscr{H}, \mathscr{K}$ are continuous, so $S, T : C(\mathscr{I}) \longrightarrow C(\mathscr{I})$ defined above are continuous. Since it is given that $\mathscr{H}, \mathscr{K}$ are nondecreasing, thus, $\mathfrak{R}$ is (S, T) closed. To show that the mappings S, T form DC-contraction, we proceed as follows:

$$\begin{aligned} |S(v)(t) - T(w)(t)|^2 &= \left| \int_0^1 g(t, \mathscr{b})\, (\mathscr{H}(\mathscr{b}, v(\mathscr{b})) - \mathscr{K}(\mathscr{b}, w(\mathscr{b})))d\mathscr{b} \right|^2 \\ &\leq \left(\int_0^1 g(t, \mathscr{b}) |\mathscr{H}(\mathscr{b}, v(\mathscr{b})) - \mathscr{K}(\mathscr{b}, w(\mathscr{b}))| d\mathscr{b} \right)^2 \\ &\leq \left(\int_0^1 g(t, \mathscr{b}) \sqrt{16 e^{-\mathscr{k}} |v(t) - w(t)|^2} d\mathscr{b} \right)^2. \end{aligned} \tag{74}$$

□ □

Since $\left(\sup \int_0^1 g(t, \mathscr{b}) d\mathscr{b}\right)^2 = 1/64$, for all $t \in \mathscr{I}$, thus, taking supremum on both sides of the above inequality, we have

$$s^2 d_*(S(v), T(w)) \leq e^{-\mathscr{k}} d_*(v, w) \forall v(\cdot), w(\cdot) \in C(\mathscr{I}). \tag{75}$$

Define the b-metric d_* on $C(\mathscr{I})$ by

$$d_*(v, w) = \begin{cases} P_b(v, w), & \text{if } v \neq w, \\ 0, & \text{if } v = w. \end{cases} \tag{76}$$

Inequality (75) can be written as

$$s^2 P_b(S(v), T(w)) \leq e^{-\mathscr{k}} P_b(v, w) \forall v(\cdot), w(\cdot) \in C(\mathscr{I}). \tag{77}$$

Defining the functions $\mathscr{C}$, F, and D by $\mathscr{C}(t) = e^{-\mathscr{k}} t$, $F(t) = \ln t$, and $D(t) = e^{F(t)}$, respectively, for all $t \in [0, \infty)$, we have

$$\begin{aligned} &\mathscr{k} + F\left(s^2 P_b(S(v), T(w))\right) \leq F(P_b(v, w)), \\ &e^{\mathscr{k}} \cdot e^{F\left(s^2 P_b(S(v), T(w))\right)} \leq e^{F(P_b(v, w))}, \\ &e^{F\left(s^2 P_b(S(v), T(w))\right)} \leq e^{-\tau} e^{F(P_b(v, w))}, \\ &D\left(s^2 P_b(S(v), T(w))\right) \leq \mathscr{C}(D(P_b(v, w))). \end{aligned} \tag{78}$$

Hence, applying Theorem 17, we say that the boundary value problems (66) and (67) have a common solution in $C(\mathscr{I})$.

Data Availability

No data were used to support this study.

Conflicts of Interest

All authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to this work.

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