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Research Article

A New Estimate for the Homogenization Method for Second-Order Elliptic Problem with Rapidly Oscillating Periodic Coefficients

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In this paper, we will investigate a multiscale homogenization theory for a second-order elliptic problem with rapidly oscillating periodic coefficients of the form $(\partial/\partial x_i)(a^{ij}(x/\varepsilon, x)(\partial u^\varepsilon(x)/\partial x_j)) = f(x)$. Noticing the fact that the classic homogenization theory presented by Oleinik has a high demand for the smoothness of the homogenization solution u^0, we present a new estimate for the homogenization method under the weaker smoothness that homogenization solution u^0 satisfies than the classical homogenization theory needs.

1. Introduction

Many people investigated the second-order elliptic problem with a fixed boundary. As far as we know, there is not any work related to the elliptic problem with periodic boundary (see [1–3]). In this article, we will consider the following multiscale elliptic model problem:

$$\begin{cases} L_\varepsilon u^\varepsilon(x) \equiv \dfrac{\partial}{\partial x_i}\left(a^{ij}\left(\dfrac{x}{\varepsilon}, x\right)\dfrac{\partial u^\varepsilon}{\partial x_j}\right) = f(x), & \text{in } \Omega, \\ u^\varepsilon(x) = g(x), & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Here, $\Omega \subset \mathfrak{R}^n (n \geq 1)$ is a bounded domain, and the matrix of coefficients $a^{ij}(\xi, x)$: $\mathfrak{R}^n \longrightarrow \mathfrak{R}^{n\times n}$ is symmetric and satisfies the following conditions:

$$\begin{gathered} \gamma|\xi|^2 \leq a^{ij}(\xi, x)\xi_i\xi_j \leq \gamma^{-1}|\xi|^2, \xi \in \mathfrak{R}^n, \text{for some } \gamma \in (0, 1] \\ a^{ij}\left(\xi + \xi'\right) = a^{ij}(\xi), \xi \in \mathfrak{R}^n, \xi' \in Z^n, 1 \leq i, j \leq n \end{gathered} \tag{2}$$

Assume that $Q = [0, 1]^n$. By the homogenization method, Oleinik et al. (see [4, 5]) obtained the 1-order approximation $\tilde{u}(x)$ of u^ε as follows:

$$\tilde{u}(x) = u^0(x) + \varepsilon N^k\left(x, \frac{x}{\varepsilon}\right)\frac{\partial u^0(x)}{\partial x_k}, \tag{3}$$

where $N^k(x, \xi)$ is a 1-periodic function and satisfies the following equations:

$$\begin{gathered} \begin{cases} \dfrac{\partial}{\partial \xi_i}\left(a^{ij}(x, \xi)\dfrac{\partial N^k(x, \xi)}{\partial \xi_j}\right) = -\dfrac{\partial a^{ik}(x, \xi)}{\partial \xi_i}, & \text{in } \mathfrak{R}^n, \\ \displaystyle\int_Q N^k(\xi, x)d\xi = 0, \end{cases} \\ a\wedge^{ij}(x) = \int_Q\left(a^{ij}(x, \xi) + a^{ik}(x, \xi)\frac{\partial N^j(x, \xi)}{\partial \xi_k}\right)d\xi, \end{gathered} \tag{4}$$

and the homogenization solution u^0 satisfies the problem as follows:

$$\begin{cases} L_0 u^0(x) \equiv \dfrac{\partial}{\partial x_i}\left(a\wedge^{ij}(x)\dfrac{\partial u^0}{\partial x_j}\right) = f(x), & \text{in } \Omega, \\ u^0(x) = g(x), & \text{on } \partial\Omega. \end{cases} \quad (5)$$

Oleinik et al. (see [5], p. 28) proved the following result.

We end this section with the details of some notations. Throughout this paper, the Einstein summation convention is used: summation is taken over repeated indices, and $\rho(x, \partial\Omega)$ denotes the distance between x and $\partial\Omega$.

2. Some Useful Lemmas

Lemma 1. *Under the assumption that $u^0 \in H^2(\Omega)$, there holds*

$$\|u^\varepsilon - \tilde{u}\|_{H^1(\Omega)} \le c\varepsilon^{1/2}\|u^0\|_{H^2(\Omega)}. \quad (6)$$

There are numerous literatures discussing the homogenization method (see [1, 2, 4–9]). There also are many works (see [3, 10–16]) discussing the numerical methods of the multiscale homogenization problem. We observe that most of them are based on the assumption $u^0 \in H^2(\Omega)$, which is unrealistic for some problems. For example, when $f \notin L^2(\Omega)$. Let $\tilde{u}_i(x) = (\partial u^0(x)/\partial x_i) + (\partial N^k(\xi, x)/\partial \xi_i)(\partial u^0(x)/\partial x_k)$. As far as we know, it is the first time for us to estimate $\tilde{u}_i(x) - (\partial u^\varepsilon(x)/\partial x_i)$ under the assumption that the homogenization solution u^0 belongs to the Sobolev space $H^{1+s}(\Omega)$ for the case that $0 < s < 1$.

Lemma 2. *Assume that $u \in H^{1+s}(\Omega) \cap W^{1,\infty}(K_{2r})$. Then,*

$$\|\nabla(u - u_r)\|_{L^2(\Omega)} \le c\left(r^s\|u\|_{H^{1+s}(\Omega)} + r^{1/2}\|\nabla u\|_{L^\infty(K_{2r})}\right), \quad (7)$$

$$\left\|\nabla^2 u_r\right\|_{L^2(\Omega)} \le c\left(r^{s-1}\|u\|_{H^{1+s}(\Omega)} + r^{-1/2}\|\nabla u\|_{L^\infty(K_{2r})}\right). \quad (8)$$

Proof. One observes that $\|\nabla(u - u_r)\|^2_{L^2(\Omega)}$ can be split into

$$\|\nabla(u - u_r)\|^2_{L^2(\Omega)} = \|\nabla(u - u_r)\|^2_{L^2(\Omega\setminus K_r)} + \|\nabla(u - u_r)\|^2_{L^2(K_r)}. \quad (9)$$

We first estimate $\|\nabla(u - u_r)\|^2_{L^2(\Omega\setminus K_r)}$. Assume that $x \in \Omega \setminus K_r$ and $B(x, r) = \{y \in \Omega : |x - y| \le r\}$. Note that the definition of $\omega_r(z)$ implies $\int_\Omega \omega_r(x - y)dy = 1$. By the definitions of $\omega_r(z)$ and $u_r(x)$, we have, for any $1 \le i \le n$,

$$\begin{aligned} \frac{\partial u_r(x)}{\partial x_i} &= \int_\Omega \frac{\partial \omega_r(x - y)}{\partial x_i}u(y)dy = -\int_\Omega \frac{\partial \omega_r(x - y)}{\partial y_i}u(y)dy \\ &= \int_\Omega \omega_r(x - y)\frac{\partial u(y)}{\partial y_i}dy = \int_{B(x,r)} \omega_r(x - y)\frac{\partial u(y)}{\partial y_i}dy. \end{aligned} \quad (10)$$

Using (10), we obtain

$$\frac{\partial u_r(x)}{\partial x_i} - \frac{\partial u(x)}{\partial x_i} = \int_{B(x,r)} \omega_r(x - y)\left(\frac{\partial u(y)}{\partial y_i} - \frac{\partial u(x)}{\partial x_i}\right)dy. \quad (11)$$

Furthermore, from the definition of $\omega_r(z)$ and (11), it follows that

$$\left\|\frac{\partial(u - u_r)}{\partial x_i}\right\|^2_{L^2(\Omega\setminus K_r)} = \int_{\Omega\setminus K_r}\left[\int_{B(x,r)} \omega_r(x - y)\left(\frac{\partial u(y)}{\partial y_i} - \frac{\partial u(x)}{\partial x_i}\right)dy\right]^2 dx, \quad (12)$$

$$\le cr^{-2n}\int_{\Omega\setminus K_r}\left[\int_{B(x,r)}\left|\frac{\partial u(y)}{\partial y_i} - \frac{\partial u(x)}{\partial x_i}\right|dy\right]^2 dx. \quad (13)$$

Note that

$$\int_{B(x,r)}\left|\frac{\partial u(y)}{\partial y_i} - \frac{\partial u(x)}{\partial x_i}\right|dy \le cr^{s+n/2}\int_{B(x,r)}\left|\frac{\partial u(y)}{\partial y_i} - \frac{\partial u(x)}{\partial x_i}\right||x - y|^{-s-n/2}dy. \quad (14)$$

This, together with (13), gives

$$\begin{aligned} \left\|\frac{\partial(u - u_r)}{\partial x_i}\right\|^2_{L^2(\Omega\setminus K_r)} &\le cr^{2s-n}\int_{\Omega\setminus K_r}\left[\int_{B(x,r)}\left|\frac{\partial u(y)}{\partial y_i} - \frac{\partial u(x)}{\partial x_i}\right||x - y|^{-s-n/2}dy\right]^2 dx \\ &\le cr^{2s}\int_{\Omega\setminus K_r}\int_{B(x,r)}\left|\frac{\partial u(y)}{\partial y_i} - \frac{\partial u(x)}{\partial x_i}\right|^2|x - y|^{-2s-n}dydx \\ &\le cr^{2s}\|u\|^2_{H^{1+s}(\Omega)}. \end{aligned} \quad (15)$$

Next we estimate $\|\nabla(u - u_r)\|^2_{L^2(K_r)}$. Assume that $x \in K_r$. Set $\bar{\omega}_r(x - y) = \omega_r(x - y)/\int_{B(x,r)}\omega_r(x - y)dy$. Let $\bar{x} = x$ or $x + \Delta x$. We have

$$\begin{aligned} u_r(\bar{x}) &= \int_{B(\bar{x},r)} \bar{\omega}_r(\bar{x} - y)u(y)dy \\ &= u(\bar{x}) + \int_{B(\bar{x},r)} \bar{\omega}_r(\bar{x} - y)(u(y) - u(x))dy. \end{aligned} \quad (16)$$

Let $|\Delta x| \le r$. Note that $\bar{\omega}_r(z) = 0$ whenever $|z| \ge r$. By (16), one observes that $u_r(x + \Delta x) - u_r(x)$ can be decomposed into

$$\begin{aligned}
u_r(x+\Delta x)-u_r(x) &= \int_{B(x+\Delta x,r)} \bar{\omega}_r(x+\Delta x-y)(u(y)-u(x))dy \\
&\quad - \int_{B(x,r)} \bar{\omega}_r(x-y)(u(y)-u(x))dy \\
&= \int_{B(x+\Delta x,r)} - \int_{B(x,r)} \bar{\omega}_r(x+\Delta x-y)(u(y) \\
&\quad - u(x))dy + \int_{B(x,r)} [\bar{\omega}_r(x+\Delta x-y) \\
&\quad - \bar{\omega}_r(x-y)](u(y)-u(x))dy = I_1 + I_2.
\end{aligned} \tag{17}$$

We need estimates I_1 and I_2. Assume that $y \in B(x+\Delta x, r) \setminus B(x,r)$. Note that $x+\Delta x \in \Omega$. One observes that $\int_{B(x+\Delta x,r/2)} \boxtimes dy \geq cr^n$. Then, we have

$$\int_{B(x+\Delta x,r/2)} \omega_r(x-y)dy \geq c. \tag{18}$$

By the definition of $\bar{\omega}_r(x)$ and (18), we have

$$|\bar{\omega}_r(x+\Delta x-y)| \leq cr^{-n}. \tag{19}$$

Note that

$$|u(y)-u(x)| \leq cr\|u\|_{W^{1,\infty}\left(K_{2r+|\Delta x|}\right)}. \tag{20}$$

Inserting (19) and (20) into (17), we have

$$|I_1| \leq cr^{n-1}|\Delta x| r^{-n} r \|u\|_{W^{1,\infty}\left(K_{2r+|\Delta x|}\right)} \leq c|\Delta x| \|u\|_{W^{1,\infty}\left(K_{2r+|\Delta x|}\right)}. \tag{21}$$

We turn now to the estimation of I_2. We split $\bar{\omega}_r(x+\Delta x-y) - \bar{\omega}_r(x-y)$ into

$$\begin{aligned}
\bar{\omega}_r(x+\Delta x-y) - \bar{\omega}_r(x-y) &= \frac{\omega_r(x+\Delta x-y)}{\int_{B(x+\Delta x,r)} \omega_r(x+\Delta x-y)dy} - \frac{\omega_r(x-y)}{\int_{B(x,r)} \omega_r(x-y)dy} \\
&= \omega_r(x+\Delta x-y)\left[\left(\int_{B(x+\Delta x,r)} \omega_r(x+\Delta x-y)dy\right)^{-1} \right. \\
&\quad \left. - \left(\int_{B(x,r)} \omega_r(x-y)dy\right)^{-1}\right] \\
&\quad + \frac{[\omega_r(x+\Delta x-y)-\omega_r(x-y)}{\int_{B(x,r)} \omega_r(x-y)dy} = J_1 + J_2.
\end{aligned} \tag{22}$$

We need to estimate the two items of the right-hand side of (22). Note that

$$\begin{aligned}
&\left|\int_{B(x+\Delta x,r)} \omega_r(x+\Delta x-y)dy - \int_{B(x,r)} \omega_r(x-y)dy\right| \\
&\quad \leq \left|\int_{B(x+\Delta x,r)} - \int_{B(x,r)} \omega_r(x+\Delta x-y)dy\right| \\
&\qquad + \int_{B(x,r)} [\omega_r(x+\Delta x-y) - \omega_r(x-y)]dy| \leq cr^{n-1}|\Delta x| r^{-n} \\
&\qquad + cr^n r^{-n-1}|\Delta x| \leq cr^{-1}|\Delta x|.
\end{aligned} \tag{23}$$

By (22) and (23), we have

$$|J_1| \leq cr^{-n} cr^{-1}|\Delta x| \leq cr^{-n-1}|\Delta x|. \tag{24}$$

To estimate J_2, we have

$$|J_2| \leq cr^{-1}|\Delta x| r^{-n} \leq cr^{-n-1}|\Delta x|. \tag{25}$$

Plugging the above two estimates into (22), we obtain

$$|\bar{\omega}_r(x+\Delta x-y) - \bar{\omega}_r(x-y)| \leq cr^{-n-1}|\Delta x|. \tag{26}$$

This, together with (17), gives

$$|I_2| \leq cr^n r^{-n-1}|\Delta x| r \|u\|_{W^{1,\infty}(K_{2r})} \leq c|\Delta x| \|u\|_{W^{1,\infty}(K_{2r})}. \tag{27}$$

Inserting (21) and (27) into (17), we have

$$|u_r(x+\Delta x) - u_r(x)| \leq c|\Delta x| \|u\|_{W^{1,\infty}\left(K_{2r+|\Delta x|}\right)}. \tag{28}$$

Furthermore, let $\Delta x \longrightarrow 0$, we have

$$\|u_r\|_{W^{1,\infty}(K_r)} \leq c\|u\|_{W^{1,\infty}(K_{2r})}, \tag{29}$$

where we have used (28). Then, (7) follows by combining (15) and (29). We turn now to the estimation of $\|\nabla^2 u_r\|_{L^2(\Omega)}$. We decompose $\|\nabla^2 u_r\|_{L^2(\Omega)}$ into

$$\left\|\nabla^2 u_r\right\|^2_{L^2(\Omega)} = \left\|\nabla^2 u_r\right\|^2_{L^2(\Omega \setminus K_r)} + \left\|\nabla^2 u_r\right\|^2_{L^2(K_r)}. \tag{30}$$

We first estimate $\|\nabla^2 u_r\|^2_{L^2(\Omega \setminus K_r)}$. Assume that $x \in \Omega \setminus K_r$. By (10), we have, for any $1 \leq i, j \leq n$,

$$\begin{aligned}
\frac{\partial^2 u_r(x)}{\partial x_i \partial x_j} &= \frac{\partial\left(\int_\Omega \omega_r(x-y)(\partial u(y)/\partial y_i)dy\right)}{\partial x_j} \\
&= \int_\Omega \frac{\partial \omega_r(x-y)}{\partial x_j} \frac{\partial u(y)}{\partial y_i} dy.
\end{aligned} \tag{31}$$

Note that $x \in \Omega \setminus K_r$. By the definition of $\omega_r(x-y)$, we have $\int_\Omega (\partial \omega_r(x-y)/\partial x_j)dy = 0$. Then, by (28) and (31), we have

$$\frac{\partial^2 u_r(x)}{\partial x_i \partial x_j} = \frac{\partial\left(\int_\Omega \omega_r(x-y)(\partial u(y)/\partial y_i)dy\right)}{\partial x_j} = \int_\Omega \frac{\partial \omega_r(x-y)}{\partial x_j}\left(\frac{\partial u(y)}{\partial y_i} - \frac{\partial u(x)}{\partial x_i}\right)dy. \tag{32}$$

Finally, similarly to (15), by (32), we have

$$\left\|\nabla^2 u_r\right\|_{L^2(\Omega\setminus K_r)} \le cr^{s-1}\|u\|_{H^{1+s}(\Omega)}. \tag{33}$$

We turn now to the estimation of $\|\nabla^2 u_r\|_{L^2(K_r)}$. Similarly to (17), we have

$$\begin{aligned} u_r(x+2\Delta x) - 2u_r(x+\Delta x) + u_r(x) = \int_{B(x,2r)} &[\bar{\omega}_r(x+2\Delta x - y) \\ &- 2\bar{\omega}_r(x+\Delta x - y) \\ &+ \bar{\omega}_r(x-y)](u(y)-u(x))dy. \end{aligned} \tag{34}$$

Note that the definition of $\omega_r(z)$ implies $\|\bar{\omega}_r\|_{W^{2,1}(\Re^n)} \le c\, r^{-2}$. Therefore, let $\Delta x \longrightarrow 0$, from (34), it follows that

$$\begin{aligned} \|u_r\|_{W^{2,\infty}(K_r)} &\le c\|\bar{\omega}_r\|_{W^{2,1}(\Re^n)} cr\|u\|_{W^{1,\infty}(K_{2r})} \\ &\le cr^{-2}r\|u\|_{W^{1,\infty}(K_{2r})} \le cr^{-1}\|u\|_{W^{1,\infty}(K_{2r})}. \end{aligned} \tag{35}$$

The desired result (8) follows by combining (33) and (35). □

3. A New Estimate for Multiscale Homogenization Method

In this section, we give the main results as follows.

Theorem 3. *Assume that $K_r = \{x \in \Omega \mid \rho(x, \partial\Omega) \le r\}$ and $Q = [0,1]^n$. Assume also that $N^k \in W^{1,\infty}(Q)$ and $u^0 \in H^{1+s}(\Omega) \cap W^{1,\infty}(K_\varepsilon)$ for some $0 < s < 1$. Then,*

$$\left\|\frac{\partial u^\varepsilon}{\partial x_i} - \tilde{u}_i\right\|_{L^2(\Omega)} \le c\left(\varepsilon^{1/2}\left\|u^0\right\|_{W^{1,\infty}(K_\varepsilon)} + \varepsilon^s\left\|u^0\right\|_{H^{1+s}(\Omega)}\right). \tag{36}$$

Assume that $\chi(z) \in C^\infty(\Re^n)$ is the cutoff function satisfying $0 \le \chi(z) \le 1$, and $\chi(z) = 1$ if $|z| \le 1/2$, and $\chi(z) = 0$ if $|z| \ge 1$. Let $\omega_r(z) = \chi(z/r)/\int_{\Re^n}\chi(y/r)dy$. One observes that $\int_{B(0,r)}\omega_r(z)dz = 1$ and $\|\omega_r\|_{W^{k,\infty}(\Re^n)} \le cr^{-k-n}$ for all $k \ge 0$. Set $u_r(x) = \int_\Omega \omega_r(x-y)u(y)dy / \int_\Omega \omega_r(x-y)dy$. In the process of proving Theorem 3, we need the above Lemma 2.

Based on Lemma 2, we can prove Theorem 3 as follows:

Proof. Assume that $\omega_r(z)$ is defined as in Lemma 2. Set

$$\begin{aligned} \bar{u}_r^0(x) &= \frac{\int_\Omega \omega_r(x-y)u^0(y)dy}{\int_\Omega \omega_r(x-y)dy}, \\ f_r(x) &= \frac{\partial}{\partial x_j}\left(\hat{a}_{ij}(x)\frac{\partial \bar{u}_r^0(x)}{\partial x_i}\right). \end{aligned} \tag{37}$$

We introduce $u_r^\varepsilon(x)$ by the following problem:

$$\begin{cases} L_\varepsilon u_r^\varepsilon(x) = f_r(x), & \text{in } \Omega, \\ u_r^\varepsilon(x) = \bar{u}_r^0(x), & \text{on } \partial\Omega. \end{cases} \tag{38}$$

One observes that $u_r^0(x)$ and $\tilde{u}_r(x)$ are the homogenization solution of (38) and the 1-order approximation of $u_r^\varepsilon(x)$, respectively. We decompose $\partial u^\varepsilon(x)/\partial x_i - \tilde{u}_i(x)$ into

$$\begin{aligned} \frac{\partial u^\varepsilon(x)}{\partial x_i} - \tilde{u}_i(x) = \frac{\partial(u^\varepsilon - u_r^\varepsilon)(x)}{\partial x_i} &+ \frac{\partial(u_r^\varepsilon - \tilde{u}_r)(x)}{\partial x_i} \\ &+ \left(\frac{\partial \tilde{u}_r(x)}{\partial x_i} - \tilde{u}_i(x)\right). \end{aligned} \tag{39}$$

We first estimate $\nabla(u^\varepsilon - u_r^\varepsilon)(x)$. Let $B_1^\varepsilon(x) = (u^\varepsilon - u_r^\varepsilon)(x)$. Note that $B_1^\varepsilon(x)$ satisfies the following problem:

$$\begin{cases} L_\varepsilon B_1^\varepsilon(x) = f(x) - f_r(x), & x \in \Omega, \\ B_1^\varepsilon(x) = g(x) - \bar{u}_r^0(x), & x \in \partial\Omega. \end{cases} \tag{40}$$

One observes that $B_1^\varepsilon(x)$ can be split into

$$B_1^\varepsilon(x) = e_1(x) + e_2^\varepsilon(x), \tag{41}$$

where $e_1(x) = (u^0 - \bar{u}_r^0)(x)$ and $e_2^\varepsilon(x)$ satisfies the following problem:

$$\begin{cases} L_\varepsilon e_2^\varepsilon(x) = (f - f_r)(x) - \dfrac{\partial}{\partial x_i}\left(a_{ij}\left(x, \dfrac{x}{\varepsilon}\right)\dfrac{\partial e_1(x)}{\partial x_j}\right), & \text{in } \Omega, \\ e_2^\varepsilon(x) = 0, & \text{on } \partial\Omega. \end{cases} \tag{42}$$

From the combination of the definition of $\bar{u}_r^0$ and (7), it follows that

$$\left\|\nabla\left(\bar{u}_r^0 - u^0\right)\right\|_{L^2(\Omega)} \le c\left(r^s\left\|u^0\right\|_{H^{1+s}(\Omega)} + r^{1/2}\left\|u^0\right\|_{W^{1,\infty}(K_{2r})}\right). \tag{43}$$

To estimate $e_2^\varepsilon(x)$, by (8) and the definitions of $\bar{u}_r^0(x)$ and $f_r(x)$, one observes that

$$\begin{aligned} \|\nabla e_2^\varepsilon\|_{L^2(\Omega)} &\le \|f - f_r\|_{H^{-1}(\Omega)} + \left\|\nabla\left(\bar{u}_r^0 - u^0\right)\right\|_{L^2(\Omega)} \\ &\le c\left\|\nabla\left(\bar{u}_r^0 - u^0\right)\right\|_{L^2(\Omega)} \le cr^s\left\|u^0\right\|_{H^{1+s}(\Omega)} \\ &\quad + cr^{1/2}\left\|u^0\right\|_{W^{1,\infty}(K_{2r})}. \end{aligned} \tag{44}$$

Combining (39), (41), (43), and (44), we have

$$\|\nabla(u^\varepsilon - u_r^\varepsilon)\|_{L^2(\Omega)} = \|\nabla B_1^\varepsilon\|_{L^2(\Omega)} \le c\Big(r^s\|u^0\|_{H^{1+s}(\Omega)} + r^{1/2}\|u^0\|_{W^{1,\infty}(K_{2r})}\Big). \quad (45)$$

Next, we estimate $(u_r^\varepsilon - \tilde{u}_r)(x)$. Set $B_2^\varepsilon(x) = (u_r^\varepsilon - \tilde{u}_r)(x)$. By the method of asymptotic expansion (see [7], p. 27), one finds that $B_2^\varepsilon(x)$ can be split into $B_2^\varepsilon(x) = w_r^\varepsilon(x) + \theta_r^\varepsilon(x)$, where $w_r^\varepsilon(x)$ and $\theta_r^\varepsilon(x)$ are defined by

$$\begin{cases} L_\varepsilon w_r^\varepsilon(x) \equiv \dfrac{\partial}{\partial x_i}\left(a_{ij}\left(x, \dfrac{x}{\varepsilon}\right)\dfrac{\partial w_r^\varepsilon(x)}{\partial x_j}\right) = \dfrac{\partial F_{r,i}(x)}{\partial x_i}, & \text{in } \Omega, \\ w_r^\varepsilon(x) = 0, & \text{on } \partial\Omega, \end{cases}$$

$$\begin{cases} L_\varepsilon \theta_r^\varepsilon(x) = 0, & \text{in } \Omega, \\ \theta_r^\varepsilon(x) = -\varepsilon N^k\left(\dfrac{x}{\varepsilon}, x\right)\dfrac{\partial \bar{u}_r^0(x)}{\partial x_k}, & \text{on } \partial\Omega, \end{cases} \quad (46)$$

respectively, where

$$F_{r,i}(x) = -\left(a_{ij}\left(x, \frac{x}{\varepsilon}\right) + a_{ik}\left(x, \frac{x}{\varepsilon}\right)\frac{\partial N^j(\xi)}{\partial \xi_k} - \hat{a}_{ij}\right)\frac{\partial \bar{u}_r^0(x)}{\partial x_j} + \varepsilon a_{ij}\left(x, \frac{x}{\varepsilon}\right)N^k\left(\frac{x}{\varepsilon}\right)\frac{\partial^2 \bar{u}_r^0(x)}{\partial x_j \partial x_k}. \quad (47)$$

We first estimate $w_r^\varepsilon(x)$. Note that (8) implies

$$\|\bar{u}_r^0\|_{H^2(\Omega)} \le c\Big(r^{-1+s}\|u^0\|_{H^{1+s}(\Omega)} + r^{-1/2}\|u^0\|_{W^{1,\infty}(K_{2r})}\Big). \quad (48)$$

By the method of asymptotic expansion (see ([7], p. 27), from (48), it follows that

$$\|w_r^\varepsilon\|_{H^1(\Omega)} \le c\varepsilon\|\bar{u}_r^0\|_{H^2(\Omega)} \le c\Big(\varepsilon r^{-1+s}\|u^0\|_{H^{1+s}(\Omega)} + \varepsilon r^{-1/2}\|u^0\|_{W^{1,\infty}(K_{2r})}\Big). \quad (49)$$

Assume that $\phi(x) \in C^\infty(\Omega)$ is a cutoff function satisfying $\phi(x) = 1$ if $\rho(x, \partial\Omega) \le \varepsilon$, and $\phi(x) = 0$ if $\rho(x, \partial\Omega) \ge 2\varepsilon$, and $\|\nabla\phi\|_{L^\infty(\Omega)} \le c_2\varepsilon^{-1}$. We split $\theta_r^\varepsilon(x)$ into

$$\theta_r^\varepsilon(x) = \psi_r^\varepsilon(x) + \hat{\psi}_r^\varepsilon(x), \quad (50)$$

where $\psi_r^\varepsilon(x) = -\varepsilon N^k(x/\varepsilon, x)(\partial \bar{u}_r^0(x)/\partial x_k)\phi(x)$ and $\hat{\psi}_r^\varepsilon(x)$ satisfies the following problem:

$$\begin{cases} L_\varepsilon \hat{\psi}_r^\varepsilon(x) = -\dfrac{\partial}{\partial x_i}\left(a_{ij}\left(x, \dfrac{x}{\varepsilon}\right)\dfrac{\partial \psi_r(x)}{\partial x_j}\right), & x \in \Omega, \\ \hat{\psi}_r^\varepsilon(x) = 0, & x \in \partial\Omega. \end{cases} \quad (51)$$

To estimate $\psi_r^\varepsilon(x)$, one has

$$\|\psi_r^\varepsilon\|_{H^1(\Omega)} \le c\varepsilon r^{-1}\|\bar{u}_r^0\|_{H^1(K_\varepsilon)} + c\varepsilon\|\bar{u}_r^0\|_{H^2(K_\varepsilon)}. \quad (52)$$

We now estimate $\|\bar{u}_r^0\|_{H^1(K_\varepsilon)}$. Assume that $r = \varepsilon$ and V_{K_ε} denotes the volume of K_ε if $n = 3$, or the area of K_ε if $n = 2$. One observes that

$$\|\bar{u}_r^0\|_{H^1(K_\varepsilon)} \le c\sqrt{V_{K_\varepsilon}}\|\bar{u}_r^0\|_{W^{1,\infty}(K_\varepsilon)} \le c\varepsilon^{1/2}\|\bar{u}_r^0\|_{W^{1,\infty}(K_\varepsilon)} \le c\varepsilon^{1/2}\|u^0\|_{W^{1,\infty}(K_\varepsilon)}. \quad (53)$$

The combination of (8), (52), and (53) gives

$$\|\psi_r^\varepsilon\|_{H^1(\Omega)} \le c\Big(\varepsilon^{1/2}\|u^0\|_{W^{1,\infty}(K_{2\varepsilon})} + \varepsilon^s\|u^0\|_{H^{1+s}(\Omega)}\Big). \quad (54)$$

Using (51) and (54), we derive

$$\|\hat{\psi}_r^\varepsilon\|_{H^1(\Omega)} \le c\|\psi_r^\varepsilon\|_{H^1(\Omega)} \le c\Big(\varepsilon^{1/2}\|u^0\|_{W^{1,\infty}(K_{2\varepsilon})} + \varepsilon^s\|u^0\|_{H^{1+s}(\Omega)}\Big). \quad (55)$$

The above two estimates, together with (50), imply

$$\|\theta_r^\varepsilon\|_{H^1(\Omega)} \le c\big(\varepsilon^{1/2} + \varepsilon^{3/2}r^{-1}\big)\|u^0\|_{W^{1,\infty}(K_{2\varepsilon})}. \quad (56)$$

Furthermore, by (49) and (56), we have

$$\|u_r^\varepsilon - \tilde{u}_r\|_{H^1(\Omega)} = \|B_2^\varepsilon\|_{H^1(\Omega)} \le c\Big[\big(\varepsilon^{1/2} + \varepsilon^{3/2}\varepsilon^{-1}\big)\|u^0\|_{W^{1,\infty}(K_{2\varepsilon})} + \varepsilon\varepsilon^{-1+s}\|u^0\|_{H^{1+s}(\Omega)}\Big] \le c\Big(\varepsilon^{1/2}\|u^0\|_{W^{1,\infty}(K_{2\varepsilon})} + \varepsilon^s\|u^0\|_{H^{1+s}(\Omega)}\Big), \quad (57)$$

We next estimate $(\partial\tilde{u}_r(x)/\partial x_i) - \tilde{u}_i(x)$. Assume that $r = \varepsilon$. Note that the definitions of u_r^0 and u_r^ε imply $u_r^0(x) = \bar{u}_r^0(x)$. By (7) and (47), we have

$$\left\|\frac{\partial \tilde{u}_r}{\partial x_i} - \tilde{u}_i\right\|_{L^2(\Omega)} \le c\|\nabla(u^0 - \bar{u}_r^0)\|_{L^2(\Omega)} \le c\Big(\varepsilon^s\|u^0\|_{H^{1+s}(\Omega)} + \varepsilon^{1/2}\|u^0\|_{W^{1,\infty}(K_{2\varepsilon})}\Big). \quad (58)$$

Assume that $r = \varepsilon$. This, together with (39), (45), and (57), gives the desired result (36). □

Data Availability

The paper's data available through the email liuxiong980211@163.com or from the author's ORCID: 0000-0002-5452-653X, and other data is given to the journal of functional spaces https://orcid.org/0000-0002-5452-653X.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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