Research Article

Extragradient Method for Fixed Points in CAT(0) Spaces

Yu-Pei Lv¹, Khurram Shabbir², Sundus Shahzeen³, Farman Ali⁴, and Jeevan Kafle⁵

¹Department of Mathematics, Huzhou University, Huzhou 313000, China

²Department of Mathematics, GC University, Lahore, Pakistan

³Department of Mathematics and Statistics, The University of Lahore, Lahore, Pakistan

⁴Department of Software, Sejong University, Seoul 05006, Republic of Korea

⁵Central Department of Mathematics, Tribhuvan University, Kirtipur, Kathmandu, Nepal

Correspondence should be addressed to Jeevan Kafle; jeevan.kafle@cdmath.tu.edu.np

Received 11 June 2021; Revised 5 August 2021; Accepted 12 August 2021; Published 26 August 2021

Academic Editor: Liliana Guran

Copyright © 2021 Yu-Pei Lv et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

This paper is dedicated to construct a viscosity extragradient algorithm for finding fixed points in a CAT(0) space. The mappings we consider are nonexpansive. Strong convergence of the algorithm is obtained. The results established in this work extend and improve some recent discovers in the literature.

1. Introduction

Let X be a CAT(0) space and K be any closed convex subset of X . Let $\phi : K \times K \rightarrow \mathbb{R}$ be a bifunction with $\phi(v, v) = 0$ for all $v \in K$. The main equilibrium problem is to get $r \in K$ satisfying

$$\phi(r, w) \geq 0, \quad \text{for all } w \in K. \quad (1)$$

Finding solutions for equilibrium problems aids in solving problems in other areas of science like physics, optimization, and economics. The set of all solutions of the equilibrium problem is denoted by $EP(\phi)$, i.e.,

$$EP(\phi) = \{r \in K : \phi(r, w) \geq 0, \forall w \in K\}. \quad (2)$$

Numerous iterative algorithms for monotone equilibrium have been investigated previously; for finding the solutions, see [1, 2]. Here, we will find the iterative algorithm for pseudomonotone bifunction. The bifunction ϕ is said to be pseudomonotone if

$$\phi(v, w) \geq 0 \implies \phi(w, v) \leq 0, \quad \text{for all } v, w \in K. \quad (3)$$

An extragradient method, to solve a pseudomonotone equilibrium problem in $\mathbb{R}^n$, was introduced in [3]. The

method of extragradient is as follows: given $v_0 \in K$, find successively w_n and v_{n+1} by

$$\begin{cases} w_n = \operatorname{argmin}_{w \in K} \left\{ \lambda_n \phi(v_n, w) + \frac{1}{2} \|w - v_n\|^2 \right\}, \\ v_{n+1} = \operatorname{argmin}_{w \in K} \left\{ \lambda_n \phi(w_n, w) + \frac{1}{2} \|w - v_n\|^2 \right\}, \end{cases} \quad (4)$$

where $\{\lambda_n\} \subset (0, \infty)$ and ϕ is such that the Lipschitz (type) condition holds. In [4], the following algorithm was introduced by Anh for finding a fixed point of a nonexpansive mapping T which is also the solution of the equilibrium problem for pseudomonotone bifunction ϕ in a Hilbert space:

$$\begin{cases} w_n = \operatorname{argmin}_{w \in K} \left\{ \lambda_n \phi(v_n, w) + \frac{1}{2} \|w - v_n\|^2 \right\}, \\ v_{n+1} = \operatorname{argmin}_{w \in K} \left\{ \lambda_n \phi(w_n, w) + \frac{1}{2} \|w - v_n\|^2 \right\}, \end{cases} \quad (5)$$

where $\{\alpha_n\}, \{\lambda_n\} \subset (0, 1]$ and ϕ is such that the Lipschitz (type) condition holds. The convergence (strong) of $\{v_n\}$ with $\lim_{n \rightarrow \infty} \|v_{n+1} - v_n\| = 0$ and under certain considerations on $\{\alpha_n\}, \{\lambda_n\} \in (0, 1]$.

In 2012, an algorithm for hybrid projection was considered by Vuong et al. [5] for

$$\begin{cases} w_n = \operatorname{argmin}_{w \in K} \left\{ \lambda_n \phi(v_n, w) + \frac{1}{2} \|w - v_n\|^2 \right\}, \\ v_{n+1} = \operatorname{argmin}_{w \in K} \left\{ \lambda_n \phi(v_n, w) + \frac{1}{2} \|w - v_n\|^2 \right\}, \\ s_n = \alpha_n v_0 + (1 - \alpha_n)[\beta_n z + (1 - \beta_n)S z_n], \\ K_n = \{z \in K : \|s_n - z\| \leq \|v_n - z\|\}, \\ D_n = \{z \in K : \langle v_n - z, v_0 - v_n \rangle \geq 0\}, \\ v_{n+1} = P_{K_n \cap D_n} v_0, \end{cases} \quad (6)$$

where $\{\alpha_n\}, \{\lambda_n\} \in (0, 1]$ and ϕ is such that the Lipschitz (type) condition holds. The convergence proved was strong.

The Armijo-type method for pseudomonotone equilibrium problems was formulated in [6] in the setting of Hilbert spaces. After that, the authors in [7] presented the convergence of weak and strong types for the algorithms in order to solve the equilibrium problem. The admirable outcomes are due to Dinh and Kim in which there is no restriction on monotonicity of the bifunction. The current results of equilibrium problems are given for pseudomonotone type. For further references, see [8, 9]. To the current knowledge, the authors modified the “hybrid projection algorithm” in order to get convergence of strong type for iterative algorithms of equilibrium problems of pseudomonotone type [10, 11].

The aim of this paper is to construct an extragradient algorithm of viscosity type for finding the same element for the solution set of a pseudomonotone equilibrium problem and fixed point set of a nonexpansive mapping in the framework of a CAT(0) space and derive its strong convergence.

2. Definitions and Known Results

In this section, we present basic definitions and known results. The notions that are not defined in this paper can be seen in [12–14].

Throughout this paper, (X, d) denotes the geodesic metric space with a geodesic triangle $\Delta(v_1, v_2, v_3)$ in (X, d) , where v_1, v_2 , and v_3 represent the vertices of Δ in (X, d) . We will represent (X, d) as X henceforth. A *comparison triangle* in X is a triangle $\bar{\Delta}(v_1, v_2, v_3) := \Delta(\bar{v}_1, \bar{v}_2, \bar{v}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(v_i, v_j) = d(v_i, v_j)$ for $i, j = 1, 2, 3$.

A geodesic space is called a CAT(0) space, if all the geodesic triangles satisfy the following comparison axiom

$$d(v, w) \leq d_{\mathbb{E}^2}(\bar{v}, \bar{w}). \quad (7)$$

Let $v, w \in X$, and by Lemma 2.1(iv) of [15] for each $s \in [0, 1]$, there exists a unique point $z \in [v, w]$ such that

$$d(v, z) = sd(v, w), \quad (8)$$

$$d(w, z) = (1 - s)d(v, w), \quad (9)$$

for all $v, w \in \Delta(v_1, v_2, v_3)$ with $\bar{v}, \bar{w}$ the corresponding points of v, w in $\bar{\Delta}(v_1, v_2, v_3)$.

Lemma 1 (see [16]). *Let X be a CAT(0) space. Then the following assertions are true:*

(i) *For any $v, w, z \in X$ and $s \in [0, 1]$,*

$$d((1 - s)v \oplus sy, z) \leq (1 - s)d(v, z) + sd(w, z) \quad (10)$$

(ii) *For any $v, w, z \in X$ and $s \in [0, 1]$,*

$$d^2((1 - s)v \oplus sy, z) \leq (1 - s)^2 d^2(v, z) + s^2 d^2(w, z) - s(1 - s)d^2(v, w) \quad (11)$$

Let (X, d) be a uniquely geodesic metric space; that is, for each $v, w \in X$, there exists a unique isometry $c : [0, d(v, w)] \rightarrow X$ such that $c(0) = v$ and $c(d(v, w)) = w$, and in this case, we write $[v, w] = \{c(t) : t \in [0, d(v, w)]\}$. For each $s \in [0, 1]$, we write $(1 - s)v \oplus sw$ for the element $z \in [v, w]$ such that $d(z, v) = sd(v, w)$ and $d(z, w) = (1 - s)d(v, w)$.

Hadamard spaces are the complete CAT(0) spaces; for details, see [16]. If v, w_1, w_2 are points of a CAT(0) space and w_0 is the midpoint of the segment $[w_1, w_2]$, which we will denote by $w_1/2 \oplus w_2/2$, then the CAT(0) inequality gives

$$d^2\left(v, \frac{w_1}{2} \oplus \frac{w_2}{2}\right) \leq \frac{1}{2}d^2(v, w_1) + \frac{1}{2}d^2(v, w_2) - \frac{1}{4}d^2(w_1, w_2). \quad (12)$$

This inequality is called the (CN) inequality of Bruhat and Tits [16]. In fact, a geodesic space is a CAT(0) space if and only if it satisfies the (CN) inequality (cf. [16], page 163). Berg and Nikolaev [17] introduced the idea of the quasilinearization as follows: Let us denote the pair $(\alpha, \beta) \in X \times X$ by $\overleftarrow{\alpha\beta}$ and call it a vector. Then, quasilinearization is defined as a map

$$\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \rightarrow \mathbb{R} \quad (13)$$

defined as

$$\langle \overleftarrow{\alpha\beta}, \overleftarrow{\gamma\vartheta} \rangle = \frac{1}{2} (d^2(\alpha, \vartheta) + d^2(\beta, \gamma) - d^2(\alpha, \gamma) - d^2(\beta, \vartheta)). \quad (14)$$

It is easy to see that $\langle \overleftarrow{\alpha\beta}, \overleftarrow{\gamma\vartheta} \rangle = \langle \overleftarrow{\gamma\vartheta}, \overleftarrow{\alpha\beta} \rangle$, $\langle \overleftarrow{\alpha\beta}, \overleftarrow{\gamma\vartheta} \rangle = -\langle \overleftarrow{\beta\alpha}, \overleftarrow{\vartheta\gamma} \rangle$, and $\langle \overleftarrow{\alpha\beta}, \overleftarrow{\gamma\vartheta} \rangle + \langle \overleftarrow{\gamma\vartheta}, \overleftarrow{\alpha\beta} \rangle = \langle \overleftarrow{\alpha\beta}, \overleftarrow{\gamma\vartheta} \rangle$ for all $\alpha, \beta, \gamma, \vartheta \in X$. We say that X satisfies the Cauchy-Schwarz

inequality if

$$\langle \overleftarrow{\alpha\beta}, \overleftarrow{v\vartheta} \rangle \leq d(\alpha, \beta)d(v, \vartheta) \quad (15)$$

for all $\alpha, \beta, v, \vartheta \in X$. It is well known [17] that a geodesically connected metric space is a CAT(0) space if and only if it satisfies the Cauchy-Schwarz inequality.

In 1976, Lim gave the definition of Δ -convergence in a general metric space case (see [18]). He named this type of convergence as strong Δ -convergence, proving that CAT(0) spaces provide a natural framework for the Lim concept and, in the setting of Δ -convergence, provide many properties of the usual notion of weak convergence in Banach spaces. Let us recall this type of convergence for the case of CAT(0) spaces, as follows.

Definition 2. Let (X, d) be a CAT(0) space. A sequence $\{v_n\}$ in X is said to Δ -converge to $v \in X$ if and only if v is the unique asymptotic center of all subsequences of $\{v_n\}$. In this case, we write $\Delta - \lim_{n \rightarrow \infty} v_n = v$, and v is called the Δ -limit of $\{v_n\}$.

Note that the asymptotic center $\mathbb{A}(\{v_n\})$ of $\{v_n\}$ is the set

$$\mathbb{A}(\{v_n\}) := \{v \in X : r(v, \{v_n\}) = r(\{v_n\})\}, \quad (16)$$

and the asymptotic radius $r(\{v_n\})$ of $\{v_n\}$ is given by

$$r(\{v_n\}) := \inf_{v \in X} \{r(v, \{v_n\})\}, \quad (17)$$

where $r(v, \{v_n\}) := \limsup_{n \rightarrow \infty} d(v, v_n)$, for $\{v_n\}$ is a bounded sequence in X .

Ahmadi Kakavandi and Amini introduced in [19] another variant of weak convergence in complete CAT(0) spaces, taking into account the concept of quasilinearization.

Definition 3 (see [19]). Let (X, d) be a complete CAT(0) space. A sequence $\{v_n\}$ in X is said to w -converge to an element $v \in X$ if for each $w \in X$, $\lim_{n \rightarrow \infty} \langle \overrightarrow{vv_n}, \overrightarrow{vw} \rangle = 0$.

Obviously, the convergence in the metric implies w -convergence, and the w -convergence implies Δ -convergence (see Proposition 2.5 in [19]). But the converse is not true (see [20]). The following result proves an explicit connection between w -convergence and Δ -convergence.

Theorem 4 (see [20]). Let (X, d) be a complete CAT(0) space. Then a sequence $\{v_n\}$ in X Δ -converges to $v \in X$ if and only if, for every $w \in X$, $\limsup_{n \rightarrow \infty} \langle \overrightarrow{vv_n}, \overrightarrow{vw} \rangle \leq 0$.

Further, let us recall some definitions for the case of a CAT(0) space.

Definition 5. Let X be a CAT(0) space and $T : X \rightarrow X$ be a mapping. Then T is called nonexpansive if

$$d(T(v), T(w)) \leq d(v, w), \quad \text{for every } v, w \in K. \quad (18)$$

Definition 6. Let X be a CAT(0) space and $T : X \rightarrow X$ be a mapping. Then T is called a contraction if

$$d(T(v), T(w)) \leq \theta d(v, w), \quad \text{for every } v, w \in K, \theta \in [0, 1). \quad (19)$$

Throughout this paper, we denote by $\text{Fix}(T) = \{v \in C \mid Tv = v\}$ the set of fixed points of T .

Concerning the convexity in CAT(0) spaces, we remark that, in this type of spaces, angles exist in a strong sense, the distance function is convex, and one has both uniform convexity and orthogonal projection onto convex subsets. Moreover, CAT(0) spaces turn up to represent a real framework for convexity theory.

Remark 7. Considering the CAT(0) space case, a subset $C \subset X$ is said to be convex if C includes every geodesic segment joining any two of its points, i.e., $(1-t)x \oplus ty \in C$, for every $x, y \in C$ and $t \in (0, 1)$.

Definition 8. Let X be a CAT(0) space. A function $f : X \rightarrow (-\infty, +\infty]$ is said to be convex if

$$f(tv \oplus (1-t)w) \leq tf(v) + (1-t)f(w), \quad (20)$$

for all $v, w \in X$, and $t \in (0, 1)$.

Let X be a CAT(0) space and $K \subset X$ a convex and closed subset. Further, let a bifunction $\phi : K \times K \rightarrow \mathbb{R}$; then ϕ is called

- (1) ς -strong monotone ($\varsigma < 0$) on K if $\forall v, w \in K$, we have

$$\phi(v, w) + \phi(w, v) \leq \varsigma d^2(v, w) \quad (21)$$

- (2) monotone on K if for each $v, w \in K$, one has

$$\phi(v, w) + \phi(w, v) \leq 0 \quad (22)$$

- (3) pseudomonotone on K if for each $v, w \in K$, one has

$$\phi(v, w) \geq 0 \Rightarrow \phi(w, v) \leq 0 \quad (23)$$

The above bifunction ϕ is Lipschitz-type continuous on

K , if there exist two constants $c_1 > 0$ and $c_2 > 0$ such that

$$\phi(v, z) \leq \phi(v, w) + \phi(w, z) + c_1 d^2(v, w) + c_2 d^2(w, z), \quad \text{for every } v, w \in K. \quad (24)$$

In [19], Kakavandi and Amini define the notion of a subdifferential of a function as follows.

Definition 9 (see [19]). Let $f : X \rightarrow (-\infty, +\infty]$ be a proper function with efficient domain $\mathcal{D}(f) = \{v : f(v) < +\infty\}$, then the subdifferential of f is the multifunction $\partial f : X \rightrightarrows X^*$ defined by

$$\partial f(v^*) = \{v^* : f(z) - f(v) \geq \langle v^*, \vec{vz} \rangle, \text{ for all } z \in X\}, \quad (25)$$

when $v \in \mathcal{D}(f)$ and $\partial f(v) = \emptyset$.

We mention that X^* denote the notion of dual space of the metric space (X, d) . For more details, see [19].

In [21], Georgiou and Papadopoulos gave a strong discussion concerning the convergence types and topologies on function spaces. Then, let us consider X, Y, Z as three topological spaces. A mapping $f : X \rightarrow Y$ into a is called *weakly continuous* at $v \in X$ if for every open neighbourhood V of $f(v)$ there exists an open neighbourhood U of v such that $f(U) \subseteq \text{Cl}(V)$. $\text{Cl}(V)$ denotes the closure of V . The mapping f is weakly continuous on X , if it is weakly continuous at each point of X . In the following, $\text{WC}(Y, Z)$ denotes the set of all weakly continuous maps of Y into Z . If τ is a topology on the set $\text{WC}(Y, Z)$, then the corresponding topological space is denoted by $\text{WC}_\tau(Y, Z)$.

In this conditions, we can recall the notion of *jointly weakly continuous* function.

Definition 10. A topology τ on $\text{WC}(Y, Z)$ is called weakly jointly continuous if for every X , the weak continuity of a map $G : X \rightarrow \text{WC}_\tau(Y, Z)$ implies the weak continuity of the map $\tilde{G} : X \times Y \rightarrow Z$.

For more details and results concerning the topology and the convergence types in function spaces, see [21–23].

Further, taking into account the previous notions, let us consider the following properties of ϕ :

- (1) $\phi(v, v) = 0$ for all $v \in K$, and ϕ is taken pseudomonotone on the subset K
- (2) ϕ is taken to be Lipschitz-type continuous on the subset K
- (3) For all $v \in K$, $w \rightarrow \phi(v, w)$ is subdifferentiable and convex
- (4) $\phi(v, w)$ is taken jointly weakly continuous on $K \times K$

With conditions (6), (7), (8), and (10), the set $\text{EP}(\phi)$ is convex and closed.

Lemma 11 (see [24]). Let X be a $\text{CAT}(0)$ space. Assume that $\text{EP}(\phi) \neq \emptyset$ and $v \in K$. Further, let $w, s \in K$ be solutions of

$$w = \operatorname{argmin}_{z \in K} \left\{ \lambda \phi(v, w) + \frac{1}{2} d^2(v, z) \right\}, \quad (26)$$

$$s = \operatorname{argmin}_{a \in K} \left\{ \lambda \phi(v, w) + \frac{1}{2} d^2(w, a) \right\}, \quad (27)$$

strongly convex problems, where $\lambda > 0$, then

$$\begin{aligned} \lambda[\phi(v, z) - \phi(v, w)] &\geq \langle \overrightarrow{wv}, \overrightarrow{wz} \rangle, \quad z \in K, \\ d^2(s, x) &\leq d^2(v, x) - (1 - 2\lambda c_1) d^2(v, w) \\ &\quad - (1 - 2\lambda c_2) d^2(s, x), \quad \forall x \in \text{EP}(\phi). \end{aligned} \quad (28)$$

Lemma 12 (the demiclosedness principle). Let K be a non-empty closed convex subset of the $\text{CAT}(0)$ space X and $T : K \rightarrow K$ such that

$$\begin{aligned} v_n &\rightarrow v \in K, \\ d(v_n, T v_n) &\rightarrow 0. \end{aligned} \quad (29)$$

Then, $v = T v$. (Here, $\rightarrow$ (respectively, $\dashrightarrow$) denotes strong (respectively, weak) convergence.) The notion of weak convergence is the same as defined in [25].

Lemma 13 (see [26]). Let X be a $\text{CAT}(0)$ space and K be any closed convex subset of X . For each point $v \in X$, there exists a unique nearest point of K , denoted by $P_K v$, such that $d(v, P_K v) \leq d(v, w)$ for all $w \in K$. Such a P_K is metric projection onto K from X . Then,

- (1) for all $v \in X$ and $z \in K$, $z = P_K v$ iff

$$\langle \overrightarrow{vz}, \overrightarrow{zw} \rangle \geq 0, \quad \text{for all } w \in K \quad (30)$$

- (2) for all $v \in X$ and $z \in K$, it holds

$$d^2(P_K v, z) \leq d^2(v, z) - d^2(P_K v, v) \quad (31)$$

For more information, see Section 3 of [27].

Lemma 14. Let X be a complete $\text{CAT}(0)$ space. For all $v, w \in X$ and $0 = [(0, 1)] \in X$, the following hold:

- (1) $d^2(v \oplus w, 0) \leq d^2(v, 0) + 2\langle \overrightarrow{w0}, \overrightarrow{v0} \rangle + 2d^2(w, 0)$
- (2) $d^2(sv \oplus (1-s)w, 0) = sd^2(v, 0) + (1-s)d^2(w, 0) - s(1-s)d^2(v, w)$ for all $s \in [0, 1]$

Consider $\{\lambda_n\} \subset [\delta_1, \delta_2]$ with $0 < \delta_1 < \delta_2 < \min\{1/2c_1, 1/2c_2\}$, $\{\alpha_n\} \subset (0, 1/(2-\rho))$, $\{\beta_n\} \subset (0, 1)$, and $v_0 \in K$. Consider $n = 0$.

ALGORITHM 19

Lemma 15 (see [28]). Assume that $\{a_n\}$ is a sequence of non-negative real numbers such that

$$a_{n+1} \leq (1 - \beta_n)a_n + \delta_n, \quad \text{for all } n \geq 0, \quad (32)$$

where $\{\beta_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence with

- (1) $\sum_{n=0}^{\infty} \beta_n = \infty$
- (2) $\limsup_{n \rightarrow \infty} \delta_n / \beta_n \leq 0$ or $\sum_{n=0}^{\infty} |\delta_n| < \infty$

Then, $\lim_{n \rightarrow \infty} a_n = 0$.

Another interesting result useful in the proof of our main results is the following.

Lemma 16 (see [29]). Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that there exists subsequences $\{n_j\}$ of $\{n\}$ such that $a_{n_i} < a_{n_{i+1}}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \rightarrow \infty$ as $k \rightarrow \infty$, and the following properties are satisfied all (sufficiently large) number $k \in \mathbb{N}$:

$$\begin{aligned} a_{m_k} &< a_{m_{k+1}}, \\ a_k &< a_{m_{k+1}}. \end{aligned} \quad (33)$$

In fact, $m_k = \max\{j \leq k : a_j \leq a_{j+1}\}$.

Example 17. Let $X = R^n$ ($n \geq 2$) and $K = \{(v_1, v_2, \dots, v_n) : v_i \geq 1, i = 1, 2, \dots, n\}$. Consider a bifunction $f : K \times K \rightarrow R$ defined as $f(v, w) = 2(w_n - v_n)d(v, w)$ for each $v, w \in K$. Then we have the inequality

$$\begin{aligned} f(v, w) &= 2(w_n - v_n)d(v, w) \leq 2(w_n - r_n)d(v, r) \\ &\quad + 2(w_n - r_n)d(w, r) + 2(r_n - v_n)d(v, r) \\ &\quad + 2(r_n - v_n)d(r, w) \leq 2d(w, r)d(v, r) \\ &\quad + 2d(r, v)d(r, w) + f(v, r) + f(r, w) \leq 2d^2(v, r) \\ &\quad + 2d^2(r, w) + f(v, r) + f(r, w). \end{aligned} \quad (34)$$

Then, f is Lipschitz-type continuous on K with $c_1 = 2 = c_2$.

3. Strong Convergence of the Proposed Algorithm

Suppose that a nonexpansive mapping $T : K \rightarrow K$ and a bifunction $\phi : K \times K \rightarrow R$ satisfy the conditions (6)–(10) and $\text{Fix}(T) \cap \text{EP}(\phi) \neq \emptyset$. Further, let $p : K \rightarrow K$ be a p

-contraction. Because $P_{\text{Fix}(T) \cap \text{EP}(\phi)}p$ is a contraction mapping on K , we have $\hat{q} \in K$, such that $\hat{q} = P_{\text{Fix}(T) \cap \text{EP}(\phi)}p(\hat{q})$.

Before presenting the first result of this section, let us recall a crucial theorem given by Lim in [18], which is used in the proof of our main results.

Theorem 18 ([18], Theorem 5.2). Every bounded sequence $\{x_n\}$ in a Hadamard space (X, d) has a Δ -convergent subsequence.

The following algorithm is useful in finding a common element of a solution set of pseudomonotone equilibrium problem on ϕ and fixed point set of T .

Step 1.

$$\begin{cases} w_n = \arg\min_{w \in K} \left\{ \lambda_n \phi(v_n, w) + \frac{1}{2} d^2(w, v_n) \right\}, \\ v_n = \arg\min_{s \in K} \left\{ \lambda_n \phi(w_n, s) + \frac{1}{2} d^2(s, v_n) \right\}. \end{cases} \quad (35)$$

Step 2. If $w_n = v_n = Tv_n$, then $v_n \in \text{Fix}(T) \cap \text{EP}(\phi)$, stop the process; otherwise, go to Step 3.

Step 3. Generate $v_{n+1} = \alpha_n p(v_n) \oplus (1 - \alpha_n)(\beta_n v_n \oplus (1 - \beta_n)T s_n)$. Set $n = n + 1$ and go to Step 1. Obviously, if $w_n = v_n$ for $n \in N$, by using (26), it gives $\phi(v_n, z) \geq 0$ for all $z \in K$ and $v_n \in \text{EP}(\phi)$. It gives $v_n \in \text{Fix}(T) \cap \text{EP}(\phi)$: from $v_n = Tv_n$. Further, for convergence of the algorithm, let us consider that Step 2 is not true for $n \in N$.

Lemma 20. Consider $\{v_n\}$ to be bounded sequence. If $d(v_n, w_n) \rightarrow 0$, $d(v_n, s_n) \rightarrow 0$, $d(s_n, Ts_n) \rightarrow 0$ as $n \rightarrow \infty$, then

$$\limsup_{n \rightarrow \infty} \langle \overrightarrow{v_n \hat{q}}, \overrightarrow{p(\hat{q}) \hat{q}} \rangle \leq 0. \quad (36)$$

Proof. We consider the following bounded sequence $\{v_n\}$. Using Theorem 18, there exists $\{v_{n_k}\}$ a subsequence of $\{v_n\}$ such that $\{v_{n_k}\}$ weakly converges to $v \in K$ with

$$\limsup_{n \rightarrow \infty} \langle \overrightarrow{v_n \hat{q}}, \overrightarrow{p(\hat{q}) \hat{q}} \rangle = \lim_{k \rightarrow \infty} \langle \overrightarrow{v_{n_k} \hat{q}}, \overrightarrow{p(\hat{q}) \hat{q}} \rangle = \langle \overrightarrow{v \hat{q}}, \overrightarrow{p(\hat{q}) \hat{q}} \rangle. \quad (37)$$

Since $d(v_n, w_n) \rightarrow 0$, we have $w_{n_k} \rightarrow v$.

By (26) with $v = v_{n_k}$ and $w = w_{n_k}$, we have

$$\lambda[\phi(v_{n_k}, z) - \phi(v_{n_k}, w_{n_k})] \geq \langle \overrightarrow{w_{n_k} v_{n_k}}, \overrightarrow{y \hat{z}} \rangle, \quad \text{for all } z \in K. \quad (38)$$

Taking $k \longrightarrow \infty$ on $\{\lambda_n\}$, (6), and (10), we achieve

$$\phi(v, z) \geq 0 \text{ for all } z \in K. \quad (39)$$

This gives $v \in \text{EP}(\phi)$. Also $s_{n_k} \rightarrow v$.

By Lemma 12 with $d(v_n, w_n) \longrightarrow 0$, we obtain $v \in \text{Fix}(T)$. Then $v \in \text{Fix}(T) \cap \text{EP}(\phi)$. Using Lemma 13 with (36), the conclusion follows. $\square$

Theorem 21. Consider the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfying the conditions:

$$\begin{aligned} \lim_{n \longrightarrow \infty} \alpha_n &= 0, \\ \sum_{n=0}^{\infty} \alpha_n &= \infty, \\ 0 &< \liminf_{n \longrightarrow \infty} \beta_n \leq \limsup_{n \longrightarrow \infty} \beta_n < 1, \end{aligned} \quad (40)$$

then the sequence $\{v_n\}$ strongly converges to $\hat{q}$.

Proof. Consider $l_n = \beta_n v_n \oplus (1 - \beta_n)Ts_n \forall n \in \mathbb{N}$. Lemma 11 follows

$$\begin{aligned} d^2(l_n, \hat{q}) &\leq \beta_n d^2(v_n, \hat{q}) + (1 - \beta_n) d^2(Ts_n, \hat{q}) \leq \beta_n d^2(v_n, \hat{q}) \\ &+ (1 - \beta_n) d^2(s_n, \hat{q}) \leq \beta_n d^2(v_n, \hat{q}) + (1 - \beta_n) [d^2(v_n, \hat{q}) \\ &- (1 - \lambda_n c_1) d^2(v_n, w_n) - (1 - \lambda_n c_2) d^2(v_n, s_n)] \\ &\leq \beta_n d^2(v_n, \hat{q}) + (1 - \beta_n) (1 - \lambda_n c_1) d^2(v_n, w_n) \\ &- (1 - \beta_n) (1 - \lambda_n c_2) d^2(w_n, s_n) \leq d^2(v_n, \hat{q}). \end{aligned} \quad (41)$$

Then

$$\begin{aligned} d(v_{n+1}, \hat{q}) &\leq \alpha_n d(p(v_n), \hat{q}) + (1 - \alpha_n) d(l_n, \hat{q}) \\ &\leq \alpha_n d(p(v_n), p(\hat{q})) + \alpha_n d(p(\hat{q}), \hat{q}) \\ &+ (1 - \alpha_n) d(v_n, \hat{q}) \leq \alpha_n \rho d(v_n, \hat{q}) \\ &+ \alpha_n d(p(\hat{q}), \hat{q}) + (1 - \alpha_n) d(v_n, \hat{q}) \\ &= (1 - \alpha_n(1 - \rho)) d(v_n, \hat{q}) + \alpha_n d(p(\hat{q}), \hat{q}) \\ &\leq \max \left\{ \frac{d(p(\hat{q}), \hat{q})}{1 - \rho}, d(v_n, \hat{q}) \right\} \cdots \leq \max \left\{ \frac{d(p(\hat{q}), \hat{q})}{(1 - \rho)}, d(v_0, \hat{q}) \right\}. \end{aligned} \quad (42)$$

Hence, $\{v_n\}$ is bounded; then, $\{p(v_n)\}$, $\{w_n\}$, and $\{s_n\}$ are bounded too. On the other hand, by (41), we have

$$\begin{aligned} d^2(v_{n+1}, \hat{q}) &\leq \alpha_n d^2(p(v_n), \hat{q}) + (1 - \alpha_n) d^2(l_n, \hat{q}) \\ &\leq \alpha_n d^2(p(v_n), \hat{q}) + (1 - \alpha_n) \\ &\cdot [d^2(v_n, \hat{q}) - (1 - \beta_n) (1 - \lambda_n c_1) d^2(v_n, w_n) \\ &+ (1 - \lambda_n c_2) d^2(w_n, s_n)]. \end{aligned} \quad (43)$$

Let us consider

$$\begin{aligned} M &= \sup \{ |d^2(p(v_n), \hat{q}) - d^2(v_n, \hat{q})| + (1 - \beta_n) [(1 - \lambda_n c_1) \\ &\cdot d^2(v_n, w_n) + (1 - \lambda_n c_2) d^2(w_n, s_n)] : n \in \mathbb{N} \}. \end{aligned} \quad (44)$$

Combining (43) and (44), we get

$$\begin{aligned} (1 - \beta_n) (1 - \lambda_n c_1) d^2(v_n, w_n) + (1 - \lambda_n c_2) d^2(w_n, s_n) \\ \leq d^2(v_n, \hat{q}) - d^2(v_{n+1}, \hat{q}) + \alpha_n M. \end{aligned} \quad (45)$$

By Lemma 14 and (40), we have

$$\begin{aligned} d^2(v_{n+1}, \hat{q}) &= d^2(\alpha_n(p(v_n) - \hat{q}) \oplus (1 - \alpha_n)(l_n - \hat{q}), 0) \\ &\leq (1 - \alpha_n)^2 d^2(l_n, \hat{q}) + 2\alpha_n \langle \overrightarrow{p(v_n)\hat{q}}, \overrightarrow{v_{n+1}\hat{q}} \rangle \\ &\leq (1 - \alpha_n)^2 d^2(v_n, \hat{q}) + 2\alpha_n \langle \overrightarrow{p(v_n)p(\hat{q})}, \overrightarrow{v_{n+1}\hat{q}} \rangle \\ &+ 2\alpha_n \langle \overrightarrow{p(\hat{q})\hat{q}}, \overrightarrow{v_{n+1}\hat{q}} \rangle \leq (1 - \alpha_n)^2 d^2(v_n, \hat{q}) \\ &+ 2\alpha_n d(p(v_n), p(\hat{q})) d(v_{n+1}, \hat{q}) \\ &+ 2\alpha_n \langle \overrightarrow{p(v_n)\hat{q}}, \overrightarrow{v_{n+1}\hat{q}} \rangle \leq (1 - \alpha_n)^2 d^2(v_n, \hat{q}) \\ &+ 2\alpha_n \rho d(v_n, \hat{q}) d(v_{n+1}, \hat{q}) + 2\alpha_n \langle \overrightarrow{p(q_n)\hat{q}}, \overrightarrow{v_{n+1}\hat{q}} \rangle \\ &\leq (1 - \alpha_n)^2 d^2(v_n, \hat{q}) + \alpha_n \rho [d^2(v_n, \hat{q}) + d^2(v_{n+1}, \hat{q}) \\ &+ 2\alpha_n \langle \overrightarrow{p(q_n)\hat{q}}, \overrightarrow{v_{n+1}\hat{q}} \rangle]. \end{aligned} \quad (46)$$

Therefore,

$$\begin{aligned} d^2(v_{n+1}, \hat{q}) &\leq \frac{(1 - \alpha_n)^2 + \alpha_n \rho}{1 - \alpha_n \rho} d^2(v_n, \hat{q}) + \frac{2\alpha_n}{1 - \alpha_n \rho} \langle \overrightarrow{p(\hat{q})\hat{q}}, \overrightarrow{v_{n+1}\hat{q}} \rangle \\ &\leq \frac{1 - 2\alpha_n + \alpha_n \rho}{1 - \alpha_n \rho} d^2(v_n, \hat{q}) + \frac{\alpha_n^2}{1 - \alpha_n \rho} d^2(v_n, \hat{q}) \\ &+ \frac{2\alpha_n}{1 - \alpha_n \rho} \langle \overrightarrow{p(\hat{q})\hat{q}}, \overrightarrow{v_{n+1}\hat{q}} \rangle \leq \left(1 - \frac{2(1 - \rho)\alpha_n}{1 - \alpha_n \rho} \right) \\ &\cdot d^2(v_n, \hat{q}) + \frac{2(1 - \rho)\alpha_n^2}{1 - \alpha_n} M_0 + \frac{2\alpha_n}{1 - \alpha_n \rho} \langle \overrightarrow{p(\hat{q})\hat{q}}, \overrightarrow{v_{n+1}\hat{q}} \rangle, \end{aligned} \quad (47)$$

where $M_0 = \sup \{d^2(v_n, \hat{q}) : n \in \mathbb{N}\}$. Taking $\gamma_n = 2(1 - \rho)\alpha_n / (1 - \alpha_n \rho)$ for all $n \in \mathbb{N}$. As $\{\alpha_n\} \subset (0, 1/(2 - \rho))$, then $\{\gamma_n\} \subset (0, 1)$. By the conditions on $\{\alpha_n\}$, we get

$$\begin{aligned} \lim_{n \longrightarrow \infty} \gamma_n &= 0, \\ \sum_{n=0}^{\infty} \gamma_n &= \infty. \end{aligned} \quad (48)$$

Then, we have

$$d^2(v_{n+1}, \hat{q}) \leq (1 - \gamma_n)d^2(v_n, \hat{q}) + \frac{\alpha_n \gamma_n M_0}{2(1 - \rho)} + \frac{\gamma_n}{1 - \rho} \langle \overrightarrow{p(\hat{q})\hat{q}}, \overrightarrow{v_{n+1}\hat{q}} \rangle. \quad (49)$$

The rest of the proof will be divided into two parts. $\square$

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{d(v_n, \hat{q})\}_{n=n_0}^\infty$ is nonincreasing. In this situation, $\{d(v_n, \hat{q})\}$ is convergent. This together with the hypothesis on $\{\gamma_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, and (45) gives

$$\lim_{n \rightarrow \infty} d(v_n, w_n) = \lim_{n \rightarrow \infty} d(v_n, s_n) = 0. \quad (50)$$

On the other hand, by Lemmas 11 and 14, we have

$$\begin{aligned} d^2(v_{n+1}, \hat{q}) &= d^2(\alpha_n(p(v_n) - \hat{q}) \oplus (1 - \alpha_n)[\beta_n v_n \oplus (1 - \beta_n) \\ &\quad \cdot Ts_n - \hat{q}], 0) \leq \alpha_n d^2(p(v_n), \hat{q}) + (1 - \alpha_n) \\ &\quad \cdot d^2(\beta_n v_n \oplus (1 - \beta_n)Ts_n - \hat{q}, 0) \leq \alpha_n d^2(p(v_n), \hat{q}) \\ &\quad + (1 - \alpha_n)[\beta_n d^2(v_n, \hat{q}) + (1 - \beta_n)d^2(Ts_n, \hat{q}) \\ &\quad - \beta_n(1 - \beta_n)d^2(v_n, Ts_n)] \leq \alpha_n d^2(p(v_n), \hat{q}) \\ &\quad + (1 - \alpha_n)[\beta_n d^2(v_n, \hat{q}) + (1 - \beta_n)d^2(v_n, \hat{q}) \\ &\quad - \beta_n(1 - \beta_n)d^2(v_n, Ts_n)] \leq \alpha_n d^2(p(v_n), \hat{q}) \\ &\quad + (1 - \alpha_n)\beta_n d^2(v_n, \hat{q}) - \beta_n(1 - \beta_n) \\ &\quad \cdot (1 - \alpha_n)d^2(v_n, Ts_n). \end{aligned} \quad (51)$$

Hence,

$$\beta_n(1 - \beta_n)(1 - \alpha_n)d^2(v_n, Ts_n) \leq d^2(v_n, \hat{q}) - d^2(v_{n+1}, \hat{q}) + \alpha_n d^2(p(v_n), \hat{q}). \quad (52)$$

Then, $\{d(v_n, \hat{q})\}$ converges. Also, $\{\alpha_n\} \subset (0, 1/(2 - \rho))$, and conditions on $\{\alpha_n\}$ and $\{\beta_n\}$ gives

$$\lim_{n \rightarrow \infty} d(v_n, Ts_n) = 0. \quad (53)$$

Combining (49) and (52), we get

$$d(s_n, Ts_n) \leq d(s_n, v_n) + d(v_n, Ts_n) \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (54)$$

Lemma 20 and (49) and (53) give

$$\limsup_{n \rightarrow \infty} \langle \overrightarrow{p(q_n)\hat{q}}, \overrightarrow{v_{n+1}\hat{q}} \rangle \leq 0. \quad (55)$$

The conclusion follows from Lemma 15 and (48), (49), and (54).

Case 2. Let $\{v_{n_i}\}$ be a subsequence of $\{v_n\}$ with

$$d(v_{n_i}, \hat{q}) < d(v_{n_i}, \hat{q}) \forall i \in \mathbb{N}. \quad (56)$$

Then, by Lemma 16, there exists a subsequence $\{m_k\}$ such that $m_k \rightarrow \infty$:

$$\begin{aligned} d(v_{m_k}, \hat{q}) &< d(v_{m_k+1}, \hat{q}), \\ d(v_k, \hat{q}) &< d(v_{k+1}, \hat{q}), \\ &\forall k \in \mathbb{N}. \end{aligned} \quad (57)$$

The above expression with (44) concludes

$$\begin{aligned} (1 - \beta_{m_k})[(1 - \lambda_{m_k}c_1)d^2(v_{m_k}, w_{m_k}) + (1 - \lambda_{m_k}c_2)d^2(w_{m_k}, s_{m_k})] \\ \leq d^2(v_{m_k}, \hat{q}) - d^2(v_{m_k+1}, \hat{q}) + \alpha_{m_k}M \leq \alpha_{m_k}M \forall k \in \mathbb{N}. \end{aligned} \quad (58)$$

By the hypothesis on $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$, it follows that

$$\lim_{k \rightarrow \infty} d^2(v_{m_k}, w_{m_k}) = d^2(w_{m_k}, s_{m_k}) = 0. \quad (59)$$

Using (51), we have

$$\begin{aligned} \beta_{m_k}(1 - \beta_{m_k})[(1 - \alpha_{m_k})d^2(v_{m_k}, Ts_{m_k}) \leq d^2(v_{m_k}, \hat{q}) \\ - d^2(v_{m_k+1}, \hat{q}) + \alpha_{m_k}d^2(p(v_{m_k}), \hat{q}) \\ \leq \alpha_{m_k}d^2(p(v_{m_k}), \hat{q}) \rightarrow 0, \text{ as } k \rightarrow \infty. \end{aligned} \quad (60)$$

By the hypothesis on $\{\alpha_n\}$ and $\{\beta_n\}$, it follows that

$$\lim_{k \rightarrow \infty} d(v_{m_k}, Ts_{m_k}) = 0. \quad (61)$$

In a similar way as Case 1, we arrive at

$$\limsup_{k \rightarrow \infty} \langle \overrightarrow{p(v_{m_k} + 1)\hat{q}}, \overrightarrow{p(\hat{q})\hat{q}} \rangle \leq 0. \quad (62)$$

Note that

$$d^2(v_{m_k+1}, \hat{q}) \leq (1 - \alpha_{m_k})d^2(v_{m_k}, \hat{q}) + 2\alpha_{m_k} \langle \overrightarrow{p(\hat{q})\hat{q}}, \overrightarrow{v_{m_k} + 1\hat{q}} \rangle. \quad (63)$$

Since

$$d(v_{m_k}, \hat{q}) < d(v_{m_k+1}, \hat{q}) \text{ for all } k \in \mathbb{N}, \quad (64)$$

we have

$$\begin{aligned} \alpha_{m_k}d^2(v_{m_k}, \hat{q}) \leq d^2(v_{m_k}, \hat{q}) - d^2(v_{m_k+1}, \hat{q}) + 2\alpha_{m_k} \langle \overrightarrow{p(\hat{q})\hat{q}}, \overrightarrow{v_{m_k+1}\hat{q}} \rangle \\ + 2\alpha_{m_k} \langle \overrightarrow{p(\hat{q})\hat{q}}, \overrightarrow{v_{m_k+1}\hat{q}} \rangle. \end{aligned} \quad (65)$$

As $\{\alpha_{m_k}\} > 0$, we have

$$d^2(v_{m_k}, \hat{q}) \leq \langle \overrightarrow{p(\hat{q})\hat{q}}, \overrightarrow{v_{m_k+1}\hat{q}} \rangle. \quad (66)$$

From (61), it follows that $d(v_{m_k}, \hat{q}) \rightarrow 0$ as $k \rightarrow \infty$. By the above arguments, using relations (61) and (62) and conditions of $\{\alpha_n\}$, we obtain

$$d(v_{m_k+1}, \hat{q}) \rightarrow 0, \quad (67)$$

with $k \rightarrow \infty$.

As $d(v_k, \hat{q}) < d(v_{m_k}, \hat{q})$ for all $k \in \mathbb{N}$, we get $v_k \rightarrow \hat{q}$ when $k \rightarrow \infty$.

Corollary 22. Consider a CAT(0) space X and nonempty, convex, and closed subset K of X . Further, consider a bifunction $\phi : K \times K \rightarrow \mathbb{R}$ that fulfils conditions (6)–(10) and a nonexpensive mapping $T : K \rightarrow K$ with $\text{Fix}(T) \cap \text{EP}(\phi) \neq \emptyset$. Then, define a sequence $\{v_n\}$ as follows: for any $u, v_0 \in K$, consider

$$\begin{cases} w_n = \operatorname{argmin}_{w \in K} \left\{ \lambda_n \phi(v_n, w) + \frac{1}{2} d^2(w, v_n) \right\}, \\ s_n = \operatorname{argmin}_{s \in K} \left\{ \lambda_n \phi(v_n, s) + \frac{1}{2} d^2(s, v_n) \right\}, \\ v_{n+1} = \alpha_n u \oplus (1 - \alpha_n)(\beta_n v_n \oplus (1\beta_n)Ts_n), n \in \mathbb{N}. \end{cases} \quad (68)$$

Initially, we choose $\{\lambda_n\} \subset [\delta_1, \delta_2]$ with $0 < \delta_1 < \delta_2 < \min \{1/2c_1, 1/2c_2\}$, $\{\alpha_n\} \subset (0, 1/(2-\rho))$, $\{\beta_n\} \subset (0, 1)$, and take $v_0 \in K$. Set $n = 0$ and the sequences $\{\alpha_n\}$, $\{\beta_n\}$ as in Theorem 21, then $\{v_n\}$ converges strongly to $\hat{q} = P_{\text{Fix}(T) \cap \text{EP}(\phi)} u$.

Corollary 23. Consider a CAT(0) space X and nonempty, convex, and closed subset K of X . Further consider a bifunction $\phi : K \times K \rightarrow \mathbb{R}$ that fulfils conditions (6)–(10) such that $\text{EP}(\phi) \neq \emptyset$, the sequence $\{v_n\}$ with $u, v_0 \in K$ and

$$\begin{cases} w_n = \operatorname{argmin}_{w \in K} \left\{ \lambda_n \phi(v_n, w) + \frac{1}{2} d^2(w, v_n) \right\}, \\ s_n = \operatorname{argmin}_{s \in K} \left\{ \lambda_n \phi(s_n, w) + \frac{1}{2} d^2(s, v_n) \right\}, \\ v_{n+1} = \alpha_n u \oplus (1 - \alpha_n)(\beta_n v_n \oplus (1 - \beta_n)Ts_n), n \in \mathbb{N}, \end{cases} \quad (69)$$

where $\{\lambda_n\} \subset [\delta_1, \delta_2]$ with $0 < \delta_1 < \delta_2 < \min \{1/2c_1, 1/2c_2\}$, $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$. If the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are same as in Theorem 21, then the $\{v_n\}$ converges strongly to $\hat{q} = P_{\text{EP}(\phi)} u$.

Example 24. Let $H = \mathbb{R}^3$ and $K = \{(v_1, v_2, v_3) : v_1, v_2, v_3 = 0\}$. Let $\phi(v, w) = (w_3 - v_3)\|v\|$ for all $v = (v_1, v_2, v_3)$, $w = (w_1, w_2, w_3) \in K$. Then ϕ is Lipschitz-type continuous with the constants $c_1 = c_2 = 1$.

TABLE 1: Some v_n and $\|v_n - w_n\|$ generated by Algorithm 19.

Iteration (n)	v_1^n	v_2^n	v_3^n	$\ v_n - w_n\ $
1	2.00000	2.00000	2.00000	1.29903
2	1.90000	1.88000	0.98071	0.88973
3	1.85250	1.82360	0.36780	0.36780
4	1.82162	1.78713	0.12097	0.12096
5	1.79886	1.76032	0.03746	0.03746
6	1.78087	1.73920	0.01116	0.01116
7	1.76026	1.72181	0.00323	0.00324
8	1.75341	1.70704	0.00092	0.00092
9	1.74245	1.69424	0.00026	0.00026
10	1.73277	1.68295	0.00007	0.00007
11	1.72411	1.67285	0.00001	0.00002
12	1.71627	1.66373	0.00001	0.00001
13	1.70912	1.65541	0.00000	0.00000

It is easy to see that ϕ satisfies conditions (6)–(10). Let $Tv = (v_1, v_2, \arctan(0.1v_3))$ and $p(v) = (0.5v_1, 0.4v_2, 0.2v_3)$ for all $v = (v_1, v_2, v_3) \in K$. It follows that T is a nonexpansive mapping and p is a contraction. Take the initial point $v_1 = (2, 2, 2)$ and put the sequences $\alpha_n = 1/10n$, $\beta_n = 1/4 + 4n$, and $\gamma_n = 1/4 + 1/8n$ for all $n \in \mathbb{N}$. It is easy to see that

$$\text{Fix}(T) \cap \text{EP}(\phi) = \{(v_1, v_2, 0) : v_1, v_2 = 0\}. \quad (70)$$

We give some v_n and $\|v_n - w_n\|$ by Table 1: from Table 1, we see that after 13 iterations $v_{13} = (1.70912, 1.65541, 0.00000) \in \text{Fix}(T) \cap \text{EP}(\phi)$.

Corollary 25. Consider a CAT(0) space X and nonempty, convex, and closed subset K of X . Further consider a bifunction $\phi : K \times K \rightarrow \mathbb{R}$ that fulfils conditions (6)–(10) and a nonexpensive mapping $T : K \rightarrow K$ with $\text{Fix}(T) \cap \text{EP}(\phi) \neq \emptyset$. Then define a sequence $\{v_n\}$ as follows.

For any $u, v_0 \in K$, consider

$$\begin{cases} w_n = \operatorname{argmin}_{w \in K} \left\{ \lambda_n \phi(v_n, w) + \frac{1}{2} d^2(w, v_n) \right\}, \\ s_n = \operatorname{argmin}_{s \in K} \left\{ \lambda_n \phi(v_n, s) + \frac{1}{2} d^2(s, v_n) \right\}, \\ v_{n+1} = \alpha_n u \oplus (1 - \alpha_n)(\beta_n v_n \oplus (1\beta_n)Ts_n), n \in \mathbb{N}. \end{cases} \quad (71)$$

Initially choose $\{\lambda_n\} \subset [\delta_1, \delta_2]$ with $0 < \delta_1 < \delta_2 < \min \{1/2c_1, 1/2c_2\}$, $\{\alpha_n\} \subset (0, 1/(2-\rho))$, $\{\beta_n\} \subset (0, 1)$ and take $v_0 \in K$. Set $n = 0$ and the sequences $\{\alpha_n\}$, $\{\beta_n\}$ as in Theorem 21.

Then $\{v_n\}$ converges strongly to $\hat{q} = P_{\text{Fix}(T) \cap \text{EP}(\phi)} u$.

Corollary 26. Consider a CAT(0) space X and nonempty, convex, and closed subset K of X . Further consider a bifunction $\phi : K \times K \rightarrow \mathbb{R}$ that fulfils conditions (6)–(10) such that

$EP(\phi) \neq \emptyset$, the sequence $\{v_n\}$ with $u, v_0 \in K$ and

$$\begin{cases} w_n = \operatorname{argmin}_{w \in K} \left\{ \lambda_n \phi(v_n, w) + \frac{1}{2} d^2(w, v_n) \right\}, \\ s_n = \operatorname{argmin}_{s \in K} \left\{ \lambda_n \phi(s_n, w) + \frac{1}{2} d^2(s, v_n) \right\}, \\ v_{n+1} = \alpha_n u \oplus (1 - \alpha_n)(\beta_n v_n \oplus (1 - \beta_n)Ts_n), n \in \mathbb{N}, \end{cases} \quad (72)$$

where $\{\lambda_n\} \subset [\delta_1, \delta_2]$ with $0 < \delta_1 < \delta_2 < \min \{1/2c_1, 1/2c_2\}$, $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$. If the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are the same as in Theorem 21, then the sequence $\{v_n\}$ converges strongly to $\hat{q} = P_{EP(\phi)}u$.

Remark 27. Our Theorem 21 is an analog of Theorem 3.1 of Wang et al. [30] for the Hilbert space case.

Remark 28. Our results can be extended to any $\text{CAT}(\kappa)$ space with $\kappa \leq 0$, since any $\text{CAT}(\kappa')$ space is a $\text{CAT}(\kappa)$ space, for any $\kappa' > \kappa$ (see [17]).

4. Conclusions

It is well known that finding solutions for equilibrium problems play an important role in solving problems in other areas of science like physics, optimization, and economics. In this paper, we construct a new viscosity extragradient algorithm in order to find fixed points in a $\text{CAT}(0)$ space for the case of nonexpansive mappings. Also, we prove a strong convergence of the proposed algorithm, and we give some examples to support our results.

Data Availability

All data required for this research is included within the paper.

Conflicts of Interest

There are no competing interests among the authors.

Authors' Contributions

Yu-Pei Lv and Farman Ali contributed equally to this work.

References

- [1] S.-S. Chang, H. W. Joseph Lee, and K. K. Chan, "A new method for solving equilibrium problem fixed point problem and variational inequality problem with application to optimization," *Nonlinear Analysis: Theory Methods & Applications*, vol. 70, no. 9, pp. 3307–3319, 2009.
- [2] P. L. Combettes and S. A. Hirstoaga, "Equilibrium programming using proximal-like algorithms," *Mathematical Programming*, vol. 78, no. 1, pp. 29–41, 1996.
- [3] D. Q. Tran, M. L. Dung, and V. H. Nguyen, "Extragradient algorithms extended to equilibrium problem," *Optimization*, vol. 57, pp. 749–776, 2008.
- [4] P. N. Anh, "A hybrid extragradient method extended to fixed point problems and equilibrium problems," *Optimization*, vol. 62, no. 2, pp. 271–283, 2013.
- [5] P. T. Vuong, J. J. Strodiot, and V. H. Nguyen, "Extragradient methods and linesearch algorithms for solving Ky Fan inequalities and fixed point problems," *Journal of Optimization Theory and Applications*, vol. 155, no. 2, pp. 605–627, 2012.
- [6] P. N. Anh and H. A. Le Thi, "An Armijo-type method for pseudomonotone equilibrium problems and its applications," *Journal of Global Optimization*, vol. 57, no. 3, pp. 803–820, 2013.
- [7] B. V. Dinh and D. S. Kim, "Projection algorithms for solving nonmonotone equilibrium problems in Hilbert space," *Journal of Computational and Applied Mathematics*, vol. 302, pp. 106–117, 2016.
- [8] P. N. Anh, "Strong convergence theorems for nonexpansive mappings and Ky Fan inequalities," *Journal of Optimization Theory and Applications*, vol. 154, no. 1, pp. 303–320, 2012.
- [9] P. N. Anh and L. T. H. An, "The subgradient extragradient method extended to equilibrium problems," *Optimization*, vol. 64, no. 2, pp. 225–248, 2015.
- [10] P. N. Anh, "A hybrid extragradient method for pseudomonotone equilibrium problems and fixed point problems," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 36, pp. 107–116, 2013.
- [11] D. V. Hieu, L. D. Muu, and P. K. Anh, "Parallel hybrid extragradient methods for pseudomonotone equilibrium problems and nonexpansive mappings," *Numerical Algorithms*, vol. 73, pp. 197–217, 2016.
- [12] M. R. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*, vol. 319, Springer Science & Business Media, 2013.
- [13] R. Espínola and A. Fernández-León, "CAT(k)-spaces, weak convergence and fixed points," *Journal of Mathematical Analysis and Applications*, vol. 353, no. 1, pp. 410–427, 2009.
- [14] K. J. Horadam, *Hadamard Matrices and Their Applications*, Princeton University Press, 2012.
- [15] S. Dhompongsa and B. Panyanak, "On Δ -convergence theorems in $\text{CAT}(0)$ spaces," *Computers and Mathematics with Applications*, vol. 56, no. 10, pp. 2572–2579, 2008.
- [16] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local. I. Donne'es radicielles value'es*, vol. 41, Institut des Hautes Études Scientifiques. Publications Mathématiques, 1972.
- [17] I. D. Berg and I. G. Nikolaev, "Quasilinearization and curvature of Aleksandrov spaces," *Geometriae Dedicata*, vol. 133, no. 1, pp. 195–218, 2008.
- [18] T. C. Lim, "Remarks on some fixed point theorems," *Proceedings of the American Mathematical Society*, vol. 60, no. 1, pp. 179–182, 1976.
- [19] B. Ahmadi Kakavandi and M. Amini, "Duality and subdifferential for convex functions on complete $\text{CAT}(0)$ metric spaces," *Nonlinear Analysis*, vol. 73, no. 10, pp. 3450–3455, 2010.
- [20] B. A. Kakavandi, "Weak topologies in complete $\text{CAT}(0)$ metric spaces," *Proceedings of the American Mathematical Society*, vol. 141, no. 3, pp. 1029–1039, 2013.
- [21] D. N. Georgiou and B. K. Papadopoulos, "Weakly continuous, weakly ϑ -continuous, super-continuous and topologies on function spaces," *Scientiae Mathematicae Japonicae Online*, vol. 4, pp. 315–328, 2001.

- [22] A. di Concilio, "Exponential law and θ -continuous functions," *Quaestiones Mathematicae*, vol. 8, no. 2, pp. 131–142, 1985.
- [23] R. Arens and J. Dugundji, "Topologies for function spaces," *Pacific Journal of Mathematics*, vol. 1, no. 1, pp. 5–31, 1951.
- [24] K. O. Aremu, L. O. Jolaoso, C. Izuchukwu, and O. T. Mewomo, "Approximation of common solution of finite family of monotone inclusion and fixed point problems for demicontractive multivalued mappings in CAT(0) spaces," *Ricerche di Matematica*, vol. 69, no. 1, pp. 13–34, 2020.
- [25] S. Dhompongsa, W. A. Kirk, and B. Panyanak, "Nonexpansive set-valued mappings in metric and Banach spaces," *Journal of nonlinear and convex analysis*, vol. 8, pp. 35–45, 2007.
- [26] H. Dehghan and J. Rooin, "Metric projection and convergence theorems for nonexpansive mapping in Hadamard spaces," 2014, <https://arxiv.org/abs/1410.1137>.
- [27] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, Marcel Dekker, New York, 1984.
- [28] H. K. Xu, "Iterative algorithms for nonlinear operators," *Journal of the London Mathematical Society*, vol. 66, no. 1, article 240256, pp. 240–256, 2002.
- [29] P. E. Maingé, "Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization," *Set-Valued Analysis*, vol. 16, no. 7-8, pp. 899–912, 2008.
- [30] S. Wang, M. Zhao, P. Kumam, and Y. J. Cho, "A viscosity extragradient method for an equilibrium problem and fixed point problem in Hilbert space," *Journal of Fixed Point Theory and Applications*, vol. 20, no. 1, p. 19, 2018.