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Research Article

Two Nonmonotonic Self-Adaptive Strongly Convergent Projection-Type Methods for Solving Pseudomonotone Variational Inequalities

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The primary objective of this study is to introduce two novel extragradient-type iterative schemes for solving variational inequality problems in a real Hilbert space. The proposed iterative schemes extend the well-known subgradient extragradient method and are used to solve variational inequalities involving the pseudomonotone operator in real Hilbert spaces. The proposed iterative methods have the primary advantage of using a simple mathematical formula for step size rule based on operator information rather than the Lipschitz constant or another line search method. Strong convergence results for the suggested iterative algorithms are well-established for mild conditions, such as Lipschitz continuity and mapping monotonicity. Finally, we present many numerical experiments that show the effectiveness and superiority of iterative methods.

1. Introduction

The primary objective of this research is to investigate the iterative methodologies used to estimate the solution of variational inequalities in a real Hilbert space. To establish the convergence analysis theorems, the following conditions need to be satisfied:

Condition 1. The solution set of the problem (VIP) denoted by Ω and it is nonempty.

Condition 2. A mapping $\mathscr{L} : \mathscr{E} \longrightarrow \mathscr{E}$ is said to be pseudo-monotone if

$$\langle \mathscr{L}(p_1), p_2 - p_1 \rangle \geq 0 \Rightarrow \langle \mathscr{L}(p_2), p_1 - p_2 \rangle \leq 0, \quad \forall p_1, p_2 \in \mathscr{A} (\text{PM}). \tag{1}$$

Condition 3. A mapping $\mathscr{L} : \mathscr{E} \longrightarrow \mathscr{E}$ is said to be *Lipschitz continuous* with constant $L > 0$ if

$$\|\mathscr{L}(p_1) - \mathscr{L}(p_2)\| \leq L\|p_1 - p_2\|, \quad \forall p_1, p_2 \in \mathscr{A} (\text{LC}). \tag{2}$$

Condition 4. A mapping $\mathscr{L} : \mathscr{E} \longrightarrow \mathscr{E}$ is said to be *weakly sequentially continuous* if $\{\mathscr{L}(u_n)\}$ converges weakly to $\mathscr{L}(u)$ for each sequence $\{u_n\}$ converges weakly to an element u.

Let $\mathscr{E}$ be any real Hilbert space and $\mathscr{A}$ be any nonempty convex closed subset of a Hilbert space $\mathscr{E}$. Assume that $\mathscr{L} : \mathscr{E} \longrightarrow \mathscr{E}$ be an arbitrary mapping. The *variational inequality problem* for an operator $\mathscr{L}$ on $\mathscr{A}$ is defined in the following manner [1, 2]:

$$\text{Find } q^* \in \mathscr{A} \text{ such that } \langle \mathscr{L}(q^*), y - q^* \rangle \geq 0, \quad \forall y \in \mathscr{A} \ (\text{VIP}). \tag{3}$$

Let Ω stand for the solution set for the problem (VIP). The mathematical model of variational inequalities covers many mathematical problems, such as partial differential equations, optimization, optimal control, mechanics, finance, and mathematical programming (see for details [3–9]) and others in [10–20]. Since it is a fundamental problem in the applied sciences and nonlinear functional analysis, many researchers are investigating not only the stability and existence of solutions to such problems but also iterative methods for solving them numerically. In order to solve variational inequalities numerically, projection iterative methods are a very important tool. Many researchers have provided various projection method extensions and modifications to solve the problem (VIP) (see [21–33]). The extragradient method described below was developed by Korpelevich [25] and Antipin [34]. Their method takes the form of

$$\begin{cases} u_1 \in \mathscr{A}, \\ p_n = P_{\mathscr{A}}[u_n - \delta \mathscr{L}(u_n)], \\ u_{n+1} = P_{\mathscr{A}}[u_n - \delta \mathscr{L}(p_n)], \end{cases} \tag{4}$$

where $0 < \delta < 1/L$. For each iteration of the above iterative scheme, two projections on the feasible set $\mathscr{A}$ are required to be figured out. Of course, if the feasible set $\mathscr{A}$ has a complicated framework, this can affect the method's computational effectiveness. The first one is to follow the subgradient extragradient method designed by Censor et al. [22] to overcome this deficiency. This method is in the form of

$$\begin{cases} u_1 \in \mathscr{A}, \\ p_n = P_{\mathscr{A}}[u_n - \delta \mathscr{L}(u_n)], \\ u_{n+1} = P_{\mathscr{Z}_n}[u_n - \delta \mathscr{L}(p_n)], \end{cases} \tag{5}$$

where $0 < \delta < 1/L$ and

$$\mathscr{Z}_n = \{z \in \mathscr{Z} : \langle u_n - \delta \mathscr{L}(u_n) - p_n, z - p_n \rangle \le 0\}. \tag{6}$$

It is a key point to note that the above-mentioned well-established methods have two major drawbacks. The first is the fixed constant step size, which needs knowledge or approximation of the appropriate operator Lipschitz constant, and also is only weakly convergent in Hilbert spaces. Using a fixed step size can be difficult in terms of computation, affecting the method convergence rate and efficiency.

Hence, a natural question arises:

"Is it possible to propose two new strongly convergent subgradient extragradient algorithms with a nonmonotone self-adaptive step size rule to solve the problem (VIP)?"

The primary objective of this study is to introduce two new strongly convergent subgradient extragradient methods for enhancing the convergence rate of an iterative sequence. The answer to the above question is given in this study, which would be the subgradient extragradient algorithms, which set up a strong convergent iterative sequence by letting a variable nonmonotone step size rule. The suggested methods are employed to solve variational inequality problems involving pseudomonotone and Lipschitz regular operators in real Hilbert space. The proposed methods are based on the projection method [22] as well as the methods proposed in [26, 35]. The established method only needs to compute one projection onto the feasible set and one projection onto the half-space for each iteration. The iterative sequences established by the proposed method strongly converge to some solution of the underlined problem in the framework of some appropriate conditions on control parameters. A number of numerical examples are also added to elaborate on the computational effectiveness of the new methods over some existing methods presented in [36, 37].

The paper is arranged in the following manner: In Section 2, we provide some basic identities and preliminary results that were used in this paper. Section 3 includes the proposed methods and proves their convergence analysis. Finally, Section 4 presents some numerical results to illustrate the convergence and the effectiveness of the proposed methods.

2. Preliminaries

This section contains a number of important identities, as well as useful lemmas and definitions. For all $u, y \in \mathscr{Z}$, we have

$$\|u + y\|^2 = \|u\|^2 + 2\langle u, y \rangle + \|y\|^2. \tag{7}$$

A metric projection $P_{\mathscr{A}}(p_1)$ of an element $p_1 \in \mathscr{Z}$ is evaluated by

$$P_{\mathscr{A}}(p_1) = \operatorname{argmin}\{\|p_1 - p_2\| : p_2 \in \mathscr{A}\}. \tag{8}$$

Next, we list some of the important identities that are used to prove the convergence analysis.

Lemma 1 (see [38]). *Let $P_{\mathscr{A}} : \mathscr{Z} \longrightarrow \mathscr{A}$ be a metric projection on set $\mathscr{A}$. For each $p_1, p_2 \in \mathscr{Z}$ and $\ell \in \mathbb{R}$, then the following inequalities are satisfied:*

(i) $p_3 = P_{\mathscr{A}}(p_1)$ is true if and only if

$$\langle p_1 - p_3, p_2 - p_3 \rangle \le 0, \quad \forall p_2 \in \mathscr{A}, \tag{9}$$

(ii)

$$\|p_1 - P_{\mathscr{A}}(p_2)\|^2 + \|P_{\mathscr{A}}(p_2) - p_2\|^2 \le \|p_1 - p_2\|^2, \quad p_1 \in \mathscr{A}, p_2 \in \mathscr{Z}, \tag{10}$$

(iii)

$$\|p_1 - P_{\mathscr{A}}(p_1)\| \le \|p_1 - p_2\|, \quad p_2 \in \mathscr{A}, p_1 \in \mathscr{X}, \tag{11}$$

(iv)

$$\|\ell p_1 + (1-\ell)p_2\|^2 = \ell\|p_1\|^2 + (1-\ell)\|p_2\|^2 - \ell(1-\ell)\|p_1 - p_2\|^2, \tag{12}$$

(v)

$$\|p_1 + p_2\|^2 \le \|p_1\|^2 + 2\langle p_2, p_1 + p_2\rangle. \tag{13}$$

Lemma 2 (see [39]). *Assuming that* $\{c_n\} \subset [0,+\infty)$, *a sequence meets the following criteria:*

$$c_{n+1} \le (1-d_n)c_n + d_n e_n, \quad \forall n \in \mathbb{N}. \tag{14}$$

Moreover, $\{d_n\} \subset (0,1)$ *and* $\{e_n\} \subset \mathbb{R}$ *are two sequences such that*

$$\lim_{n\longrightarrow+\infty} d_n = 0, \sum_{n=1}^{+\infty} d_n = +\infty \text{and} \limsup_{n\longrightarrow+\infty} e_n \le 0. \tag{15}$$

Then, $\lim_{n\longrightarrow+\infty} c_n = 0$.

Lemma 3 (see [40]). *Assume that a sequence* $\{c_n\}$ *of real numbers and there is* $\{n_i\}$ *subsequence of* $\{n\}$ *such that* $c_{n_i} < c_{n_{i+1}}, \forall i \in \mathbb{N}$.*(102)*

Thus, there exists a natural nondecreasing sequence $\{m_j\}$ *with* $m_j \longrightarrow +\infty$ *as* $j \longrightarrow +\infty$ *and satisfies the following criteria for* $j \in \mathbb{N}$:

$$c_{m_j} \le c_{m_{j+1}}, c_j \le c_{m_{j+1}}. \tag{16}$$

Indeed, $m_j = \max\{j \le j : c_j \le c_{j+1}\}$.

Lemma 4 (see [41]). *Assume that* $\mathscr{L} : \mathscr{A} \longrightarrow \mathscr{X}$ *is a pseudomonotone and continuous mapping. Then,* q^* *is a solution of the problem (VIP) if and only if* q^* *is a solution of the following problem:*

$$\textit{Find } u \in \mathscr{A} \textit{ such that } \langle \mathscr{L}(y), y-u\rangle \ge 0, \quad \forall y \in \mathscr{A}. \tag{17}$$

3. Main Results

In this part of the research article, we propose two new methods and the corresponding strong convergence theorems. Both methods are presented in the following manner. The first method is of the following form.

Assume that $g : \mathscr{X} \longrightarrow \mathscr{X}$ is a contraction having constant $\xi \in [0,1)$. The second major contribution of this study work is as follows. The second main algorithm has the following form.

Lemma 5. *A sequence* $\{\delta_n\}$ *generated by (3.1) is convergent to* δ *and satisfies the following inequality:*

$$\min\left\{\frac{\mu}{L}, \delta_1\right\} \le \delta \le \delta_1 + P \quad \textit{where } P = \sum_{n=1}^{+\infty} \varphi_n. \tag{18}$$

Proof. Let $\langle \mathscr{L}(u_n) - \mathscr{L}(p_n), q_n - p_n\rangle > 0$ such that

$$\frac{\mu(\|u_n - p_n\|^2 + \|q_n - p_n\|^2)}{2\langle \mathscr{L}(u_n) - \mathscr{L}(p_n), q_n - p_n\rangle} \ge \frac{2\mu\|u_n - p_n\|\|q_n - p_n\|}{2\|\mathscr{L}(u_n) - \mathscr{L}(p_n)\|\|q_n - p_n\|} \ge \frac{2\mu\|u_n - p_n\|\|q_n - p_n\|}{2L\|u_n - p_n\|\|q_n - p_n\|} \ge \frac{\mu}{L}. \tag{19}$$

By using mathematical induction on the definition of δ_{n+1}, we have

$$\min\left\{\frac{\mu}{L}, \delta_1\right\} \le \delta_n \le \delta_1 + P. \tag{20}$$

Let

$$[\delta_{n+1} - \delta_n]^+ = \max\{0, \delta_{n+1} - \delta_n\}, \quad [\delta_{n+1} - \delta_n]^- = \max\{0, -(\delta_{n+1} - \delta_n)\}. \tag{21}$$

Due to expression of $\{\delta_n\}$, we can write

$$\sum_{n=1}^{+\infty} (\delta_{n+1} - \delta_n)^+ = \sum_{n=1}^{+\infty} \max\{0, \delta_{n+1} - \delta_n\} \le P < +\infty. \tag{22}$$

Thus, $\sum_{n=1}^{+\infty} (\delta_{n+1} - \delta_n)^+$ is convergent. Next, we have to prove the convergence of the following series:

$$\sum_{n=1}^{+\infty} (\delta_{n+1} - \delta_n)^-. \tag{23}$$

Let $\sum_{n=1}^{+\infty} (\delta_{n+1} - \delta_n)^- = +\infty$. Thus, we have $\delta_{n+1} - \delta_n = (\delta_{n+1} - \delta_n)^+ - (\delta_{n+1} - \delta_n)^-$. Thus, we have

$$\delta_{k+1} - \delta_1 = \sum_{n=0}^{k} (\delta_{n+1} - \delta_n) = \sum_{n=0}^{k} (\delta_{n+1} - \delta_n)^+ - \sum_{n=0}^{k} (\delta_{n+1} - \delta_n)^-. \tag{24}$$

Letting $k \longrightarrow +\infty$ in (24), we obtain $\delta_k \longrightarrow -\infty$ as $k \longrightarrow \infty$. This is a contradiction. Due to the convergence of the series $\sum_{n=0}^{k} (\delta_{n+1} - \delta_n)^+$ and $\sum_{n=0}^{k} (\delta_{n+1} - \delta_n)^-$ taking $k \longrightarrow +\infty$ in (24), we obtain $\lim_{n\longrightarrow\infty} \delta_n = \delta$. This completes the proof.□□ □

Lemma 6. *Let* $\mathscr{L} : \mathscr{X} \longrightarrow \mathscr{X}$ *be an operator satisfies the criteria Condition 1–Condition 4. For a given* $q^* \in \Omega \neq \emptyset$, *we* *have*

$$\|q_n - q^*\|^2 \le \|u_n - q^*\|^2 - \left(1 - \frac{\mu\delta_n}{\delta_{n+1}}\right)\|u_n - p_n\|^2 - \left(1 - \frac{\mu\delta_n}{\delta_{n+1}}\right)\|q_n - p_n\|^2. \tag{25}$$

Proof. We have to evaluate

$$\begin{aligned}\|q_n - q^*\|^2 &= \left\|P_{\mathscr{Z}_n}[u_n - \delta_n \mathscr{L}(p_n)] - q^*\right\|^2 \\ &= \left\|P_{\mathscr{Z}_n}[u_n - \delta_n \mathscr{L}(p_n)] + [u_n - \delta_n \mathscr{L}(p_n)] \right.\\ &\quad \left. - [u_n - \delta_n \mathscr{L}(p_n)] - q^*\right\|^2 \\ &= \|[u_n - \delta_n \mathscr{L}(p_n)] - q^*\|^2 + \left\|P_{\mathscr{Z}_n}[u_n - \delta_n \mathscr{L}(p_n)] \right.\\ &\quad \left. - [u_n - \delta_n \mathscr{L}(p_n)]\right\|^2 + 2\left\langle P_{\mathscr{Z}_n}[u_n - \delta_n \mathscr{L}(p_n)] \right.\\ &\quad \left. - [u_n - \delta_n \mathscr{L}(p_n)], [u_n - \delta_n \mathscr{L}(p_n)] - q^*\right\rangle.\end{aligned} \tag{26}$$

It is given that $q^* \in \Omega \subset \mathscr{A} \subset \mathscr{Z}_n$ such that

$$\begin{aligned}&\left\|P_{\mathscr{Z}_n}[u_n - \delta_n \mathscr{L}(p_n)] - [u_n - \delta_n \mathscr{L}(p_n)]\right\|^2 \\ &\quad + \left\langle P_{\mathscr{Z}_n}[u_n - \delta_n \mathscr{L}(p_n)] - [u_n - \delta_n \mathscr{L}(p_n)], [u_n - \delta_n \mathscr{L}(p_n)] \right.\\ &\quad \left. - q^*\right\rangle = \left\langle [u_n - \delta_n \mathscr{L}(p_n)] - P_{\mathscr{Z}_n}[u_n - \delta_n \mathscr{L}(p_n)], q^* \right.\\ &\quad \left. - P_{\mathscr{Z}_n}[u_n - \delta_n \mathscr{L}(p_n)]\right\rangle \le 0.\end{aligned} \tag{27}$$

Furthermore, it implies that

$$\begin{aligned}&\left\langle P_{\mathscr{Z}_n}[u_n - \delta_n \mathscr{L}(p_n)] - [u_n - \delta_n \mathscr{L}(p_n)], [u_n - \delta_n \mathscr{L}(p_n)] - q^*\right\rangle \\ &\quad \le -\left\|P_{\mathscr{Z}_n}[u_n - \delta_n \mathscr{L}(p_n)] - [u_n - \delta_n \mathscr{L}(p_n)]\right\|^2.\end{aligned} \tag{28}$$

By using expressions (26) and (28), we obtain

$$\begin{aligned}\|q_n - q^*\|^2 &\le \|u_n - \delta_n \mathscr{L}(p_n) - q^*\|^2 - \left\|P_{\mathscr{Z}_n}[u_n - \delta_n \mathscr{L}(p_n)] \right.\\ &\quad \left. - [u_n - \delta_n \mathscr{L}(p_n)]\right\|^2 \\ &\le \|u_n - q^*\|^2 - \|u_n - q_n\|^2 + 2\delta_n \langle \mathscr{L}(p_n), q^* - q_n\rangle.\end{aligned} \tag{29}$$

Since q^* is the solution of problem (VIP), we have

$$\langle \mathscr{L}(q^*), y - q^*\rangle \ge 0, \quad \forall y \in \mathscr{A}. \tag{30}$$

We have condition on mapping $\mathscr{L}$ with feasible set $\mathscr{A}$, we get

$$\langle \mathscr{L}(y), y - q^*\rangle \ge 0, \quad \forall y \in \mathscr{A}. \tag{31}$$

By substituting $y = p_n \in \mathscr{A}$, we get

$$\langle \mathscr{L}(p_n), p_n - q^*\rangle \ge 0. \tag{32}$$

Therefore, we have

$$\begin{aligned}\langle \mathscr{L}(p_n), q^* - q_n\rangle &= \langle \mathscr{L}(p_n), q^* - p_n\rangle + \langle \mathscr{L}(p_n), p_n - q_n\rangle \\ &\le \langle \mathscr{L}(p_n), p_n - q_n\rangle.\end{aligned} \tag{33}$$

From (29) and (33), we get

$$\begin{aligned}\|q_n - q^*\|^2 &\le \|u_n - q^*\|^2 - \|u_n - q_n\|^2 + 2\delta_n \langle \mathscr{L}(p_n), p_n - q_n\rangle \\ &\le \|u_n - q^*\|^2 - \|u_n - p_n + p_n - q_n\|^2 \\ &\quad + 2\delta_n \langle \mathscr{L}(p_n), p_n - q_n\rangle \\ &\le \|u_n - q^*\|^2 - \|u_n - p_n\|^2 - \|p_n - q_n\|^2 \\ &\quad + 2\langle u_n - \delta_n \mathscr{L}(p_n) - p_n, q_n - p_n\rangle.\end{aligned} \tag{34}$$

Since $q_n = P_{\mathscr{Z}_n}[u_n - \delta_n \mathscr{L}(p_n)]$ and from δ_{n+1}, we have

$$\begin{aligned}&2\langle u_n - \delta_n \mathscr{L}(p_n) - p_n, q_n - p_n\rangle = 2\langle u_n - \delta_n \mathscr{L}(u_n) - p_n, q_n - p_n\rangle \\ &\quad + 2\delta_n \langle \mathscr{L}(u_n) - \mathscr{L}(p_n), q_n - p_n\rangle \\ &\le \frac{\delta_n}{\delta_{n+1}} 2\delta_{n+1} \langle \mathscr{L}(u_n) - \mathscr{L}(p_n), q_n - p_n\rangle \\ &\le \frac{\mu\delta_n}{\delta_{n+1}}\|u_n - p_n\|^2 + \frac{\mu\delta_n}{\delta_{n+1}}\|q_n - p_n\|^2.\end{aligned} \tag{35}$$

Combining expressions (34) and (35), we obtain

$$\begin{aligned}\|q_n - q^*\|^2 &\le \|u_n - q^*\|^2 - \|u_n - p_n\|^2 - \|p_n - q_n\|^2 \\ &\quad + \frac{\delta_n}{\delta_{n+1}}\left[\mu\|u_n - p_n\|^2 + \mu\|q_n - p_n\|^2\right] \\ &\le \|u_n - q^*\|^2 - \left(1 - \frac{\mu\delta_n}{\delta_{n+1}}\right)\|u_n - p_n\|^2 \\ &\quad - \left(1 - \frac{\mu\delta_n}{\delta_{n+1}}\right)\|q_n - p_n\|^2.\end{aligned} \tag{36}$$

□□ □

Lemma 7. *Let $\mathscr{L} : \mathscr{Z} \longrightarrow \mathscr{Z}$ be a mapping meet the items Condition 1–Condition 4. If there exists a subsequence $\{u_{n_k}\}$ convergent weakly to $\hat{u}$ and $\lim_{k\longrightarrow\infty}\|u_{n_k} - p_{n_k}\| = 0$, then $\hat{u}$ is the solution of (VIP).*

Proof. We need to prove that $\hat{u} \in \Omega$. Indeed, we have

$$p_{n_k} = P_{\mathscr{A}}\left[u_{n_k} - \delta_{n_k} \mathscr{L}\left(u_{n_k}\right)\right], \tag{37}$$

that is equivalent to

$$\left\langle u_{n_k} - \delta_{n_k} \mathscr{L}\left(u_{n_k}\right) - p_{n_k}, y - p_{n_k}\right\rangle \le 0, \quad \forall y \in \mathscr{A}. \tag{38}$$

The above inequality implies that

$$\left\langle u_{n_k} - p_{n_k}, y - p_{n_k} \right\rangle \le \delta_{n_k} \left\langle \mathscr{L}\left(u_{n_k}\right), y - p_{n_k} \right\rangle, \quad \forall y \in \mathscr{A}. \tag{39}$$

Thus, we obtain

$$\frac{1}{\delta_{n_k}} \left\langle u_{n_k} - p_{n_k}, y - p_{n_k} \right\rangle + \left\langle \mathscr{L}\left(u_{n_k}\right), p_{n_k} - u_{n_k} \right\rangle \le \left\langle \mathscr{L}\left(u_{n_k}\right), y - u_{n_k} \right\rangle, \quad \forall y \in \mathscr{A}. \tag{40}$$

Since $\min\{\mu/L, \delta_1\} \le \delta \le \delta_1 + P$ and $\{u_{n_k}\}$ is a bounded sequence. By taking $\lim_{k\longrightarrow\infty}\|u_{n_k} - p_{n_k}\| = 0$ and $k \longrightarrow \infty$ in (18), we obtain

$$\liminf_{k\longrightarrow\infty} \left\langle \mathscr{L}\left(u_{n_k}\right), y - u_{n_k} \right\rangle \ge 0, \quad \forall y \in \mathscr{A}. \tag{41}$$

Moreover, we have

$$\begin{aligned}\left\langle \mathscr{L}\left(p_{n_k}\right), y - p_{n_k} \right\rangle &= \left\langle \mathscr{L}\left(p_{n_k}\right) - \mathscr{L}\left(u_{n_k}\right), y - u_{n_k} \right\rangle \\ &\quad + \left\langle \mathscr{L}\left(u_{n_k}\right), y - u_{n_k} \right\rangle + \left\langle \mathscr{L}\left(p_{n_k}\right), u_{n_k} - p_{n_k} \right\rangle.\end{aligned} \tag{42}$$

Since $\lim_{k\longrightarrow\infty}\|u_{n_k} - p_{n_k}\| = 0$ and $\mathscr{L}$ is L-Lipschitz continuity on $\mathscr{X}$, we have

$$\lim_{k\longrightarrow\infty} \left\| \mathscr{L}\left(u_{n_k}\right) - \mathscr{L}\left(p_{n_k}\right) \right\| = 0, \tag{43}$$

which together with (42) and (43), we obtain

$$\liminf_{k\longrightarrow\infty} \left\langle \mathscr{L}\left(p_{n_k}\right), y - p_{n_k} \right\rangle \ge 0, \quad \forall y \in \mathscr{A}. \tag{44}$$

Next, we have

$$\left\langle \mathscr{L}\left(u_{n_i}\right), y - u_{n_i} \right\rangle + \varepsilon_k \ge 0, \quad \forall i \ge m_k. \tag{45}$$

Due to $\{\varepsilon_k\}$ decreasing, this implies that $\{m_k\}$ is increasing.□□ □

Case I. Suppose that a subsequence $u_{n_{m_{k_j}}}$ of $u_{n_{m_k}}$ such as $\mathscr{L}(u_{n_{m_{k_j}}}) = 0$ $(\forall j)$. Let $j \longrightarrow \infty$, we get

$$\langle \mathscr{L}(\hat{u}), y - \hat{u} \rangle = \lim_{j\longrightarrow\infty} \left\langle \mathscr{L}\left(u_{n_{m_{k_j}}}\right), y - \hat{u} \right\rangle = 0. \tag{46}$$

Then, $\hat{u} \in \mathscr{A}$ which further implies that $\hat{u} \in \Omega$.

Case II. Suppose that there exists $N_0 \in \mathbb{N}$ such that for all $n_{m_k} \ge N_0$, $\mathscr{L}(u_{n_{m_k}}) \ne 0$. Consider that

$$\Xi_{n_{m_k}} = \frac{\mathscr{L}\left(u_{n_{m_k}}\right)}{\left\|\mathscr{L}\left(u_{n_{m_k}}\right)\right\|^2}, \quad \forall n_{m_k} \ge N_0. \tag{47}$$

Due to the above definition, we obtain

$$\left\langle \mathscr{L}\left(u_{n_{m_k}}\right), \Xi_{n_{m_k}} \right\rangle = 1, \quad \forall n_{m_k} \ge N_0. \tag{48}$$

Moreover, expressions (45) and (48) for all $n_{m_k} \ge N_0$, we have

$$\left\langle \mathscr{L}\left(u_{n_{m_k}}\right), y + \varepsilon_k \Xi_{n_{m_k}} - u_{n_{m_k}} \right\rangle \ge 0. \tag{49}$$

Due to the pseudomonotonicity of $\mathscr{L}$ for $n_{m_k} \ge N_0$, we have

$$\left\langle \mathscr{L}\left(y + \varepsilon_k \Xi_{n_{m_k}}\right), y + \varepsilon_k \Xi_{n_{m_k}} - u_{n_{m_k}} \right\rangle \ge 0. \tag{50}$$

For all $n_{m_k} \ge N_0$, we have

$$\begin{aligned}\left\langle \mathscr{L}(y), y - u_{n_{m_k}} \right\rangle &\ge \left\langle \mathscr{L}(y) - \mathscr{L}\left(y + \varepsilon_k \Xi_{n_{m_k}}\right), y + \varepsilon_k \Xi_{n_{m_k}} - u_{n_{m_k}} \right\rangle \\ &\quad - \varepsilon_k \left\langle \mathscr{L}(y), \Xi_{n_{m_k}} \right\rangle.\end{aligned} \tag{51}$$

Let us consider that $\mathscr{L}(\hat{u}) \ne 0$; we obtain

$$\|\mathscr{L}(\hat{u})\| \le \liminf_{k\longrightarrow\infty} \left\|\mathscr{L}\left(u_{n_k}\right)\right\|. \tag{52}$$

Thus, we obtain

$$0 \le \lim_{k\longrightarrow\infty} \left\|\varepsilon_k \Xi_{n_{m_k}}\right\| = \lim_{k\longrightarrow\infty} \frac{\varepsilon_k}{\left\|\mathscr{L}\left(u_{n_{m_k}}\right)\right\|} \le \frac{0}{\|\mathscr{L}(\hat{u})\|} = 0. \tag{53}$$

By letting $k \longrightarrow \infty$ in (51), we obtain

$$\langle \mathscr{L}(y), y - \hat{u} \rangle \ge 0, \quad \forall y \in \mathscr{A}. \tag{54}$$

Due to the Minty Lemma in [41], we infer that $\hat{u} \in \Omega$.

Theorem 8. *Let $\mathscr{L} : \mathscr{X} \longrightarrow \mathscr{X}$ be a mapping that satisfies the conditions Condition 1–Condition 4. Then, $\{u_n\}$ sequence generated by Algorithm 1 strongly converges to an element $q^* \in \Omega$.*

Step 0: take $u_1 \in \mathscr{A}, \delta_1 > 0$ and select a nonnegative sequence of real numbers $\{\varphi_n\}$ such that $\sum_{n=1}^{+\infty}\varphi_n < +\infty$. Moreover, $\{\phi_n\} \subset (a, b) \subset (0, 1 - \psi_n)$ and $\{\psi_n\} \subset (0, 1)$ meet the following criteria:
$\lim_{n\longrightarrow+\infty}\psi_n = 0$ and $\sum_{n=1}^{+\infty}\psi_n = +\infty$.
Step 1: evaluate
$p_n = P_{\mathscr{A}}(u_n - \delta_n\mathscr{L}(u_n))$.
If $u_n = p_n$, then STOP. Otherwise, go to Step 2.
Step 2: firstly, construct a half-space
$\mathscr{Z}_n = \{z \in \mathscr{Z} : \langle u_n - \delta_n\mathscr{L}(u_n) - p_n, z - p_n\rangle \le 0\}$,
and compute
$q_n = P_{\mathscr{Z}_n}(u_n - \delta_n\mathscr{L}(p_n))$.
Step 3: evaluate
$u_{n+1} = (1 - \phi_n - \psi_n)u_n + \phi_n q_n$.
Step 4: evaluate

$$\delta_{n+1} = \begin{cases} \min\{\delta_n + \varphi_n, (\mu\|u_n - p_n\|^2 + \mu\|q_n - p_n\|^2/2[\langle\mathscr{L}(u_n) - \mathscr{L}(p_n), q_n - p_n\rangle])\} & \text{if}\langle\mathscr{L}(u_n) - \mathscr{L}(p_n), q_n - p_n\rangle > 0, \\ \varphi_n + \delta_n, & \text{otherwise.} \end{cases} \quad (3.1)$$

Set $n := n + 1$ and go back to Step 1.

Algorithm 1: Nonmonotonic explicit Mann-type subgradient extragradient method.

Proof. Since $\delta_n \longrightarrow \delta$, there exists a fixed number $\varepsilon \in (0, 1 - \mu)$ such that

$$\lim_{n\longrightarrow\infty}\left(1 - \frac{\mu\delta_n}{\delta_{n+1}}\right) = 1 - \mu > \varepsilon > 0, \quad \forall n \ge n_0. \tag{55}$$

Thus, expression (36) gives that

$$\|q_n - q^*\|^2 \le \|u_n - q^*\|^2, \quad \forall n \ge n_0. \tag{56}$$

It is given that $q^* \in \Omega$; we obtain

$$\begin{aligned}\|u_{n+1} - q^*\| &= \|(1 - \phi_n - \psi_n)u_n + \phi_n q_n - q^*\| \\ &= \|(1 - \phi_n - \psi_n)(u_n - q^*) + \phi_n(q_n - q^*) - \psi_n q^*\| \\ &\le \|(1 - \phi_n - \psi_n)(u_n - q^*) + \phi_n(q_n - q^*)\| + \psi_n\|q^*\|.\end{aligned} \tag{57}$$

Next, we have to evaluate the following:

$$\begin{aligned}&\|(1 - \phi_n - \psi_n)(u_n - q^*) + \phi_n(q_n - q^*)\|^2 \\ &\quad = (1 - \phi_n - \psi_n)^2\|u_n - q^*\|^2 + \phi_n^2\|q_n - q^*\|^2 \\ &\qquad + 2\langle(1 - \phi_n - \psi_n)(u_n - q^*), \phi_n(q_n - q^*)\rangle \\ &\quad \le (1 - \phi_n - \psi_n)^2\|u_n - q^*\|^2 + \phi_n^2\|q_n - q^*\|^2 \\ &\qquad + 2\phi_n(1 - \phi_n - \psi_n)\|u_n - q^*\|\|q_n - q^*\|,\end{aligned} \tag{58}$$

$$\begin{aligned}&\le (1 - \phi_n - \psi_n)^2\|q_n - q^*\|^2 + \phi_n^2\|q_n - q^*\|^2 \\ &\qquad + \phi_n(1 - \phi_n - \psi_n)\|u_n - q^*\|^2 + \phi_n(1 - \phi_n - \psi_n)\|q_n - q^*\|^2 \\ &\quad \le (1 - \phi_n - \psi_n)(1 - \psi_n)\|u_n - q^*\|^2 + \phi_n(1 - \psi_n)\|q_n - q^*\|^2.\end{aligned} \tag{59}$$

Substituting (56) into (59), we obtain

$$\begin{aligned}&\|(1 - \phi_n - \psi_n)(u_n - q^*) + \phi_n(q_n - q^*)\|^2 \\ &\quad \le (1 - \phi_n - \psi_n)(1 - \psi_n)\|u_n - q^*\|^2 + \phi_n(1 - \psi_n)\|u_n - q^*\|^2 \\ &\quad = (1 - \psi_n)^2\|u_n - q^*\|^2.\end{aligned} \tag{60}$$

Next, we have

$$\|(1 - \phi_n - \psi_n)(u_n - q^*) + \phi_n(q_n - q^*)\| \le (1 - \psi_n)\|u_n - q^*\|. \tag{61}$$

From expressions (57) and (61), we obtain

$$\begin{aligned}\|u_{n+1} - q^*\| &\le (1 - \psi_n)\|u_n - q^*\| + \psi_n\|q^*\| \le \max\{\|u_n - q^*\|, \|q^*\|\} \\ &\le \max\{\|u_{n_0} - q^*\|, \|q^*\|\}.\end{aligned} \tag{62}$$

Thus, from the above relation, we obtain that $\{u_n\}$ is bounded sequence. From sequence $\{u_{n+1}\}$, we can write

$$\begin{aligned}\|u_{n+1} - q^*\|^2 &= \|(1 - \phi_n - \psi_n)u_n + \phi_n q_n - q^*\|^2 \\ &= \|(1 - \phi_n - \psi_n)(u_n - q^*) + \phi_n(q_n - q^*) - \psi_n q^*\|^2 \\ &= \|(1 - \phi_n - \psi_n)(u_n - q^*) + \phi_n(q_n - q^*)\|^2 \\ &\quad + \psi_n^2\|q^*\|^2 - 2\langle(1 - \phi_n - \psi_n)(u_n - q^*) \\ &\quad + \phi_n(q_n - q^*), \psi_n q^*\rangle.\end{aligned} \tag{63}$$

By the use of expression (59), we have

$$\begin{aligned}&\|(1 - \phi_n - \psi_n)(u_n - q^*) + \phi_n(q_n - q^*)\|^2 \\ &\quad \le (1 - \phi_n - \psi_n)(1 - \psi_n)\|u_n - q^*\|^2 + \phi_n(1 - \psi_n)\|q_n - q^*\|^2.\end{aligned} \tag{64}$$

Combining expressions (63) and (64) (for some $K_2 > 0$), we obtain

$$\begin{aligned}\|u_{n+1}-q^*\|^2 &\le (1-\phi_n-\psi_n)(1-\psi_n)\|u_n-q^*\|^2+\phi_n(1-\psi_n)\|q_n-q^*\|^2+\psi_n K_2\\ &\le (1-\phi_n-\psi_n)(1-\psi_n)\|u_n-q^*\|^2+\psi_n K_2+\phi_n(1-\psi_n)\\ &\quad\cdot\left[\|u_n-q^*\|^2-\left(1-\frac{\mu\delta_n}{\delta_{n+1}}\right)\|u_n-p_n\|^2-\left(1-\frac{\mu\delta_n}{\delta_{n+1}}\right)\|q_n-p_n\|^2\right]\\ &= (1-\psi_n)^2\|u_n-q^*\|^2+\psi_n K_2-\phi_n(1-\psi_n)\\ &\quad\cdot\left[\left(1-\frac{\mu\delta_n}{\delta_{n+1}}\right)\|u_n-p_n\|^2+\left(1-\frac{\mu\delta_n}{\delta_{n+1}}\right)\|q_n-p_n\|^2\right]\\ &\le \|u_n-q^*\|^2+\psi_n K_2-\phi_n(1-\psi_n)\\ &\quad\cdot\left[\left(1-\frac{\mu\delta_n}{\delta_{n+1}}\right)\|u_n-p_n\|^2+\left(1-\frac{\mu\delta_n}{\delta_{n+1}}\right)\|q_n-p_n\|^2\right].\end{aligned} \tag{65}$$

□

The remainder of the proof is now split into two parts:

Case 1. Suppose that there exists a fixed number $n_1 \in \mathbb{N}$ $(n_1 \ge n_0)$ such that

$$\|u_{n+1}-q^*\| \le \|u_n-q^*\|, \quad \forall n \ge n_1. \tag{66}$$

Then, $\lim_{n\longrightarrow\infty}\|u_n-q^*\|$ exists. From (65), we have

$$\begin{aligned}&\phi_n(1-\psi_n)\left[\left(1-\frac{\mu\delta_n}{\delta_{n+1}}\right)\|u_n-p_n\|^2+\left(1-\frac{\mu\delta_n}{\delta_{n+1}}\right)\|q_n-p_n\|^2\right]\\ &\quad\le \|u_n-q^*\|^2+\psi_n K_2-\|u_{n+1}-q^*\|^2.\end{aligned} \tag{67}$$

Due to the existence of $\lim_{n\longrightarrow+\infty}\|u_n-q^*\|$ and $\psi_n \longrightarrow 0$, we infer that

$$\lim_{n\longrightarrow\infty}\|u_n-p_n\| = \lim_{n\longrightarrow\infty}\|q_n-p_n\| = 0. \tag{68}$$

It follows that

$$\lim_{n\longrightarrow\infty}\|u_n-q_n\| \le \lim_{n\longrightarrow\infty}\|u_n-p_n\| + \lim_{n\longrightarrow\infty}\|p_n-q_n\| = 0. \tag{69}$$

It follows from expression (69) and $\psi_n \longrightarrow 0$ that

$$\begin{aligned}\|u_{n+1}-u_n\| &= \|(1-\phi_n-\psi_n)u_n+\phi_n q_n-u_n\|\\ &= \|u_n-\psi_n u_n+\phi_n q_n-\phi_n u_n-u_n\|\\ &\le \phi_n\|q_n-u_n\|+\psi_n\|u_n\|,\end{aligned} \tag{70}$$

which implies that

$$\|u_{n+1}-u_n\| \longrightarrow 0 \text{ as } n \longrightarrow +\infty. \tag{71}$$

We have $q^* = P_\Omega(0)$ and by using Lemma 1 (i), we can write

$$\langle 0-q^*, y-q^*\rangle \le 0, \quad \forall y \in \Omega. \tag{72}$$

By using expression (72) and Lemma 7, we obtain

$$\limsup_{n\longrightarrow\infty}\langle q^*, q^*-u_n\rangle = \limsup_{k\longrightarrow\infty}\langle q^*, q^*-u_{n_k}\rangle = \langle q^*, q^*-\hat{u}\rangle \le 0. \tag{73}$$

Due to $\lim_{n\longrightarrow\infty}\|u_{n+1}-u_n\| = 0$. It gives that

$$\limsup_{n\longrightarrow\infty}\langle q^*, q^*-u_{n+1}\rangle \le \limsup_{n\longrightarrow\infty}\langle q^*, q^*-u_n\rangle + \limsup_{n\longrightarrow\infty}\langle q^*, u_n-u_{n+1}\rangle \le 0. \tag{74}$$

Next, we assume that

$$t_n = (1-\phi_n)u_n+\phi_n q_n. \tag{75}$$

Thus, we have

$$\begin{aligned}u_{n+1} &= t_n-\psi_n u_n = (1-\psi_n)t_n-\psi_n(u_n-t_n)\\ &= (1-\psi_n)t_n-\psi_n\phi_n(u_n-q_n),\end{aligned} \tag{76}$$

where

$$u_n-t_n = u_n-(1-\phi_n)u_n-\phi_n q_n = \phi_n(u_n-q_n). \tag{77}$$

Thus, we obtain

$$\begin{aligned}\|u_{n+1}-q^*\|^2 &= \|(1-\psi_n)t_n+\phi_n\psi_n(q_n-u_n)-q^*\|^2\\ &= \|(1-\psi_n)(t_n-q^*)+[\phi_n\psi_n(q_n-u_n)-\psi_n q^*]\|^2\\ &\le (1-\psi_n)^2\|t_n-q^*\|^2+2\langle\phi_n\psi_n(q_n-u_n)\\ &\quad-\psi_n q^*, (1-\psi_n)(t_n-q^*)+\phi_n\psi_n(q_n-u_n)-\psi_n q^*\rangle\\ &= (1-\psi_n)^2\|t_n-q^*\|^2+2\langle\phi_n\psi_n(q_n-u_n)\\ &\quad-\psi_n q^*, t_n-\psi_n t_n-\psi_n(u_n-t_n)-q^*\rangle\\ &= (1-\psi_n)\|t_n-q^*\|^2+2\phi_n\psi_n\langle q_n-u_n, u_{n+1}-q^*\rangle\\ &\quad+2\psi_n\langle q^*, q^*-u_{n+1}\rangle\\ &\le (1-\psi_n)\|t_n-q^*\|^2+2\phi_n\psi_n\|q_n-u_n\|\|u_{n+1}\\ &\quad-q^*\|+2\psi_n\langle q^*, q^*-u_{n+1}\rangle.\end{aligned} \tag{78}$$

Next, we have to compute

$$\begin{aligned}\|t_n-q^*\|^2 &= \|(1-\phi_n)u_n+\phi_n q_n-q^*\|^2\\ &= \|(1-\phi_n)(u_n-q^*)+\phi_n(q_n-q^*)\|^2\\ &= (1-\phi_n)^2\|u_n-q^*\|^2+\phi_n^2\|q_n-q^*\|^2\\ &\quad+2\langle(1-\phi_n)(u_n-q^*), \phi_n(q_n-q^*)\rangle\\ &\le (1-\phi_n)^2\|u_n-q^*\|^2+\phi_n^2\|q_n-q^*\|^2\\ &\quad+2\phi_n(1-\phi_n)\|u_n-q^*\|\|q_n-q^*\|\\ &\le (1-\phi_n)^2\|u_n-q^*\|^2+\phi_n^2\|q_n-q^*\|^2\\ &\quad+\phi_n(1-\phi_n)\|u_n-q^*\|^2+\phi_n(1-\phi_n)\|q_n-q^*\|^2\\ &= (1-\phi_n)\|u_n-q^*\|^2+\phi_n\|q_n-q^*\|^2\\ &\le (1-\phi_n)\|u_n-q^*\|^2+\phi_n\|u_n-q^*\|^2 = \|u_n-q^*\|^2.\end{aligned} \tag{79}$$

Step 0: take $u_1 \in \mathcal{A}, \delta_1 > 0$ and select a nonnegative sequence of real number $\{\varphi_n\}$ such that $\sum_{n=1}^{+\infty}\varphi_n < +\infty$. Moreover, $\{\psi_n\} \subset (0,1)$ meet the following criteria:
$\lim_{n\longrightarrow+\infty}\psi_n = 0$ and $\sum_{n=1}^{+\infty}\psi_n = +\infty$.
Step 1: evaluate
$p_n = P_{\mathcal{A}}(u_n - \delta_n \mathcal{L}(u_n))$.
If $u_n = p_n$, then STOP. Otherwise, go to Step 2.
Step 2: evaluate
$\mathcal{Z}_n = \{z \in \mathcal{Z} : \langle u_n - \delta_n \mathcal{L}(u_n) - p_n, z - p_n\rangle \le 0\}$,
and compute
$q_n = P_{\mathcal{Z}_n}(u_n - \delta_n \mathcal{L}(p_n))$.
Step 3: evaluate
$u_{n+1} = \psi_n g(u_n) + (1-\psi_n) q_n$.
Step 4: evaluate

$$\delta_{n+1} = \begin{cases} \min\left\{\delta_n + \varphi_n, \left(\mu\|u_n - p_n\|^2 + \mu\|q_n - p_n\|^2/2[\langle \mathcal{L}(u_n) - \mathcal{L}(p_n), q_n - p_n\rangle]\right)\right\} & \text{if } \langle \mathcal{L}(u_n) - \mathcal{L}(p_n), q_n - p_n\rangle > 0, \\ \varphi_n + \delta_n, & \text{otherwise.} \end{cases} \quad (3.2)$$

Set $n := n+1$ and go back to Step 1.

Algorithm 2: Nonmonotonic explicit viscosity-type subgradient extragradient method.

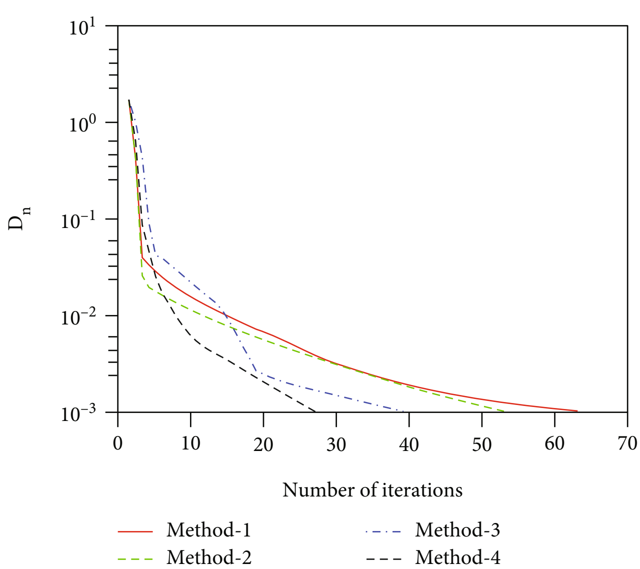

Figure 1: Algorithmic descriptions of Algorithm 1 and Algorithm 2 as well Algorithm 2 in [37] and Algorithm 1 in [36] when $m = 5$.

From expressions (78) and (79), we have

$$\|u_{n+1} - q^*\|^2 \le (1-\psi_n)\|u_n - q^*\|^2 + \psi_n[2\phi_n\|q_n - u_n\|\|u_{n+1} - q^*\| + 2\langle q^*, q^* - u_{n+1}\rangle]. \quad (80)$$

By the use of expressions (74) and (80) and Lemma 2, we can derive that $\|u_n - q^*\| \longrightarrow 0$ as $n \longrightarrow +\infty$.

Case 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ in order that

$$\|u_{n_i} - q^*\| \le \|u_{n_{i+1}} - q^*\|, \quad \forall i \in \mathbb{N}. \quad (81)$$

By Lemma 3, we have

$$\|u_{m_k} - q^*\| \le \|u_{m_{k+1}} - q^*\| \quad \text{and} \quad \|u_k - q^*\| \le \|u_{m_{k+1}} - q^*\|, \quad \forall k \in \mathbb{N}. \quad (82)$$

By the use of expression (67), we have

$$\phi_{m_k}\left(1-\psi_{m_k}\right)\left[\left(1 - \frac{\mu\delta_{m_k}}{\delta_{m_k+1}}\right)\left\|u_{m_k} - p_{m_k}\right\|^2 + \left(1 - \frac{\mu\delta_{m_k}}{\delta_{m_k+1}}\right)\left\|q_{m_k} - p_{m_k}\right\|^2\right] \le \|u_{m_k} - q^*\|^2 + \psi_{m_k}K_2 - \|u_{m_k+1} - q^*\|^2. \quad (83)$$

Due to $\psi_{m_k} \longrightarrow 0$, we can deduce as follows:

$$\lim_{k\longrightarrow\infty}\left\|u_{m_k} - p_{m_k}\right\| = \lim_{k\longrightarrow\infty}\left\|q_{m_k} - p_{m_k}\right\| = 0. \quad (84)$$

It follows that

$$\lim_{k\longrightarrow\infty}\left\|u_{m_k} - q_{m_k}\right\| \le \lim_{k\longrightarrow\infty}\left\|u_{m_k} - p_{m_k}\right\| + \lim_{k\longrightarrow\infty}\left\|p_{m_k} - q_{m_k}\right\| = 0. \quad (85)$$

Further, it implies that

$$\begin{aligned} \|u_{m_k+1} - u_{m_k}\| &= \left\|\left(1 - \phi_{m_k} - \psi_{m_k}\right)u_{m_k} + \phi_{m_k}q_{m_k} - u_{m_k}\right\| \\ &= \left\|u_{m_k} - \psi_{m_k}u_{m_k} + \phi_{m_k}q_{m_k} - \phi_{m_k}u_{m_k} - u_{m_k}\right\| \\ &\le \phi_{m_k}\left\|q_{m_k} - u_{m_k}\right\| + \psi_{m_k}\left\|u_{m_k}\right\| \longrightarrow 0. \end{aligned} \quad (86)$$

Similar to Case 1, we obtain

$$\limsup_{k\longrightarrow\infty}\left\langle q^*, u_{m_k+1} - q^*\right\rangle \le 0. \quad (87)$$

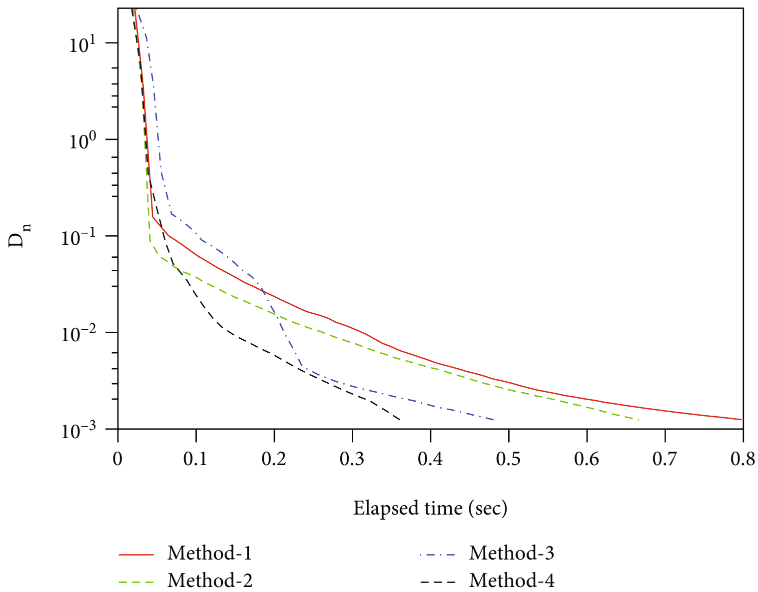

Figure 2: Algorithmic descriptions of Algorithm 1 and Algorithm 2 with Algorithm 2 in [37] and Algorithm 1 in [36] when $m = 5$.

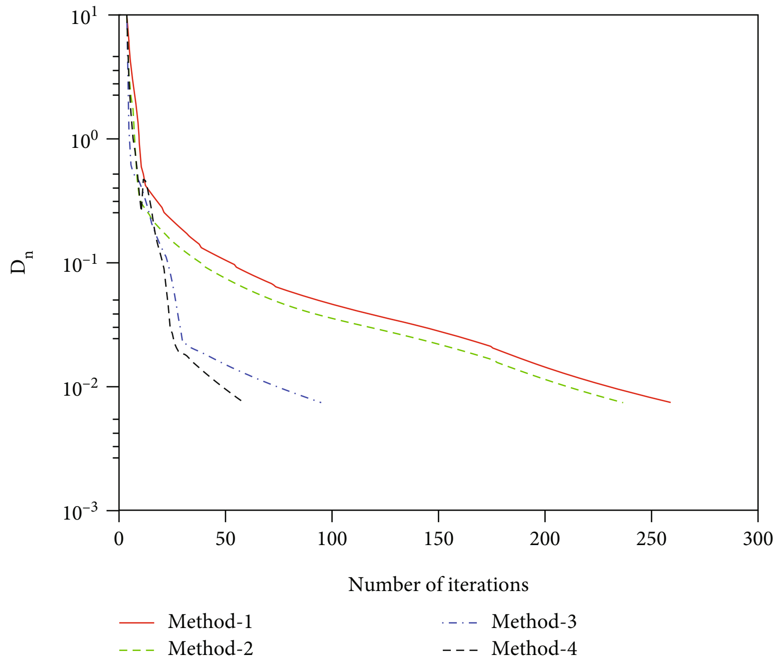

Figure 3: Algorithmic descriptions of Algorithm 1 and Algorithm 2 with Algorithm 2 in [37] and Algorithm 1 in [36] when $m = 10$.

By the use of expressions (80) and (82), we have

$$\begin{aligned} \left\|u_{m_k+1} - q^*\right\|^2 &\le \left(1 - \psi_{m_k}\right)\left\|u_{m_k} - q^*\right\|^2 + \psi_{m_k} \\ &\quad \cdot \left[2\phi_{m_k}\left\|q_{m_k} - u_{m_k}\right\|\left\|u_{m_k+1} - q^*\right\| + 2\left\langle q^*, q^* - u_{m_k+1}\right\rangle\right] \\ &\le \left(1 - \psi_{m_k}\right)\left\|u_{m_{k+1}} - q^*\right\|^2 + \psi_{m_k} \\ &\quad \cdot \left[2\phi_{m_k}\left\|q_{m_k} - u_{m_k}\right\|\left\|u_{m_k+1} - q^*\right\| + 2\left\langle q^*, q^* - u_{m_k+1}\right\rangle\right]. \end{aligned} \tag{88}$$

It follows that

$$\left\|u_{m_k+1} - q^*\right\|^2 \le 2\phi_{m_k}\left\|q_{m_k} - u_{m_k}\right\|\left\|u_{m_k+1} - q^*\right\| + 2\left\langle q^*, q^* - u_{m_k+1}\right\rangle. \tag{89}$$

Since $\psi_{m_k} \longrightarrow 0$ and $\|u_{m_k} - q^*\|$ is a bounded sequence with (87) and (89), we have

$$\left\|u_{m_k+1} - q^*\right\|^2 \longrightarrow 0, \text{as} k \longrightarrow \infty. \tag{90}$$

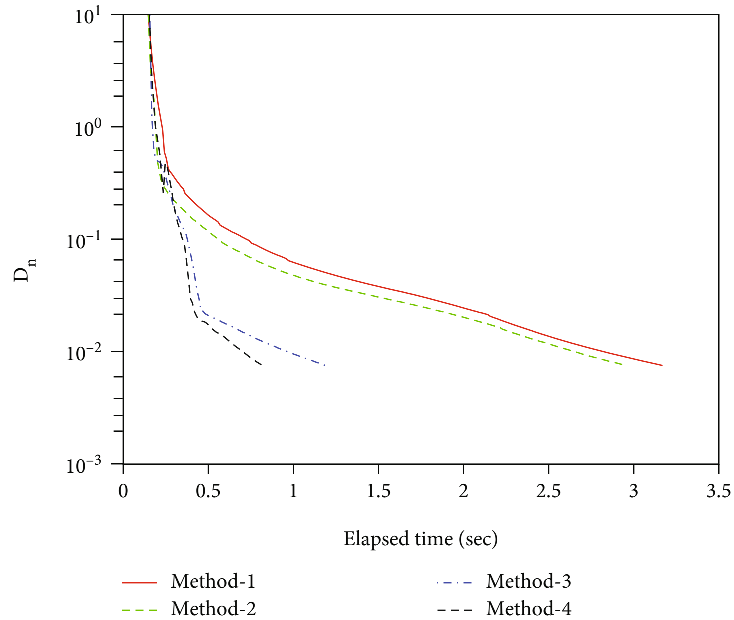

FIGURE 4: Algorithmic descriptions of Algorithm 1 and Algorithm 2 with Algorithm 2 in [37] and Algorithm 1 in [36] when $m = 10$.

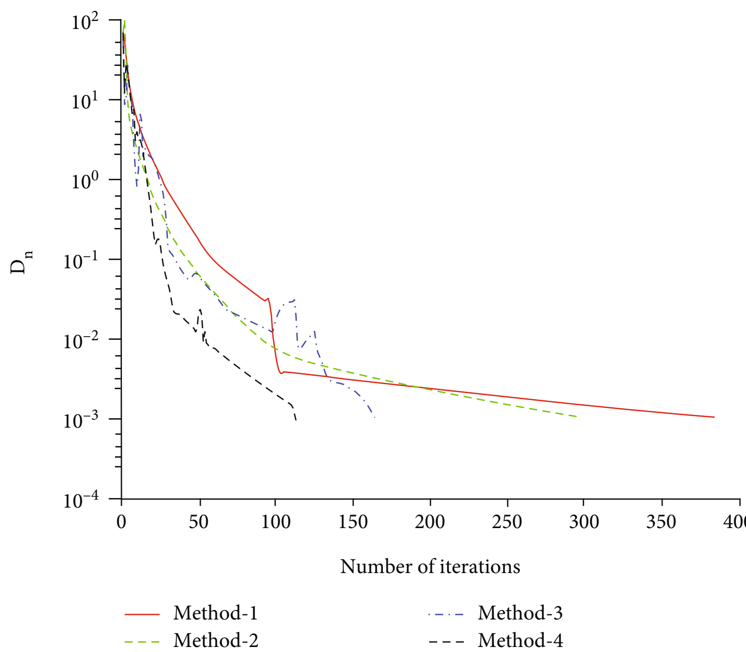

FIGURE 5: Algorithmic descriptions of Algorithm 1 and Algorithm 2 with Algorithm 2 in [37] and Algorithm 1 in [36] when $m = 20$.

The above expression implies that

$$\lim_{k \longrightarrow \infty} \|u_k - q^*\|^2 \leq \lim_{k \longrightarrow \infty} \left\|u_{m_k+1} - q^*\right\|^2 \leq 0. \quad (91)$$

From the above discussions, we have $u_n \longrightarrow q^*$ as $n \longrightarrow \infty$.

Theorem 9. *Assume that $\mathscr{L} : \mathscr{E} \longrightarrow \mathscr{E}$ is a mapping that meets the conditions Condition 1–Condition 4. Then, the $\{u_n\}$ sequence generated by Algorithm 2 strongly converges to an element $q^* \in \Omega$.*

Proof. Since $\delta_n \longrightarrow \delta$, there exists a positive number $\varepsilon \in (0, 1 - \mu)$ such that

$$\lim_{n \longrightarrow \infty} \left(1 - \frac{\mu \delta_n}{\delta_{n+1}}\right) = 1 - \mu > \varepsilon > 0. \quad (92)$$

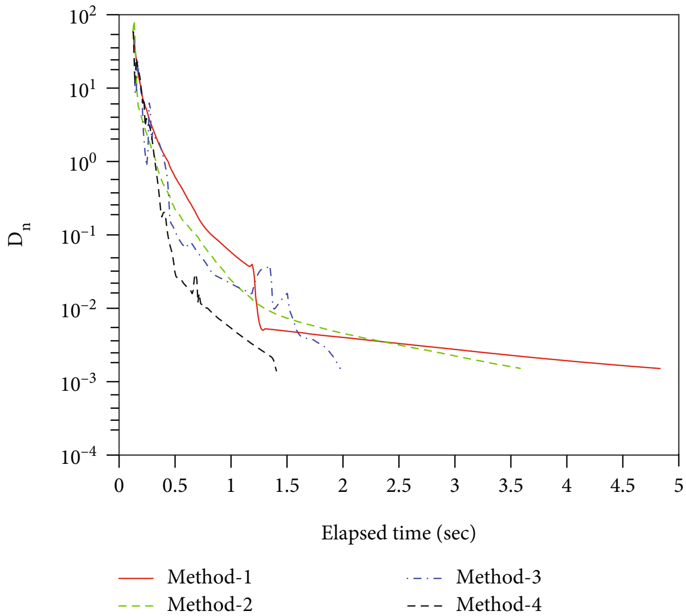

Figure 6: Algorithmic descriptions of Algorithm 1 and Algorithm 2 with Algorithm 2 in [37] and Algorithm 1 in [36] when $m = 20$.

Table 1: Algorithmic study for Example 11.

m	Method-1		Method-2		Method-3		Method-4	
	Iter.	Time	Iter.	Time	Iter.	Time	Iter.	Time
5	68	0.798005200	57	0.665091600	42	0.480156900	29	0.360117900
10	264	3.218157200	241	2.998657400	95	1.121305500	58	0.733156900
20	384	4.884210700	297	3.593486400	164	1.930182900	113	1.339267400
30	649	8.356499700	545	7.850765000	308	3.986678100	196	2.593555200
40	1138	16.96909990	997	15.25866440	357	4.784712300	238	3.410400500
50	2467	37.42977650	17962	24.32523500	558	10.32619485	487	6.132444757

Table 2: Algorithmic study for Example 12.

m	Method-1		Method-2		Method-3		Method-4	
	Iter.	Time	Iter.	Time	Iter.	Time	Iter.	Time
1	482	0.294028700	406	0.211852600	349	0.166720900	308	0.166098200
t^2	268	0.136482920	196	0.112184600	155	0.103728300	112	0.094193700
e^t	371	0.145282000	311	0.123552000	255	0.099748260	211	0.073510000
$\sin(t)$	431	0.212849200	389	0.201682200	279	0.112940100	239	0.102749200

Thus, there exists a number $N_1 \in \mathbb{N}$ such that

$$\left(1 - \frac{\mu\delta_n}{\delta_{n+1}}\right) > \varepsilon > 0, \quad \forall n \geq N_1. \tag{93}$$

From expression (36), we obtain

$$\|q_n - q^*\|^2 \leq \|u_n - q^*\|^2, \quad \forall n \geq N_1. \tag{94}$$

Next, we have

$$\begin{aligned}\|u_{n+1} - q^*\| &= \|\psi_n g(u_n) + (1 - \psi_n)q_n - q^*\| \\ &= \|\psi_n[g(u_n) - q^*] + (1 - \psi_n)[q_n - q^*]\| \\ &= \|\psi_n[g(u_n) + g(q^*) - g(q^*) - q^*] + (1 - \psi_n)[q_n - q^*]\| \\ &\leq \psi_n\|g(u_n) - g(q^*)\| + \psi_n\|g(q^*) - q^*\| + (1 - \psi_n)\|q_n - q^*\| \\ &\leq \psi_n\xi\|u_n - q^*\| + \psi_n\|g(q^*) - q^*\| + (1 - \psi_n)\|q_n - q^*\|.\end{aligned} \tag{95}$$

From expressions (94) and (95) and $\psi_n \subset (0, 1)$, we

Table 3: Example 13: algorithmic description of Algorithm 2 in [37] and $u_1 = (1, 2, 3, 4)^T$.

Iter. (n)	u_1	u_2	u_3	u_4
1	4.59999999902338	18.5464999999117	18.7004999999153	17.5944999993118
2	-6.26760004899506	16.7378886580863	16.9671141650914	15.8548101409442
3	-5.91418181483705	16.0308795300640	16.2986437617166	15.2919549387799
4	-5.38921479466356	4.78801650028211	4.64461284974750	5.38714304326419
5	1.00580406596249	4.73241274186297	4.62870240188635	5.25179596894124
6	1.23013013951725	4.70677542647729	4.63592929873170	4.97142857141815
7	1.44950907993550	4.70213655930865	4.65912535719577	4.97499999984847
8	1.66717886007189	4.71660015705847	4.69751098404819	4.97777776498426
9	1.88563318471514	4.74871960483109	4.75038581190807	4.97999999900850
10	2.10707812245062	4.79781281512387	4.81759412391964	4.98181818166635
11	2.33359792353143	4.86377467908333	4.89941458277316	4.98333333327664
12	2.56667054007920	4.94667114193612	4.98265426879577	4.98294766642007
13	2.79924540130681	4.96031745289142	4.97036949333673	4.98462062077961
14	3.03033454990915	4.96124045379347	4.97356261422028	4.98546003308369
15	3.26012741095086	4.96358686948872	4.97505259464489	4.98655189412727
⋮	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮
1086	4.99924843119192	4.99944280475227	4.99963717745358	4.99983154929598
1087	4.99924912189630	4.99944331684910	4.99963751094444	4.99983170418246
1088	4.99924981133227	4.99944382800546	4.99963784382278	4.99983185878434
1089	4.99925049950333	4.99944433822395	4.99963817609027	4.99983201310240
1090	4.99925118641296	4.99944484750715	4.99963850774860	4.99983216713743
1091	4.99925187206462	4.99944535585762	4.99963883879944	4.99983232089020
1092	4.99925255646177	4.99944586327793	4.99963916924446	4.99983247436150
1093	4.99925323960783	4.99944636977061	4.99963949908532	4.99983262755208
1094	4.99925392150626	4.99944687533823	4.99963982832368	4.99983278046273
1095	4.99925460216044	4.99944737998330	4.99964015696118	4.99983293309420
1096	4.99925528157380	4.99944788370835	4.99964048499947	4.99983308544727
1097	4.99925595974971	4.99944838651589	4.99964081244018	4.99983323752269
1098	4.99925663669156	4.99944888840844	4.99964113928495	4.99983338932122
1099	4.99925731240271	4.99944938938848	4.99964146553540	4.99983354084361
1100	4.99925798688652	4.99944988945849	4.99964179119316	4.99983369209063
1101	4.99925866014632	4.99945038862097	4.99964211625983	4.99983384306301
1102	4.99925933218544	4.99945088687837	4.99964244073702	4.99983399376152
1103	4.99926000300721	4.99945138423316	4.99964276462634	4.99983414418688
1104	4.99926067261492	4.99945188068778	4.99964308792938	4.99983429433984
1105	4.99926134101187	4.99945237624468	4.99964341064774	4.99983444422115
1106	4.99926200820134	4.99945287090629	4.99964373278299	4.99983459383154
1107	4.99926267418660	4.99945336467504	4.99964405433671	4.99983474317174
CPU time is seconds	8.339245			

obtain

$$\begin{aligned}\|u_{n+1}-q^*\| &\le \psi_n\xi\|u_n-q^*\| + \psi_n\|g(q^*)-q^*\| + (1-\psi_n)\|u_n-q^*\| \\ &= [1-\psi_n+\xi\psi_n]\|u_n-q^*\| + \psi_n(1-\xi)\frac{\|g(q^*)-q^*\|}{(1-\xi)} \\ &\le \max\left\{\|u_n-q^*\|, \frac{\|g(q^*)-q^*\|}{(1-\xi)}\right\} \le \max\left\{\|u_{N_1}-q^*\|, \frac{\|g(q^*)-q^*\|}{(1-\xi)}\right\}.\end{aligned} \tag{96}$$

Therefore, we infer that the $\{u_n\}$ is a bounded sequence. Now, we are in a position to use the Banach contraction theorem for the existence of a unique fixed point $q^* \in \Omega$ such that

$$q^* = P_\Omega(g(q^*)). \tag{97}$$

Due to the projection mapping, we can write

$$\langle g(q^*) - q^*, y - q^* \rangle \leq 0, \quad \forall y \in \Omega. \tag{98}$$

From Lemma 1 and expression (36), we have

$$\begin{aligned}
\|u_{n+1} - q^*\|^2 &= \|\psi_n g(u_n) + (1-\psi_n) q_n - q^*\|^2 \\
&= \|\psi_n [g(u_n) - q^*] + (1-\psi_n)[q_n - q^*]\|^2 \\
&= \psi_n \|g(u_n) - q^*\|^2 + (1-\psi_n)\|q_n - q^*\|^2 \\
&\quad - \psi_n (1-\psi_n)\|g(u_n) - q_n\|^2 \\
&\leq \psi_n \|g(u_n) - q^*\|^2 + (1-\psi_n) \\
&\quad \cdot \left[\| \|u_n - q^*\|^2 - \left(1 - \frac{\mu\delta_n}{\delta_{n+1}}\right)\|u_n - p_n\|^2 \right. \\
&\quad \left. - \left(1 - \frac{\mu\delta_n}{\delta_{n+1}}\right)\|q_n - p_n\|^2 \right] - \psi_n(1-\psi_n)\|g(u_n) - q_n\|^2 \\
&\leq \psi_n \|g(u_n) - q^*\|^2 + \|u_n - q^*\|^2 - (1-\psi_n) \\
&\quad \cdot \left(1 - \frac{\mu\delta_n}{\delta_{n+1}}\right)\|u_n - p_n\|^2 - (1-\psi_n)\left(1 - \frac{\mu\delta_n}{\delta_{n+1}}\right)\|q_n - p_n\|^2.
\end{aligned} \tag{99}$$

The above expression implies that

$$\begin{aligned}
&(1-\psi_n)\left(1 - \frac{\mu\delta_n}{\delta_{n+1}}\right)\|u_n - p_n\|^2 + (1-\psi_n)\left(1 - \frac{\mu\delta_n}{\delta_{n+1}}\right)\|q_n - p_n\|^2 \\
&\quad \leq \psi_n \|g(u_n) - q^*\|^2 + \|u_n - q^*\|^2 - \|u_{n+1} - q^*\|^2.
\end{aligned} \tag{100}$$

□ □

Case 1. Suppose that there exists a fixed number $N_2 \in \mathbb{N}$ $(N_2 \geq N_1)$ such that

$$\|u_{n+1} - q^*\| \leq \|u_n - q^*\|, \forall n \geq N_2. \tag{101}$$

Then, $\lim_{n\longrightarrow\infty}\|u_n - q^*\|$ exists and let $\lim_{n\longrightarrow\infty}\|u_n - q^*\| = l$. By the use of expression (99), we have

$$\begin{aligned}
&(1-\psi_n)\left(1 - \frac{\mu\delta_n}{\delta_{n+1}}\right)\|u_n - p_n\|^2 + (1-\psi_n)\left(1 - \frac{\mu\delta_n}{\delta_{n+1}}\right)\|q_n - p_n\|^2 \\
&\quad \leq \psi_n \|g(u_n) - q^*\|^2 + \| \|u_n - q^*\|^2 - \|u_{n+1} - q^*\|^2.
\end{aligned} \tag{102}$$

By the of existence of a limit of a sequence $\lim_{n\longrightarrow\infty}\|u_n - q^*\|$ and $\psi_n \longrightarrow 0$, we obtain

$$\lim_{n\longrightarrow\infty}\|u_n - p_n\| = \lim_{n\longrightarrow\infty}\|q_n - p_n\| = 0. \tag{103}$$

Furthermore, it implies that

$$\lim_{n\longrightarrow\infty}\|u_n - p_n\| \leq \lim_{n\longrightarrow\infty}\|u_n - p_n\| + \lim_{n\longrightarrow\infty}\|q_n - p_n\| = 0. \tag{104}$$

Thus, we have

$$\begin{aligned}
\|u_{n+1} - u_n\| &= \|\psi_n g(u_n) + (1-\psi_n) q_n - u_n\| \\
&= \|\psi_n [g(u_n) - u_n] + (1-\psi_n)[q_n - u_n]\| \\
&\leq \psi_n \|g(u_n) - u_n\| + (1-\psi_n)\|q_n - u_n\| \longrightarrow 0.
\end{aligned} \tag{105}$$

Thus, we have

$$\lim_{n\longrightarrow\infty}\|u_{n+1} - u_n\| = 0. \tag{106}$$

By using expression (97) and Lemma 7, we obtain

$$\begin{aligned}
\limsup_{n\longrightarrow\infty}\langle g(q^*) - q^*, u_n - q^* \rangle &= \limsup_{k\longrightarrow\infty}\langle g(q^*) - q^*, u_{n_k} - q^* \rangle \\
&= \langle g(q^*) - q^*, \hat{u} - q^* \rangle \leq 0.
\end{aligned} \tag{107}$$

By the use of $\lim_{n\longrightarrow\infty}\|u_{n+1} - u_n\| = 0$, we have

$$\begin{aligned}
\limsup_{n\longrightarrow\infty}\langle g(q^*) - q^*, u_{n+1} - q^* \rangle &\leq \limsup_{n\longrightarrow\infty}\langle g(q^*) - q^*, u_{n+1} - u_n \rangle \\
&\quad + \limsup_{n\longrightarrow\infty}\langle g(q^*) - q^*, u_n - q^* \rangle \leq 0.
\end{aligned} \tag{108}$$

By using Lemma 1 and expression (94), we have

$$\begin{aligned}
\|u_{n+1} - q^*\|^2 &= \|\psi_n g(u_n) + (1-\psi_n) q_n - q^*\|^2 \\
&= \|\psi_n [g(u_n) - q^*] + (1-\psi_n)[q_n - q^*]\|^2 \\
&\leq (1-\psi_n)^2\|q_n - q^*\|^2 + 2\psi_n \langle g(u_n) - q^*, (1-\psi_n)[q_n - q^*] \\
&\quad + \psi_n [g(u_n) - q^*] \rangle \\
&= (1-\psi_n)^2\|q_n - q^*\|^2 + 2\psi_n \langle g(u_n) - g(q^*) \\
&\quad + g(q^*) - q^*, u_{n+1} - q^* \rangle \\
&= (1-\psi_n)^2\|q_n - q^*\|^2 + 2\psi_n \langle g(u_n) - g(q^*), u_{n+1} - q^* \rangle \\
&\quad + 2\psi_n \langle g(q^*) - q^*, u_{n+1} - q^* \rangle \\
&\leq (1-\psi_n)^2\|q_n - q^*\|^2 + 2\psi_n \xi \|u_n - q^*\|^2 \|u_{n+1} - q^*\| \\
&\quad + 2\psi_n \langle g(q^*) - q^*, u_{n+1} - q^* \rangle \\
&\leq \left(1 + \psi_n^2 - 2\psi_n\right)\|u_n - q^*\|^2 + 2\psi_n \xi \|u_n - q^*\|^2 \\
&\quad + 2\psi_n \langle g(q^*) - q^*, u_{n+1} - q^* \rangle \\
&= (1 - 2\psi_n)\|u_n - q^*\|^2 + \psi_n^2 \|u_n - q^*\|^2 + 2\psi_n \xi \|u_n - q^*\|^2 \\
&\quad + 2\psi_n \langle g(q^*) - q^*, u_{n+1} - q^* \rangle = [1 - 2\psi_n (1-\xi)]\|u_n - q^*\|^2 \\
&\quad + 2\psi_n (1-\xi)\left[\frac{\psi_n \|u_n - q^*\|^2}{2(1-\xi)} + \frac{\langle g(q^*) - q^*, u_{n+1} - q^* \rangle}{1-\xi}\right].
\end{aligned} \tag{109}$$

It is clear from expressions (108) and (109) such that

$$\limsup_{n\longrightarrow\infty}\left[\frac{\psi_n \|u_n - q^*\|^2}{2(1-\xi)} + \frac{\langle g(q^*) - q^*, u_{n+1} - q^* \rangle}{1-\xi}\right] \leq 0. \tag{110}$$

By choosing $n \geq N_3 \in \mathbb{N}$ $(N_3 \geq N_2)$ large enough such that $2\gamma_n(1-\xi) < 1$. By the use of (109) and (110) and

Table 4: Example 13: algorithmic description of Algorithm 1 in [36] and $u_1 = (1, 2, 3, 4)^T$.

Iter. (n)	u_1	u_2	u_3	u_4
1	4.54999999902338	18.4464999999117	18.5504999999153	17.3944999993118
2	-6.24222797440616	13.5241851137974	13.6213471992304	12.5302619081771
3	-0.659876119446077	4.68238127646957	4.65194330093169	5.00329162462314
4	0.946802477625882	4.61084600770298	4.58119171017774	4.92362497958044
5	1.04934390009719	4.55766490194321	4.52868493067142	4.86118347395020
6	1.14854226167386	4.51824332123436	4.48986816365673	4.81270779537308
7	1.24574590629948	4.48946698164860	4.46164839905812	4.77550686438977
8	1.34146056234125	4.46933050144783	4.44203357836196	4.74729888976619
9	1.43627311838568	4.45647220663972	4.42967080774664	4.72665515468439
10	1.53065237050934	4.44994299224969	4.42361803309317	4.71255603858976
11	1.62498530558409	4.44907178330946	4.42320908822800	4.70425974095071
12	1.71962000624484	4.45338248767389	4.42797158878528	4.70123046557113
13	1.81488364798492	4.46254250802635	4.43757564752582	4.70308628647431
14	1.91109217493918	4.47632857351891	4.45180046474299	4.70956149682268
15	2.00855776273081	4.49460389018728	4.47051224149727	4.72048089550204
⋮	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮
1081	4.99949078231531	4.99949078231531	4.99949078231531	4.99949078231531
1082	4.99949130039142	4.99949130039142	4.99949130039142	4.99949130039142
1083	4.99949181741442	4.99949181741442	4.99949181741442	4.99949181741442
1084	4.99949233338752	4.99949233338752	4.99949233338752	4.99949233338752
1085	4.99949284831392	4.99949284831392	4.99949284831392	4.99949284831392
1086	4.99949336219679	4.99949336219679	4.99949336219679	4.99949336219679
1087	4.99949387503931	4.99949387503931	4.99949387503931	4.99949387503931
1088	4.99949438684463	4.99949438684463	4.99949438684463	4.99949438684463
1089	4.99949489761590	4.99949489761590	4.99949489761590	4.99949489761590
1090	4.99949540735624	4.99949540735624	4.99949540735624	4.99949540735624
1091	4.99949591606878	4.99949591606878	4.99949591606878	4.99949591606878
1092	4.99949642375661	4.99949642375661	4.99949642375661	4.99949642375661
1093	4.99949693042284	4.99949693042284	4.99949693042284	4.99949693042284
1094	4.99949743607054	4.99949743607054	4.99949743607054	4.99949743607054
1095	4.99949794070278	4.99949794070278	4.99949794070278	4.99949794070278
1096	4.99949844432262	4.99949844432262	4.99949844432262	4.99949844432262
1097	4.99949894693310	4.99949894693310	4.99949894693310	4.99949894693310
1098	4.99949944853725	4.99949944853725	4.99949944853725	4.99949944853725
1099	4.99949994913809	4.99949994913809	4.99949994913809	4.99949994913809
1000	4.99950044873863	4.99950044873863	4.99950044873863	4.99950044873863
1001	4.99950094734186	4.99950094734186	4.99950094734186	4.99950094734186
CPU time is seconds	7.690413			

through Lemma 2, we conclude that $\|u_n - q^*\| \longrightarrow 0$ as $n \longrightarrow \infty$.

Case 2. Consider that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\left\|u_{n_i} - q^*\right\| \le \left\|u_{n_{i+1}} - q^*\right\|, \quad \forall i \in \mathbb{N}. \tag{111}$$

Then, using Lemma 3, there exists a sequence $\{m_k\} \subset \mathbb{N}$ as $\{m_k\} \longrightarrow \infty$, such that

$$\left\|u_{m_k} - q^*\right\| \le \left\|u_{m_{k+1}} - q^*\right\|, \left\|u_k - q^*\right\| \le \left\|u_{m_{k+1}} - q^*\right\|, \quad \text{for all} k \in \mathbb{N}. \tag{112}$$

Table 5: Example 13: algorithmic description of Algorithm 1 and $u_1 = (1, 2, 3, 4)^T$.

Iter. (n)	u_1	u_2	u_3	u_4
1	4.76666666563579	19.3990833332401	19.4727499999106	18.2164166659403
2	-6.89912059339238	18.1614280199001	18.2430063869481	16.8357425433078
3	-6.79443152520300	17.8896263780042	17.9698451056013	16.5860356593549
4	-6.48577908884055	4.74876581346600	4.68957409065068	5.74296770759381
5	0.967259458426478	4.73386092054218	4.67600506823831	5.28600264404358
6	1.95405193843723	6.82802561081780	6.99483339988724	5.16669743240466
7	3.19683502766684	5.04658834550239	5.03534741320422	5.05670610709957
8	4.88155078605583	4.96308402800923	4.96500619808367	4.96133883492052
9	1.29027081990860	6.20942957200619	6.32344981434144	6.10583077439888
10	-16.8977269679284	39.2523493068962	39.5416762684644	38.9810641968152
11	-16.8686988161152	39.2288250269955	39.5165446165549	38.9590470564150
12	-17.5936196810597	5.25777948675977	5.05059553613937	5.45334598035476
13	0.953804394536662	5.24340705918321	5.03779221686998	5.23720308394482
14	-2.97340727848293	17.2929482826102	17.1772480445095	17.2896868859967
15	-2.98444635460549	17.2911791507813	17.1759609970061	17.2879313433204
⋮	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮
718	4.99913721960184	4.99966942735891	4.99966941266818	4.99966942693145
719	4.99913841886068	4.99966988620944	4.99966987154072	4.99966988578262
720	4.99913961479023	4.99967034378793	4.99967032914114	4.99967034336174
721	4.99914080740432	4.99967080009965	4.99967078547474	4.99967079967410
722	4.99914199671672	4.99967125514986	4.99967124054676	4.99967125472495
723	4.99914318274110	4.99967170894380	4.99967169436244	4.99967170851951
724	4.99914436549112	4.99967216148664	4.99967214692697	4.99967216106299
725	4.99914554498034	4.99967261278354	4.99967259824550	4.99967261236052
726	4.99914672122220	4.99967306283966	4.99967304832317	4.99967306241727
727	4.99914789423003	4.99967351166012	4.99967349716513	4.99967351123835
728	4.99914906401720	4.99967395924998	4.99967394477643	4.99967395882883
729	4.99915023059699	4.99967440561429	4.99967439116212	4.99967440519377
730	4.99915139398255	4.99967485075809	4.99967483632724	4.99967485033819
731	4.99915255418697	4.99967529468638	4.99967528027678	4.99967529426709
732	4.99915371122326	4.99967573740412	4.99967572301572	4.99967573698545
733	4.99915486510439	4.99967617891628	4.99967616454901	4.99967617849822
734	4.99915601584329	4.99967661922774	4.99967660488156	4.99967661881030
735	4.99915716345274	4.99967705834342	4.99967704401826	4.99967705792660
736	4.99915830794551	4.99967749626818	4.99967748196397	4.99967749585196
737	4.99915944933426	4.99967793300684	4.99967791872353	4.99967793259123
738	4.99916058763160	4.99967836856424	4.99967835430177	4.99967836814924
739	4.99916172285006	4.99967880294516	4.99967878870346	4.99967880253076
740	4.99916285500216	4.99967923615434	4.99967922193337	4.99967923574055
741	4.99916398410028	4.99967966819653	4.99967965399622	4.99967966778334
742	4.99916511015677	4.99968009907643	4.99968008489673	4.99968009866384
743	4.99916623318388	4.99968052879874	4.99968051463958	4.99968052838674
744	4.99916735319382	4.99968095736811	4.99968094322945	4.99968095695672
745	4.99916847019876	4.99968138478917	4.99968137067095	4.99968138437837
CPU time is seconds	5.182248			

Table 6: Example 13: algorithmic description of Algorithm 2 and $u_1 = (1, 2, 3, 4)^T$.

Iter. (n)	u_1	u_2	u_3	u_4
1	3.72399999925560	14.5721099999327	14.9072699999354	14.2820299994755
2	-2.15456883397507	14.1819518247169	14.5351508101264	13.8743472579798
3	-1.91884564644980	14.7335505679521	15.0699505703768	14.4407029303704
4	-3.75031965270586	8.60066516780067	8.66301230690689	8.54719173870520
5	-0.610354905863696	5.97030538364658	5.97335233684089	5.96773839611341
6	-0.760180052633688	6.39612802072305	6.39915756275061	6.39357570169468
7	0.883426357410548	5.46334299554989	5.45041815549462	5.47423303492034
8	2.24275125763655	5.11576078694906	5.12103138003741	5.11131429071216
9	1.88904098349673	6.05244056925146	6.05529370932879	6.05002755957568
10	2.94173536203857	5.29641543454450	5.29701704593678	5.29590634994195
11	3.95963752720103	5.10693325887312	5.10709889135520	5.10679307211866
12	4.05539113558902	5.22425012478815	5.22438849354865	5.22413300798106
13	4.48640280826840	5.09245571277969	5.09251889856043	5.09240223042630
14	4.63687195165905	5.07427886151574	5.07432105870254	5.07424314405336
15	4.78911903854907	5.03528185258641	5.03530457387877	5.03526262019362
⋮	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮
616	4.99913861830495	4.99967028584338	4.99967032490328	4.99967025278006
617	4.99914001259341	4.99967081825953	4.99967085725320	4.99967078525225
618	4.99914140237539	4.99967134895898	4.99967138788665	4.99967131600757
619	4.99914278767268	4.99967187795002	4.99967191681191	4.99967184505430
620	4.99914416850695	4.99967240524090	4.99967244403722	4.99967237240068
621	4.99914554489976	4.99967293083979	4.99967296957076	4.99967289805489
622	4.99914691687247	4.99967345475484	4.99967349342067	4.99967342202507
623	4.99914828444637	4.99967397699410	4.99967401559502	4.99967394431929
624	4.99914964764256	4.99967449756562	4.99967453610183	4.99967446494558
625	4.99915100648201	4.99967501647738	4.99967505494909	4.99967498391194
626	4.99915236098562	4.99967553373728	4.99967557214471	4.99967550122626
627	4.99915371117408	4.99967604935322	4.99967608769656	4.99967601689643
628	4.99915505706797	4.99967656333301	4.99967660161249	4.99967653093029
629	4.99915639868777	4.99967707568443	4.99967711390025	4.99967704333559
631	4.99915773605379	4.99967758641521	4.99967762456757	4.99967755412008
632	4.99915906918624	4.99967809553302	4.99967813362214	4.99967806329143
633	4.99916039810519	4.99967860304549	4.99967864107156	4.99967857085726
634	4.99916172283057	4.99967910896021	4.99967914692344	4.99967907682517
635	4.99916304338221	4.99967961328470	4.99967965118530	4.99967958120268
636	4.99916435977982	4.99968011602645	4.99968015386462	4.99968008399728
637	4.99916567204293	4.99968061719291	4.99968065496885	4.99968058521642
638	4.99916698019104	4.99968111679145	4.99968115450536	4.99968108486747
639	4.99916828424343	4.99968161482945	4.99968165248151	4.99968158295780
CPU time is seconds	4.517220			

Thus, we have

$$\left(1-\psi_{m_k}\right)\left(1-\frac{\mu\delta_{m_k}}{\delta_{m_k+1}}\right)\left\|u_{m_k}-p_{m_k}\right\|^2+\left(1-\psi_{m_k}\right)\left(1-\frac{\mu\delta_{m_k}}{\delta_{m_k+1}}\right)\left\|q_{m_k}-p_{m_k}\right\|^2 \le \psi_{m_k}\left\|g\left(u_{m_k}\right)-q^*\right\|^2+\left\|u_{m_k}-q^*\right\|^2-\left\|u_{m_k+1}-q^*\right\|^2. \tag{113}$$

Since $\psi_{m_k} \longrightarrow 0$ implies that

$$\lim_{k\longrightarrow\infty}\left\|u_{m_k}-p_{m_k}\right\| = \lim_{k\longrightarrow\infty}\left\|q_{m_k}-p_{m_k}\right\| = 0. \tag{114}$$

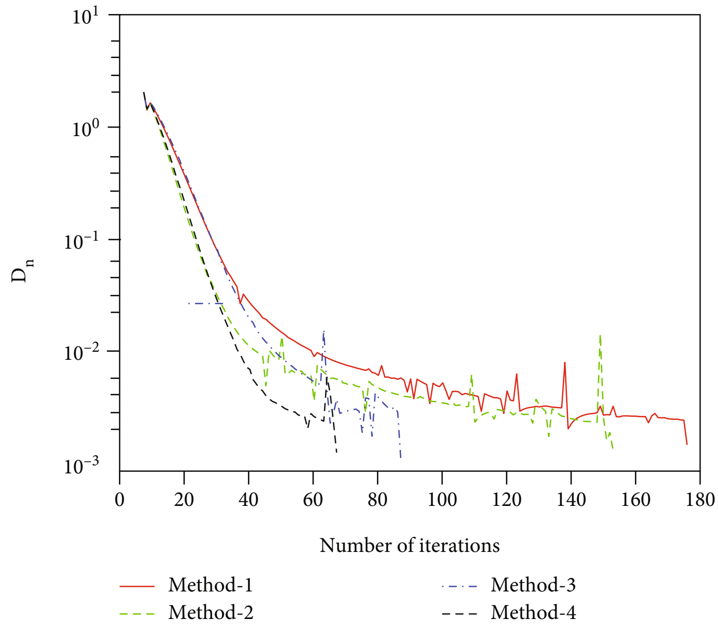

Figure 7: Algorithmic descriptions of Algorithm 1 and Algorithm 2 as well as Algorithm 2 in [37] and Algorithm 1 in [36] when $u_1 = [1.5,1.7]^T$.

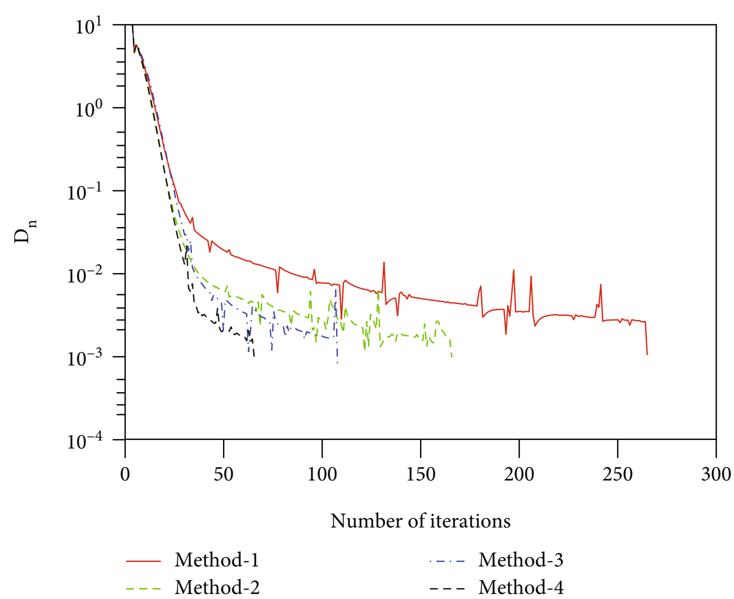

Figure 8: Algorithmic descriptions of Algorithm 1 and Algorithm 2 as well as Algorithm 2 in [37] and Algorithm 1 in [36] when $u_1 = [0, 0]^T$.

Next, we obtain

$$\begin{aligned}\left\|u_{m_k+1}-u_{m_k}\right\| &= \left\|\psi_{m_k} g\left(u_{m_k}\right)+\left(1-\psi_{m_k}\right) q_{m_k}-u_{m_k}\right\| \\ &= \left\|\psi_{m_k}\left[g\left(u_{m_k}\right)-u_{m_k}\right]+\left(1-\psi_{m_k}\right)\left[q_{m_k}-u_{m_k}\right]\right\| \\ &\leq \psi_{m_k}\left\|g\left(u_{m_k}\right)-u_{m_k}\right\|+\left(1-\psi_{m_k}\right)\left\|q_{m_k}-u_{m_k}\right\| \longrightarrow 0.\end{aligned} \tag{115}$$

Similar to the Case 1, we can write

$$\limsup_{k \longrightarrow \infty}\left\langle g\left(q^*\right)-q^*, u_{m_k+1}-q^*\right\rangle \leq 0. \tag{116}$$

By using (109) and (112), we have

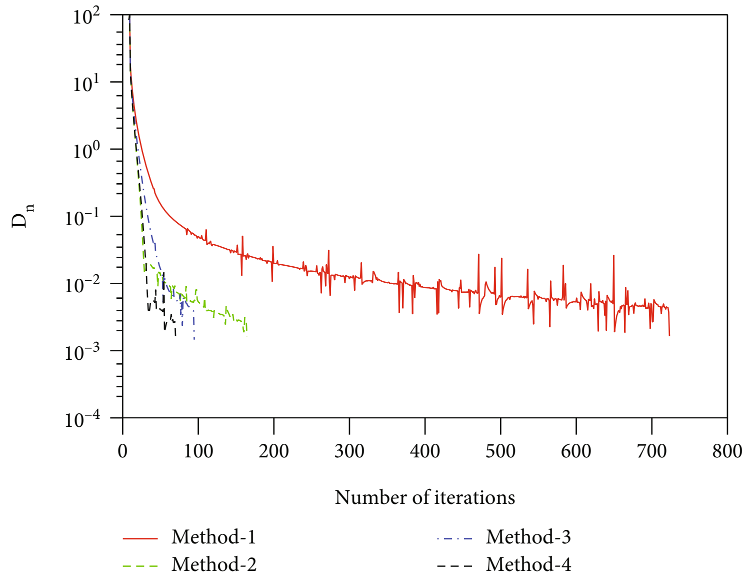

FIGURE 9: Algorithmic descriptions of Algorithm 1 and Algorithm 2 as well as Algorithm 2 in [37] and Algorithm 1 in [36] when $u_1 = [10, 10]^T$.

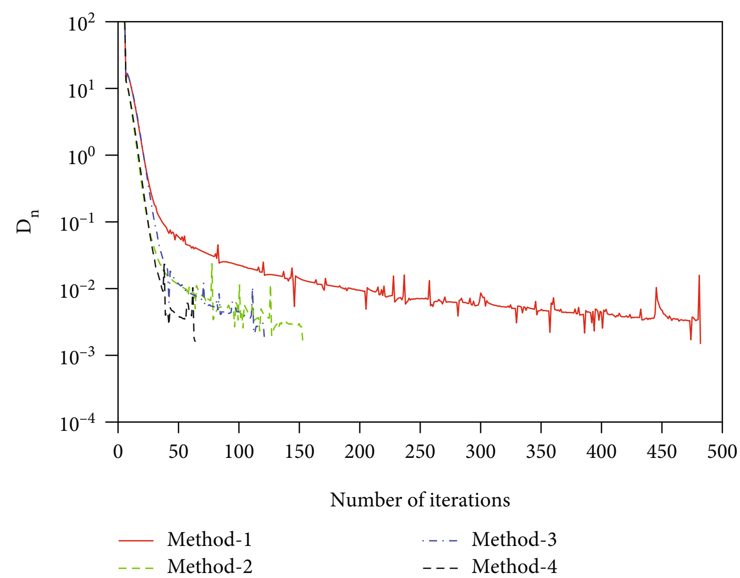

FIGURE 10: Algorithmic descriptions of Algorithm 1 and Algorithm 2 as well as Algorithm 2 in [37] and Algorithm 1 in [36] when $u_1 = [-5,-5]^T$.

TABLE 7: Algorithmic study for Example 14.

u_1	Method-1		Method-2		Method-3		Method-4	
	Iter.	Time	Iter.	Time	Iter.	Time	Iter.	Time
$[1.5,1.7]^T$	170	9.38124300	147	8.166378500	81	5.082575400	61	3.489680300
$[0, 0]^T$	267	14.7065173	166	9.695011100	107	5.894192000	64	3.753339200
$[10, 10]^T$	729	40.4305133	160	8.596874100	89	4.762929600	64	3.520466200
$[-5,-5]^T$	483	28.03565270	150	8.570527900	118	6.676780500	60	3.331014400

$$\begin{aligned}\|u_{m_k+1}-q^*\|^2 &\le \left[1-2\psi_{m_k}(1-\xi)\right]\|u_{m_k}-q^*\|^2+2\psi_{m_k}(1-\xi)\\ &\quad\cdot\left[\frac{\psi_{m_k}\|u_{m_k}-q^*\|^1}{2(1-\xi)}+\frac{\langle g(q^*)-q^*,u_{m_k+1}-q^*\rangle}{1-\xi}\right]\\ &\le \left[1-2\psi_{m_k}(1-\xi)\right]\|u_{m_k+1}-q^*\|^2+2\psi_{m_k}(1-\xi)\\ &\quad\cdot\left[\frac{\psi_{m_k}\|u_{m_k}-q^*\|^2}{2(1-\xi)}+\frac{\langle g(q^*)-q^*,u_{m_k+1}-q^*\rangle}{1-\xi}\right].\end{aligned} \tag{117}$$

Moreover, we have

$$\|u_{m_k+1}-q^*\|^2 \le \frac{\psi_{m_k}\|u_{m_k}-q^*\|^2}{2(1-\xi)}+\frac{\langle g(q^*)-q^*,u_{m_k+1}-q^*\rangle}{1-\xi}. \tag{118}$$

Since $\psi_{m_k}\longrightarrow 0$ and $\|u_{m_k}-q^*\|$ is a bounded sequence. Then, the expressions (116) and (118) imply that

$$\|u_{m_k+1}-q^*\|^2\longrightarrow 0,\quad \text{as} k\longrightarrow\infty. \tag{119}$$

The above expression implies that

$$\lim_{k\longrightarrow\infty}\|u_k-q^*\|^2\le\lim_{k\longrightarrow\infty}\|u_{m_k+1}-q^*\|^2\le 0. \tag{120}$$

Consequently, $u_n\longrightarrow q^*$ as $n\longrightarrow\infty$. This completes the proof of the theorem. □

Remark 10.

(i) Two nonmonotonic explicit extragradient-type methods for finding an approximate solution of variational inequalities involving pseudomonotone mapping in a real Hilbert space have been established

(ii) Two strongly convergent results, corresponding to the proposed algorithms have been proven

(iii) It is important to note that these methods use nonmonotonic step size rules that use an operator value rather than the Lipschitz constant of an operator

4. Numerical Illustrations

In this section, computational results of the proposed methods are described and compared to existing related work in the literature. All computations are done in MATLAB R2018b and run on an HP i-5 Core(TM)i5-6200 8.00 GB (7.78 GB usable) RAM laptop.

Example 11. Let a mapping $\mathscr{L}:\mathbb{R}^m\longrightarrow\mathbb{R}^m$ be defined by

$$\mathscr{L}(u)=Mu+q, \tag{121}$$

with $q\in\mathbb{R}^m$ and

$$M=NN^T+B+D. \tag{122}$$

During this experiment, we have chosen $N=\text{rand}\,(m)$ to be a random matrix and $B=0.5K-0.5K^T$ to be a skew-symmetric matrix with $K=\text{rand}\,(m)$, and $D=\text{diag}\,(\text{rand}\,(m,1))$ is a diagonal matrix. The feasible set $\mathscr{A}$ is defined in the subsequent sense:

$$\mathscr{A}=\{u\in\mathbb{R}^m : Qu\le b\}, \tag{123}$$

where $Q=\text{rand}\,(100,m)$ and $b=\text{rand}\,(100,1)$. It is obvious that $\mathscr{L}$ is monotone and Lipschitz continuous by $L=\|M\|$. The starting point is $u_1=(2,2,\cdots,2)$ and $D_n=\|u_n-p_n\|\le 1$ 0^{-3}. The numerical results of these methods are shown in Figures 1–6 and Table 1. The control conditions are taken in the following manner: (1) Algorithm 2 in [37] (shortly, Method-1): $\delta_1=0.12,\mu=0.55,\phi_n=1/50(n+2)$; (2) Algorithm 1 in [36] (shortly, Method-2): $\delta_1=0.12,\gamma=0.55,\phi_n=1/2(n+2),f(u)=u/3$; (3) Algorithm 1 (shortly, Method-3): $\delta_1=0.12,\mu=0.55,\psi_n=1/2(n+2),\phi_n=(5/10)(1-\psi_n),\varphi_n=100/(n+1)^2$; and (4) Algorithm 2 (shortly, Method-4): $\delta_1=0.12,\mu=0.55,\psi_n=1/2(n+2),f(u)=u/3$, $\varphi_n=100/(n+1)^2$.

Example 12. Let $\mathscr{E}=L^2([0,1])$ be a Hilbert space having an inner product

$$\langle u,y\rangle=\int_0^1 u(t)y(t)dt,\quad \forall u,y\in\mathscr{E}, \tag{124}$$

and the induced norm is determined as follows:

$$\|u\|=\sqrt{\int_0^1|u(t)|^2dt}. \tag{125}$$

Let $\mathscr{A}:=\{u\in L^2([0,1]):\|u\|\le 1\}$ be a unit ball. A mapping $\mathscr{L}:\mathscr{A}\longrightarrow\mathscr{E}$ is defined by

$$\mathscr{L}(u)(t)=\int_0^1(u(t)-H(t,s)f(u(s)))ds+g(t), \tag{126}$$

where

$$H(t,s)=\frac{2tse^{(t+s)}}{e\sqrt{e^2-1}},f(u)=\cos u,g(t)=\frac{2te^t}{e\sqrt{e^2-1}}. \tag{127}$$

It can easily be seen that $\mathscr{L}$ is Lipschitz-continuous with the constant $L=2$ and monotone. The starting point for this experiment is taken differently, and $D_n=\|u_n-p_n\|\le 10^{-3}$. The numerical results of these methods are shown in Table 2. The control conditions are taken as follows: (1) Algorithm 2 in [37] (shortly, Method-1): $\delta_1=0.33,\mu=0.75$ $,\phi_n=1/100(n+2)$; (2) Algorithm 1 in [36] (shortly, Method-2): $\delta_1=0.33,\gamma=0.75,\phi_n=1/100(n+2),f(u)=u/2$

; (3) Algorithm 1 (shortly, Method-3): $\delta_1 = 0.33, \mu = 0.75$, $\psi_n = 1/100(n+2), \phi_n = (7/10)(1-\psi_n), \varphi_n = 100/(n+1)^2$; and (4) Algorithm 2 (shortly, Method-4): $\delta_1 = 0.33, \mu = 0.75, \psi_n = 1/100(n+2), f(u) = u/2, \varphi_n = 100/(n+1)^2$.

Example 13. Let a mapping $\mathscr{L} : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be defined by

$$\mathscr{L}(u) = \begin{pmatrix} u_1 + u_2 + u_3 + u_4 - 4u_2u_3u_4 \\ u_1 + u_2 + u_3 + u_4 - 4u_1u_3u_4 \\ u_1 + u_2 + u_3 + u_4 - 4u_1u_2u_4 \\ u_1 + u_2 + u_3 + u_4 - 4u_1u_2u_3 \end{pmatrix}. \tag{128}$$

Moreover, the constraint set $\mathscr{A}$ is defined by

$$\mathscr{A} = \left\{ u \in \mathbb{R}^4 : 1 \le u_i \le 5, i = 1, 2, 3, 4 \right\}. \tag{129}$$

It is clear to see that $\mathscr{L}$ is not monotone on the set $\mathscr{A}$. The starting point for this experiment is $u_1 = (1, 2, 3, 4)^T$ and $D_n = \|u_n - p_n\| \le 10^{-3}$. The numerical results of these methods are shown in Tables 3–6. The control conditions are taken as follows: (1) Algorithm 2 in [37] (shortly, Method-1): $\delta_1 = 0.05, \mu = 0.33, \phi_n = 1/20(n+2)$; (2) Algorithm 1 in [36] (shortly, Method-2): $\delta_1 = 0.05, \gamma = 0.33, \phi_n = 1/20(n+2), f(u) = u/4$; (3) Algorithm 1 (shortly, Method-3): $\delta_1 = 0.05, \mu = 0.33, \psi_n = 1/20(n+2), \phi_n = (6/10)(1-\psi_n), \varphi_n = 100/(n+1)^2$; and (4) Algorithm 2 (shortly, Method-4): $\delta_1 = 0.05, \mu = 0.33, \psi_n = 1/20(n+2), f(u) = u/4$, $\varphi_n = 100/(n+1)^2$.

Example 14. This test problem is taken from [42]. Let a mapping $\mathscr{L} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by

$$\mathscr{L}(u) = \begin{pmatrix} 0.5u_1u_2 - 2u_2 - 10^7 \\ -4u_1 - 0.1u_2^2 - 10^7 \end{pmatrix}, \tag{130}$$

where

$$\mathscr{A} = \left\{ u \in \mathbb{R}^2 : (u_1 - 2)^2 + (u_2 - 2)^2 \le 1 \right\}. \tag{131}$$

We can easily observe that the mapping $\mathscr{L}$ is not monotone on $\mathscr{A}$ but pseudomonotone and Lipschitz continuous with $L = 5$. This problem has a unique solution that is $u^* = (2.707, 2.707)^T$. The starting point for this experiment is taken differently, and $D_n = \|u_n - p_n\| \le 10^{-3}$. The numerical results of these methods are shown in Figures 7–10 and Table 7. The control conditions are taken as follows: (1) Algorithm 2 in [37] (shortly, Method-1): $\delta_1 = 0.333, \mu = 0.90, \phi_n = 1/3(n+2)$; (2) Algorithm 1 in [36] (shortly, Method-2): $\delta_1 = 0.333, \gamma = 0.90, \phi_n = 1/2(n+2), f(u) = u/5$; (3) Algorithm 1 (shortly, Method-3): $\delta_1 = 0.333, \mu = 0.90$, $\psi_n = 1/2(n+2), \phi_n = (5/10)(1-\psi_n), \varphi_n = 100/(n+1)^2$; and (4) Algorithm 2 (shortly, Method-4): $\delta_1 = 0.333, \mu = 0.90, \psi_n = 1/2(n+2), f(u) = u/5, \varphi_n = 100/(n+1)^2$.

5. Conclusion

Two nonmonotonic explicit extragradient-type methods for finding an approximate solution of variational inequalities involving pseudomonotone mapping in a real Hilbert space have been established. Two strongly convergent results, corresponding to the proposed algorithms have been proven. The numerical results were interpreted to demonstrate that the proposed algorithms worked numerically better than current methods. According to these numerical findings, the nonmonotone variable step size rule improves the efficiency of the iterative sequence in this case.

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

No potential conflict of interest was reported by the authors.

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