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Research Article

Approximation of Mixed Euler-Lagrange σ-Cubic-Quartic Functional Equation in Felbin's Type f-NLS

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In this research paper, the authors present a new mixed Euler-Lagrange σ-cubic-quartic functional equation. For this introduced mixed type functional equation, the authors obtain general solution and investigate the various stabilities related to the Ulam problem in Felbin's type of fuzzy normed linear space (f-NLS) with suitable counterexamples. This approach leads us to approximate the Euler-Lagrange σ-cubic-quartic functional equation with better estimation.

1. Introduction

One of the famous questions concerning the stability of homomorphisms was raised by Ulam [1] in 1940. The author Hyers [2] provided a partial answer to Ulam's question in 1941, and then, a generalized solution to Ulam's question was given by Rassias [3] in 1978, which is called Hyers-Ulam-Rassias stability or generalized Hyers-Ulam stability. The generalization of Hyers stability result by Rassias [4] is called Ulam-Gavruta-Rassias stability. Later, Ravi et al. [5] investigated the stability using mixed powers of norms which is called Rassias stability.

Definition 1 (see [6]). A fuzzy subset ξ on $\mathbb{R}$ *is* said to be a fuzzy real number when it satisfies two axioms:

(N_1)There exists $\tau_0 \in \mathbb{R}$ such that $\xi(\tau_0) = 1$

(N_2)For each $\gamma \in (0, 1]$, $[\xi]_\gamma = [\xi_\gamma^-, \xi_\gamma^+]$, where $-\infty < \xi_\gamma^- \leq \xi_\gamma^+ < +\infty$

Note that $[\xi]_\gamma = \{\tau : \xi(\tau) \geq \gamma\}$ is γ-level set. We show the set of all fuzzy real numbers by $\Lambda(\mathbb{R})$. Also, ξ is said to be a nonnegative fuzzy real number when $\xi \in \Lambda(\mathbb{R})$ and $\xi(\tau) = 0$ for $\tau < 0$. We show the set of all nonnegative fuzzy real numbers by $\Lambda^*(\mathbb{R})$.

We define $\bar{0}$ as

$$\bar{0}(\tau) = \begin{cases} 1, & \tau = 0, \\ 0, & \tau \neq 0. \end{cases} \tag{1}$$

Definition 2 (see [6]). We define $\oplus, \ominus, \otimes, \oslash$ *on* $\Lambda(\mathbb{R}) \times \Lambda(\mathbb{R})$ *as*

(i) $(\xi \otimes \Xi)(\tau) = \sup_{q \in \mathbb{R}} \{\xi(q) \wedge \Xi(\tau - q)\}, \tau \in \mathbb{R}$

(ii) $(\xi \ominus \Xi)(\tau) = \sup_{q \in \mathbb{R}} \{\xi(q) \wedge \Xi(q - \tau)\}, \tau \in \mathbb{R}$

(iii) $(\xi \otimes \Xi)(\tau) = \sup_{q \in \mathbb{R}, q \neq 0} \{\xi(s) \wedge \Xi(\tau / q)\}, \tau \in \mathbb{R}$

(iv) $(\xi \oslash \Xi)(\tau) = \sup_{q \in \mathbb{R}} \{\xi(q\tau) \wedge \Xi(q)\}, \tau \in \mathbb{R}$

$\bar{0}, \bar{1} \in \Lambda(\mathbb{R})$ are additive and multiplicative identities, respectively. We also define $\ominus \xi$ as $\bar{0} - \xi$; so, $\xi \ominus \Xi = \xi \oplus (\ominus \xi)$.

Definition 3 (see [6]). For $l \in \mathbb{R}/0$, the notation $l \odot \xi$ shows fuzzy scalar multiplication and defied as $(l \odot \xi)(\tau) = \xi(\tau/l)$ *and* $0 \odot \xi = \bar{0}$.

Definition 4 (see [7]). Consider the vector space S and the left and right norms $L, R : [0,1]^2 \longrightarrow [0,1]$ which are symmetric and nondecreasing functions satisfying $L(0,0) = 0$, $R(1,1) = 1$. *So,* $\|\cdot\| : S \longrightarrow \Lambda^*(R)$ *is said to be a fuzzy norm and* $(S, \|\cdot\|, L, R)$ *is a fuzzy normed linear space (in short f-NLS) if*

$(N1)\|s\| = \bar{0}$ if and only if $s = 0$

$(N2)\|\lambda s\| = |\bar{\lambda}| \odot \|s\|$ for all $s \in S$ and $\lambda \in (-\infty, \infty)$

$(N3)$for all $t, s \in S$:

$(N3L)$if $q \le \|s\|_1^-$, $\tau \le \|t\|_1^-$ and $q + \tau \le \|t + s\|_1^-$, then $\|s + t\|(q + \tau) \ge L(\|s\|(q), \|t\|(\tau))$

$(N3R)$if $q \ge \|s\|_1^-$, $\tau \ge \|t\|_1^-$ and $q + \tau \ge \|s + t\|_1^-$, then $\|s + t\|(q + \tau) \le R(\|s\|(q), \|t\|(\tau))$,where $[\|s\|]_\alpha = [\|s\|_\alpha^-, \|s\|_\alpha^+]$ for $s \in S$ and $\alpha \in (0;1]$.

Lemma 5 (see [8]). *Consider f-NLS* $(S, \|\cdot\|, L, R)$, *and let*

(R1)$R(c,d) \le \max(c,d)$

(R2)$\forall \gamma \in (0,1], \exists \zeta (0,\gamma]$ *in which* $R(\zeta, h) \le \gamma$ *for every* $h \in (0,\gamma)$

(R3)$\lim_{c \to 0^+} R(c,c) = 0$

So, (R1) $\Rightarrow$ *(R2)* $\Rightarrow$ *(R3). The converse is not true.*

Lemma 6 (see [8]). *Consider f-NLS* $(S, \|\cdot\|, L, R)$. *Then,*

(1) if $R(c,d) \le \max(c,d)$, *then for all* $\gamma \in (0,1]$, $\|s + t\|_\gamma^+ \le \|s\|_\gamma^+ + \|t\|_\gamma^+$ *for all* $s, t \in S$

(2) (R2) implies that, for every $\gamma \in (0,1]$, *there exists* $\zeta \in (0,\gamma]$ *such that* $\|s + t\|_\gamma^+ \le \|s\|_\zeta^+ + \|t\|_\gamma^+$ *for every* $s, t \in S$

(3) $\lim_{c \to 0^+} R(c,c) = 0$, *implies that for every* $\gamma \in (0,1]$, *there exists* $\zeta \in (0,\gamma]$ *such that* $\|s + t\|_\gamma^+ \le \|s\|_\zeta^+ + \|t\|_\zeta^+$ *for every* $s, t \in S$

Lemma 7. *Consider f-NLS* $(S, \|\cdot\|, L, R)$ *and let*

(L1)$L(c,d) \ge \min(c,d)$

(L2)$\forall \gamma \in (0,1], \exists \zeta (\gamma, 1]$ *such that* $L(\zeta, \eta) \ge \gamma$ *for all* $\eta \in (\gamma, 1)$

(L3)$\lim_{d \to 1^-} L(d,d) = 1$

So, (L1) $\Rightarrow$ *(L2)* $\Rightarrow$ *(L3), but not conversely.*

Lemma 8. *Consider f-NLS* $(S, \|\cdot\|, L, R)$, *then*

(1) $L(c,d) \ge \min(c,d)$, *implying that* $\forall \gamma \in (0,1]$, $\|s + t\|_\alpha^- \le \|s\|_\gamma^- + \|t\|_\gamma^-$ *for every* $s, t \in S$

(2) (L2) implies that for every $\gamma \in (0,1]$, *there exists* $\zeta \in (\gamma, 1]$ *such that* $\|s + t\|_\gamma^- \le \|s\|_\zeta^- + \|t\|_\gamma^-$ *for every* $s, t \in S$

(3) $\lim_{c \to 1^-} L(c,c) = 1$, *implying that for every* $\gamma \in (0,1]$, *there exists* $\zeta \in (\gamma, 1]$ *such that* $\|s + t\|_\gamma^- \le \|s\|_\zeta^- + \|t\|_\zeta^-$ *for every* $s, t \in S$

Lemma 9 (see [7]). *Consider f-NLS* $(S, \|\cdot\|, L, R)$. *Then,*

(1) $R(c,d) \ge \max(c,d)$ *and for all* $\gamma \in (0,1]$, *implying that* $\|s + t\|_\gamma^+ \le \|s\|_\gamma^+ + \|t\|_\gamma^+$ *for all* $s, t \in S$, *then (N3R)*

(2) $L(c,d) \le \min(c,d)$ *and for all* $\gamma \in (0,1]$, *implying that* $\|s + t\|_\gamma^- \le \|s\|_\gamma^- + \|t\|_\gamma^-$ *for all* $s, t \in S$, *so (N3L)*

Definition 10 (see [7]). Consider f-NLS $(S, \|\cdot\|, L, R)$ *and let* $\lim_{c \to 0^+} R(c,c) = 0$. A sequence $\{s_m\}_{m=1}^\infty \subseteq S$ converges to $s \in S$, denoted by $\lim_{m \to \infty} s_m = s$, *if* $\lim_{m \to \infty} \|s_m - s\|_\gamma^+ = 0$ for every $\gamma \in (0,1]$, and is called a Cauchy sequence if $\lim_{m,n \to \infty} \|s_m - s_n\|_\gamma^+ = 0$ for every $\gamma \in (0,1]$. A subset $E \subseteq S$ is said to be complete if every Cauchy sequence in E converges in E. A f-NLS is called a fuzzy Banach space (f-BS) if it is complete.

Lemma 11. *Consider f-NLS* $(S, \|\cdot\|, L, R)$ *which satisfies* $(R2)$. *Then,*

(1) $\|\cdot\|_\gamma^+$ *is continuous from* U *into* $\mathbb{R}$ *at* $s \in S$ *for every* $\gamma \in (0,1]$

(2) For every $m \in \mathbb{Z}^+$ *and* $\{s_j\}_{j=1}^m$, *we have* $\forall \gamma \in (0,1], \exists \zeta \in (0,\gamma]$; $\|\sum_{j=1}^m s_j\|_\gamma^+ \le \sum_{j=1}^m \|s_j\|_\zeta^+$

Recently, the stability problems of several functional equations (FEs) have been extensively investigated by a number of authors [4, 9–20] in Felbin type f-NLS. Our method helps to solve some new problems of stability and approximation of functional equations [21–28] in Felbin type f-NLS.

Motivated from the above historical developments in the field of FEs, the authors introduce a new mixed Euler-Lagrange σ*-cubic-quartic functional equation (FE)*

$$\begin{aligned}
&\pi(t + \sigma s) + \pi(\sigma t + s) + \pi(t - \sigma s) + \pi(s - \sigma t) \\
&\quad = \sigma^2 \{2\pi(t + s) + \pi(t - s) + \pi(s - t)\} \\
&\qquad - 2(\sigma^4 - 1)\{\pi(t) + \pi(s)\} + \frac{1}{4}\sigma^2(\sigma^2 - 1)\{\pi(2t) + \pi(2s)\},
\end{aligned} \tag{2}$$

where $\sigma \in \mathbb{R} - \{0, \pm 1\}$. *For this mixed type FE, authors obtain the general solution and investigate the various stabilities related to Ulam problem [1] in Felbin's type f-NLS with suitable counterexamples.*

2. General Solution of Euler-Lagrange σ-Cubic-Quartic FE

Theorem 12. *Consider* π *satisfies (2) and odd that is* $\pi(-t) = -\pi(t)$, *then a mapping* $\pi : T \longrightarrow S$ *is cubic.*

Proof. Assume π satisfies (2). Putting $t=s=0$ in (2), we get $\pi(0)=0$. Setting (t,s) by $(t,0)$ in (2), we obtain

$$2\pi(t)+\pi(\sigma t)+\pi(-\sigma t)=\sigma^2(3\pi(t)+\pi(-t))-2(\sigma^4-1)\pi(t)+\frac{1}{4}\sigma^2(\sigma^2-1)\pi(2t), \tag{3}$$

for all $t\in T, \sigma\in\mathbb{R}-\{0,\pm1\}$ and by assuming $\pi(-t)=-\pi(t)$ in (3) which leads

$$\pi(2t)=8\pi(t), \quad \forall t\in T. \tag{4}$$

Thus, π is cubic.

Theorem 13. *If π satisfies (2) and even that is $\pi(-t)=\pi(t)$, then a mapping $\pi: T\longrightarrow S$ is quartic.*

Proof. Assume π holds (2). Putting $t=s=0$ in (2), we get $\pi(0)=0$. Setting (t,s) by $(t,0)$ in (2), we arrive

$$2\pi(\sigma t)+2\pi(t)=4\sigma^2\pi(t)-2(\sigma^4-1)\pi(t)+\frac{1}{4}\sigma^2(\sigma^2-1)\pi(2t), \quad \forall t\in T. \tag{5}$$

Allowing $\sigma=2$ in (5), we arrive $\pi(2t)=16\pi(t)$. Using $\pi(-t)=\pi(t)$ and $\pi(2t)=16\pi(t)$ in (5), we get

$$\pi(\sigma t)=\sigma^4\pi(t), \tag{6}$$

for all $t\in T, \sigma\in\mathbb{R}-\{0,\pm1\}$. Thus, π is quartic.

3. Generalized Hyers-Ulam-Rassias Stability of a Euler-Lagrange σ-Cubic-Quartic FE

Consider the following abbreviation

$$\begin{aligned}G\pi(t,s)&=\pi(t+\sigma s)+\pi(\sigma t+s)+\pi(t-\sigma s)+\pi(s-\sigma t)\\&\quad-\sigma^2\{2\pi(t+s)+\pi(t-s)+\pi(s-t)\}\\&\quad+2(\sigma^4-1)\{\pi(t)+\pi(s)\}\\&\quad-\frac{1}{4}\sigma^2(\sigma^2-1)\{\pi(2t)+\pi(2s)\}, \quad \forall t,s\in T,\end{aligned} \tag{7}$$

and the integer $\sigma\neq 0,\pm1$.

Theorem 14. *Consider the odd mapping $\pi: T\longrightarrow S$ for which we can find $\Phi: T\times T\longrightarrow \Lambda^*(\mathbb{R})$ for a linear space T and a fuzzy Banach space (f-BS) S where*

$$\sum_{i=0}^{\infty}\frac{\left(\Phi(2^it,2^is)\right)_\gamma^+}{2^{3i}}<\infty, \quad \forall t,s\in T, \tag{8}$$

$$\|G\pi(t,s)\|\leq\Phi(t,s), \quad \forall t,s\in T. \tag{9}$$

So, we can find a unique cubic function $\Theta: T\longrightarrow S$ such that

$$\|\pi(t)-\Theta(t)\|_\gamma^+\leq\frac{4}{8\sigma^2(\sigma^2-1)}\sum_{i=0}^{\infty}\frac{\left(\Phi(2^it,0)\right)_\zeta^+}{2^{3i}}, \quad \forall t\in T, \tag{10}$$

for all $\gamma\in(0,1], \zeta\in(0,\gamma]$, where

$$\Theta(t):=\lim_{\sigma\to\infty}\frac{\pi(2^\sigma t)}{2^{3\sigma}}. \tag{11}$$

Proof. Putting $s=0$ in (9) implies that

$$\left\|2\sigma^2(\sigma^2-1)\pi(t)-\frac{1}{4}\sigma^2(\sigma^2-1)\pi(2t)\right\|\leq\Phi(t,0), \quad \forall t\in T. \tag{12}$$

Multiply both sides of equation (12) by $4/\sigma^2(\sigma^2-1)$, so we get

$$\|\pi(2t)-8\pi(t)\|\leq\frac{4}{\sigma^2(\sigma^2-1)}\odot\Phi(t,0), \quad \forall t\in T. \tag{13}$$

Again multiplying (13) by $1/2^{3\sigma+3}$ and replacing t by $2^\sigma t$, we obtain

$$\left\|\frac{\pi(2^{\sigma+1}t)}{2^{3(\sigma+1)}}-\frac{\pi(2^\sigma t)}{2^{3\sigma}}\right\|\leq\frac{4}{8\sigma^2(\sigma^2-1)}\frac{1}{2^{3\sigma}}\odot\Phi(2^\sigma t,0), \quad \forall t\in T, \tag{14}$$

and it leads to

$$\left\|\frac{\pi(2^{\sigma+1}t)}{2^{3(\sigma+1)}}-\frac{\pi(2^lt)}{2^{3l}}\right\|_\gamma^+\leq\frac{4}{8\sigma^2(\sigma^2-1)}\sum_{i=l}^{\sigma}\frac{1}{2^{3i}}\left(\Phi(2^it,0)\right)_\zeta^+, \tag{15}$$

$\forall t\in T$ with $\sigma\geq l$, nonnegative integers. Now, (8) and (15) imply that the sequence $\{\pi(2^\sigma t)/2^{3\sigma}\}$ is fuzzy Cauchy in S. So, the sequence $\{\pi(2^\sigma t)/2^{3\sigma}\}$ converges, which let us to define the mapping $\Theta: T\longrightarrow S$ by

$$\Theta(t):=\lim_{\sigma\to\infty}\frac{\pi(2^\sigma t)}{2^{3\sigma}}, \quad \forall t\in T. \tag{16}$$

Considering $l=0$ and allowing $\sigma\longrightarrow\infty$ in (15), we obtain

$$\|\pi(t)-\Theta(t)\|_\gamma^+\leq\frac{4}{8\sigma^2(\sigma^2-1)}\sum_{i=0}^{\infty}\frac{\left(\Phi(2^it,0)\right)_\zeta^+}{2^{3i}}, \quad \forall t\in T, \tag{17}$$

and it gives (10). Using (8) and (9), we have

$$\begin{aligned}
&\|\Theta(t+\sigma s)+\Theta(\sigma t+s)+\Theta(t-\sigma s)+\Theta(s-\sigma t)\\
&\quad-\sigma^2\{2\Theta(t+s)+\Theta(t-s)+\Theta(s-t)\}\\
&\quad+2\left(\sigma^4-1\right)\{\Theta(t)+\Theta(s)\}-\frac{1}{4}\sigma^2\left(\sigma^2-1\right)\{\Theta(2t)+\Theta(2s)\}\|_{\gamma}^{+}\\
&\le\lim_{\sigma\to\infty}\frac{(\Phi(2^\sigma t,2^\sigma s))_\gamma^+}{2^{3\sigma}}=0,\quad\forall t,s\in T,
\end{aligned}\tag{18}$$

which implies that $\Theta:T\longrightarrow S$ is cubic. Suppose that $\Theta':T\longrightarrow S$ is a cubic mapping satisfying (10) and implies

$$\begin{aligned}
\left\|\Theta(t)-\Theta'(t)\right\|&\le\lim_{\sigma\to\infty}\frac{1}{2^{3\sigma}}\frac{4}{\sigma^2(\sigma^2-1)}\sum_{i=0}^{\infty}\frac{\left(\Phi\left(2^i2^\sigma t,0\right)\right)_\zeta^+}{2^{3i}}\\
&\le\lim_{\sigma\to\infty}\frac{4}{\sigma^2(\sigma^2-1)}\sum_{i=\sigma}^{\infty}\frac{\left(\Phi\left(2^it,0\right)\right)_\zeta^+}{2^{3i}}=0,\quad\forall t\in T,
\end{aligned}\tag{19}$$

$\Theta=\Theta'$, which shows the uniqueness of Θ.

Theorem 15. *Consider $\pi:T\longrightarrow S$ and let there exist a function $\Phi:T\times S\longrightarrow\Lambda^*(\mathbb{R})$ such that*

$$\begin{gathered}
\sum_{i=1}^{\infty}2^{3i}\left(\Phi\left(\frac{t}{2^i},\frac{s}{2^i}\right)\right)_\gamma^+<\infty,\\
\|G\pi(t,s)\|\le\Phi(t,s),\quad\forall t,s\in T,
\end{gathered}\tag{20}$$

for a linear space T and a fuzzy Banach space (f-BS) S. So, we can find a unique cubic mapping $\Theta:T\longrightarrow S$, such that

$$\|\pi(t)-\Theta(t)\|_\gamma^+\le\frac{4}{\sigma^2(\sigma^2-1)}\sum_{i=1}^{\infty}2^{3i}\left(\Phi\left(\frac{t}{2^i},0\right)\right)_\zeta^+,\quad\forall t\in T,\tag{21}$$

where

$$\Theta(t):=\lim_{\sigma\to\infty}\left\{2^{3\sigma}\pi\left(\frac{t}{2^\sigma}\right)\right\}.\tag{22}$$

The following corollary gives the Hyers-Ulam, Hyers-Ulam-Rassias, and Rassias stabilities of (2).

Corollary 16. *Consider $\pi:T\longrightarrow S$ and let there be real numbers δ and ρ such that*

$$\|G\pi(t,s)\|_\gamma^+\preceq\begin{cases}\delta,\\ \delta\otimes\{\|t\|^\rho\oplus\|s\|^\rho\}, & \rho\ne3,\\ \delta\otimes\left\{\|t\|^\rho\otimes\|s\|^\rho+\left\{\|t\|^{2\rho}\oplus\|s\|^{2\rho}\right\}\right\}, & 2\rho\ne3,\forall t,s\in T,\end{cases}\tag{23}$$

then there is a unique cubic mapping $Q:T\longrightarrow S$ such that

$$\|\pi(t)-Q(t)\|_\gamma^+\preceq\begin{cases}\dfrac{4\delta_\zeta^+}{7\sigma^2(\sigma^2-1)},\\ \dfrac{4\delta_\zeta^+\left(\|t\|^\rho\right)_\zeta^+}{\sigma^2(\sigma^2-1)\mid 2^3-2^\rho\mid},\\ \dfrac{4\delta_\zeta^+\left(\|t\|^{2\rho}\right)_\zeta^+}{\sigma^2(\sigma^2-1)\mid 2^3-2^{2\rho}\mid},\forall t\in T.\end{cases}\tag{24}$$

In the next example, we consider the unstability of FE (2) for $p=3$ in Corollary 16.

Example 17. Define the mapping $\Phi:T\times T\longrightarrow\Lambda^*(\mathbb{R})$ as

$$\Phi(t)=\begin{cases}\delta\otimes t^3, & \text{if } |t|<1,\\ \delta, & \text{o.w.},\end{cases}\tag{25}$$

in which $\delta>0$ is a fuzzy real number. Define $\pi:T\longrightarrow S$ as

$$\pi(t)=\sum_{\sigma=0}^{\infty}\frac{\Phi(2^\sigma t)}{\left(2^3\right)^\sigma},\quad\forall t\in T.\tag{26}$$

So,

$$\begin{aligned}
&|\pi(t+\sigma s)+\pi(\sigma t+s)+\pi(t-\sigma s)+\pi(s-\sigma t)\\
&\quad-\sigma^2\{2\pi(t+s)+\pi(t-s)+\pi(s-t)\}\\
&\quad+2\left(\sigma^4-1\right)\{\pi(t)+\pi(s)\}-\frac{1}{4}\sigma^2\left(\sigma^2-1\right)\{\pi(2t)+\pi(2s)\}|\\
&\le32\left(\sigma^2-1\right)\delta\left(|t|^3+|s|^3\right),\quad\forall t,s\in T.
\end{aligned}\tag{27}$$

As a result, there does not exist a cubic mapping $\Theta:T\longrightarrow S$ and a constant $\zeta>0$ such that

$$|\pi(t)-\Theta(t)|\le\zeta\otimes|t|^3,\quad\forall t\in T.\tag{28}$$

Proof. The below inequality

$$|\pi(t)|\le\sum_{\sigma=0}^{\infty}\frac{|\Phi(2^\sigma t)|}{\left|2^{3\sigma}\right|}=\sum_{\sigma=0}^{\infty}\frac{\delta}{2^{3\sigma}}=\frac{2^3\delta}{2^3-1},\tag{29}$$

showing the boundedness of π. Now, we show that π satisfies (27).

Let $t=s=0$, then (27) is trivial. If $|t|^3+|s|^3\ge1/m^3$, then the left-hand side of (27) is less than $((28(\sigma^2-1)/7)\delta)$. If $0<|t|^3+|s|^3<1/m^3$. So, we can find a positive integer r such that

$$\frac{1}{\left(2^3\right)^{r+1}}\le|t|^3+|s|^3<\frac{1}{\left(2^3\right)^r},\tag{30}$$

so that

$$\left(2^3\right)^{r-1}t^3 < \frac{1}{2^3},\quad \left(2^3\right)^{r-1}s^3 < \frac{1}{2^3}, \tag{31}$$

and therefore, for each $\sigma = 0, 1, \cdots, r-1$, we have

$$\begin{aligned}
&2^\sigma(t+\sigma s), 2^\sigma(\sigma t+s), 2^\sigma(t-\sigma s), 2^\sigma(s-\sigma t), 2^\sigma(t+s), 2^\sigma(t-s),\\
&\quad m^\sigma(s-t), m^\sigma(t), m^\sigma(s), m^\sigma(2t), m^\sigma(2s) \in (-1,1),\\
&\Phi(2^\sigma(t+\sigma s)) + \Phi(2^\sigma(\sigma t+s)) + \Phi(2^\sigma(t-\sigma s)) + \Phi(2^\sigma(s-\sigma t))\\
&\quad - \sigma^2\{2\Phi(2^\sigma(t+s)) + \Phi(2^\sigma(t-s)) + \Phi(2^\sigma(s-t))\}\\
&\quad + 2\left(\sigma^4-1\right)\{\Phi(2^\sigma(t)) + \Phi(2^\sigma(s))\}\\
&\quad - \frac{1}{4}\sigma^2\left(\sigma^2-1\right)\{\Phi(2^\sigma(2t)) + \Phi(2^\sigma(2s))\},
\end{aligned} \tag{32}$$

for $\sigma = 0, 1, \cdots, r-1$. From (26) and (30), we have

$$\begin{aligned}
&|\pi(t+\sigma s) + \pi(\sigma t+s) + \pi(t-\sigma s) + \pi(s-\sigma t)\\
&\quad - \sigma^2\{2\pi(t+s) + \pi(t-s) + \pi(s-t)\}\\
&\quad + 2\left(\sigma^4-1\right)\{\pi(t)+\pi(s)\} - \frac{1}{4}\sigma^2\left(\sigma^2-1\right)\{\pi(2t)+\pi(2s)\}|\\
&\le \sum_{\sigma=r}^{\infty}\frac{1}{2^{3\sigma}}|\Phi(2^\sigma(t+\sigma s)) + \Phi(2^\sigma(\sigma t+s)) + \Phi(2^\sigma(t-\sigma s))\\
&\quad + \Phi(2^\sigma(s-\sigma t)) - \sigma^2\{2\Phi(2^\sigma(t+s)) + \Phi(2^\sigma(t-s))\\
&\quad + \Phi(2^\sigma(s-t))\} + 2\left(\sigma^4-1\right)\{\Phi(2^\sigma(t)) + \Phi(2^\sigma(s))\}\\
&\quad - \frac{1}{4}\sigma^2\left(\sigma^2-1\right)\{\Phi(2^\sigma(2t)) + \Phi(2^\sigma(2s))\}|\\
&= 32\left(\sigma^2-1\right)\delta\left(|t|^3+|s|^3\right).
\end{aligned} \tag{33}$$

Thus, π satisfies (27) for all $t, s \in T$ with $0 < |t|^3 + |s|^3 < 1/2^3$. Corollary 16 shows $\Theta(t) = rt^3$ for any t in T, and so,

$$|\pi(t)| \le (\zeta \oplus |r|) \otimes |t|^3. \tag{34}$$

But we can choose a positive integer l with $l\delta > \zeta \oplus |r|$. If $t \in (0, 1/2^{l-1})$, then $2^\sigma t \in (0,1)$ for all $\sigma = 0, 1, \cdots, l-1$. For this t, we have

$$\pi(t) = \sum_{\sigma=0}^{\infty}\frac{\Phi(2^\sigma t)}{2^{3\sigma}} \ge \sum_{\sigma=0}^{l-1}\frac{\delta(2^\sigma t)^3}{2^{3\sigma}} = l\delta t^3 > (\zeta \oplus |r|) \otimes t^3, \tag{35}$$

which contradicts (34). Therefore, the functional equation (2) is not stable in the sense of Ulam, Hyers, and Rassias if $\rho = 3$.

Theorem 18. *Consider the even mapping $\pi : T \longrightarrow S$ for which we can find $\Phi : T \times T \longrightarrow \Lambda^*(\mathbb{R})$ such that*

$$\sum_{i=0}^{\infty}\frac{\left(\Phi\left(\sigma^i t, \sigma^i s\right)\right)_\gamma^+}{\sigma^{4i}} < \infty, \quad \forall t, s \in T, \tag{36}$$

$$\|G\pi(t,s)\| \le \Phi(t,s), \quad \forall t, s \in T, \tag{37}$$

and all $\gamma \in (0,1]$. So, we can find a unique quartic mapping $Q : T \longrightarrow S$ and $\forall \gamma \in (0,1], \exists \zeta \in (0,\gamma]$, such that

$$\|\pi(t) - Q(t)\|_\gamma^+ \le \frac{1}{2\sigma^4}\sum_{i=0}^{\infty}\frac{\left(\Phi\left(\sigma^i t, 0\right)\right)_\zeta^+}{\sigma^{4i}}, \quad \forall t, \in T, \tag{38}$$

where

$$Q(t) \coloneqq \lim_{m\to\infty}\frac{\pi(\sigma^m t)}{\sigma^{4m}}. \tag{39}$$

Proof. Putting $s = 0$ in (37), we get

$$\left\|2\left(\pi(\sigma t) - \sigma^4\pi(t)\right)\right\| \le \Phi(t,0), \quad \forall t \in T. \tag{40}$$

Multiply (40) by 1/2, we obtain

$$\left\|\pi(\sigma t) - \sigma^4\pi(t)\right\| \le \frac{1}{2} \odot \Phi(t,0), \quad \forall t \in T. \tag{41}$$

Replacing t by $\sigma^m t$ and multiplying (41) by $1/\sigma^{4m+4}$, we obtain

$$\left\|\frac{\pi\left(\sigma^{m+1}t\right)}{\sigma^{4(m+1)}} - \frac{\pi(\sigma^m t)}{\sigma^{4m}}\right\| \le \frac{1}{2\sigma^4}\frac{1}{\sigma^{4m}}e\Phi(\sigma^m t, 0), \quad \forall t \in T. \tag{42}$$

Therefore, for all $\gamma \in (0,1]$, there is $\zeta \in (0,\gamma]$ such that

$$\left\|\frac{\pi\left(\sigma^{m+1}t\right)}{\sigma^{4(m+1)}} - \frac{\pi\left(\sigma^l t\right)}{\sigma^{4l}}\right\|_\gamma^+ \le \frac{1}{2\sigma^4}\sum_{i=l}^{m}\frac{1}{\sigma^{4i}}\left(\Phi\left(\sigma^i t, 0\right)\right)_\zeta^+, \quad \forall t \in T, \tag{43}$$

with $m \ge l$. From (36) and (43) and because S is a f-BS, we have the sequence $\{\pi(\sigma^m t)/\sigma^{4m}\}$ which is a fuzzy Cauchy in S and converges $\forall t \in T$. Now, we define $Q : T \longrightarrow S$ by

$$Q(t) \coloneqq \lim_{m\to\infty}\frac{\pi(\sigma^m t)}{\sigma^{4m}}, \quad \forall t \in T. \tag{44}$$

Assuming $l = 0$ and allowing the limit as $m \longrightarrow \infty$ in (43), we have

$$\|\pi(t) - Q(t)\|_\gamma^+ \le \frac{1}{2\sigma^4}\sum_{i=0}^{\infty}\frac{\left(\Phi\left(\sigma^i t, 0\right)\right)_\zeta^+}{\sigma^{4i}}, \quad \forall t \in T. \tag{45}$$

Therefore, we obtain (38). From (36) and (37), we have

$$\begin{aligned}&\|Q(t+\sigma s)+Q(\sigma t+s)+Q(t-\sigma s)+Q(s-\sigma t)\\&\quad-\sigma^2\{2Q(t+s)+Q(t-s)+Q(s-t)\}\\&\quad+2(\sigma^4-1)\{Q(t)+Q(s)\}-\frac{1}{4}\sigma^2(\sigma^2-1)\{Q(2t)+Q(2s)\}\|_\gamma^+\\&\leq\lim_{m\to\infty}\frac{(\Phi(\sigma^m t,k^m s))_\gamma^+}{\sigma^{4m}}=0,\quad\forall t,s\in T,\end{aligned}\tag{46}$$

and hence, the mapping $Q:T\longrightarrow S$ is quartic. Letting $Q':T\longrightarrow S$ be a quartic mapping fulfills (38), and we have

$$\begin{aligned}\|Q(t)-Q'(t)\|&\leq\lim_{\sigma\to\infty}\frac{1}{\sigma^{4m}}\frac{1}{2\sigma^4}\sum_{i=0}^{\infty}\frac{(\Phi(\sigma^i\sigma^m t,0))_\zeta^+}{\sigma^{4i}}\\&\leq\lim_{m\to\infty}\frac{1}{2\sigma^4}\sum_{i=m}^{\infty}\frac{(\Phi(\sigma^i t,0))_\zeta^+}{\sigma^{4i}}=0,\end{aligned}\tag{47}$$

for all $t\in T$, $Q=Q'$, and hence, Q is unique.

Theorem 19. *Consider $\pi:T\longrightarrow S$ for which we can find a mapping $\Phi:T\times T\longrightarrow\Lambda^*(\mathbb{R})$ such that*

$$\begin{aligned}&\sum_{i=1}^{\infty}\sigma^{4i}\left(\Phi\left(\frac{t}{\sigma^i},\frac{s}{\sigma^i}\right)\right)_\gamma^+<\infty,\quad\forall t,s\in T,\\&\|G\pi(t,s)\|\leq\Phi(t,s),\quad\forall t,s\in T,\end{aligned}\tag{48}$$

and all $\gamma\in(0,1]$. So, we can find a unique quartic mapping $\pi:T\longrightarrow S$ and $\forall\gamma\in(0,1]$, $\exists\zeta\in(0,\gamma]$, such that

$$\|\pi(t)-Q(t)\|_\gamma^+\leq\frac{1}{2\sigma^4}\sum_{i=1}^{\infty}\sigma^{4i}\left(\Phi\left(\frac{t}{\sigma^i},0\right)\right)_\zeta^+,\quad\forall t\in T,\tag{49}$$

where

$$Q(t):=\lim_{m\to\infty}\left\{\sigma^{3m}\pi\left(\frac{t}{\sigma^m}\right)\right\}.\tag{50}$$

Corollary 20. *Consider $\pi:T\longrightarrow S$ and let there be real numbers δ and ρ such that*

$$\|G\pi(t,s)\|_\gamma^+\preceq\begin{cases}\delta,\\\delta\otimes\{\|t\|^\rho\oplus\|s\|^\rho\}, & \rho\neq4,\\\delta\otimes\{\|t\|^\rho\otimes\|s\|^\rho+\{\|t\|^{2\rho}\oplus\|s\|^{2\rho}\}\}, & \rho\neq2,\forall t,s\in T,\end{cases}\tag{51}$$

so we can find a unique quartic mapping $Q:T\longrightarrow S$ satisfying

$$\|\pi(t)-Q(t)\|_\gamma^+\preceq\begin{cases}\dfrac{\delta_\zeta^+}{2|\sigma^4-1|},\\[2mm]\dfrac{\delta_\zeta^+(\|t\|^\rho)_\zeta^+}{|\sigma^4-\sigma^\rho|},\\[2mm]\dfrac{\delta_\zeta^+(\|t\|^{2\rho})_\zeta^+}{2|\sigma^4-\sigma^{2\rho}|},\forall t\in T.\end{cases}\tag{52}$$

In the next example, we show that the FE (2) is not stable for $\rho=4$ in Corollary 20.

Example 21. Letting $\Phi:T\times T\longrightarrow\Lambda^*(\mathbb{R})$ be a mapping defined by

$$\Phi(t)=\begin{cases}\delta\otimes t^4, & \text{if } |t|<1,\\\delta, & \text{o.w.},\end{cases}\tag{53}$$

where $\delta>0$ is a fuzzy real number and $\pi:T\longrightarrow S$ is defined by

$$\pi(t)=\sum_{\sigma=0}^{\infty}\frac{\Phi(\sigma^m t)}{(\sigma^4)^m},\quad\forall t\in T,\tag{54}$$

a linear space T and a fuzzy Banach space (f-BS) S. Then π fulfills the functional inequality

$$\begin{aligned}&|\pi(t+\sigma s)+\pi(\sigma t+s)+\pi(t-\sigma s)+\pi(s-\sigma t)\\&\quad-\sigma^2\{2\pi(t+s)+\pi(t-s)+\pi(s-t)\}\\&\quad+2(\sigma^4-1)\{\pi(t)+\pi(s)\}-\frac{1}{4}\sigma^2(\sigma^2-1)\{\pi(2t)+\pi(2s)\}|\\&\leq\frac{7\sigma^{10}+9\sigma^{12}}{2(\sigma^4-1)}\delta(|t|^4+|s|^4),\quad\forall t,s\in T.\end{aligned}\tag{55}$$

So, we cannot find a quartic mapping $Q:T\longrightarrow S$ and a constant $\zeta>0$ such that

$$|\pi(t)-Q(t)|\leq\zeta\otimes|t|^4,\quad\forall t\in T.\tag{56}$$

4. Conclusion

In our work, we have obtained the general solution of a new generalized mixed Euler-Lagrange σ-cubic-quartic functional equation and studied its generalized Hyers-Ulam-Rassias, Hyers-Ulam, Hyers-Ulam-Rassias, and Rassias stabilities in fuzzy normed linear space using Felbin's concept. Moreover, some counterexamples show both stability and unstability of FE (2) in f-BS.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors conceived the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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