Research Article

On Impulsive Boundary Value Problem with Riemann-Liouville Fractional Order Derivative

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Our manuscript is devoted to investigating a class of impulsive boundary value problems under the concept of the Riemann-Liouville fractional order derivative. The subject problem is of implicit type. We develop some adequate conditions for the existence and uniqueness of a solution to the proposed problem. For our required results, we utilize the classical fixed point theorems from Banach and Schaefer. It is to be noted that the impulsive boundary value problem under the fractional order derivative of the Riemann-Liouville type has been very rarely considered in literature. Finally, to demonstrate the obtained results, we provide some pertinent examples.

1. Introduction

The fractional order differential equations (abbreviated as FODEs) are the generalization of the ordinary differential equations of the integer order. In the 17th century (1665), the great mathematicians Newton, Leibnitz and L'Hospital introduced for the first time the idea of fractional order differential equations (FODEs). Later on, in 1823, another mathematician by the name of Lacroix, introduced the fractional derivative [1] of simple power function. Furthermore, this area has been studied by many researchers because it has significant applications in various fields of science and technology in mathematical modeling of different fields of Science and Technology. For instance, some phenomena including the diffusion process [2], some chemical processes of electrochemistry [3], infectious disease in biology [4], signal and image processing [5], dynamic processes [6], and systems control theory [7] can be excellently described by using FODEs instead of the ordinary derivative. For further applications of FODEs, we refer to [8–13] and the references therein.

On the other hand, an interesting and important branch recently got warm attention known as impulsive differential

equations (IDEs). In recent times, the said area has been increasingly used to model many physical and social phenomena in social sciences in a very interesting way. Currently in the said area, significant contribution has been done by various researchers like Simeonov and Bainov [14], Benchohra et al. [15], Lakshmikantham et al. [16], and Samoilenko and Perestyuk [17]. Benchohra and Slimani [18] has initiated the study of FODEs under impulsive conditions by using fractional derivatives of the Caputo and Riemann-Liouville type with order $\alpha \in (0, 1)$. In addition, some applications of IDEs have been studied in various scientific disciplines such as biology, geography, engineering, dynamics, physics, geology, and management sciences. In terms of the important applications of IDEs, due to important applications of IDEs this field has a lot of significance and concentration (see [19–22]). For general research and significance, we refer some more important publications [23–26]. Many researchers have recently studied nonlinear FODEs with different kinds of boundary and initial conditions. Boundary value problems have significant applications in various fields of dynamics and fluid mechanics as well as engineering disciplines. Here, it is

remarkable that problems under integral boundary conditions have some important applications in fluid mechanics, chemical engineering, thermoelasticity, flow of groundwater, population dynamics, and more (see [27–29]). Furthermore, we also refer some significance of FODEs under integral boundary conditions as discussed in [27–31].

Recently, due to increasing applications of FODEs to model real world problems more comprehensively, researchers are taking keen interest to investigate different areas of fractional calculus. In particular, the use of FODEs in mathematical modeling of infectious diseases and other biological phenomena have got more attention. Various researchers have studied the fractional predator-prey pathogen model, the n -predator-prey model with herd behavior, etc. (see [32–34]). Further, there is also modeling of the interaction between tumor growth and the immune system, edge-detecting techniques, an infectious disease on a pre-predator model, etc., (we refer to [35–40]).

As stated earlier, the area devoted to IDEs with a fractional order has many applications. These differential equations can be modeled to those evolutionary processes which are subjected to abrupt changes in their states. Recently, some authors have used IDEs for the mathematical modeling of certain biological events. It is remarkable that impulsive differential equations are using in mathematical models which give rise to some important dual-layered impulsive systems. The said systems will open new doors in the future to develop a general mathematical theory for the said systems. For instance, the author of [41] has obtained very interesting results in this regard for various kinds of biological models of infectious diseases. Here, we remark that a very basic and important qualitative problem in the investigation of IDEs with a fractional order concerns the existence theory of solutions. For these purposes, researchers have used the classical fixed point theory and some tools of nonlinear analysis. For instance, in [42], the authors have applied fixed point results to develop the corresponding existence theory of solutions by using the Caputo derivative of the fractional order. In the same line, in [43], the authors have used the Picard-type analysis to investigate the stochastic-type IDEs of a fractional order by using the Caputo operator. In all these papers, the Caputo operator has been increasingly used. It is to be noted that the fractional order derivative of the Riemann-Liouville type has been very rarely used in IDEs.

Authors [44] have established existence theory for fractional order IDEs with initial conditions by using the fixed point theory. The authors in [45] investigated the following problem of IDEs under the fractional order derivative of the Riemann-Liouville type as

$$\begin{cases} {}^{RL}D^{\alpha_k} r(z) = f(z, r(z)), & z \in [0, T], z \neq z_m, 1 < \alpha_m \leq 2, z \in \mathcal{J}, z \neq z_m, \\ \Delta r(z_m) = \psi_m(r(z_m)), & m = 1, 2, 3, \dots, q, \\ \Delta^* r(z_m) = \psi_m^*(r(z_m)), & m = 1, 2, 3, \dots, q, \\ r(0) = 0, \quad D^{\alpha_0-1} r(0) = \beta, \end{cases} \quad (1)$$

where $\beta \in \mathcal{R}$ and $f : [0, T] \times \mathcal{R} \longrightarrow \mathcal{R}$ is a continuous function. They developed sufficient conditions for the existence of at least one solution to the considered problem by using a fixed point approach.

Motivated from the said work given in (1), we are interested in studying a class of nonlinear implicit fractional order IDEs under the Riemann-Liouville derivative with the Riemann-Liouville-type integral boundary conditions as

$$\begin{cases} {}^{RL}D^{\alpha_k} r(z) = f\left(z, r(z), {}^{RL}D^{\alpha} r(z)\right), & 1 < \alpha \leq 2, z \in \mathcal{J}, z \neq z_m, \\ \Delta r(z_m) = \psi_m(r(z_m)), & m = 1, 2, 3, \dots, q, \\ \Delta^* r(z_m) = \psi_m^*(r(z_m)), & m = 1, 2, 3, \dots, q, \\ I^{1-\alpha} r(0) = 0, \quad I^{2-\alpha} r(1) = 0, \end{cases} \quad (2)$$

where ${}^{RL}D$ is denoted as the Riemann-Liouville fractional order derivative, $\mathcal{J} = [0, 1]$, $f : \mathcal{J} \times \mathcal{R} \times \mathcal{R} \longrightarrow \mathcal{R}$ is a continuous function. Furthermore, $\psi_m, \psi_m^* : \mathcal{R} \longrightarrow \mathcal{R}$ are continuous functions for $m = 1, 2, \dots, q$ and $\Delta r(z_m) = I_{z_m}^{1-\alpha} r(z_m^+) - I_{z_m}^{1-\alpha} r(z_m^-)$, $\Delta^* r(z_m) = I_{z_m}^{2-\alpha} r(z_m^+) - I_{z_m}^{2-\alpha} r(z_m^-)$ with $r(z_m^+) = \lim_{h \rightarrow 0^+} r(z_m + h)$, $r(z_m^-) = \lim_{h \rightarrow 0^-} r(z_m + h)$, $m = 1, 2, \dots, q$, for $0 = z_0 < z_1 < z_2 \dots < z_{q+1} = 1$. And also, where $I^{1-\alpha}, I^{2-\alpha}$ are denoted as the Riemann-Liouville integral of fractional order $1 - \alpha < 0$, $2 - \alpha > 0$ on $\mathcal{J}$, respectively. To establish the required results, we utilize the Schaefer fixed point theorem to investigate sufficient conditions for the existence of at least one solution to the problem under consideration (2). Furthermore, the criterion of uniqueness is derived by using the Banach contraction theorem. For the demonstration of our results, we provide some concrete examples.

2. Preliminaries

In this section, we provide some important results, basic definitions, and lemmas from the literature of fractional calculus [1, 3, 10, 11], which are needed in this manuscript.

Let $\mathcal{J} = [0, 1]$, $\mathcal{J}_0 = (z_0, z_1]$, and $\mathcal{J}_m = (z_m, z_{m+1}]$ for $m = 1, 2, 3 \dots q$. Suppose that $PC(\mathcal{J}, \mathcal{R}) = \{r : \mathcal{J} \longrightarrow \mathcal{R} ; r \in \mathcal{C}((z_m, z_{m+1}], \mathcal{R}), m = 0, 1, 2, \dots, q+1\}$ and $r(z_m^+)$ and $r(z_m^-)$ exist with $r(z_m^-) = \{r(z_m), m = 1, 2, \dots, q\}$. Note that $PC(\mathcal{J}, \mathcal{R})$ is a Banach space of piece-wise continuous function with norm $\|r\| = \sup_{z \in \mathcal{J}} |r(z)|$.

Definition 1. The integral of the Riemann-Liouville fractional order $\beta > 0$ of a continuous function $f : (0, \infty) \longrightarrow \mathcal{R}$ is defined by

$$I^{\beta} f(z) = \int_0^z \frac{(z-s)^{\beta-1}}{\Gamma(\beta)} f(s) ds, \quad z \in (0, 1). \quad (3)$$

Therefore, the right side is point-wise defined on $(0, \infty)$, where Γ is the symbol of gamma function defined as $\Gamma(\beta) = \int_0^{\infty} e^{-s} s^{\beta-1} ds$.

Definition 2. The derivative of the Riemann-Liouville fractional order $\beta > 0$, of a continuous function $f \in C[0, 1] \cap L[0, 1]$, $b > 0$ is defined by

$${}^{RL}\mathbf{D}^\beta f(z) = \left(\frac{d}{dz}\right)^n \int_0^z \frac{(z-s)^{n-\beta-1}}{\Gamma(n-\beta)} f(s) ds, \quad n-1 < \beta < n, z \in (0, 1), \quad (4)$$

where $n = [\beta] + 1$, $[\beta]$ represent the whole part of the real number β ; therefore, the right side is point-wise defined on $(0, \infty)$.

Definition 3. If the function $f : (c, d) \longrightarrow \mathcal{R}$ is at least n -times differentiable, then the Caputo fractional derivative of order $\beta > 0$ is defined as

$${}^C\mathbf{D}_c^\beta f(z) = \int_c^z \frac{(z-s)^{n-\beta-1}}{\Gamma(n-\beta)} f^n(s) ds, \quad n-1 < \beta < n, z \in (c, d), \quad (5)$$

where $n = [\beta] + 1$.

Lemma 4 [11]. Suppose $\beta > 0$, and $r \in C(b, d) \cap L(b, d)$. Then, FODE

$${}^{RL}\mathbf{D}^\beta r(z) = 0, \quad (6)$$

has a unique solution given by

$$r(z) = d_1(z-b)^{\beta-1} + d_2(z-b)^{\beta-2} + \dots + d_n(z-b)^{\beta-n}, \quad (7)$$

where $d_i \in \mathcal{R}$, $i = 1, 2, \dots, n$, and $n-1 < \beta < n$.

Lemma 5 [11]. In particular, $\beta > 0$ and $r \in C[0, 1] \cap L[0, 1]$. We have

$$I^\beta \mathbf{D}^\beta r(z) = r(z) + a_1 z^{\beta-1} + a_2 z^{\beta-2} + \dots + a_n z^{\beta-n}, \quad (8)$$

where $a_k \in \mathcal{R}$, $k = 1, 2, \dots, n$, and $n-1 < \beta < n$.

3. Main Works

To convert our considered problem in to an impulsive fractional integral equation, the given Lemma is provided.

Lemma 6. The solution of the given linear IDE of the fractional order with the Riemann-Liouville derivative

$$\begin{cases} {}^{RL}\mathbf{D}^\alpha r(z) = \sigma(z), & 1 < \alpha \leq 2, z \in \mathcal{J}, z \neq z_m, \\ \Delta r(z_m) = \psi_m(r(z_m)), & m = 1, 2, 3, \dots, q, \\ \Delta^* r(z_m) = \psi_m^*(r(z_m)), & m = 1, 2, 3, \dots, q, \\ I^{1-\alpha} r(0) = 0, & I^{2-\alpha} r(1) = 0, \end{cases} \quad (9)$$

is given by

$$r(z) = \begin{cases} \int_0^z \frac{(z-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds + z_m \left[\int_{z_m}^1 (1-s)^2 \sigma(s) ds + \left((1-z_m) \left(\int_0^{z_m} (z_m-s) \sigma(s) ds - \psi_m(r(z_m)) \right) \right) \right. \\ \quad \left. + \left(\int_0^{z_m} (z_m-s)^2 \sigma(s) ds - \psi_m^*(r(z_m)) \right) \right] z^{\alpha-2}, & z \in [0, z_1], \\ \int_{z_m}^z \frac{(z-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds - [(z-z_m)^{\alpha-1} + ((z-z_m)^{\alpha-1} + z_m(z-z_m)^{\alpha-2})(1-z_m)] \\ \quad \times \left(\int_{z_{m-1}}^{z_m} (z_m-s) \sigma(s) ds - \psi_m(r(z_m)) \right) - [(z-z_m)^{\alpha-2} + ((z-z_m)^{\alpha-1} + z_m(z-z_m)^{\alpha-2})] \\ \quad \times \left(\int_{z_{m-1}}^{z_m} (z_m-s)^2 \sigma(s) ds - \psi_m^*(r(z_m)) \right) - ((z-z_m)^{\alpha-1} + z_m(z-z_m)^{\alpha-2}) \left(\int_{z_m}^1 (1-s)^2 \sigma(s) ds \right), & z \in (z_m, z_{m+1}]. \end{cases} \quad (10)$$

Proof. Suppose $r(z)$ is a solution to Problem (9); then, taking the Riemann-Liouville integral on both sides to using Lemma 5, there exist some constants $c_0, c_1 \in \mathcal{R}$ such that

$$r(z) = \int_0^z \frac{(z-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds - c_0 z^{\alpha-1} - c_1 z^{\alpha-2}, \quad z \in [0, z_1]. \quad (11)$$

Again taking the Riemann-Liouville integral to using Lemma (9), for some constant $d_0, d_1 \in \mathcal{R}$, we have

$$r(z) = \int_{z_1}^z \frac{(z-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds - d_0 (z-z_1)^{\alpha-1} - d_1 (z-z_1)^{\alpha-2}, \quad z \in (z_1, z_2]. \quad (12)$$

Now, by using the impulsive conditions, we have $\Delta r(z_1) = I_{z_1}^{1-\alpha} r(z_1^+) - I_{z_1}^{1-\alpha} r(z_1^-) = \psi_1(r(z_1))$ and $\Delta^* r(z_1) = I_{z_1}^{2-\alpha} r(z_1^+) - I_{z_1}^{2-\alpha} r(z_1^-) = \psi_1^*(r(z_1))$, and we find that

$$-d_0 = \int_0^{z_1} (z_1-s) \sigma(s) ds - c_0 - c_1 z_1^{-1} + \psi_1(r(z_1)),$$

$$-d_1 = \int_0^{z_1} (z_1-s)^2 \sigma(s) ds - c_0 z_1 - c_1 + \psi_1^*(r(z_1)). \quad (13)$$

Thus, putting the values in (12), we have

$$\begin{aligned}
 r(z) = & \int_{z_1}^z \frac{(z-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds - (z-z_1)^{\alpha-1} \\
 & \cdot \left(\int_0^{z_1} (z_1-s) \sigma(s) ds - \psi_1(r(z_1)) \right) - (z-z_1)^{\alpha-2} \\
 & \cdot \left(\int_0^{z_1} (z_1-s)^2 \sigma(s) ds - \psi_1^*(r(z_1)) \right) \\
 & - c_0 \{ (z-z_1)^{\alpha-1} + z_1(z-z_1)^{\alpha-2} \} \\
 & - c_1 \{ z_1^{-1} (z-z_1)^{\alpha-1} + (z-z_1)^{\alpha-2} \}, \quad z \in (z_1, z_2].
 \end{aligned} \tag{14}$$

The above process can be repeated in this way until we obtain the solution $r(z)$ for $z \in (z_m, z_{m+1}]$ as

$$\begin{aligned}
 r(z) = & \int_{z_m}^z \frac{(z-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds - (z-z_m)^{\alpha-1} \\
 & \cdot \left(\int_{z_{m-1}}^{z_m} (z_m-s) \sigma(s) ds - \psi_m(r(z_m)) \right) - (z-z_m)^{\alpha-2} \\
 & \cdot \left(\int_{z_{m-1}}^{z_m} (z_m-s)^2 \sigma(s) ds - \psi_m^*(r(z_m)) \right) \\
 & - c_0 \{ (z-z_m)^{\alpha-1} + z_m(z-z_m)^{\alpha-2} \} \\
 & - c_1 \{ z_m^{-1} (z-z_m)^{\alpha-1} + (z-z_m)^{\alpha-2} \}, \quad z \in (z_m, z_{m+1}].
 \end{aligned} \tag{15}$$

Now, applying boundary condition $I^{1-\alpha} r(0) = 0$ and $I^{2-\alpha} r(1) = 0$ to get the values of constant c_0 and c_1 , we have

$$c_0 = 0,$$

$$\begin{aligned}
 c_1 = & z_m \int_{z_m}^1 (1-s)^2 \sigma(s) ds - z_m(1-z_m) \\
 & \cdot \left(\int_0^{z_m} (z_m-s) \sigma(s) ds - \psi_m(r(z_m)) \right) \\
 & - z_m \left(\int_0^{z_m} (z_m-s)^2 \sigma(s) ds - \psi_m^*(r(z_m)) \right).
 \end{aligned} \tag{16}$$

The values of c_0, c_1 putting in (11) and (15), one can obtain (10). On the contrary, suppose $r(z)$ is a solution of the impulsive fractional integral equation (10). Following the direct calculation, we see that (10) satisfies the problem (9). $\square$

For simplification, we use the following notations:

$$\sigma_1 = \sup_{z \in [0,1]} |(z-z_m)^{\alpha-1} + \{(z-z_m)^{\alpha-1} + z_m(z-z_m)^{\alpha-2}\}(1-z_m)|,$$

$$\sigma_2 = \sup_{z \in [0,1]} |(z-z_m)^{\alpha-2} + \{(z-z_m)^{\alpha-1} + z_m(z-z_m)^{\alpha-2}\}|,$$

$$\sigma_3 = \sup_{z \in [0,1]} |(z-z_m)^{\alpha-1} + z_m(z-z_m)^{\alpha-2}|. \tag{17}$$

For the existence and uniqueness of the solution, we use some fixed point theorems. To transform the considered Problem (2) to a fixed point problem, we need to define the operator by $M : PC(\mathcal{J}, \mathcal{R}) \longrightarrow PC(\mathcal{J}, \mathcal{R})$ as

$$\begin{aligned}
 Mr(z) = & \int_{z_m}^z \frac{(z-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds - [(z-z_m)^{\alpha-1} \\
 & + ((z-z_m)^{\alpha-1} + z_m(z-z_m)^{\alpha-2})(1-z_m)] \\
 & \times \left(\int_{z_{m-1}}^{z_m} (z_m-s) \sigma(s) ds - \psi_m(r(z_m)) \right) \\
 & - [(z-z_m)^{\alpha-2} + ((z-z_m)^{\alpha-1} + z_m(z-z_m)^{\alpha-2})] \\
 & \times \left(\int_{z_{m-1}}^{z_m} (z_m-s)^2 \sigma(s) ds - \psi_m^*(r(z_m)) \right) \\
 & - ((z-z_m)^{\alpha-1} + z_m(z-z_m)^{\alpha-2}) \left(\int_{z_m}^1 (1-s)^2 \sigma(s) ds \right).
 \end{aligned} \tag{18}$$

By using Lemma 6 with $r(z) = f(z, r(z), {}^{RL}\mathbf{D}^\alpha r(z))$, Problem (2) is reduced to a fixed point problem $Mr(z) = r(z)$, where the operator M is given by (18). Therefore, Problem (2) has a solution if and only if operator M has a fixed point, where $\tau(z) = f(z, r(z), \tau(z))$ and $\tau(z) = {}^{RL}\mathbf{D}^\alpha r(z)$. The following hypotheses are satisfied:

(H₁) The function $f : \mathcal{J} \times \mathcal{R} \times \mathcal{R} \longrightarrow \mathcal{R}$ is continuous

(H₂) There exist some constants $K^* > 0$ and $0 < L^* < 1$, such that

$$\begin{aligned}
 |f(z, r(z), \tau(z)) - f(z, \bar{r}(z), \bar{\tau}(z))| \\
 \leq K^* |r(z) - \bar{r}(z)| + L^* |\tau(z) - \bar{\tau}(z)|,
 \end{aligned} \tag{19}$$

for any $r, \bar{r} \in PC(\mathcal{J}, \mathcal{R})$, $\tau, \bar{\tau} \in \mathcal{R}$, and $z \in \mathcal{J}$.

(H₃) There exists a constant $N_1^* > 0$, such that

$$|\psi_m(r(z)) - \psi_m(\bar{r}(z))| \leq N_1^* |r(z) - \bar{r}(z)|, \tag{20}$$

for each $r, \bar{r} \in PC(\mathcal{J}, \mathcal{R})$ and $m = 1, 2, 3, \dots, q$.

(H₄) There exists a constant $N_2^* > 0$, such that

$$|\psi_m^*(r(z)) - \psi_m^*(\bar{r}(z))| \leq N_2^* |r(z) - \bar{r}(z)|, \tag{21}$$

for every $r, \bar{r} \in PC(\mathcal{J}, \mathcal{R})$ and $m = 1, 2, 3, \dots, q$.

We use Banach fixed point theorem to prove that problem (2) has unique solution.

Theorem 7. Under hypotheses (H_1) – (H_4) and if the following condition holds

$$\left[\frac{K^*}{1-L^*} \left(\frac{1}{\Gamma(\alpha+1)} + \frac{\sigma_1}{2} + \frac{\sigma_2}{3} + \frac{\sigma_3}{3} \right) + (\sigma_1 N_1^* + \sigma_2 N_2^*) \right] < 1, \quad (22)$$

then, there exists a unique solution for Problem (2) on $\mathcal{J}$.

Proof. Let $r, \bar{r} \in PC(\mathcal{J}, \mathcal{R})$ for some $z \in \mathcal{J}$, we have

$$\begin{aligned} & |Mr(z) - M\bar{r}(z)| \\ & \leq \int_{z_m}^z \frac{(z-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, r(s), \tau(s)) - f(s, \bar{r}(s), \bar{\tau}(s))| ds \\ & \quad + |(z-z_m)^{\alpha-1} + \{(z-z_m)^{\alpha-1} + z_m(z-z_m)^{\alpha-2}\}(1-z_m)| \\ & \quad \times \left(\int_{z_{m-1}}^{z_m} (z_m-s) |f(s, r(s), \tau(s)) - f(s, \bar{r}(s), \bar{\tau}(s))| ds \right. \\ & \quad \left. + |\psi_m(r(z)) - \psi_m(\bar{r}(z))| \right) + |(z-z_m)^{\alpha-2} \\ & \quad + \{(z-z_m)^{\alpha-1} + z_m(z-z_m)^{\alpha-2}\}| \\ & \quad \times \left(\int_{z_{m-1}}^{z_m} (z_m-s)^2 |f(s, r(s), \tau(s)) - f(s, \bar{r}(s), \bar{\tau}(s))| ds \right. \\ & \quad \left. + |\psi_m^*(r(z)) - \psi_m^*(\bar{r}(z))| \right) + |(z-z_m)^{\alpha-1} + z_m(z-z_m)^{\alpha-2}| \\ & \quad \times \left(\int_{z_m}^1 (1-s)^2 |f(s, r(s), \tau(s)) - f(s, \bar{r}(s), \bar{\tau}(s))| ds \right), \end{aligned} \quad (23)$$

which further gives

$$\begin{aligned} & |Mr(z) - M\bar{r}(z)| \\ & \leq \int_{z_m}^z \frac{(z-s)^{\alpha-1}}{\Gamma(\alpha)} |\tau(s) - \bar{\tau}(s)| ds + |(z-z_m)^{\alpha-1} \\ & \quad + \{(z-z_m)^{\alpha-1} + z_m(z-z_m)^{\alpha-2}\}(1-z_m)| \\ & \quad \times \left(\int_{z_{m-1}}^{z_m} (z_m-s) |\tau(s) - \bar{\tau}(s)| + |\psi_m(r(z)) - \psi_m(\bar{r}(z))| \right) \\ & \quad + |(z-z_m)^{\alpha-2} + \{(z-z_m)^{\alpha-1} + z_m(z-z_m)^{\alpha-2}\}| \\ & \quad \times \left(\int_{z_{m-1}}^{z_m} (z_m-s)^2 |\tau(s) - \bar{\tau}(s)| ds + |\psi_m^*(r(z)) - \psi_m^*(\bar{r}(z))| \right) \\ & \quad + |(z-z_m)^{\alpha-1} + z_m(z-z_m)^{\alpha-2}| \left(\int_{z_m}^1 (1-s)^2 |\tau(s) - \bar{\tau}(s)| ds \right), \end{aligned} \quad (24)$$

where $\tau, \bar{\tau} \in \mathcal{C}(\mathcal{J}, \mathcal{R})$ are given by

$$\begin{aligned} \tau(z) &= f(z, r(z), \tau(z)), \\ \bar{\tau}(z) &= f(z, \bar{r}(z), \bar{\tau}(z)). \end{aligned} \quad (25)$$

By using hypothesis (H_2) , we have

$$|\tau(z) - \bar{\tau}(z)| = |f(z, r(z), \tau(z)) - f(z, \bar{r}(z), \bar{\tau}(z))| \leq K^* |r(z) - \bar{r}(z)| + L^* |\tau(z) - \bar{\tau}(z)|. \quad (26)$$

Repeating this process, we get

$$|\tau(z) - \bar{\tau}(z)| \leq \frac{K^*}{1-L^*} |r(z) - \bar{r}(z)|. \quad (27)$$

Therefore, for every $z \in \mathcal{J}$ and from (24), using hypothesis (H_3) , (H_4) , and (27), one has

$$\begin{aligned} & |Mr(z) - M\bar{r}(z)| \\ & \leq \frac{K^*}{1-L^*} \int_{z_m}^z \frac{(z-s)^{\alpha-1}}{\Gamma(\alpha)} |r(s) - \bar{r}(s)| ds + |(z-z_m)^{\alpha-1} \\ & \quad + \{(z-z_m)^{\alpha-1} + z_m(z-z_m)^{\alpha-2}\}(1-z_m)| \\ & \quad \times \left(\frac{K^*}{1-L^*} \int_{z_{m-1}}^{z_m} (z_m-s) |r(s) - \bar{r}(s)| ds + N_1^* |r(z) - \bar{r}(z)| \right) \\ & \quad + |(z-z_m)^{\alpha-2} + \{(z-z_m)^{\alpha-1} + z_m(z-z_m)^{\alpha-2}\}| \\ & \quad \times \left(\frac{K^*}{1-L^*} \int_{z_{m-1}}^{z_m} (z_m-s)^2 |r(s) - \bar{r}(s)| ds + N_2^* |r(z) - \bar{r}(z)| \right) \\ & \quad + |(z-z_m)^{\alpha-1} + z_m(z-z_m)^{\alpha-2}| \frac{K^*}{1-L^*} \\ & \quad \times \left(\int_{z_m}^1 (1-s)^2 |r(s) - \bar{r}(s)| ds \right). \end{aligned} \quad (28)$$

Upon further simplification, (29) yields

$$\begin{aligned} & \|Mr - M\bar{r}\| \\ & \leq \frac{K^*}{1-L^*} \frac{(z-z_m)^\alpha}{\Gamma(\alpha+1)} \|r - \bar{r}\| + \sigma_1 \left(\frac{K^*}{1-L^*} \frac{(z_m-z_{m-1})^2}{2} \|r - \bar{r}\| \right. \\ & \quad \left. + N_1^* \|r - \bar{r}\| \right) + \sigma_2 \left(\frac{K^*}{1-L^*} \frac{(z_m-z_{m-1})^3}{3} \|r - \bar{r}\| + N_2^* \|r - \bar{r}\| \right) \\ & \quad + \sigma_3 \frac{K^*}{1-L^*} \left(\frac{(1-z_m)^3}{3} \|r - \bar{r}\| \right). \end{aligned} \quad (29)$$

Hence, from (29), we have

$$\begin{aligned} & \|Mr - M\bar{r}\| \\ & \leq \left[\frac{K^*}{1-L^*} \left(\frac{1}{\Gamma(\alpha+1)} + \frac{\sigma_1}{2} + \frac{\sigma_2}{3} + \frac{\sigma_3}{3} \right) + (\sigma_1 N_1^* + \sigma_2 N_2^*) \right] \\ & \quad \times \|r - \bar{r}\|. \end{aligned} \quad (30)$$

By (22), operator M is a contraction. Thus, according to Banach's contraction principle, operator M has a unique fixed point which is the unique solution to Problem (2). $\square$

Next, we will prove that Problem (2) has at least one solution for this, and we use Schaefer's fixed point theorem. Let the given hypotheses hold true:

(H₅) There exist $x, y, h \in PC(\mathcal{J}, \mathcal{R})$, with

$$\begin{aligned} x^* &= \sup_{z \in [0,1]} x(z), \\ y^* &= \sup_{z \in [0,1]} y(z), \\ h^* &= \sup_{z \in [0,1]} |h(z)| < 1, \end{aligned} \quad (31)$$

such that

$$|f(z, r(z), \tau(z))| \leq x(z) + y(z)|r(z)| + h(z)|\tau(z)|, \quad (32)$$

for $z \in \mathcal{J}$, $r \in PC(\mathcal{J}, \mathcal{R})$, and $\tau \in \mathcal{R}$.

(H₆) The function $\psi_m : PC(\mathcal{J}, \mathcal{R}) \longrightarrow \mathcal{R}$ is continuous, and there exist constants $A_1^*, B_1^* > 0$ such that

$$|\psi_m(r(z))| \leq A_1^*|r(z)| + B_1^*, \quad (33)$$

for every $r \in PC(\mathcal{J}, \mathcal{R})$, $m = 1, \dots, q$.

(H₇) The function $\psi_m^* : PC(\mathcal{J}, \mathcal{R}) \longrightarrow \mathcal{R}$ is continuous, and there exist constants $A_2^*, B_2^* > 0$ such that

$$|\psi_m^*(r(z))| \leq A_2^*|r(z)| + B_2^*, \quad (34)$$

for every $r \in PC(\mathcal{J}, \mathcal{R})$, $m = 1, \dots, q$.

Theorem 8. *If the hypotheses (H₁), (H₂), (H₅)-(H₇) are satisfied, then Problem (2) has at least one solution.*

Proof. The proof is performed in several steps.

(Step 1) The operator M is continuous

Assume $\{r_n\}$ be a sequence such that $r_n \longrightarrow r$ on $PC(\mathcal{J}, \mathcal{R})$.

For $z \in \mathcal{J}$, we have

$$\begin{aligned} & |Mr_n(z) - Mr(z)| \\ & \leq \int_{z_m}^z \frac{(z-s)^{\alpha-1}}{\Gamma(\alpha)} |\tau_n(s) - \tau(s)| ds + |(z-z_m)^{\alpha-1} \\ & \quad + \{(z-z_m)^{\alpha-1} + z_m(z-z_m)^{\alpha-2}\}(1-z_m)| \\ & \quad \times \left(\int_{z_{m-1}}^{z_m} (z_m-s) |\tau_n(s) - \tau(s)| ds + |\psi_m(r_n(z)) - \psi_m(r(z))| \right) \\ & \quad + |(z-z_m)^{\alpha-2} + \{(z-z_m)^{\alpha-1} + z_m(z-z_m)^{\alpha-2}\}| \\ & \quad \times \left(\int_{z_{m-1}}^{z_m} (z_m-s)^2 |\tau_n(s) - \tau(s)| ds + |\psi_m^*(r_n(z)) - \psi_m^*(r(z))| \right) \\ & \quad + |(z-z_m)^{\alpha-1} + z_m(z-z_m)^{\alpha-2}| \left(\int_{z_m}^1 (1-s)^2 |\tau_n(s) - \tau(s)| ds \right), \end{aligned} \quad (35)$$

where $\tau_n(z), \tau(z) \in PC(\mathcal{J}, \mathcal{R})$ are given by

$$\begin{aligned} \tau_n(z) &= f(z, r_n(z), \tau_n(z)), \\ \tau(z) &= f(z, r(z), \tau(z)). \end{aligned} \quad (36)$$

Now, from assumption (H₂), we have

$$\begin{aligned} |\tau_n(z) - \tau(z)| &= |f(z, r_n(z), \tau_n(z)) - f(z, r(z), \tau(z))| \\ &\leq K^* \|r_n - r\| + L^* |\tau_n(z) - \tau(z)|. \end{aligned} \quad (37)$$

Repeating this process, we get

$$|\tau_n(z) - \tau(z)| \leq \frac{K^*}{1-L^*} \|r_n - r\|. \quad (38)$$

Since $r_n \longrightarrow r$, $\tau_n(z) \longrightarrow \tau(z)$ as $n \longrightarrow \infty$ for every $z \in \mathcal{J}$. We know that every convergent sequence is bounded, so for this, let $\zeta > 0$ such that for every $z \in \mathcal{J}$, we have $|\tau_n(z)| \leq \zeta$ and $|\tau(z)| \leq \zeta$. Then, we have

$$\begin{aligned} (z-s)^{\alpha-1} |\tau_n(z) - \tau(z)| &\leq (z-s)^{\alpha-1} [|\tau_n(z)| + |\tau(z)|] \\ &\leq 2\zeta (z-s)^{\alpha-1}, \end{aligned}$$

$$\begin{aligned} (z_m-s)^{\alpha-1} |\tau_n(z) - \tau(z)| &\leq (z_m-s)^{\alpha-1} [|\tau_n(z)| + |\tau(z)|] \\ &\leq 2\zeta (z_m-s)^{\alpha-1}, \end{aligned} \quad (39)$$

for every $z \in \mathcal{J}$; the function $s \longrightarrow 2\zeta(z-s)^{\alpha-1}$ and $s \longrightarrow 2\zeta(z_m-s)^{\alpha-1}$ are integrable on $[0, 1]$, upon the use of these facts and the Lebesgue-dominated convergence theorem in (35). After using assumptions (H₅)-(H₇), we see that

$$|Mr_n(z) - Mr(z)| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \quad (40)$$

and hence, we have

$$\|Mr_n - Mr\| \longrightarrow 0, \quad n \longrightarrow \infty. \quad (41)$$

Therefore, operator M is continuous.

(Step 2) The operator M assigns bounded sets to bounded sets on $PC(\mathcal{J}, \mathcal{R})$. Just prove it for any $\xi^* > 0$, there exists a positive constant E^* , such that for every $r \in D = \{r \in PC(\mathcal{J}, \mathcal{R}), \|r\| \leq \xi^*\}$, we have $\|Mr\| \leq E^*$. To derive this result for each $z \in \mathcal{J}$, one has

$$\begin{aligned}
|Mr(z)| &\leq \int_{z_m}^z \frac{(z-s)^{\alpha-1}}{\Gamma(\alpha)} |\tau(s)| ds + |(z-z_m)^{\alpha-1} \\
&\quad + \{(z-z_m)^{\alpha-1} + z_m(z-z_m)^{\alpha-2}\}(1-z_m)| \\
&\quad \times \left(\int_{z_{m-1}}^{z_m} (z_m-s) |\tau(s)| ds + |\psi_m(r(z_m))| \right) \\
&\quad + |(z-z_m)^{\alpha-2} + \{(z-z_m)^{\alpha-1} + z_m(z-z_m)^{\alpha-2}\}| \\
&\quad \times \left(\int_{z_{m-1}}^{z_m} (z_m-s)^2 |\tau(s)| ds + |\psi_m^*(r(z_m))| \right) \\
&\quad + |(z-z_m)^{\alpha-1} + z_m(z-z_m)^{\alpha-2}| \left(\int_{z_m}^1 (1-s)^2 |\tau(s)| ds \right),
\end{aligned} \tag{42}$$

where $\tau(z) \in \mathcal{C}(\mathcal{J}, \mathcal{R})$ is given by

$$\tau(z) = f(z, r(z), \tau(z)). \tag{43}$$

By hypothesis (H_5) and for every $z \in \mathcal{J}$, we have

$$\begin{aligned}
|\tau(z)| &= |f(z, r(z), \tau(z))| \leq x(z) + y(z)|r(z)| + h(z)|\tau(z)| \\
&\leq x(z) + y(z)\|r\| + h(z)|\tau(z)| \\
&\leq x(z) + y(z)\xi^* + h(z)|\tau(z)| \leq x^* + y^*\xi^* + h^*|\tau(z)|.
\end{aligned} \tag{44}$$

Thus, we have

$$|\tau(z)| \leq \frac{x^* + y^*\xi^*}{1 - h^*} := R^*. \tag{45}$$

Therefore, from (42) by using (45), one has

$$\begin{aligned}
|Mr(z)| &\leq \frac{R^*}{\Gamma(\alpha+1)} + \frac{\sigma_1 R^*}{2} + A_1^* \xi^* + B_1^* + \frac{\sigma_2 R^*}{3} \\
&\quad + A_2^* \xi^* + B_2^* + \frac{\sigma_3 R^*}{3}.
\end{aligned} \tag{46}$$

Hence, one has

$$\begin{aligned}
\|Mr\| &\leq \left[R^* \left(\frac{1}{\Gamma(\alpha+1)} + \frac{\sigma_1}{2} + \frac{\sigma_2}{3} + \frac{\sigma_3}{3} \right) + (A_1^* + A_2^*) \xi^* + B_1^* + B_2^* \right] \\
&:= E^*, \\
\|Mr\| &\leq E^*.
\end{aligned} \tag{47}$$

Therefore, the operator M is bounded.

(Step 3) The operator M assigns bounded sets to equi-continuous sets of $PC(\mathcal{J}, \mathcal{R})$. Let $z_1, z_2 \in \mathcal{J}$, and $z_1 < z_2$. D is a bounded set as in Step 2, and let $r \in D$; then, we have

$$\begin{aligned}
|Mr(z_2) - Mr(z_1)| &\leq \left| \int_{z_m}^{z_2} \frac{(z_2-s)^{\alpha-1}}{\Gamma(\alpha)} \tau(s) ds - \int_{z_m}^{z_1} \frac{(z_1-s)^{\alpha-1}}{\Gamma(\alpha)} \tau(s) ds \right| \\
&\quad + |[(z_2-z_m)^{\alpha-1} + \{(z_2-z_m)^{\alpha-1} + z_m(z_2-z_m)^{\alpha-2}\}(1-z_m)] \\
&\quad - [(z_1-z_m)^{\alpha-1} + \{(z_1-z_m)^{\alpha-1} + z_m(z_1-z_m)^{\alpha-2}\}(1-z_m)]| \\
&\quad \times \left(\int_{z_{m-1}}^{z_m} (z_m-s) |\tau(s)| ds + |\psi_m(r(z_m))| \right) \\
&\quad + |[(z_2-z_m)^{\alpha-2} + \{(z_2-z_m)^{\alpha-1} + z_m(z_2-z_m)^{\alpha-2}\}] \\
&\quad - [(z_1-z_m)^{\alpha-2} + \{(z_1-z_m)^{\alpha-1} + z_m(z_1-z_m)^{\alpha-2}\}]| \\
&\quad \times \left(\int_{z_{m-1}}^{z_m} (z_m-s)^2 |\tau(s)| ds + |\psi_m^*(r(z_m))| \right) \\
&\quad + |[(z_2-z_m)^{\alpha-1} + z_m(z_2-z_m)^{\alpha-2}] \\
&\quad - [(z_1-z_m)^{\alpha-1} + z_m(z_1-z_m)^{\alpha-2}]| \left(\int_{z_m}^1 (1-s)^2 |\tau(s)| ds \right).
\end{aligned} \tag{48}$$

Using (45) and hypotheses (H_6) and (H_7) in (48), we obtain

$$\begin{aligned}
|Mr(z_2) - Mr(z_1)| &\leq \frac{R^*}{\Gamma(\alpha+1)} ((z_2-z_m)^\alpha - (z_1-z_m)^\alpha) \\
&\quad + \left[\frac{R^*(z_m-z_{m-1})^2}{2} + A_1^* \xi^* + B_1^* \right] \\
&\quad \times ((z_2-z_m)^{\alpha-1} - (z_1-z_m)^{\alpha-1}) z_m \\
&\quad + \left[\frac{R^*(z_m-z_{m-1})^2}{2} + A_1^* \xi^* + B_1^* \right] \\
&\quad \times ((z_2-z_m)^{\alpha-2} - (z_1-z_m)^{\alpha-2}) z_m (1-z_m) \\
&\quad + \left[\frac{R^*(z_m-z_{m-1})^3}{3} + A_2^* \xi^* + B_2^* \right] \\
&\quad \times ((z_2-z_m)^{\alpha-2} - (z_1-z_m)^{\alpha-2}) (1+z_m) \\
&\quad + \left[\frac{R^*(z_m-z_{m-1})^3}{3} + A_2^* \xi^* + B_2^* \right] \\
&\quad \times ((z_2-z_m)^{\alpha-1} - (z_1-z_m)^{\alpha-1}) + \left[\frac{R^*(1-z_m)^3}{3} \right] \\
&\quad \times ((z_2-z_m)^{\alpha-1} - (z_1-z_m)^{\alpha-1}) + \left[\frac{R^*(1-z_m)^3}{3} \right] \\
&\quad \times ((z_2-z_m)^{\alpha-2} - (z_1-z_m)^{\alpha-2}) (z_m).
\end{aligned} \tag{49}$$

Clearly, in the inequality (49), the right hand side tends to zero as $z_1 \rightarrow z_2$. Hence, $|Mr(z_2) - Mr(z_1)| \rightarrow 0$ as $z_1 \rightarrow z_2$. As a consequence of the passage from Step 1 to Step 3 combined with the Arzellá-Ascoli theorem, we

conclude that $M : PC(\mathcal{J}, \mathcal{R}) \longrightarrow PC(\mathcal{J}, \mathcal{R})$ is completely continuous.

(Step 4) In the last step, we need to show that the set $F = \{r(z) \in PC(\mathcal{J}, \mathcal{R}) : r(z) = \mu M(r(z)), \text{ for some } 0 < \mu < 1\}$ is bounded. Let $r \in F$; then, $r(z) = \mu M(r(z))$ for some $0 < \mu < 1$. Therefore, for every $z \in \mathcal{J}$, we have

$$\begin{aligned} r(z) &= \mu M(r(z)) \\ &= \mu \int_{z_m}^z \frac{(z-s)^{\alpha-1}}{\Gamma(\alpha)} \tau(s) ds \\ &\quad - \mu \left[((z-z_m)^{\alpha-1} + ((z-z_m)^{\alpha-1} + z_m(z-z_m)^{\alpha-2}))(1-z_m) \right] \\ &\quad \times \left(\int_{z_{m-1}}^{z_m} (z_m-s) \tau(s) ds - \psi_m(r(z_m)) \right) \\ &\quad - \mu \left[(z-z_m)^{\alpha-2} + ((z-z_m)^{\alpha-1} + z_m(z-z_m)^{\alpha-2}) \right] \\ &\quad \times \left(\int_{z_{m-1}}^{z_m} (z_m-s)^2 \tau(s) ds - \psi_m^*(r(z_m)) \right) \\ &\quad - \mu \left((z-z_m)^{\alpha-1} + z_m(z-z_m)^{\alpha-2} \right) \left(\int_{z_m}^1 (1-s)^2 \tau(s) ds \right). \end{aligned} \quad (50)$$

Now, we have

$$\begin{aligned} |r(z)| &= \mu |M(r(z))| \\ &\leq \int_{z_m}^z \frac{(z-s)^{\alpha-1}}{\Gamma(\alpha)} |\tau(s)| ds + |(z-z_m)^{\alpha-1} \\ &\quad + \{(z-z_m)^{\alpha-1} + z_m(z-z_m)^{\alpha-2}\}(1-z_m)| \\ &\quad \times \left(\int_{z_{m-1}}^{z_m} (z_m-s) |\tau(s)| ds + |\psi_m(r(z_m))| \right) \\ &\quad + |(z-z_m)^{\alpha-2} + \{(z-z_m)^{\alpha-1} + z_m(z-z_m)^{\alpha-2}\}| \\ &\quad \times \left(\int_{z_{m-1}}^{z_m} (z_m-s)^2 |\tau(s)| ds + |\psi_m^*(r(z_m))| \right) \\ &\quad + |(z-z_m)^{\alpha-1} + z_m(z-z_m)^{\alpha-2}| \left(\int_{z_m}^1 (1-s)^2 |\tau(s)| ds \right). \end{aligned} \quad (51)$$

Using (45) and hypotheses (H_6) and (H_7) in (51), we get

$$\begin{aligned} |r(z)| &\leq \frac{R^*}{\Gamma(\alpha+1)} + \frac{\sigma_1 R^*}{2} + A_1^* \xi^* + B_1^* + \frac{\sigma_2 R^*}{3} \\ &\quad + A_2^* \xi^* + B_2^* + \frac{\sigma_3 R^*}{3}. \end{aligned} \quad (52)$$

Hence, one has from where

$$\begin{aligned} \|r\| &\leq \left[R^* \left(\frac{1}{\Gamma(\alpha+1)} + \frac{\sigma_1}{2} + \frac{\sigma_2}{3} + \frac{\sigma_3}{3} \right) + (A_1^* + A_2^*) \xi^* + B_1^* + B_2^* \right] \\ &:= Z^*, \\ \|r\| &\leq Z^*. \end{aligned} \quad (53)$$

Hence, the given set F is bounded as a result of the Schaefer fixed point theorem, and we conclude that operator M has at least one fixed point. Hence, the corresponding Problem (2) has at least one solution. $\square$

4. Examples

Here, we provide two pertinent examples to verify the previous results.

Example 1. Consider the following IDE under the Riemann-Liouville-type integral boundary condition and the Riemann-Liouville fractional order derivative

$$\begin{cases} {}^{RL}D^\alpha r(z) = \frac{z + \cos(r(z)) + \cos({}^{RL}D^\alpha r(z))}{90 + z^2}, & 1 < \alpha \leq 2, z \in [0, 1], z \neq \frac{1}{5}, \\ \Delta r\left(\frac{1}{5}\right) = \psi_1\left(r\left(\frac{1}{5}\right)\right) = \frac{\sin(r(1/5))}{30}, \\ \Delta^* r\left(\frac{1}{5}\right) = \psi_1^*\left(r\left(\frac{1}{5}\right)\right) = \frac{e^{-r(1/5)}}{15}, \\ I^{1-\alpha} r(0) = 0, I^{2-\alpha} r(1) = 0. \end{cases} \quad (54)$$

where $\alpha = (1/2)$, we set

$$f(z, r(z), \tau(z)) = \frac{z + \cos(r(z)) + \cos(\tau(z))}{90 + z^2}, \quad r(z) \in PC(\mathcal{J}, \mathcal{R}), \quad \tau(z) \in \mathcal{R}, \text{ and } z \in [0, 1]. \quad (55)$$

Clearly f is a jointly continuous function.

Now for every $r(z), \bar{r}(z) \in PC(\mathcal{J}, \mathcal{R})$, $\tau(z), \bar{\tau}(z) \in \mathcal{R}$, and $z \in [0, 1]$, we have

$$\begin{aligned} |f(z, r(z), \tau(z)) - f(z, \bar{r}(z), \bar{\tau}(z))| &= \left| \frac{z + \cos(r(z)) + \cos(\tau(z))}{90 + z^2} - \frac{z + \cos(\bar{r}(z)) + \cos(\bar{\tau}(z))}{90 + z^2} \right| \\ &= \left| \frac{\cos(r(z)) - \cos(\bar{r}(z)) + \cos(\tau(z)) - \cos(\bar{\tau}(z))}{90 + z^2} \right| \\ &\leq \left| \frac{\cos(r(z)) - \cos(\bar{r}(z))}{90 + z^2} \right| + \left| \frac{\cos(\tau(z)) - \cos(\bar{\tau}(z))}{90 + z^2} \right|, \end{aligned}$$

$$|f(z, r(z), \tau(z)) - f(z, \bar{r}(z), \bar{\tau}(z))| \leq \frac{1}{90} (|r - \bar{r}| + |\tau - \bar{\tau}|). \quad (56)$$

Which satisfies hypothesis (H_2) with $K^* = L^* = (1/90)$. Now, we set

$$\Delta r\left(\frac{1}{5}\right) = \psi_1\left(r\left(\frac{1}{5}\right)\right) = \frac{\sin(r(1/5))}{30}, \quad r \in PC(\mathcal{J}, \mathcal{R}). \quad (57)$$

Then, for $r(z), \bar{r}(z) \in PC(\mathcal{J}, \mathcal{R})$, we have

$$\begin{aligned} & \left| \psi_1 \left(r \left(\frac{1}{5} \right) \right) - \psi_1 \left(\bar{r} \left(\frac{1}{5} \right) \right) \right| \\ &= \left| \frac{\sin(r(1/5))}{30} - \frac{\sin(\bar{r}(1/5))}{30} \right| \leq \frac{1}{30} |r - \bar{r}|. \end{aligned} \quad (58)$$

Therefore, with $N_1^* = (1/30)$, hypothesis (H_3) is satisfied. Next, we set

$$\Delta^* r \left(\frac{1}{5} \right) = \psi_1^* \left(r \left(\frac{1}{5} \right) \right) = \frac{e^{-(r(1/5))}}{15}, \quad r \in PC(\mathcal{J}, \mathcal{R}). \quad (59)$$

Then, for $r(z), \bar{r}(z) \in PC(\mathcal{J}, \mathcal{R})$, we have

$$\left| \psi_1^* \left(r \left(\frac{1}{5} \right) \right) - \psi_1^* \left(\bar{r} \left(\frac{1}{5} \right) \right) \right| = \left| \frac{e^{-(r(1/5))}}{15} - \frac{e^{-(\bar{r}(1/5))}}{15} \right| \leq \frac{1}{15} |r - \bar{r}|. \quad (60)$$

Thus, with $N_2^* = (1/15)$, hypothesis (H_4) is satisfied. Further, we need to satisfy the given condition of Theorem 7, by

$$\begin{aligned} & \left[\frac{K^*}{1-L^*} \left(\frac{1}{\Gamma(\alpha+1)} + \frac{\sigma_1}{2} + \frac{\sigma_2}{3} + \frac{\sigma_3}{3} \right) + (\sigma_1 N_1^* + \sigma_2 N_2^*) \right] \\ &= \frac{1}{90} \left[\left(\frac{1}{\Gamma(\alpha+1)} + \frac{35}{20} + \frac{55}{30} + \frac{28}{30} \right) + \frac{35}{300} + \frac{55}{150} \right] < 1. \end{aligned} \quad (61)$$

Therefore, all the hypotheses and conditions of Theorem 7 are satisfied. Therefore, the considered problem (54) has a unique solution on $\mathcal{J}$.

Example 2. Consider another example of IDE under the Riemann-Liouville-type integral boundary condition and the Riemann-Liouville fractional order derivative:

$$\begin{cases} {}^{RL}\mathbf{D}^\alpha r(z) = \frac{e^{-z} + e^{-2z} \sin(\sqrt{r(z)}) + \sin(\sqrt{{}^{RL}\mathbf{D}^\alpha r(z)})}{35 + z^3}, & 1 < \alpha \leq 2, z \in [0, 1], z \neq \frac{1}{7}, \\ \Delta r \left(\frac{1}{7} \right) = \psi_1 \left(r \left(\frac{1}{7} \right) \right) = \frac{\tan(r(1/7))}{25 + \tan(r(1/7))}, \\ \Delta^* r \left(\frac{1}{7} \right) = \psi_1^* \left(r \left(\frac{1}{7} \right) \right) = \frac{e^{r(z)}}{55 + 30e^{r(z)}}, \\ I^{1-\alpha} r(0) = 0, I^{2-\alpha} r(1) = 0. \end{cases} \quad (62)$$

where $\alpha = (3/2)$ and ${}^{RL}\mathbf{D}^\alpha r(z) = \tau(z)$. We set

$$\begin{aligned} f(z, r(z), \tau(z)) &= \frac{e^{-z} + e^{-2z} \sin(\sqrt{r(z)}) + \sin(\sqrt{\tau(z)})}{35 + z^3}, \\ r(z) &\in PC(\mathcal{J}, \mathcal{R}), \tau(z) \in \mathcal{R}, z \in [0, 1]. \end{aligned} \quad (63)$$

Clearly f is a jointly continuous function.

Now for every $r(z), \bar{r}(z) \in PC(\mathcal{J}, \mathcal{R})$, $\tau(z), \bar{\tau}(z) \in \mathcal{R}$, and $z \in [0, 1]$, we have

$$\begin{aligned} & |f(z, r(z), \tau(z)) - f(z, \bar{r}(z), \bar{\tau}(z))| \\ &= \left| \frac{e^{-z} + e^{-2z} \sin(\sqrt{r(z)}) + \sin(\sqrt{\tau(z)})}{35 + z^3} \right. \\ &\quad \left. - \frac{e^{-z} + e^{-2z} \sin(\sqrt{\bar{r}(z)}) + \sin(\sqrt{\bar{\tau}(z)})}{35 + z^3} \right| \end{aligned}$$

$$\begin{aligned} & \leq \left| \frac{e^{-2z} \left\{ \sin(\sqrt{r(z)}) - \sin(\sqrt{\bar{r}(z)}) \right\}}{35 + z^3} \right| \\ & \quad + \left| \frac{\sin(\sqrt{\tau(z)}) - \sin(\sqrt{\bar{\tau}(z)})}{35 + z^3} \right| \\ & \leq \frac{e^{-2z}}{35 + z^3} \left| \sqrt{r(z)} - \sqrt{\bar{r}(z)} \right| + \frac{1}{35 + z^3} \left| \sqrt{\tau(z)} - \sqrt{\bar{\tau}(z)} \right|, \\ & |f(z, r(z), \tau(z)) - f(z, \bar{r}(z), \bar{\tau}(z))| \leq \frac{1}{35} (|r - \bar{r}| + |\tau - \bar{\tau}|). \end{aligned} \quad (64)$$

This satisfies hypothesis (H_2) , with $K^* = L^* = (1/35)$.

Now, another hypotheses for every $r(z) \in PC(\mathcal{J}, \mathcal{R})$, $\tau(z) \in \mathcal{R}$, and $z \in [0, 1]$, we have

$$\begin{aligned} & |f(z, r(z), \tau(z))| \\ &= \left| \frac{e^{-z} + e^{-2z} \sin(\sqrt{r(z)}) + \sin(\sqrt{\tau(z)})}{35 + z^3} \right| \\ &\leq \frac{e^{-z}}{35 + z^3} + \frac{e^{-2z}}{35 + z^3} |\sqrt{r(z)}| + \frac{1}{35 + z^3} |\sqrt{\tau(z)}| \\ &\leq \frac{e^{-z}}{35 + z^3} + \frac{e^{-2z}}{35 + z^3} |r(z)| + \frac{1}{35 + z^3} |\tau(z)|. \end{aligned} \quad (65)$$

Thus, hypothesis (H_5) is satisfied with $x(z) = (e^{-z}/35 + z^3)$, $y(z) = (e^{-2z}/35 + z^3)$, and $h(z) = (1/35 + z^3)$. Now, we set

$$\Delta r\left(\frac{1}{7}\right) = \psi_1\left(r\left(\frac{1}{7}\right)\right) = \frac{\tan(r(1/7))}{25 + \tan(r(1/7))}, \quad r \in PC(\mathcal{J}, \mathcal{R}). \quad (66)$$

Then, for every $r \in PC(\mathcal{J}, \mathcal{R})$, we have

$$\left| \psi_1\left(r\left(\frac{1}{7}\right)\right) \right| = \left| \frac{\tan(r(1/7))}{25 + \tan(r(1/7))} \right|, \quad \leq \frac{1}{25} |r| + 1. \quad (67)$$

Therefore, hypothesis (H_6) is satisfied with $A_1^* = (1/25)$ and $B_1^* = 1$. Next, we set

$$\Delta^* r\left(\frac{1}{7}\right) = \psi_1^*\left(r\left(\frac{1}{7}\right)\right) = \frac{e^{(r(1/7))}}{55 + 30e^{(r(1/7))}}, \quad r \in PC(\mathcal{J}, \mathcal{R}). \quad (68)$$

Then, for every $r \in PC(\mathcal{J}, \mathcal{R})$, we have

$$\left| \psi_1^*\left(r\left(\frac{1}{7}\right)\right) \right| = \left| \frac{e^{(r(1/7))}}{55 + 30e^{(r(1/7))}} \right|, \quad \leq \frac{1}{55} |r| + \frac{1}{30}. \quad (69)$$

Thus, hypothesis (H_7) is satisfied with $A_2^* = (1/55)$ and $B_2^* = (1/30)$. Therefore, all of the hypotheses of Theorem 8 are satisfied, and therefore, the considered problem (62) has at least one solution on $\mathcal{J}$.

5. Conclusion

IDEs of the fractional order have received proper attention due to their important applications in various fields of applied sciences. In the past, most studies have been done using the Caputo-type fractional derivatives to handle IDEs. In very few papers, investigating IDEs of the fractional order was done using the Riemann-Liouville derivative. Therefore, we have established successfully some important results devoted to the existence theory of a solution to the considered nonlinear implicit IDE with the Riemann-Liouville-type integral boundary conditions under the Riemann-Liouville fractional order derivative. The corresponding results for the existence and unique-

ness of the solution have been archived by utilizing the classical Schiefer and Banach contraction fixed point theorems. For the demonstration of our results, we have enriched the paper by providing two pertinent examples.

Data Availability

Data used to support the findings of this study are included within the article.

Conflicts of Interest

No conflict of interest exist.

Authors' Contributions

Equal contribution has been done by all the authors.

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