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Research Article

Toeplitz Operators with IMO^s Symbols between Generalized Fock Spaces

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In this paper, we study the mapping properties of Toeplitz operators T_f associated with IMO^s symbols f acting between two generalized Fock spaces F^p_φ, where $1 < s \leq \infty$. We characterize bounded or compact Toeplitz operators T_f from one generalized Fock space F^p_φ to another F^q_φ, respectively, in four cases.

1. Introduction

Let $\mathbb{C}^n$ be the n-dimensional complex Euclidean space and $\omega_0 = dd^c|\cdot|^2$ be the Euclidean Kähler form on $\mathbb{C}^n$, where d denotes the usual exterior derivative and $d^c = (\sqrt{-1}/4)(\bar{\partial} - \partial)$. In what follows, $\varphi \in C^2(\mathbb{C}^n)$ is assumed to be real-valued such that

$$c_1\omega_0 \leq dd^c\varphi \leq c_2\omega_0, \tag{1}$$

for two positive constants c_1 and c_2.

Given $1 \leq p < \infty$, the space L^p_φ consists of all measurable functions f on $\mathbb{C}^n$ such that

$$\|f\|_{p,\varphi} = \left(\int_{\mathbb{C}^n} \left|f(z)e^{-\varphi(z)}\right|^p dV(z)\right)^{1/p} < \infty, \tag{2}$$

where dV is the usual Lebesgue measure on $\mathbb{C}^n$.

Let $H(\mathbb{C}^n)$ be the class of all entire functions on $\mathbb{C}^n$. For $1 \leq p < \infty$; the generalized Fock space is defined by

$$F^p_\varphi = H(\mathbb{C}^n) \bigcap L^p_\varphi, \tag{3}$$

and

$$F^\infty_\varphi = \left\{f \in H(\mathbb{C}^n): \|f\|_{\infty,\varphi} = \sup_{z\in\mathbb{C}^n} |f(z)| e^{-\varphi(z)} < \infty\right\}. \tag{4}$$

It is well known that F^p_φ is a Banach space under the norm $\|\cdot\|_{p,\varphi}$ when $1 \leq p \leq \infty$. The generalized Fock spaces F^p_φ have been studied by many mathematics researchers (refer to [1–4] and the references therein). For a particular choice of φ, F^p_φ corresponds to some spaces that we are familiar with. For example, when $\varphi = (\alpha/2)|z|^2$ with $\alpha > 0$, it becomes the classical Fock space studied in [5]. When $\varphi(z) = (1/2)|z|^2 - m \ln |z|$, it becomes the Fock-Sobolev space introduced in [6].

It is clear that F^2_φ is a Hilbert space with inner product

$$\langle f, g\rangle_{F^2_\varphi} = \int_{\mathbb{C}^n} f(z)\bar{g}(z)e^{-2\varphi(z)}dV(z). \tag{5}$$

For $z, w \in \mathbb{C}^n$, the reproducing kernel of F^2_φ is given by $K_z(w) = K(w, z)$ and the projection P from L^2_φ to F^2_φ is given by

$$Pf(z) = \int_{\mathbb{C}^n} f(w)K(z, w)e^{-2\varphi(w)}dV(w). \tag{6}$$

For $z \in \mathbb{C}^n$, we denote $k_z = K_z/\|K_z\|_{2,\varphi}$ by the normalized reproducing kernel for F^2_φ. By Proposition 4.2 in [4], we see that $k_z \longrightarrow 0$ weakly in F^p_φ as $z \longrightarrow \infty$ for $1 \leq p \leq \infty$.

Given $z \in \mathbb{C}^n$ and $r > 0$, write $B(z, r) = \{w \in \mathbb{C}^n : |w - z| < r\}$. For a locally Lebesgue integrable function f on $\mathbb{C}^n$

(written as $f \in L^1_{loc}$), the averaging function $\widehat{f}_r(z)$ on $\mathbb{C}^n$ is defined by

$$\widehat{f}_r(z) = \frac{1}{V(B(z,r))}\int_{B(z,r)} f(w)dV(w), \tag{7}$$

and the mean oscillation of f at z is given by

$$\mathrm{MO}_r(f)(z) = \frac{1}{V(B(z,r))}\int_{B(z,r)} \left|f(w) - \widehat{f}_r(z)\right| dV(w). \tag{8}$$

For $1 < s \le \infty$ and $r > 0$, the space IMO^s_r, s-th integrable mean oscillation, is defined to be the class of all $f \in L^1_{loc}$ such that

$$\|f\|_{\mathrm{IMO}^s_r} = \|MO_r(f)\|_{L^s} < \infty, \tag{9}$$

where $L^s = L^s(\mathbb{C}^n, dV)$.

For $r > 0$, denote BMO_r by the space of functions on $\mathbb{C}^n$ with bounded mean oscillation, consisting of those functions $f \in L^1_{loc}$ such that

$$\|f\|_{\mathrm{BMO}_r} = \sup_{z\in\mathbb{C}^n} MO_r(f)(z) < \infty. \tag{10}$$

The spaces IMO^s_r and BMO_r are both independent of r (see Remark 5 below and Lemma 3.1 of [2]); thus, we will respectively write them IMO^s and BMO for simplicity. From the definition, it is easy to see that $\mathrm{IMO}^\infty = \mathrm{BMO}$. Hence, IMO^s spaces are generalizations of BMO spaces.

Given $f \in \mathrm{IMO}^s$, the Toeplitz operator T_f on F^p_φ is defined by

$$T_f g(z) = \int_{\mathbb{C}^n} f(w)g(w)K(z,w)e^{-2\varphi(w)}dV(w),\ z \in \mathbb{C}^n. \tag{11}$$

For any f on $\mathbb{C}^n$ with $f|k_z|^2 \in L^1_\varphi$ and $z \in \mathbb{C}^n$, the Berezin transform of f is given by

$$\tilde{f}(z) = \int_{\mathbb{C}^n} f(w)|k_z(w)|^2 e^{-2\varphi(w)}dV(w). \tag{12}$$

It is easy to check that $\tilde{f}(z) = \langle T_f k_z, k_z\rangle$.

For $1 < s \le \infty$ and $r > 0$, let IA^s_r be the space of all functions $f \in L^1_{loc}$ with

$$\|f\|_{IA^s_r} = \left\|\widehat{|f|}_r\right\|_{L^s} < \infty. \tag{13}$$

Lemma 2.3 of [1] shows that the space IA^s_r is independent of r, and we write it IA^s for simplicity. Moreover,

$$f \in IA^s \text{ if and only if } \widetilde{|f|} \in L^s. \tag{14}$$

Given a continuous function f on $\mathbb{C}^n$, the oscillation of f at z is defined to be

$$\omega_r(f)(z) = \sup\{|f(w) - f(z)|: |w - z| < r\}. \tag{15}$$

For $1 < s \le \infty$ and $r > 0$, let IO^s_r denote the space of continuous functions on $\mathbb{C}^n$ such that

$$\|f\|_{IO^s_r} = \|\omega_r(f)\|_{L^s} < \infty. \tag{16}$$

It follows from Lemma 2.1 in [7] that the space IO^s_r is independent of the choice of r, we simply write IO^s instead of IO^s_r.

The BMO space in the Bergman metric was first introduced by Zhu in [8] and further studied by Berger et al. in [9]. Over the past few decades, a great number of research work have been done on the boundedness and compactness of Toeplitz operators induced by BMO symbols on Bergman spaces or Fock spaces (see [10–14] and the references therein). The IMO^s space was firstly introduced and studied by Hu et al. in [2]. Recently, Wang [12] studied the the boundedness and compactness of Toeplitz operators induced by IMO^s symbols between two classical Fock spaces. Inspired by his work, we proceed to extend the results to the generalized Fock spaces. Moreover, the largest generalized Fock spaces F^∞_φ are taken into account and some interesting results are obtained.

In this paper, we aim to answer when the Toeplitz operators T_f with $f \in \mathrm{IMO}^s$ between two generalized Fock spaces are bounded or compact, where $1 < s \le \infty$. Specifically, we characterize the boundedness and compactness of Toeplitz operators T_f from F^p_φ to F^q_φ in four cases, i.e., $1 < p \le q < \infty$, $1 < q < p < \infty$, $1 < p < \infty$ and $q = \infty$, and $p = \infty$ and $1 < q < \infty$. As we see, the proofs of the main results in [12] rely heavily on the explicit formula of the reproducing kernel. Unfortunately, we do not know whether there is a similar explicit formula on the generalized Fock space setting. Therefore, some different techniques and methods are needed to show our main conclusions on the generalized Fock space.

Throughout this paper, for each $1 < p < \infty$, we write p' for its conjugate exponent, that is, $(1/p) + (1/p') = 1$. We will use the same letter C to denote various constants which may change at different occurrences. We denote $f \lesssim g$ whenever there exists a positive number C, independent of the essential variables, such that $f \le Cg$, and $f \simeq g$ if both $f \lesssim g$ and $g \lesssim f$ hold.

2. Preliminaries

We begin with some estimates for the reproducing kernel $K(\cdot,\cdot)$ of F^2_φ; results were proved in [4].

Lemma 1. *The reproducing kernel $K(\cdot,\cdot)$ for F^2_φ satisfies:*

(1) For $z, w \in \mathbb{C}^n$, there exists some $\theta > 0$ such that

$$|K(z,w)|e^{-\varphi(z)}e^{-\varphi(w)} \lesssim e^{-\theta|z-w|}. \quad (17)$$

(2) For $z \in \mathbb{C}^n$, there exists some $r > 0$ such that

$$|K(z,w)|e^{-\varphi(z)}e^{-\varphi(w)} \simeq 1, \text{for } w \in B(z,r). \quad (18)$$

(3) For $z \in \mathbb{C}^n$ and $1 \le p \le \infty$,

$$\|K(\cdot,z)\|_{p,\varphi} \simeq e^{\varphi(z)} \simeq \sqrt{K(z,z)}. \quad (19)$$

The following lemma from Lemma 2.4 in [1] is called the atomic decomposition for F^p_φ.

Lemma 2. *For $1 \le p \le \infty$, let $\{a_j\}_{j=1}^{\infty}$ be an r-lattice and $\{\lambda_j\}_{j=1}^{\infty} \in l^p$ (the p-summable sequence space) and set*

$$f(z) = \sum_{j=1}^{\infty} \lambda_j k_{a_j}(z), \quad z \in \mathbb{C}^n. \quad (20)$$

Then, $f \in F^p_\varphi$ and there exists a constant $C > 0$ such that

$$\|f\|_{p,\varphi} \le C\left\|\{\lambda_j\}_{j=1}^{\infty}\right\|_{l^p}. \quad (21)$$

Lemma 3. *For any $z \in \mathbb{C}^n$ and $w \in B(z,r)$, where r is taken as in (18), there exists some constant $C > 0$ depending only on r such that*

$$\frac{1}{V(B(z,r))} \le C|k_z(w)|^2 e^{-2\varphi(w)}. \quad (22)$$

Proof. By estimate (18), for $w \in B(z,r)$, we have

$$|k_z(w)|^2 e^{-2\varphi(w)} \simeq 1. \quad (23)$$

Since $V(B(z,r)) \simeq r^{2n}$, we have

$$\frac{1}{V(B(z,r))} \lesssim \max\left\{\frac{1}{r^{2n}}, 1\right\} \simeq \max\left\{\frac{1}{r^{2n}}, 1\right\}|k_z(w)|^2 e^{-2\varphi(w)}. \quad (24)$$

This completes the proof. □ □

Proposition 4. *Suppose $f \in L^1_{loc}$ and $1 < s \le \infty$. Then, the following statements are equivalent:*

(1) $f \in IMO^s_r$ for some (or any) $r > 0$.

(2) $f \in IMO^s$.

(3) $f = f_1 + f_2$, where $f_1 \in IO^s$ and $f_2 \in IA^s$.

(4) f acting on $\mathbb{C}^n$ with $f|k_z|^2 \in L^1_\varphi$ satisfies

$$\int_{\mathbb{C}^n}\left|f(w) - \tilde{f}(z)\right||k_z(w)|^2 e^{-2\varphi(w)} dV(w) \in L^s. \quad (25)$$

Proof. The equivalences of (1) ⇔ (2) ⇔ (3) follow from Theorem 2.2 of [7].

(3) ⇒ (4). For $f \in IA^s$, by the triangle inequality, we obtain

$$\begin{aligned}
&\int_{\mathbb{C}^n}\left|f(w) - \tilde{f}(z)\right||k_z(w)|^2 e^{-2\varphi(w)} dV(w) \\
&\le \int_{\mathbb{C}^n}|f(w)||k_z(w)|^2 e^{-2\varphi(w)} dV(w) \\
&+ \int_{\mathbb{C}^n}\left|\tilde{f}(z)\right||k_z(w)|^2 e^{-2\varphi(w)} dV(w) \le 2\widetilde{|f|}(z).
\end{aligned} \quad (26)$$

Since $f \in IA^s$ if and only if $\widetilde{|f|} \in L^s$ by (14), we obtain (25) whenever $f \in IA^s$.

Meanwhile, for $f \in IO^s$, by using Lemma 2.1, the changing of variables, Lemma 4.2 in [7], and Fubini's theorem, we deduce that

$$\begin{aligned}
&\int_{\mathbb{C}^n}\left|f(w) - \tilde{f}(z)\right||k_z(w)|^2 e^{-2\varphi(w)} dV(w) \\
&\lesssim \int_{\mathbb{C}^n}\left|f(w) - \tilde{f}(z)\right|e^{-2\theta|z-w|} dV(w) \\
&= \int_{\mathbb{C}^n}\left|f(z-w) - \tilde{f}(z)\right|e^{-2\theta|w|} dV(w) \\
&= \int_{\mathbb{C}^n}\left|f(z-w) - \int_{\mathbb{C}^n} f(\xi)|k_z(\xi)|^2 e^{-2\varphi(\xi)} dV(\xi)\right|e^{-2\theta|w|} dV(w) \\
&\le \int_{\mathbb{C}^n}\int_{\mathbb{C}^n}|f(z-w) - f(\xi)||k_z(\xi)|^2 e^{-2\varphi(\xi)} dV(\xi) e^{-2\theta|w|} dV(w) \\
&\lesssim \int_{\mathbb{C}^n}\int_{\mathbb{C}^n}|f(z-w) - f(\xi)|e^{-2\theta|z-\xi|} dV(\xi) e^{-2\theta|w|} dV(w) \\
&= \int_{\mathbb{C}^n}\int_{\mathbb{C}^n}|f(z-w) - f(z-\xi)|e^{-2\theta|\xi|} dV(\xi) e^{-2\theta|w|} dV(w) \\
&\lesssim \int_{\mathbb{C}^n}\int_{\mathbb{C}^n}\left((1 + |w-\xi|)\int_0^1 \omega(f)(z - w + t(w-\xi)) dt\right) \\
&\cdot e^{-2\theta|w|} dV(w) e^{-2\theta|\xi|} dV(\xi) = \int_{\mathbb{C}^n}\int_{\mathbb{C}^n}\int_0^1 (1 + |w-\xi|)\omega(f) \\
&\cdot (z - w + t(w-\xi)) dt e^{-2\theta|w|} dV(w) e^{-2\theta|\xi|} dV(\xi).
\end{aligned} \quad (27)$$

Let $dV(t,w,\xi) \coloneqq dt e^{-2\theta|w|} dV(w) e^{-2\theta|\xi|} dV(\xi)$ and apply Hölder's inequality; then, the last integral above is less than or equal to C times

$$\left(\int_{\mathbb{C}^n\times\mathbb{C}^n\times[0,1]}(1+|w-\xi\,|)^s \omega(f)^s (z - w + t(w-\xi)) dV(t,w,\xi)\right)^{1/s}. \quad (28)$$

For $|w-\xi\,| \ge 1$, the integral above is less than or equal to

C times:

$$\left(\int_{\mathbb{C}^n\times\mathbb{C}^n\times[0,1]}\left(|w|^s+|\xi|^s\right)\omega(f)^s(z-w+t(w-\xi))dV(t,w,\xi)\right)^{1/s}. \tag{29}$$

It follows by Fubini's theorem that

$$\begin{aligned}\int_{\mathbb{C}^n}\left(\int_{\mathbb{C}^n}\left|f(w)-\tilde f(z)\right||k_z(w)|^2e^{-2\varphi(w)}dV(w)\right)^s dV(z)\\ \lesssim\int_{\mathbb{C}^n}\int_{\mathbb{C}^n\times\mathbb{C}^n\times[0,1]}\left(|w|^s+|\xi|^s\right)\omega(f)^s(z-w+t(w-\xi))\\ \cdot dV(t,w,\xi)dV(z)=\int_{\mathbb{C}^n\times\mathbb{C}^n\times[0,1]}\left(|w|^s+|\xi|^s\right)\\ \cdot dV(t,w,\xi)\int_{\mathbb{C}^n}\omega(f)^s(z-w+t(w-\xi))dV(z)\lesssim\|f\|^s_{IO^s}.\end{aligned} \tag{30}$$

For $|w-\xi|<1$, the integral in (28) is less than or equal to C times:

$$\left(\int_{\mathbb{C}^n\times\mathbb{C}^n\times[0,1]}\omega(f)^s(z-w+t(w-\xi))dV(t,w,\xi)\right)^{1/s}. \tag{31}$$

Thus, we obtain

$$\begin{aligned}\int_{\mathbb{C}^n}\left(\int_{\mathbb{C}^n}\left|f(w)-\tilde f(z)\right||k_z(w)|^2e^{-2\varphi(w)}dV(w)\right)^s dV(z)\\ \lesssim\int_{\mathbb{C}^n}\int_{\mathbb{C}^n\times\mathbb{C}^n\times[0,1]}\omega(f)^s(z-w+t(w-\xi))dV(t,w,\xi)dV(z)\\ =\int_{\mathbb{C}^n\times\mathbb{C}^n\times[0,1]}dV(t,w,\xi)\int_{\mathbb{C}^n}\omega(f)^s(z-w+t(w-\xi))dV(z)\lesssim\|f\|^s_{IO^s}.\end{aligned} \tag{32}$$

Hence, (3) $\Rightarrow$ (4) holds.

(4) $\Rightarrow$ (1). Suppose f on $\mathbb{C}^n$ with $f|k_z|^2\in L^1_\varphi$ satisfies condition (25), we are to show $f\in \mathrm{IMO}^s$.

Lemma 3 implies that

$$\begin{aligned}MO_r(f)(z)&=\frac{1}{V(B(z,r))}\int_{B(z,r)}\left|f(w)-\hat f_r(z)\right|dV(w)\\ &\le\frac{1}{V(B(z,r))}\int_{B(z,r)}\left|f(w)-\tilde f(z)\right|dV(w)\\ &\quad+\left|\hat f_r(z)-\tilde f(z)\right|\le\frac{2}{V(B(z,r))}\int_{B(z,r)}\left|f(w)-\tilde f(z)\right|\\ &\quad\cdot dV(w)\lesssim\int_{\mathbb{C}^n}\left|f(w)-\tilde f(z)\right||k_z(w)|^2e^{-2\varphi(w)}dV(w),\end{aligned} \tag{33}$$

and this shows that $MO_r(f)(z)\in L^s$; hence, $f\in\mathrm{IMO}^s_r$, which gives (4) $\Rightarrow$ (1). The proof ends here. □ □

Remark 5. It follows from Proposition 4 that the spaces IMO^s_r are independent of r. Although the equivalences of (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) hold for all $0<s\le\infty$, it is unknown whether they are equivalent to (4) in the case of $0<s\le 1$. We tried to prove it, but there were two difficulties. Firstly, the techniques of Hölder's inequality or Jensen's inequality are not available when $0<s\le1$. Secondly, the function $\omega(f)$ in Proposition 4 is not subharmonic, which makes the method of lattice invalid in this case.

Corollary 6. *Let $1<s\le\infty$. If $f\in IMO^s$, then $|\widetilde{f|}(z)-|\tilde f(z)|\in L^s$ for all $z\in\mathbb{C}^n$.*

Proof. For $z\in\mathbb{C}^n$, by the triangle inequality, we get

$$\begin{aligned}|\tilde f|(z)-\left|\tilde f(z)\right|&=\int_{\mathbb{C}^n}\left(|f(w)|-\left|\tilde f(z)\right|\right)|k_z(w)|^2e^{-2\varphi(w)}dV(w)\\ &\le\int_{\mathbb{C}^n}\left|f(w)-\tilde f(z)\right||k_z(w)|^2e^{-2\varphi(w)}dV(w).\end{aligned} \tag{34}$$

Taking into account Proposition 4, we obtain

$$\int_{\mathbb{C}^n}\left|f(w)-\tilde f(z)\right||k_z(w)|^2e^{-2\varphi(w)}dV(w)\in L^s, \tag{35}$$

thus, we have $|\widetilde{f|}(z)-|\tilde f(z)|\in L^s$ for all $z\in\mathbb{C}^n$. This completes the proof. □ □

Lemma 7 (see [1], Lemma 3.2). *Let $1\le p<\infty$. A subset $E\subset F^p_\varphi$ is relatively compact if and only if for any $\varepsilon>0$, there exists some $R>0$ such that*

$$\sup_{f\in E}\int_{|z|\ge R}\left|f(z)e^{-\varphi(z)}\right|^p dV(z)<\varepsilon. \tag{36}$$

Lemma 8 (see [4], Proposition 2.3). *Suppose $1\le p<\infty$ and $r>0$. Then, for $f\in H(\mathbb{C}^n)$ and $z\in\mathbb{C}^n$, we have*

$$\left|f(z)e^{-\varphi(z)}\right|^p\lesssim\frac{1}{r^{2n}}\int_{B(z,r)}\left|f(w)e^{-\varphi(w)}\right|^p dV(w). \tag{37}$$

Lemma 9. *Suppose $f\in IMO^s$ with $1<s\le\infty$. Then, for any $g\in H(\mathbb{C}^n)$, there exists some constant $C>0$ such that*

$$\int_{\mathbb{C}^n}\left|g(z)e^{-\varphi(z)}\right|^p|f(z)|dV(z)\le C\int_{\mathbb{C}^n}\left|g(z)e^{-\varphi(z)}\right|^p|\hat f|_r(z)dV(z), \tag{38}$$

for $r>0$ and $1\le p<\infty$.

Proof. By Lemma 8 and Fubini's theorem, we have

$$\int_{\mathbb{C}^n}\left|g(z)e^{-\varphi(z)}\right|^p|f(z)|dV(z) \lesssim \frac{1}{r^{2n}}\int_{\mathbb{C}^n}\int_{B(z,r)}\left|g(w)e^{-\varphi(w)}\right|^p dV(w)|f(z)|dV(z) = \int_{\mathbb{C}^n}\left|g(w)e^{-\varphi(w)}\right|^p\frac{1}{r^{2n}}\int_{B(w,r)}|f(z)|dV \cdot (z)dV(w) \lesssim \int_{\mathbb{C}^n}\left|g(w)e^{-\varphi(w)}\right|^p\widehat{|f|}_r(w)dV(w), \tag{39}$$

which completes the proof. □ □

Lemma 10. *Let $1 \le p \le q < \infty$. Suppose E is relatively compact in F^p_φ, then E is relatively compact in F^q_φ.*

Proof. Suppose E is relatively compact in F^p_φ. Given any $\varepsilon > 0$, by Lemma 7, there exists some $R > 0$ such that

$$\sup_{f\in E}\int_{|z|\ge R}\left|f(z)e^{-\varphi(z)}\right|^p dV(z) < \varepsilon. \tag{40}$$

Lemma 8 gives

$$|f(z)| \lesssim \|f\|_{p,\varphi}e^{\varphi(z)}, \quad \text{for } f \in E. \tag{41}$$

Thus,

$$\int_{|z|\ge R}\left|f(z)e^{-\varphi(z)}\right|^q dV(z) = \int_{|z|\ge R}\left||f(z)|^p f(z)\right|^{q-p} e^{-q\varphi(z)}dV(z) \le \|f\|^{q-p}_{p,\varphi}\int_{|z|\ge R}\left|f(z)e^{-\varphi(z)}\right|^p dV(z) < \|f\|^{q-p}_{p,\varphi}\varepsilon, \tag{42}$$

for $f \in E$. Note that by the proof of Lemma 2.7 in [1], the fact that E is relatively compact in F^p_φ implies that E is bounded in F^p_φ. It follows that there is a constant $C > 0$ such that

$$\sup_{f\in E}\int_{|z|\ge R}\left|f(z)e^{-\varphi(z)}\right|^q dV(z) < C\varepsilon. \tag{43}$$

An application of Lemma 7 implies that E is relatively compact in F^q_φ, and the lemma is proved. □ □

3. Toeplitz Operators with IMO^s Symbols

In this section, we will derive necessary and sufficient conditions for which Toeplitz operators T_f with IMO^s symbols are bounded or compact from F^p_φ to F^q_φ, respectively, in four cases, where $1 < s \le \infty$.

Case 1. $1 < p \le q < \infty$.

The following theorems are the main results in the case of $1 < p \le q < \infty$.

Theorem 11. *Let $1 < p \le q < \infty$ and $f \in IMO^\infty$. Then, $T_f : F^p_\varphi \longrightarrow F^q_\varphi$ is bounded if and only if $\tilde{f} \in L^\infty$.*

Proof. Assume that $\tilde{f} \in L^\infty$, then T_f is bounded on F^p_φ by Theorem 3.2 in [1]. That is, for any function $g \in F^p_\varphi$, we have $T_f g \in F^p_\varphi$ with

$$\|T_f g\|_{p,\varphi} \lesssim \|g\|_{p,\varphi}. \tag{44}$$

Observe that $F^p_\varphi \subset F^q_\varphi$ when $1 < p \le q < \infty$. Thus,

$$\|T_f g\|_{q,\varphi} \lesssim \|T_f g\|_{p,\varphi} \lesssim \|g\|_{p,\varphi}. \tag{45}$$

This implies that $T_f : F^p_\varphi \longrightarrow F^q_\varphi$ is bounded.

Conversely, if $T_f : F^p_\varphi \longrightarrow F^q_\varphi$ is bounded for $1 < p \le q < \infty$. Note that $\|k_z\|_{p,\varphi} \simeq 1$ by estimate (19). Applying Hölder's inequality, we get

$$|\tilde{f}(z)| = \left|\left\langle T_f k_z, k_z\right\rangle\right| \le \|T_f k_z\|_{q,\varphi}\|k_z\|_{q',\varphi} \lesssim \|T_f\|_{F^p_\varphi\longrightarrow F^q_\varphi}\|k_z\|_{p,\varphi} \simeq \|T_f\|_{F^p_\varphi\longrightarrow F^q_\varphi}. \tag{46}$$

It follows that $\tilde{f} \in L^\infty$.

This completes the proof. □ □

Furthermore, we equivalently characterize the compact Toeplitz operators T_f with $f \in \mathrm{IMO}^\infty$.

Theorem 12. *Let $1 < p \le q < \infty$ and $f \in IMO^\infty$. Then, $T_f : F^p_\varphi \longrightarrow F^q_\varphi$ is compact if and only if*

$$\lim_{z\longrightarrow\infty}\sup_{w\in B(z,r)}\left|\left\langle T_f k_z, k_w\right\rangle_{F^2_\varphi}\right| = 0, \text{for any } r > 0. \tag{47}$$

Proof. For $1 < p \le q < \infty$, since $k_z \longrightarrow 0$ weakly in F^p_φ as $z \longrightarrow \infty$ and $T_f : F^p_\varphi \longrightarrow F^q_\varphi$ is compact by assumption, we deduce that

$$\lim_{z\longrightarrow\infty}\left\|T_f k_z\right\|_{q,\varphi} = 0. \tag{48}$$

Thus, by Hölder's inequality, we have

$$\sup_{w\in B(z,r)}\left|\left\langle T_f k_z, k_w\right\rangle_{F^2_\varphi}\right| \le \sup_{w\in B(z,r)}\|T_f k_z\|_{q,\varphi}\|k_w\|_{q',\varphi} \lesssim \|T_f k_z\|_{q,\varphi} \longrightarrow 0, \tag{49}$$

for any $r > 0$ as $z \longrightarrow \infty$.

Conversely, suppose

$$\lim_{z\longrightarrow\infty}\sup_{w\in B(z,r)}\left|\left\langle T_f k_z, k_w\right\rangle_{F^2_\varphi}\right| = 0, \tag{50}$$

for any $r > 0$, then by Theorem 3.2 in [1], we see that T_f is compact on F^p_φ, namely, T_f maps any bounded set in F^p_φ to a relatively compact set in F^p_φ. Lemma 10 implies that a relatively compact set in F^p_φ is also relatively compact in F^q_φ for $1 < p \le q < \infty$. Therefore, T_f maps any bounded set in F^p_φ to a relatively compact set in F^q_φ. This shows T_f is compact from F^p_φ to F^q_φ.

This completes the proof. ☐ ☐

Case 2. $1 < q < p < \infty$.

Before we show the main result in the case of $1 < q < p < \infty$, we need to introduce some notations firstly. Recall that the Rademacher functions are defined by

$$r_0(t) = \begin{cases} 1, & 0 \le t - [t] < \dfrac{1}{2}, \\ -1, & \dfrac{1}{2} \le t - [t] < 1, \end{cases} \quad (51)$$
$$r_n(t) = r_0(2^n t), \quad n \ge 1,$$

where $[t]$ denotes the largest integer less than or equal to t.

The Khinchine's inequality [15] is given as follows.

For $0 < p < \infty$, there exist positive constants C_1 and C_2 depending only on p such that, for all natural numbers m and all complex numbers $b_1, b_2, \cdots, b_m$,

$$C_1\left(\sum_{j=1}^{m}\left|b_j\right|^2\right)^{p/2} \le \int_0^1\left|\sum_{j=1}^{m} b_j r_j(t)\right|^p dt \le C_2\left(\sum_{j=1}^{m}\left|b_j\right|^2\right)^{p/2}. \quad (52)$$

Given some $r > 0$, a sequence $\{a_j\}_{j=1}^{\infty}$ in $\mathbb{C}^n$ is called an r-lattice if the balls $\{B(a_j, r)\}_{j=1}^{\infty}$ cover $\mathbb{C}^n$ and $\{B(a_j, (r/4))\}_{j=1}^{\infty}$ are pairwise disjoint. For any $\delta > 0$, it is elementary to prove that there exists some positive integer N (depending only on r and δ) such that each $z \in \mathbb{C}^n$ belongs to at most N balls of $\{B(a_j, \delta)\}_{j=1}^{\infty}$. Moreover, given $r > 0$, it is easy to choose a_j such that $\{a_j\}_{j=1}^{\infty}$ forms an r-lattice.

Theorem 13. *Let $1 < q < p < \infty$ and set $s = pq/(p-q)$. Then, for $f \in IMO^s$, the following statements are equivalent:*

(1) $T_f : F^p_\varphi \longrightarrow F^q_\varphi$ is bounded.

(2) $T_f : F^p_\varphi \longrightarrow F^q_\varphi$ is compact.

(3) $\tilde{f} \in L^s$.

Proof. The implication (2) ⇒ (1) is trivial, it suffices to prove the implications (1) ⇒ (3) and (3) ⇒ (2).

(1) ⇒ (3). Given any $r > 0$, let $\{a_j\}_{j=1}^{\infty}$ be an r-lattice. For $z \in \mathbb{C}^n$ and $1 < p < \infty$, let $\{\lambda_j\}_{j=1}^{\infty} \in l^p$, by Lemma 2, we see that the function

$$g_t(z) = \sum_{j=1}^{\infty} \lambda_j r_j(t) k_{a_j}(z) \quad (53)$$

belongs to F^p_φ and $\|g_t\|_{p,\varphi} \lesssim \|\{\lambda_j\}_{j=1}^{\infty}\|_{l^p}$. By assumption, $T_f : F^p_\varphi \longrightarrow F^q_\varphi$ is bounded; it follows that

$$\left\|T_f g_t\right\|_{q,\varphi} \le \left\|T_f\right\|_{F^p_\varphi \longrightarrow F^q_\varphi} \|g_t\|_{p,\varphi} \lesssim \left\|T_f\right\|_{F^p_\varphi \longrightarrow F^q_\varphi}\left\|\{\lambda_j\}_{j=1}^{\infty}\right\|_{l^p}. \quad (54)$$

Applying Fubini's theorem, Khinchine's inequality, and Lemma 8, we obtain

$$\begin{aligned}
\int_0^1 \left\|T_f g_t\right\|_{q,\varphi}^q dt &= \int_0^1\int_{\mathbb{C}^n}\left|\sum_{j=1}^{\infty}\lambda_j r_j(t) T_f k_{a_j}(z)\right|^q e^{-q\varphi(z)} dV(z)\,dt \\
&= \int_{\mathbb{C}^n}\int_0^1\left|\sum_{j=1}^{\infty}\lambda_j r_j(t) T_f k_{a_j}(z)\right|^q dt\, e^{-q\varphi(z)} dV(z) \\
&\gtrsim \int_{\mathbb{C}^n}\left(\sum_{j=1}^{\infty}\left|\lambda_j\right|^2\left|T_f k_{a_j}(z)\right|^2\right)^{q/2} e^{-q\varphi(z)} dV(z) \\
&\gtrsim \sum_{i=1}^{\infty}\int_{B(a_i,r)}\left(\sum_{j=1}^{\infty}\left|\lambda_j\right|^2\left|T_f k_{a_j}(z)\right|^2\right)^{q/2} e^{-q\varphi(z)} dV(z) \\
&\ge \sum_{i=1}^{\infty}|\lambda_i|^q\int_{B(a_i,r)}\left|T_f k_{a_i}(z)\right|^q e^{-q\varphi(z)} dV(z) \\
&\gtrsim \sum_{i=1}^{\infty}|\lambda_i|^q\left|T_f k_{a_i}(a_i)\right|^q e^{-q\varphi(a_i)}.
\end{aligned} \quad (55)$$

By the reproducing property and estimate (19), we have

$$\begin{aligned}
\left|T_f k_{a_i}(a_i)\right|^q e^{-q\varphi(a_i)} &= \left|\left\langle T_f k_{a_i}, K_{a_i}\right\rangle_{F^2_\varphi}\right|^q e^{-q\varphi(a_i)} \\
&= \left|\left\langle T_f k_{a_i}, k_{a_i}\right\rangle_{F^2_\varphi}\right|^q e^{-q\varphi(a_i)}\|K_{a_i}\|_{p,\varphi}^q \simeq \left|\tilde{f}(a_i)\right|^q.
\end{aligned} \quad (56)$$

This together with (54) and (55) gives

$$\begin{aligned}
\sum_{i=1}^{\infty}|\lambda_i|^q\left|\tilde{f}(a_i)\right|^q &\lesssim \left\|T_f\right\|_{F^p_\varphi \longrightarrow F^q_\varphi}^q\left\|\{\lambda_j\}_{j=1}^{\infty}\right\|_{l^p}^q \\
&= \left\|T_f\right\|_{F^p_\varphi \longrightarrow F^q_\varphi}^q\left\|\{|\lambda_j|^q\}_{j=1}^{\infty}\right\|_{l^{p/q}}.
\end{aligned} \quad (57)$$

Since $p/q > 1$, the classical duality $(l^{p/q})^* \simeq l^{p/(p-q)}$ now implies

$$\sum_{i=1}^{\infty}\left|\tilde{f}(a_i)\right|^s \lesssim \left\|T_f\right\|_{F^p_\varphi \longrightarrow F^q_\varphi}^s. \quad (58)$$

Choose a point $w_i \in B(\bar{a_i}, r)$ such that

$$\left|\tilde{f}(w_i)\right| = \sup_{z\in B(a_i,r)} \left|\tilde{f}(z)\right|. \tag{59}$$

Then, we obtain

$$\int_{\mathbb{C}^n} \left|\tilde{f}(z)\right|^s dV(z) \le \sum_{i=1}^{\infty} \int_{B(a_i,r)} \left|\tilde{f}(z)\right|^s dV(z) \lesssim \sum_{i=1}^{\infty} \left|\tilde{f}(w_i)\right|^s. \tag{60}$$

Note that the set $\{w_i\}_i$ is a finite union of δ-lattices and (58) holds for any r-lattice, this along with (58) implies that $\tilde{f} \in L^s$.

$(3) \Rightarrow (2)$. Assume that $\tilde{f} \in L^s$. Since $s = pq/(p-q) > 1$, by Corollary 6, we have $\widetilde{|f|} \in L^s$. Notice that

$$0 \le (\operatorname{Re} f)^+, (\operatorname{Re} f)^-, (\operatorname{Im} f)^+, (\operatorname{Im} f)^- \le |f|, \tag{61}$$

we get $\widetilde{(\operatorname{Re} f)}^+$, $\widetilde{(\operatorname{Re} f)}^-$, $\widetilde{(\operatorname{Im} f)}^+$, and $\widetilde{(\operatorname{Im} f)}^-$ are all in L^s. By Theorem 3.6 in [1], Toeplitz operators induced by these non-negative symbols $(\operatorname{Re} f)^+$, $(\operatorname{Re} f)^-$, $(\operatorname{Im} f)^+$, and $(\operatorname{Im} f)^-$ are all compact from F^p_φ to F^q_φ. Consequently, T_f is compact from F^p_φ to F^q_φ, since

$$f = \operatorname{Re} f + i \operatorname{Im} f = \left[(\operatorname{Re} f)^+ - (\operatorname{Re} f)^-\right] + i\left[(\operatorname{Im} f)^+ - (\operatorname{Im} f)^-\right]. \tag{62}$$

This completes the proof. □ □

Case 3. $1 < p < \infty$, $q = \infty$.

We now state the main result in the case of $1 < p < \infty$, $q = \infty$.

Theorem 14. *Let $1 < p < \infty$ and $f \in IMO^\infty$. Then, the following statements are equivalent:*

(1) $T_f : F^p_\varphi \longrightarrow F^\infty_\varphi$ is bounded.

(2) $T_f : F^p_\varphi \longrightarrow F^\infty_\varphi$ is compact.

(3) $\tilde{f} \in L^\infty$.

Proof. The implication $(2) \Rightarrow (1)$ is trivial; we are to prove the implications $(1) \Rightarrow (3)$ and $(3) \Rightarrow (2)$.

$(1) \Rightarrow (3)$. Assume that $T_f : F^p_\varphi \longrightarrow F^\infty_\varphi$ is bounded. For any $z \in \mathbb{C}^n$, since $\|k_z\|_{p,\varphi} \simeq 1$ by the estimate (19), we have

$$\begin{aligned}\left|\tilde{f}(z)\right| &= \left|\left\langle T_f k_z, k_z\right\rangle_{F^2_\varphi}\right| \le \int_{\mathbb{C}^n} \left|T_f k_z(w)\right| |k_z(w)| e^{-2\varphi(w)} dV(w) \\ &\le \|T_f k_z\|_{\infty,\varphi} \int_{\mathbb{C}^n} |k_z(w)| e^{-\varphi(w)} dV(w) \\ &\lesssim \|T_f\|_{F^p_\varphi \longrightarrow F^\infty_\varphi} \|k_z\|_{p,\varphi} \|k_z\|_{1,\varphi} \simeq \|T_f\|_{F^p_\varphi \longrightarrow F^\infty_\varphi}.\end{aligned} \tag{63}$$

Therefore, $\tilde{f} \in L^\infty$.

$(3) \Rightarrow (2)$. Suppose $\tilde{f} \in L^\infty$, we have $\widetilde{|f|} \in L^\infty$ by Corollary 6, equivalently, $\widehat{|f|}_r \in L^\infty$ by (14). Assume that the sequence $g_j \longrightarrow 0$ weakly in F^p_φ as $j \longrightarrow \infty$, we just need to show that

$$\lim_{j\longrightarrow\infty} \left\|T_f g_j\right\|_{\infty,\varphi} \longrightarrow 0, \tag{64}$$

since F^p_φ is reflexive for $1 < p < \infty$.

For any $\varepsilon > 0$, there exists an $R > 0$ large enough such that

$$\left(\int_{|w|>R} |k_z(w)|^{p'} e^{-p'\varphi(w)} dV(w)\right)^{1/p'} < \varepsilon, \tag{65}$$

since $k_z \in F^{p'}_\varphi$.

In views of Proposition 4.1 in [4], our assumption on $\{g_j\}_j$ implies that $\|g_j\|_{p,\varphi} \lesssim 1$ and $g_j \longrightarrow 0$ uniformly on compact subsets of $\mathbb{C}^n$ as $j \longrightarrow \infty$. This shows

$$\int_{|w|\le R} \left|g_j(w)\right| |k_z(w)| e^{-2\varphi(w)} dV(w) < \varepsilon, \tag{66}$$

for arbitrary $\varepsilon > 0$ and fixed R above.

Combing the estimate (19), Lemma 9 (to the weight 2φ), Hölder's inequality, (65), and (66), we get

$$\begin{aligned}|T_f g_j(z)| e^{-\varphi(z)} &\le \int_{\mathbb{C}^n} |f(w)| \left|g_j(w)\right| |K(z,w)| e^{-2\varphi(w)} dV(w) e^{-\varphi(z)} \simeq \int_{\mathbb{C}^n} \\ &\quad \cdot |f(w)| \left|g_j(w)\right| |k_z(w)| e^{-2\varphi(w)} dV(w) \\ &\lesssim \int_{\mathbb{C}^n} \left|g_j(w)\right| |k_z(w)| e^{-2\varphi(w)} \widehat{|f|}_r(w) dV(w) \\ &\lesssim \left\|\widehat{|f|}_r\right\|_{L^\infty} \int_{\mathbb{C}^n} \left|g_j(w)\right| |k_z(w)| e^{-2\varphi(w)} dV(w) \\ &\lesssim \int_{|w|\le R} |g_j(w)| |k_z(w)| e^{-2\varphi(w)} dV(w) + \int_{|w|>R} \\ &\quad \cdot \left|g_j(w)\right| |k_z(w)| e^{-2\varphi(w)} dV(w) \\ &\lesssim \varepsilon + \|g_j\|_{p,\varphi} \left(\int_{|w|>R} |k_z(w)|^{p'} e^{-p'\varphi(w)} dV(w)\right)^{1/p'} \\ &\lesssim \varepsilon\end{aligned} \tag{67}$$

as $j \longrightarrow \infty$, which yields (64). Hence, $T_f : F^p_\varphi \longrightarrow F^\infty_\varphi$ is compact.

This finishes the proof. □ □

Remark 15. It is interesting and surprising that the boundedness and compactness of T_f from F^p_φ to F^∞_φ are equivalent when $1 < p < \infty$, which is quite different from that in the case of $1 < p \le q < \infty$.

We will end this section by proving the last case.

Case 4. $p = \infty$, $1 < q < \infty$.

The following theorem is the main result in the case of $p = \infty$, $1 < q < \infty$.

Theorem 16. *Let $1 < q < \infty$ and $f \in IMO^q$. Then, the following statements are equivalent:*

(1) $T_f : F^\infty_\varphi \longrightarrow F^q_\varphi$ is bounded.

(2) $T_f : F^\infty_\varphi \longrightarrow F^q_\varphi$ is compact.

(3) $\tilde{f} \in L^q$.

Proof. It is trivial that $(2) \Rightarrow (1)$, it remains to prove the implications $(1) \Rightarrow (3)$ and $(3) \Rightarrow (2)$. The proof is similar to that of Theorem 13, but we include it here for the sake of completeness.

$(1) \Rightarrow (3)$. Let $\{\lambda_j\}_{j=1}^\infty$ be any bounded sequence on $\mathbb{C}^n$ and $\{a_j\}_{j=1}^\infty$ be an r-lattice. For $z \in \mathbb{C}^n$, set

$$g_t(z) = \sum_{j=1}^\infty r_j(t)\lambda_j k_{a_j}(z). \tag{68}$$

By Lemma 2, we have $g_t \in F^\infty_\varphi$ with

$$\|g_t\|_{\infty,\varphi} \lesssim \left\| \{\lambda_j\}_{j=1}^\infty \right\|_{l^\infty}. \tag{69}$$

Since $T_f : F^\infty_\varphi \longrightarrow F^q_\varphi$ is bounded by hypothesis, we have $T_f g_t \in F^q_\varphi$. Applying Khinchine's inequality and Fubini's theorem, we deduce that

$$\begin{aligned}
&\int_{\mathbb{C}^n} \left(\sum_{j=1}^\infty \left|\lambda_j T_f k_{a_j}(z)\right|^2 \right)^{q/2} e^{-q\varphi(z)} dV(z) \\
&\lesssim \int_{\mathbb{C}^n} \int_0^1 \left| \sum_{j=1}^\infty r_j(t)\lambda_j T_f k_{a_j}(z) \right|^q dt e^{-q\varphi(z)} dV(z) \\
&= \int_0^1 \left\| T_f \left(\sum_{j=1}^\infty r_j(t)\lambda_j k_{a_j} \right) \right\|_{q,\varphi}^q dt \\
&\lesssim \int_0^1 \|T_f\|^q_{F^\infty_\varphi \longrightarrow F^q_\varphi} \|g_t\|^q_{\infty,\varphi} dt \lesssim \|T_f\|^q_{F^\infty_\varphi \longrightarrow F^q_\varphi} \|\{\lambda_j\}_{j=1}^\infty\|^q_{l^\infty}.
\end{aligned} \tag{70}$$

By Lemma 8 and the estimate (56), we get

$$\begin{aligned}
&\int_{\mathbb{C}^n} \left(\sum_{j=1}^\infty \left|\lambda_j T_f k_{a_j}(z)\right|^2 \right)^{q/2} e^{-q\varphi(z)} dV(z) \\
&\cdot \left\| \sum_{i=1}^\infty \int_{B(a_i,r)} \left(\sum_{j=1}^\infty \left|\lambda_j T_f k_{a_j}(z)\right|^2 \right)^{q/2} e^{-q\varphi(z)} dV(z) \right. \\
&\gtrsim \sum_{i=1}^\infty \int_{B(a_i,r)} |\lambda_i T_f k_{a_i}(z)|^q e^{-q\varphi(z)} dV(z) \| \sum_{i=1}^\infty |\lambda_i|^q \left|T_f k_{a_i}(a_i)\right|^q e^{-q\varphi(a_i)} \\
&\simeq \sum_{i=1}^\infty |\lambda_i|^q \left|\tilde{f}(a_i)\right|^q.
\end{aligned} \tag{71}$$

This along with the estimate (70) yields

$$\sum_{i=1}^\infty |\lambda_i|^q \left|\tilde{f}(a_i)\right|^q \lesssim \|T_f\|^q_{F^\infty_\varphi \longrightarrow F^q_\varphi} \|\{\lambda_i\}_{i=1}^\infty\|^q_{l^\infty}. \tag{72}$$

Take $b_i = |\lambda_i|^q$ for each i. Then, $\{b_i\}_{i=1}^\infty \in l^\infty$ and

$$\sum_{i=1}^\infty b_i \left|\tilde{f}(a_i)\right|^q \lesssim \|T_f\|^q_{F^\infty_\varphi \longrightarrow F^q_\varphi} \|\{b_i\}_{i=1}^\infty\|_{l^\infty}. \tag{73}$$

It follows that

$$\sum_{i=1}^\infty \left|\tilde{f}(a_i)\right|^q \lesssim \|T_f\|^q_{F^\infty_\varphi \longrightarrow F^q_\varphi}. \tag{74}$$

Choose one point $w_i \in \overline{B(a_i, r)}$ such that

$$\left|\tilde{f}(w_i)\right| = \sup_{z \in B(a_i,r)} \left|\tilde{f}(z)\right|. \tag{75}$$

Then, we obtain

$$\int_{\mathbb{C}^n} \left|\tilde{f}(z)\right|^q dV(z) \le \sum_{i=1}^\infty \int_{B(a_i,r)} \left|\tilde{f}(z)\right|^q dV(z) \lesssim \sum_{i=1}^\infty \left|\tilde{f}(w_i)\right|^q. \tag{76}$$

Note that the set $\{w_i\}_i$ is finite union of some δ-lattices and (74) holds for any r-lattice, this along with (74) implies that $\tilde{f} \in L^q$.

$(3) \Rightarrow (2)$. Assume that $\tilde{f} \in L^q$, by Corollary 6, we have $\widetilde{|f|} \in L^q$. Since

$$0 \le (\operatorname{Re} f)^+, (\operatorname{Re} f)^-, (\operatorname{Im} f)^+, (\operatorname{Im} f)^- \le |f|, \tag{77}$$

we see that $\widetilde{(\operatorname{Re} f)^+}$, $\widetilde{(\operatorname{Re} f)^-}$, $\widetilde{(\operatorname{Im} f)^+}$, and $\widetilde{(\operatorname{Im} f)^-}$ are all in L^q. By Theorem 2.5 in [3], Toeplitz operators induced by these nonnegative symbols $(\operatorname{Re} f)^+$, $(\operatorname{Re} f)^-$, $(\operatorname{Im} f)^+$, and $(\operatorname{Im} f)^-$ are all compact from F^∞_φ to F^q_φ. Notice that

$$f = \operatorname{Re} f + i \operatorname{Im} f = \left[(\operatorname{Re} f)^{+} - (\operatorname{Re} f)^{-}\right] + i\left[(\operatorname{Im} f)^{+} - (\operatorname{Im} f)^{-}\right], \tag{78}$$

we conclude that T_f is compact from F_{φ}^{∞} to F_{φ}^{q}.

This finishes the proof. □ □

Data Availability

No data were used.

Conflicts of Interest

The authors declare that they have no competing interests.

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