

## Research Article

# Hilfer-Hadamard Nonlocal Integro-Multipoint Fractional Boundary Value Problems

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This paper is concerned with the existence and uniqueness of solutions for a new class of boundary value problems, consisting by Hilfer-Hadamard fractional differential equations, supplemented with nonlocal integro-multipoint boundary conditions. The existence of a unique solution is obtained via Banach contraction mapping principle, while the existence results are established by applying Schaefer and Krasnoselskii fixed point theorems as well as Leray-Schauder nonlinear alternative. Examples illustrating the main results are also constructed.

## 1. Introduction

The fractional calculus has always been an interesting research topic for many years. This is because fractional differential equations describe many real-world process related to memory and hereditary properties of various materials more accurately as compared to classical-order differential equations. Fractional differential equations arise in lots of engineering and clinical disciplines which include biology, physics, chemistry, economics, signal and image processing, and control theory (see the monographs and papers in [1–8]).

Various types of fractional derivatives were introduced among which the following Riemann-Liouville and Caputo derivatives are the most widely used ones.

(1) *Riemann-Liouville derivative*. For  $n - 1 < \alpha < n$ , the derivative of  $u$  is

$${}^{\text{RL}}D_a^\alpha u(t) := D^n I_a^{n-\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} u(s) ds \quad (1)$$

(2) *Caputo derivative*. For  $n - 1 < \alpha < n$ , the derivative of  $u$  is

$${}^{\text{C}}D_a^\alpha u(t) := I_a^{n-\alpha} D^n u(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} \left(\frac{d}{ds}\right)^n u(s) ds, \quad (2)$$

where  $D^n = (d/dt)^n$ . Both Riemann-Liouville and Caputo derivatives are defined via fractional integral, the

Riemann-Liouville fractional integral, which is defined by

$$I_a^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} u(s) ds, \quad n-1 < \alpha < n \quad (3)$$

A generalization of derivatives of both Riemann-Liouville and Caputo was given by R. Hilfer in [9], known as the Hilfer fractional derivative of order  $\alpha$  and a type  $\beta \in [0, 1]$ , which interpolates between the Riemann-Liouville and Caputo derivatives, since it is reduced to the Riemann-Liouville and Caputo fractional derivatives when  $\beta=0$  and  $\beta=1$ , respectively. The Hilfer fractional derivative of order  $\alpha$  and parameter  $\beta$  of a function  $u$  is defined by

$${}^H D_a^{\alpha, \beta} u(t) = I_a^{\beta(n-\alpha)} D^n I_a^{(1-\beta)(n-\alpha)} u(t), \quad (4)$$

where  $n-1 < \alpha < n, 0 \leq \beta \leq 1, t > a, D = d/dt$ . Some properties and applications of the Hilfer derivative are given in [10, 11] and references cited therein. Initial value problems involving Hilfer fractional derivatives were studied by several authors (see, for example, [12–15] and references therein). Nonlocal boundary value problems for Hilfer fractional derivative were studied in [16, 17].

The Hadamard fractional calculus contains fractional derivative and integral with respect to the logarithmic function while someone say that it is generalized of the derivative  $(t(d/dt))^\alpha$ , where  $\alpha$  is an arbitrary order. The Hadamard calculus can be obtained by replacing as  $d/dt \rightarrow td/dt, (t-s)^{(\cdot)} \rightarrow (\log_e t - \log_e s)^{(\cdot)}$ , and  $ds \rightarrow (1/s)ds$  in (1)–(3). In the same way, the concept of Hilfer-Hadamard derivative is arrived by the modified definition in (4). Existence and uniqueness of solutions for system of Hilfer-Hadamard sequential fractional differential equations with two point boundary conditions were studied in [18].

In this paper, we study existence and uniqueness of solutions for boundary value problems for Hilfer-Hadamard fractional differential equations with nonlocal integro-multipoint boundary conditions,

$$\begin{cases} {}^{HH}D_1^{\alpha, \beta} x(t) = f(t, x(t)), & t \in [1, T], \\ x(1) = 0, & \sum_{i=1}^m \theta_i x(\xi_i) = \lambda {}^H I_1^\delta x(\eta), \end{cases} \quad (5)$$

where  ${}^{HH}D_1^{\alpha, \beta}$  is the Hilfer-Hadamard fractional derivative of order  $\alpha \in (1, 2]$  and type  $\beta \in [0, 1]$ ,  $\theta_i, \lambda \in \mathbb{R}, i = 1, 2, \dots, m$ , are given constants, and  $f : [1, T] \times \mathbb{R} \rightarrow \mathbb{R}$  is a given continuous function,  ${}^H I_1^\delta$  is the Hadamard fractional integral of order  $\delta > 0$ , and  $\eta, \xi_i \in (1, T), i = 1, 2, \dots, m$ , are given points.

Existence and uniqueness results are established by using classical fixed point theorems. We make use of Banach's fixed point theorem to obtain the uniqueness result, while Schaefer and Krasnoselskii's fixed point theorem [19] as well

as nonlinear alternative of Leray-Schauder type [20] is applied to obtain the existence results for the problem (5).

The paper is constructed as follows: in Section 2, we recall some basic facts needed in our study. The main results are proved in Section 3. Examples illustrating the main results are presented in Section 4.

## 2. Preliminaries

In this section, some basic definitions, lemmas, and theorems are mentioned.

*Definition 1* (Hadamard fractional integral [2]). The Hadamard fractional integral of order  $\alpha > 0$  for a function  $f : [a, \infty) \rightarrow \mathbb{R}$  is defined as

$${}^H I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left( \log \frac{t}{\tau} \right)^{\alpha-1} \frac{f(\tau)}{\tau} d\tau, \quad t > a, \quad (6)$$

provided the integral exists, where  $\log(\cdot) = \log_e(\cdot)$ .

*Definition 2* (Hadamard fractional derivative [2]). The Hadamard fractional derivative of order  $\alpha > 0$ , applied to the function  $f : [a, \infty) \rightarrow \mathbb{R}$ , is defined as follows:

$${}^H D_a^\alpha f(t) = \delta^n ({}^H I_a^{n-\alpha} f)(t), \quad n = [\alpha] + 1, \quad (7)$$

where  $\delta^n = (t(d/dt))^n$  and  $[\alpha]$  denotes the integer part of the real number  $\alpha$ .

*Definition 3* (Hilfer-Hadamard fractional derivative [11]). Let  $n-1 < \alpha < n$  and  $0 \leq \beta \leq 1, f \in L^1(a, b)$ . The Hilfer-Hadamard fractional derivative of order  $\alpha$  and type  $\beta$  of  $f$  is defined as

$$\begin{aligned} ({}^{HH}D_a^{\alpha, \beta} f)(t) &= \left( {}^H I_a^{\beta(n-\alpha)} \delta^n {}^H I_a^{(n-\alpha)(1-\beta)} f \right)(t) \\ &= \left( {}^H I_a^{\beta(n-\alpha)} \delta^n {}^H I_a^{(n-\gamma)} f \right)(t) \\ &= \left( {}^H I_a^{\beta(n-\alpha)} {}^H D_a^\gamma f \right)(t), \quad \gamma = \alpha + n\beta - \alpha\beta, \end{aligned} \quad (8)$$

where  ${}^H I_a^{(\cdot)}$  and  ${}^H D_a^{(\cdot)}$  is the Hadamard fractional integral and derivative defined by (6) and (7), respectively.

The Hilfer-Hadamard fractional derivative may be viewed as interpolating between Hadamard and Caputo-Hadamard fractional derivatives. Indeed, for  $\beta=0$ , this derivative reduces to the Hadamard fractional derivative while for  $\beta=1$ , it leads the Caputo-Hadamard derivative defined by

$${}^C D_a^\alpha f(t) = \left( {}^H I_a^{n-\alpha} \delta^n f \right)(t), \quad n = [\alpha] + 1. \quad (9)$$

We recall the following known theorem by Kilbas et al. [2] which will be used in the following.

**Theorem 4** ([2]). Let  $\alpha > 0$ ,  $0 \leq \beta \leq 1$ ,  $\gamma = \alpha + n\beta - \alpha\beta$ ,  $n = [\alpha] + 1$ , and  $0 < a < b < \infty$ . If  $f \in L^1(a, b)$  and  $({}^H I_a^{\alpha-n\gamma} f)(t) \in AC_{\delta}^n[a, b]$ , then

$$\begin{aligned} & {}^H I_a^\alpha \left( {}^{HH} D_a^{\alpha,\beta} f \right) (t) = {}^H I_a^\gamma \left( {}^{HH} D_a^\gamma f \right) (t) \\ & = f(t) - \sum_{j=0}^{n-1} \frac{\left( \delta^{(n-j-1)} \left( {}^H I_a^{\alpha-n\gamma} f \right) \right) (a)}{\Gamma(\gamma-j)} \left( \log \frac{t}{a} \right)^{\gamma-j-1} \end{aligned} \quad (10)$$

Since  $\gamma \in [\alpha, n]$ , then the  $\Gamma(\gamma - j)$  exists for all  $j = 1, 2, \dots, n - 1$ .

Finally, we will use the following well-known fixed point theorems on Banach space for proving the existence and uniqueness of the solutions to Hilfer-Hadamard fractional boundary value problem (5).

**Theorem 5** (Banach’s contraction principle [21]). Let  $X$  be a Banach space,  $D \subset X$  be closed, and  $T : D \rightarrow D$  be a contraction (i.e., there exists a constant  $L \in (0, 1)$  such that for any  $x, y \in X$ ,  $|Tx - Ty| \leq L|x - y|$ ). Then,  $T$  has a unique fixed point on  $X$ .

**Theorem 6** (Krasnoselskii’s fixed point theorem)[19]. Let  $Y$  be a bounded, closed, convex, and nonempty subset of a Banach space  $X$ . Let  $F_1$  and  $F_2$  be the operators satisfying the following conditions: (i)  $F_1 y_1 + F_2 y_2 \in Y$  whenever  $y_1, y_2 \in Y$ ; (ii)  $F_1$  is compact and continuous; and (iii)  $F_2$  is a contraction mapping. Then, there exists  $y \in Y$  such that  $y = F_1 y + F_2 y$ .

**Theorem 7** (Schaefer fixed point theorem [22]). Let  $T : E \rightarrow E$  be a completely continuous operator (i.e., a continuous map  $T$  restricted to any bounded set in  $E$  is compact). Let  $\varepsilon(T) = \{x \in E : x = \lambda T(x), 0 \leq \lambda \leq 1\}$ . Then, either the set  $\varepsilon(T)$  is unbounded or  $T$  has at least one fixed point.

**Theorem 8** (Nonlinear alternative for single-valued maps) ([20]). Let  $E$  be a Banach space,  $C$  a closed, convex subset of  $E$ ,  $U$  an open subset of  $C$ , and  $0 \in U$ . Suppose that  $\mathcal{A} : \bar{U} \rightarrow C$  is continuous and compact (that is,  $\mathcal{A}(\bar{U})$  is a relatively compact subset of  $C$ ) map. Then, either

- (i)  $\mathcal{A}$  has a fixed point in  $\bar{U}$ , or
- (ii) there is  $ax \in \partial U$  (the boundary of  $U$  in  $C$ ) and  $\lambda \in (0, 1)$  with  $x = \lambda \mathcal{A}(x)$

### 3. Main Results

In this section, we prove existence and uniqueness of solutions for nonlinear Hilfer-Hadamard fractional boundary value problem (5). Firstly, we start by proving a basic lemma concerning a linear variant of the boundary value problem (5), which will be used to transform the boundary value problem (5) into an equivalent integral equation. In this case,  $n = [\alpha] + 1 = 2$ ; then, we have  $\gamma = \alpha + (2 - \alpha)\beta$ .

**Lemma 9.** Let  $h \in C([1, T], \mathbb{R})$  and  $\Lambda \neq 0$ , where

$$\Lambda = \sum_{i=1}^m \theta_i (\log \xi_i)^{\gamma-1} - \lambda \frac{\Gamma(\gamma)}{\Gamma(\gamma+\delta)} (\log \eta)^{\gamma+\delta-1}. \quad (11)$$

Then,  $x$  is a solution of the following linear Hilfer-Hadamard fractional differential equation:

$${}^{HH} D_1^{\alpha,\beta} x(t) = h(t), \quad 1 < \alpha \leq 2, \quad t \in [1, T], \quad (12)$$

supplemented with the boundary conditions in (5), if and only if

$$\begin{aligned} x(t) = & {}^H I_1^\alpha h(t) + \frac{(\log t)^{\gamma-1}}{\Lambda} \left\{ \lambda \left( {}^H I_1^{\alpha+\delta} h \right) (\eta) - \sum_{i=1}^m \theta_i \left( {}^H I_1^\alpha h \right) (\xi_i) \right\} = \frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{\tau} \right)^{\alpha-1} \frac{h(\tau)}{\tau} d\tau \\ & + \frac{(\log t)^{\gamma-1}}{\Lambda} \left\{ \lambda \frac{1}{\Gamma(\alpha+\delta)} \int_1^\eta \left( \log \frac{\eta}{\tau} \right)^{\alpha+\delta-1} \frac{h(\tau)}{\tau} d\tau - \sum_{i=1}^m \theta_i \frac{1}{\Gamma(\alpha)} \int_1^{\xi_i} \left( \log \frac{\xi_i}{\tau} \right)^{\alpha-1} \frac{h(\tau)}{\tau} d\tau \right\}, \quad t \in [1, T]. \end{aligned} \quad (13)$$

*Proof.* By taking the Hadamard fractional integral of order  $\alpha$  from 1 to  $t$  on both sides of (12) and using Theorem 4, it follows that

$$x(t) - \sum_{j=0}^1 \frac{\left( \delta^{(2-j-1)} \left( {}^H I_1^{2-\gamma} x \right) \right) (1)}{\Gamma(\gamma-j)} (\log t)^{\gamma-j-1} = {}^H I_1^\alpha h(t). \quad (14)$$

Then, we have

$$\begin{aligned} x(t) - & \frac{\delta \left( {}^H I_1^{2-\gamma} x \right) (1)}{\Gamma(\gamma)} (\log t)^{\gamma-1} \\ & - \frac{\left( {}^H I_1^{2-\gamma} x \right) (1)}{\Gamma(\gamma-1)} (\log t)^{\gamma-2} = {}^H I_1^\alpha h(t). \end{aligned} \quad (15)$$

Equation (15) can be rewritten by

$$x(t) = c_0(\log t)^{\gamma-1} + c_1(\log t)^{\gamma-2} + \frac{1}{\Gamma(\alpha)} \int_1^t \frac{h(s)}{s} \left(\log \frac{t}{s}\right)^{\alpha-1} ds, \tag{16}$$

where  $c_0, c_1$  are arbitrary constants. Now, the first boundary condition  $x(1) = 0$  together with (16) yields  $c_1 = 0$ , since  $\gamma \in [\alpha, 2]$ . Putting  $c_1 = 0$  in (16), we get

$$x(t) = c_0(\log t)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{h(s)}{s} ds. \tag{17}$$

Next, the second boundary condition  $\sum_{i=1}^m \theta_i x(\xi_i) = \lambda^H I_1^\delta x(\eta)$  together with (17) yields

$$c_0 = \frac{1}{\Lambda} \left\{ \lambda^H I_1^{\alpha+\delta} h(\eta) - \sum_{i=1}^m \theta_i^H I_1^\alpha h(\xi_i) \right\}. \tag{18}$$

Substituting the value of  $c_0$  in (17), we get equation (13) as desired.

The converse follows by direct computation. The proof is completed.  $\square$

Let us introduce the Banach space  $X = C([1, T], \mathbb{R})$  endowed with the norm defined by  $\|x\| := \max_{t \in [1, T]} |x(t)|$ .

In view of Lemma 9, we define an operator  $\mathcal{F} : X \rightarrow X$ , where

$$\begin{aligned} (\mathcal{F}x)(t) &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} \frac{f(\tau, x(\tau))}{\tau} d\tau \\ &+ \frac{(\log t)^{\gamma-1}}{\Lambda} \left\{ \frac{\lambda}{\Gamma(\alpha+\delta)} \int_1^\eta \left(\log \frac{\eta}{\tau}\right)^{\alpha+\delta-1} \frac{f(\tau, x(\tau))}{\tau} d\tau \right. \\ &\left. - \sum_{i=1}^m \theta_i \frac{1}{\Gamma(\alpha)} \int_1^{\xi_i} \left(\log \frac{\xi_i}{\tau}\right)^{\alpha-1} \frac{f(\tau, x(\tau))}{\tau} d\tau \right\}, \quad t \in [1, T]. \end{aligned} \tag{19}$$

In the following, for convenience, we put

$$\Omega = \frac{(\log T)^\alpha}{\Gamma(\alpha+1)} + \frac{(\log T)^{\gamma-1}}{|\Lambda|} \left\{ \frac{|\lambda|(\log \eta)^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)} + \sum_{i=1}^m |\theta_i| \frac{(\log \xi_i)^\alpha}{\Gamma(\alpha+1)} \right\}. \tag{20}$$

We need the following hypotheses in the sequel:

(H<sub>1</sub>). There exists a constant  $l > 0$  such that for all  $t \in [1, T]$  and  $u_i \in \mathbb{R}, i = 1, 2$ ,

$$|f(t, u_1) - f(t, u_2)| \leq l|u_1 - u_2|. \tag{21}$$

(H<sub>2</sub>). There exists a continuous nonnegative function  $\phi \in C([1, T], \mathbb{R}^+)$  such that

$$|f(t, u)| \leq \phi(t), \quad \text{for each } (t, u) \in [1, T] \times \mathbb{R}. \tag{22}$$

(H<sub>3</sub>). There exists a real constant  $M > 0$  such that for all  $t \in [1, T], u \in \mathbb{R}$ ,

$$|f(t, u)| \leq M. \tag{23}$$

(H<sub>4</sub>). There exist  $p \in C([1, T], \mathbb{R}^+)$  and a continuous nondecreasing function  $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that

$$|f(t, u)| \leq p(t)\psi(\|u\|) \quad \text{for each } (t, u) \in [1, T] \times \mathbb{R}. \tag{24}$$

(H<sub>5</sub>). There exists a constant  $K > 0$  such that

$$\frac{K}{\Omega \|p\| \psi(K)} > 1. \tag{25}$$

**3.1. Existence and Uniqueness Result via Banach's Fixed Point Theorem.** We prove an existence and uniqueness result based on Banach's contraction mapping principle.

**Theorem 10.** Assume that (H<sub>1</sub>) holds. Then, boundary value problem (5) has a unique solution on  $[1, T]$ , provided that  $l\Omega < 1$ , where  $\Omega$  is defined by (20).

*Proof.* We will use Banach's fixed point theorem to prove that  $\mathcal{F}$ , defined by (19), has a unique fixed point. Fixing  $N = \max_{t \in [1, T]} |f(t, 0)| < \infty$  and using hypothesis (H<sub>1</sub>), we obtain

$$\begin{aligned} |f(t, x(t))| &\leq |f(t, x(t)) - f(t, 0)| + |f(t, 0)| \\ &\leq l|x(t)| + |f(t, 0)| \leq l\|x\| + N. \end{aligned} \tag{26}$$

Choose

$$r \geq \frac{N\Omega}{1 - l\Omega}. \tag{27}$$

We divide the proof into two steps.

Step I. We show that  $\mathcal{F}(B_r) \subset B_r$ , where  $B_r = \{x \in X : \|x\| < r\}$ . Let  $x \in B_r$ . Then, we have

$$\begin{aligned} |(\mathcal{F}x)(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} \frac{|f(\tau, x(\tau))|}{\tau} d\tau \\ &\quad + \frac{(\log t)^{\gamma-1}}{|\Lambda|} \left\{ \frac{|\lambda|}{\Gamma(\alpha+\delta)} \int_1^\eta \left(\log \frac{\eta}{\tau}\right)^{\alpha+\delta-1} \frac{|f(\tau, x(\tau))|}{\tau} d\tau \right. \\ &\quad \left. + \sum_{i=1}^m |\theta_i| \frac{1}{\Gamma(\alpha)} \int_1^{\xi_i} \left(\log \frac{\xi_i}{\tau}\right)^{\alpha-1} \frac{|f(\tau, x(\tau))|}{\tau} d\tau \right\} \\ &\leq \frac{(\log T)^\alpha}{\Gamma(\alpha+1)} (l\|x\| + N) + \frac{(\log T)^{\gamma-1}}{|\Lambda|} \left\{ \frac{|\lambda|(\log \eta)^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)} \right. \\ &\quad \left. + \sum_{i=1}^m |\theta_i| \frac{(\log \xi_i)^\alpha}{\Gamma(\alpha+1)} \right\} (l\|x\| + N) \\ &= \left[ \frac{(\log T)^\alpha}{\Gamma(\alpha+1)} + \frac{(\log T)^{\gamma-1}}{|\Lambda|} \left\{ \frac{|\lambda|(\log \eta)^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)} \right. \right. \\ &\quad \left. \left. + \sum_{i=1}^m |\theta_i| \frac{(\log \xi_i)^\alpha}{\Gamma(\alpha+1)} \right\} \right] (l\|x\| + N) \\ &\leq \left[ \frac{(\log T)^\alpha}{\Gamma(\alpha+1)} + \frac{(\log T)^{\gamma-1}}{|\Lambda|} \left\{ \frac{|\lambda|(\log \eta)^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)} \right. \right. \\ &\quad \left. \left. + \sum_{i=1}^m |\theta_i| \frac{(\log \xi_i)^\alpha}{\Gamma(\alpha+1)} \right\} \right] (lr + N) = \Omega(lr + N) \leq r. \end{aligned} \tag{28}$$

Thus,

$$\|(\mathcal{F}x)\| = \max_{t \in [1, T]} |(\mathcal{F}x)(t)| \leq r, \tag{29}$$

which means that  $(\mathcal{F}B_r) \subset B_r$ .

Step II. To show that the operator  $\mathcal{F}$  is a contraction, let  $x_1, x_2 \in X$ . Then, for any  $t \in [1, T]$ , we have

$$\begin{aligned} |(\mathcal{F}x_2)(t) - (\mathcal{F}x_1)(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} \frac{|f(\tau, x_2(\tau)) - f(\tau, x_1(\tau))|}{\tau} d\tau \\ &\quad + \frac{(\log t)^{\gamma-1}}{|\Lambda|} \left\{ \frac{|\lambda|}{\Gamma(\alpha+\delta)} \int_1^\eta \left(\log \frac{\eta}{\tau}\right)^{\alpha+\delta-1} \frac{|f(s, x_2(\tau)) - f(s, x_1(\tau))|}{\tau} d\tau \right. \\ &\quad \left. + \sum_{i=1}^m |\theta_i| \frac{1}{\Gamma(\alpha)} \int_1^{\xi_i} \left(\log \frac{\xi_i}{\tau}\right)^{\alpha-1} \frac{|f(\tau, x_2(\tau)) - f(\tau, x_1(\tau))|}{\tau} d\tau \right\} \\ &\leq l\|x_2 - x_1\| \left[ \frac{(\log T)^\alpha}{\Gamma(\alpha+1)} + \frac{(\log T)^{\gamma-1}}{|\Lambda|} \left\{ \frac{|\lambda|(\log \eta)^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)} + \sum_{i=1}^m |\theta_i| \frac{(\log \xi_i)^\alpha}{\Gamma(\alpha+1)} \right\} \right]. \end{aligned} \tag{30}$$

Thus,

$$\|(\mathcal{F}x_2) - (\mathcal{F}x_1)\| = \max_{t \in [1, T]} |(\mathcal{F}x_2)(t) - (\mathcal{F}x_1)(t)| \leq l\Omega\|x_2 - x_1\|, \tag{31}$$

which, in view of  $l\Omega < 1$ , shows that the operator  $\mathcal{F}$  is a contraction. By Theorem 5, we get that the operator  $\mathcal{F}$  has a

unique fixed point. Therefore, the problem (5) has a unique solution on  $[1, T]$ . The proof is completed.  $\square$

3.2. Existence Result via Krasnoselskii’s Fixed Point Theorem.

In this subsection, we prove an existence result based on Krasnoselskii’s fixed point theorem.

**Theorem 11.** Assume that  $(H_1) - (H_2)$  hold. Then, the problem (5) has at least one solution on  $[1, T]$ , provided that

$$\frac{(\log T)^{\gamma-1}}{|\Lambda|} \left\{ \frac{|\lambda|(\log \eta)^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)} + \sum_{i=1}^m |\theta_i| \frac{(\log \xi_i)^\alpha}{\Gamma(\alpha+1)} \right\} l < 1. \tag{32}$$

*Proof.* By assumption  $(H_2)$ , we can fix  $\rho \geq \Omega\|\phi\|$ , where  $\|\phi\| = \sup_{t \in [1, T]} |\phi(t)|$  and consider  $B_\rho = \{x \in C([1, T], \mathbb{R}) : \|x\| \leq \rho\}$ . We split the operator  $\mathcal{F} : C([1, T], \mathbb{R}) \rightarrow C([1, T], \mathbb{R})$  defined by (19) as  $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$ , where  $\mathcal{F}_1$  and  $\mathcal{F}_2$  are given by

$$\begin{aligned} (\mathcal{F}_1x)(t) &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} \frac{f(\tau, x(\tau))}{\tau} d\tau, \\ (\mathcal{F}_2x)(t) &= \frac{(\log t)^{\gamma-1}}{\Lambda} \left\{ \frac{\lambda}{\Gamma(\alpha+\delta)} \int_1^\eta \left(\log \frac{\eta}{\tau}\right)^{\alpha+\delta-1} \frac{f(\tau, x(\tau))}{\tau} d\tau \right. \\ &\quad \left. - \sum_{i=1}^m \theta_i \frac{1}{\Gamma(\alpha)} \int_1^{\xi_i} \left(\log \frac{\xi_i}{\tau}\right)^{\alpha-1} \frac{f(\tau, x(\tau))}{\tau} d\tau \right\}. \end{aligned} \tag{33}$$

For any  $x, y \in B_\rho$ , we have

$$\begin{aligned} |(\mathcal{F}_1x)(t) + (\mathcal{F}_2y)(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} \frac{|f(\tau, x(\tau))|}{\tau} d\tau \\ &\quad + \frac{(\log t)^{\gamma-1}}{|\Lambda|} \left\{ \frac{|\lambda|}{\Gamma(\alpha+\delta)} \int_1^\eta \left(\log \frac{\eta}{\tau}\right)^{\alpha+\delta-1} \frac{|f(\tau, y(\tau))|}{\tau} d\tau \right. \\ &\quad \left. + \sum_{i=1}^m |\theta_i| \frac{1}{\Gamma(\alpha)} \int_1^{\xi_i} \left(\log \frac{\xi_i}{\tau}\right)^{\alpha-1} \frac{|f(\tau, y(\tau))|}{\tau} d\tau \right\} \\ &\leq \left[ \frac{(\log T)^\alpha}{\Gamma(\alpha+1)} + \frac{(\log T)^{\gamma-1}}{|\Lambda|} \left\{ \frac{|\lambda|(\log \eta)^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)} + \sum_{i=1}^m |\theta_i| \frac{(\log \xi_i)^\alpha}{\Gamma(\alpha+1)} \right\} \right] \|\phi\| \\ &= \Omega\|\phi\| \leq \rho. \end{aligned} \tag{34}$$

Hence,  $\|(\mathcal{F}_1x) + (\mathcal{F}_2y)\| \leq \rho$ , which shows that  $(\mathcal{F}_1x) + (\mathcal{F}_2y) \in B_\rho$ . It is easy to prove, using conditions  $(H_1)$  and (32), that the operator  $\mathcal{F}_2$  is a contraction mapping.

Next, the operator  $\mathcal{F}_1$  is continuous by the continuity of  $f$ . Also,  $\mathcal{F}_1$  is uniformly bounded on  $B_\rho$ , since

$$\|\mathcal{F}_1x\| \leq \frac{(\log T)^\alpha}{\Gamma(\alpha+1)} \|\phi\|. \tag{35}$$

Finally, we prove the compactness of the operator  $\mathcal{F}_1$ . For  $t_1, t_2 \in [1, T]$ ,  $t_1 < t_2$ , we have

$$\begin{aligned} & |(\mathcal{F}_1 x)(t_2) - (\mathcal{F}_1 x)(t_1)| \\ & \leq \frac{1}{\Gamma(\alpha)} \int_1^{t_1} \left[ \left( \log \frac{t_2}{\tau} \right)^{\alpha-1} - \left( \log \frac{t_1}{\tau} \right)^{\alpha-1} \right] \frac{|f(\tau, x(\tau))|}{\tau} d\tau \\ & \quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left( \log \frac{t_2}{\tau} \right)^{\alpha-1} \frac{|f(\tau, x(\tau))|}{\tau} d\tau \\ & \leq \frac{\|\phi\|}{\Gamma(\alpha+1)} [2(\log t_2 - \log t_1)^\alpha + |(\log t_2)^\alpha - (\log t_1)^\alpha|], \end{aligned} \quad (36)$$

which tends to zero, independently of  $x \in B_\rho$ , as  $t_1 \rightarrow t_2$ . Thus,  $\mathcal{F}_1$  is equicontinuous. From the Arzelà-Ascoli theorem, we conclude that the operator  $\mathcal{F}_1$  is compact on  $B_\rho$ . Thus, the hypotheses of Krasnoselskii fixed point theorem are satisfied, and therefore, there exists at least one solution on  $[1, T]$ . The proof is finished.  $\square$

**3.3. Existence Result via Schaefer's Fixed Point Theorem.** Our second existence result is based on Schaefer's fixed point theorem.

**Theorem 12.** *Assume that  $(H_3)$  holds. Then, the boundary value problem (5) has at least one solution on  $[1, T]$ .*

*Proof.* We will prove that the operator  $\mathcal{F}$ , defined by (19), has a fixed point, by using Schaefer's fixed point theorem. We divide the proof into two steps.

*Step I.* We show that the operator  $\mathcal{F} : X \rightarrow X$  is completely continuous.

We show first that  $\mathcal{F}$  is continuous. Let  $\{x_n\}$  be a sequence such that  $x_n \rightarrow x$  in  $X$ . Then, for each  $t \in [1, T]$ , we have

$$\begin{aligned} & |(\mathcal{F}x_n)(t) - (\mathcal{F}x)(t)| \\ & \leq \frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{\tau} \right)^{\alpha-1} \frac{|f(\tau, x_n(\tau)) - f(\tau, x(\tau))|}{\tau} d\tau \\ & \quad + \frac{(\log t)^{\gamma-1}}{|\Lambda|} \left\{ \frac{|\lambda|}{\Gamma(\alpha+\delta)} \int_1^\eta \left( \log \frac{\eta}{\tau} \right)^{\alpha+\delta-1} \frac{|f(\tau, x_n(\tau)) - f(\tau, x(\tau))|}{\tau} d\tau \right. \\ & \quad \left. + \sum_{i=1}^m |\theta_i| \frac{1}{\Gamma(\alpha)} \int_1^{\xi_i} \left( \log \frac{\xi_i}{\tau} \right)^{\alpha-1} \frac{|f(\tau, x_n(\tau)) - f(\tau, x(\tau))|}{\tau} d\tau \right\}. \end{aligned} \quad (37)$$

Since  $f$  is continuous, we get

$$|f(s, x_n(s)) - f(s, x(s))| \rightarrow 0 \quad \text{as } x_n \rightarrow x. \quad (38)$$

Then

$$\|(\mathcal{F}x_n) - (\mathcal{F}x)\| \rightarrow 0 \quad \text{as } x_n \rightarrow x. \quad (39)$$

Hence,  $\mathcal{F}$  is continuous.

Secondly, we show that the operator  $\mathcal{F}$  maps bounded sets into bounded sets in  $X$ . For a positive number  $R$ , let  $B_R = \{x \in X : \|x\| \leq R\}$  be a bounded ball in  $X$ . Then, for  $t \in [1, T]$ , we have

$$\begin{aligned} |(\mathcal{F}x)(t)| & \leq \frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{\tau} \right)^{\alpha-1} \frac{|f(\tau, x(\tau))|}{\tau} d\tau \\ & \quad + \frac{(\log t)^{\gamma-1}}{|\Lambda|} \left\{ \frac{|\lambda|}{\Gamma(\alpha+\delta)} \int_1^\eta \left( \log \frac{\eta}{\tau} \right)^{\alpha+\delta-1} \frac{|f(\tau, x(\tau))|}{\tau} d\tau \right. \\ & \quad \left. + \sum_{i=1}^m |\theta_i| \frac{1}{\Gamma(\alpha)} \int_1^{\xi_i} \left( \log \frac{\xi_i}{\tau} \right)^{\alpha-1} \frac{|f(\tau, x(\tau))|}{\tau} d\tau \right\} \\ & \leq \frac{(\log T)^\alpha}{\Gamma(\alpha+1)} M + \frac{(\log T)^{\gamma-1}}{|\Lambda|} \left\{ \frac{|\lambda|(\log \eta)^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)} \right. \\ & \quad \left. + \sum_{i=1}^m |\theta_i| \frac{(\log \xi_i)^\alpha}{\Gamma(\alpha+1)} \right\} M, \end{aligned} \quad (40)$$

and

$$\|\mathcal{F}x\| \leq \frac{(\log T)^\alpha}{\Gamma(\alpha+1)} M + \frac{(\log T)^{\gamma-1}}{|\Lambda|} \left\{ \frac{|\lambda|(\log \eta)^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)} + \sum_{i=1}^m |\theta_i| \frac{(\log \xi_i)^\alpha}{\Gamma(\alpha+1)} \right\} M. \quad (41)$$

Thirdly, we show that  $\mathcal{F}$  maps bounded sets into equicontinuous sets. Let  $t_1, t_2 \in [1, T]$  with  $t_1 < t_2$  and  $u \in B_R$ . Then, we have

$$\begin{aligned} |(\mathcal{F}x)(t_2) - (\mathcal{F}x)(t_1)| & \leq \frac{1}{\Gamma(\alpha)} \int_1^{t_1} \left[ \left( \log \frac{t_2}{\tau} \right)^{\alpha-1} \right. \\ & \quad \left. - \left( \log \frac{t_1}{\tau} \right)^{\alpha-1} \right] \frac{|f(\tau, x(\tau))|}{\tau} d\tau \\ & \quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left( \log \frac{t_2}{\tau} \right)^{\alpha-1} \frac{|f(\tau, x(\tau))|}{\tau} d\tau \\ & \quad + \frac{|(\log t_2)^{\gamma-1} - (\log t_1)^{\gamma-1}|}{|\Lambda|} \left\{ \frac{|\lambda|}{\Gamma(\alpha+\delta)} \int_1^\eta \left( \log \frac{\eta}{\tau} \right)^{\alpha+\delta-1} \frac{|f(\tau, x(\tau))|}{\tau} d\tau \right. \\ & \quad \left. + \sum_{i=1}^m |\theta_i| \frac{1}{\Gamma(\alpha)} \int_1^{\xi_i} \left( \log \frac{\xi_i}{\tau} \right)^{\alpha-1} \frac{|f(\tau, x(\tau))|}{\tau} d\tau \right\} \\ & \leq \frac{M}{\Gamma(\alpha+1)} [2(\log t_2 - \log t_1)^\alpha + |(\log t_2)^\alpha - (\log t_1)^\alpha|] \\ & \quad + \frac{|(\log t_2)^{\gamma-1} - (\log t_1)^{\gamma-1}|}{|\Lambda|} \left\{ \frac{|\lambda|(\log \eta)^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)} + \sum_{i=1}^m |\theta_i| \frac{(\log \xi_i)^\alpha}{\Gamma(\alpha+1)} \right\} M, \end{aligned} \quad (42)$$

which tends to zero, independently of  $x \in B_R$ , as  $t_1 \rightarrow t_2$ .



Thus, the Arzelá-Ascoli theorem applies and hence,  $\mathcal{F} : X \rightarrow X$  is completely continuous.

*Step II.* We show that the set  $\mathcal{E} = \{x \in X \mid x = \nu(\mathcal{F}x), 0 \leq \nu \leq 1\}$  is bounded. Let  $x \in \mathcal{E}$ , then  $x = \nu(\mathcal{F}x)$ . For any  $t \in [1, T]$ , we have  $x(t) = \nu(\mathcal{F}x)(t)$ . Then, in view of the hypothesis  $(H_3)$ , as in Step I, we obtain

$$|x(t)| \leq \frac{(\log T)^\alpha}{\Gamma(\alpha + 1)} M + \frac{(\log T)^{\gamma-1}}{|\Lambda|} \left\{ \frac{|\lambda|(\log \eta)^{\alpha+\delta}}{\Gamma(\alpha + \delta + 1)} + \sum_{i=1}^m |\theta_i| \frac{(\log \xi_i)^\alpha}{\Gamma(\alpha + 1)} \right\} M. \tag{43}$$

Thus,

$$\|x\| \leq \frac{(\log T)^\alpha}{\Gamma(\alpha + 1)} M + \frac{(\log T)^{\gamma-1}}{|\Lambda|} \left\{ \frac{|\lambda|(\log \eta)^{\alpha+\delta}}{\Gamma(\alpha + \delta + 1)} + \sum_{i=1}^m |\theta_i| \frac{(\log \xi_i)^\alpha}{\Gamma(\alpha + 1)} \right\} M, \tag{44}$$

which shows that the set  $\mathcal{E}$  is bounded. By Theorem 7, we get that the operator  $\mathcal{F}$  has at least one fixed point. Therefore, the boundary value problem (5) has at least one solution on  $[1, T]$ . This completes the proof.  $\square$

**3.4. Existence Result via Leray-Schauder Nonlinear Alternative.** Our final existence result is proved via Leray-Schauder nonlinear alternative.

**Theorem 13.** *Assume that  $(H_4)$  and  $(H_5)$  hold. Then, the boundary value problem (5) has at least one solution on  $[1, T]$ .*

*Proof.* As in Theorem 12, we can prove that the operator  $\mathcal{F}$  is completely continuous. We will prove that there exists an open set  $U \subseteq C([1, T], \mathbb{R})$  with  $x \neq \mu(\mathcal{F}x)$  for  $\mu \in (0, 1)$  and  $x \in \partial U$ .

Let  $x \in C([1, T], \mathbb{R})$  be such that  $x = \mu(\mathcal{F}x)$  for some  $0 < \mu < 1$ . Then, for each  $t \in [1, T]$ , we have

$$\begin{aligned} |x(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{\tau} \right)^{\alpha-1} \frac{|f(\tau, x(\tau))|}{\tau} d\tau \\ &\quad + \frac{(\log t)^{\gamma-1}}{|\Lambda|} \left\{ \frac{|\lambda|}{\Gamma(\alpha + \delta)} \int_1^\eta \left( \log \frac{\eta}{\tau} \right)^{\alpha+\delta-1} \frac{|f(\tau, x(\tau))|}{\tau} d\tau \right. \\ &\quad \left. + \sum_{i=1}^m |\theta_i| \frac{1}{\Gamma(\alpha)} \int_1^{\xi_i} \left( \log \frac{\xi_i}{\tau} \right)^{\alpha-1} \frac{|f(\tau, x(\tau))|}{\tau} d\tau \right\} \\ &\leq \left[ \frac{(\log T)^\alpha}{\Gamma(\alpha + 1)} + \frac{(\log T)^{\gamma-1}}{|\Lambda|} \left\{ \frac{|\lambda|(\log \eta)^{\alpha+\delta}}{\Gamma(\alpha + \delta + 1)} \right. \right. \\ &\quad \left. \left. + \sum_{i=1}^m |\theta_i| \frac{(\log \xi_i)^\alpha}{\Gamma(\alpha + 1)} \right\} \right] \|p\| \psi(\|x\|). \end{aligned} \tag{45}$$

Consequently,

$$U = \{x \in C([1, T], \mathbb{R}) : \|x\| < K\}. \tag{46}$$

In view of  $(H_5)$ , there is no solution  $x$  such that  $\|x\| \neq K$ . Let us set

$$U = \{x \in C([1, T], \mathbb{R}) : \|x\| < K\}. \tag{47}$$

The operator  $\mathcal{F} : \bar{U} \rightarrow C([1, T], \mathbb{R})$  is continuous and completely continuous. From the choice of  $U$ , there is no  $u \in \partial U$  such that  $x = \mu(\mathcal{F}x)$  for some  $\mu \in (0, 1)$ . Consequently, by the nonlinear alternative of Leray-Schauder type (Theorem 8), we deduce that  $\mathcal{F}$  has a fixed point  $x \in \bar{U}$  which is a solution of the boundary value problem (5). The proof is completed.  $\square$

### 4. Examples

*Example 1.* Consider the Hilfer-Hadamard nonlocal integro-multipoint fractional boundary value problem of the form

$$\begin{cases} {}^{\text{HH}}D_1^{3/2, 1/2} x(t) = f(t, x(t)), & t \in [1, 4.5], \\ x(1) = 0, & \frac{6}{29} I_1^{5/2} x\left(\frac{5}{2}\right) = \frac{1}{11} x\left(\frac{3}{2}\right) + \frac{2}{13} x(2) + \frac{3}{17} x(3) + \frac{4}{19} x\left(\frac{7}{2}\right) + \frac{5}{23} x(4). \end{cases} \tag{48}$$

Set constants  $\alpha = 3/2$ ,  $\beta = 1/2$ ,  $T = 9/2$ ,  $m = 5$ ,  $\theta_1 = 1/11$ ,  $\theta_2 = 2/13$ ,  $\theta_3 = 3/17$ ,  $\theta_4 = 4/19$ ,  $\theta_5 = 5/23$ ,  $\xi_1 = 3/2$ ,  $\xi_2 = 2$ ,  $\xi_3 = 3$ ,  $\xi_4 = 7/2$ ,  $\xi_5 = 4$ ,  $\lambda = 6/29$ ,  $\delta = 5/2$ , and  $\eta = 5/2$ . Then, from these data, we compute that  $\gamma = 7/4$ ,  $|\Lambda| \approx 1.152315230$ , and  $\Omega \approx 2.250815481$ ,

- (i) Let a nonlinear unbounded function  $f : [1, 4.5] \times \mathbb{R} \rightarrow \mathbb{R}$  be defined by

$$f(t, x) = \frac{e^{1-t}}{t+4} \left( \frac{x^2 + 2|x|}{1 + |x|} \right) + \frac{1}{2} t^2 + 1 \tag{49}$$

Now, we can verify that function  $f(t, x)$  satisfies the Lipschitz condition  $(H_1)$  with  $l = 2/5$ , that is,  $|f(t, x) - f(t, y)| \leq (2/5)|x - y|$ , for all  $t \in [1, 4.5]$  and  $x, y \in \mathbb{R}$ . Then, we obtain  $l\Omega \approx 0.9003261924 < 1$ . The conclusion can be gotten

from Theorem 10 that the problem (47), with  $f$  given by (48), has a unique solution on  $[1,4.5]$ .

(ii) If the term  $x^2$  in (48) is replaced by  $|x|$ , that is,

$$f(t, x) = \frac{3e^{1-t}}{t+4} \left( \frac{|x|}{1+|x|} \right) + \frac{1}{2}t^2 + 1, \quad (50)$$

then the nonlinear function  $f$  is bounded by a function of  $t$  by

$$|f(t, x)| \leq \frac{3e^{1-t}}{t+4} + \frac{1}{2}t^2 + 1 := \phi(t), \quad (51)$$

which is satisfied a condition  $(H_2)$ . It is easy to check that function  $f$  in (49) is fulfilled the Lipschitz condition with constant  $l = 3/5$ . Since

$$\frac{(\log T)^{\gamma-1}}{|\Lambda|} \left\{ \frac{|\lambda|(\log \eta)^{\alpha+\delta}}{\Gamma(\alpha+\delta+1)} + \sum_{i=1}^m |\theta_i| \frac{(\log \xi_i)^\alpha}{\Gamma(\alpha+1)} \right\} l \approx 0.5179201139 < 1, \quad (52)$$

we can deduce that the Hilfer-Hadamard nonlocal integro-multipoint fractional boundary value problem (47), with  $f$  given by (49), has at least one solution on  $[1,4.5]$ . In addition, we observe that Theorem 10 cannot be used in this case because  $l\Omega \approx 1.350489289 > 1$

(iii) If  $f$  is a non-Lipschitzian function as

$$f(t, x) = \frac{e^{1-t}}{t+4} \sin^{12} \left( \frac{x^2 + 2|x|}{1+|x|} \right) + \frac{1}{2}t^2 + 1, \quad (53)$$

to claim the existence of boundary value problem (47), with  $f$  given by (52), we can find that

$$|f(t, x)| \leq \frac{1}{5} + \frac{1}{2} \left( \frac{9}{2} \right)^2 + 1 = \frac{453}{40} \quad (54)$$

Hence,  $(H_3)$  is satisfied with  $M = 453/40$ . Therefore, by the benefit of Theorem 12, problem (47) with  $f$  given by (52) has at least one solution on  $[1,4.5]$ .

(iv) If  $f$  is a non-Lipschitzian function defined by

$$f(t, x) = \frac{e^{1-t}}{t+4} \left( \frac{x^{26}}{1+x^{24}} + \frac{1}{9}t + \frac{1}{2} \right), \quad (55)$$

we see that  $(H_4)$  holds as

$$|f(t, x)| \leq \frac{e^{1-t}}{t+4} (x^2 + 1), \quad (56)$$

by setting  $p(t) = (e^{1-t})/(t+4)$  and  $\psi(x) = x^2 + 1$ . Then, we obtain  $\|p\| = 1/5$  and there exists a constant  $K \in$

$(0.6273106970, 1.594106403)$  satisfying condition  $(H_5)$ . Thus, using Theorem 13, the boundary value problem (47), with  $f$  is given in (54), has at least one solution on  $[1,4.5]$

## 5. Conclusion

In this paper, we have presented the existence and uniqueness criteria for solutions for Hilfer-Hadamard fractional differential equation complemented with nonlocal integro-multipoint boundary conditions. Firstly, we convert the given nonlinear problem into a fixed point problem. Once the fixed point operator is available, we make use of Banach contraction mapping principle to obtain the uniqueness result, while the existence results are established by applying Schaefer and Krasnoselskii fixed point theorems as well as Leray-Schauder nonlinear alternative. Our results are new in the given configuration and enrich the literature on boundary value problems involving Hilfer-Hadamard fractional differential equations.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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