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Research Article

Semianalytical Solutions of Some Nonlinear-Time Fractional Models Using Variational Iteration Laplace Transform Method

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In this work, we combined two techniques, the variational iteration technique and the Laplace transform method, in order to solve some nonlinear-time fractional partial differential equations. Although the exact solutions may exist, we introduced the technique VITM that approximates the solutions that are difficult to find. Even a single iteration best approximates the exact solutions. The fractional derivatives being used are in the Caputo-Fabrizio sense. The reliability and efficiency of this newly introduced method is discussed in details from its numerical results and their graphical approximations. Moreover, possible consequences of these results as an application of fixed-point theorem are placed before the experts as an open problem.

1. Introduction and Preliminaries

Almost all the phenomena in science and engineering are naturally modeled in the form of nonlinear differential equations, like Korteweg-de Vries equation [1, 2], nonlinear Schrödinger equation [3–5], alternating current power flow model [6], Richards equation for unsaturated water flow [7–10], Burger equation [11], and gravitational general theory [12].

Recently, the above-mentioned and other nonlinear model equations are solved by using more than one semianalytical and numerical methods, like the Laplace transform method (LTM) [13, 14], variational iteration method (VIM) [15, 16], Newton-Raphson formula (NRF) [17], Adomian decomposition method (ADM) [18], homotopy analysis method (HAM) [19, 20], homotopy perturbation method (HPM) [21], spectral collocation technique [22] and the equation presented in [23].

Nowadays, the techniques of fractional calculus are being employed successfully for better understanding of complex natural phenomena, which not only agree with the ordinary calculus techniques but also give the best results and understanding of the phenomena. Laplace transform is a powerful tool, which has been used in the past decades to solve the ODEs with constant and variable coefficients as well as to solve PDEs. Similarly, in these days, the variational iteration technique, developed by the Chinese mathematician He [15], is also a reliable technique (which was originally developed to solve differential and integrodifferential equations) to solve PDEs. The main drawback of the variational iteration method is that one may have difficulty in calculating the Lagrange multiplier. Currently, much attention is being paid in combining more than one technique to solve a model especially nonlinear models, to get better and rapid results. In this direction the work has been started, and it is observed that the results obtained by combining more than one technique are much better than that of a single technique as discussed in [24, 25].

In the current paper, two techniques, variational iteration technique and the Laplace transform, are being utilized, and the combined technique, the variational iteration transform method (VITM), is employed to handle the nonlinear fractional order partial differential equations, like the Korteweg-de Vries equation [26], Schrödinger equation [27], and Burger equation [28]. The rapid convergence of the method proves that it is a more reliable technique now more than ever than the existing one to solve FPDEs, and it introduces a new significant improvement. In the proceeding sections, method description along with the validity of the results obtained by the technique is presented.

In the study of the fractional differential equations, the Caputo-Fabrizio fractional derivative [29] will be considered. The Caputo-Fabrizio fractional derivative is the most recent fractional derivative which is more effective than the other fractional derivatives present in the literature, in dealing with the initial value problems. First, let us recall the some definitions from the area of fractional calculus.

(1) *Riemann-Liouville Fractional Derivative*. The Riemann-Liouville fractional derivative of a function $f(t)$ is defined to be

$$D^{\alpha} f(t) = \left(\frac{d}{dt}\right)^{\varepsilon+1} \int_a^t (t-\tau)^{\varepsilon-\alpha} f(\tau) d\tau, \tag{1}$$

where $\varepsilon \le \alpha < \varepsilon + 1$ or

$$D^{\alpha} f(t) = \frac{1}{\Gamma(\kappa-\alpha)} \frac{d^{\kappa}}{dt^{\kappa}} \int_a^t (t-\tau)^{\kappa-\alpha-1} f(\tau) d\tau, \tag{2}$$

where $\kappa - 1 \le \alpha < \kappa$. Both κ and ε are integers.

(2) *Caputo's Fractional Derivative*. Caputo's fractional derivative of $f(t)$ is given by

$$D_a^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{f'(t-\tau)}{(\tau-a)^{\alpha}} d\tau, \tag{3}$$

such that $\alpha \in [0, 1]$.

(3) *Caputo-Fabrizio Derivative*. Let us recall one of the most recent definitions of the fractional derivative Caputo-Fabrizio derivative, as follows. Let $\mathscr{F}(t) \in H^1(a, b), b > a$; then, the Caputo-Fabrizio time fractional derivative of $\mathscr{F}(t)$ is defined as

$$D_t^{\alpha} \mathscr{F}(t) = \frac{\mathscr{M}(a)}{(1-\alpha)} \int_a^t \mathscr{F}'(t) \exp\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right] d\tau, \tag{4}$$

where $\alpha \epsilon [0, 1]$ and $\mathscr{M}(a)$ is a normalization function that is $\mathscr{M}(0) = \mathscr{M}(1) = 1$.

(4) *Laplace Transform*. Let $f(t)$ be a function; then, its Laplace transform is defined as

$$\mathscr{L}\{f(t)\} = \mathscr{F}(s) = \int_0^{\infty} e^{-st} f(t) dt, \tag{5}$$

and the Laplace transform of $f'(t)$ is given by

$$\mathscr{L}\left\{D_t^{(\alpha+n)} f'(t)\right\} = \frac{s^{(n+1)} \mathscr{L}[f(t)] - s^n f(0) - s^{n-1} f'(0) - \cdots - f^{(n)}(0)}{s + \alpha(1-s)}. \tag{6}$$

2. Methodology of VITM

This section is devoted to present the methodology of the proposed technique. Then, let us consider the general time fractional partial differential equation.

$$D_t^{\alpha} \mathscr{X}(\varphi, t) + L(\mathscr{X}(\varphi, t)) + N(\mathscr{X}(\varphi, t), \mathscr{Y}(\varphi, t), \mathscr{Z}(\varphi, t)) = \mathscr{F}(\varphi, t), \tag{7}$$

subject to

$$\mathscr{X}(\varphi, 0) = \mathscr{X}_0, \tag{8}$$

where $L(\mathscr{X}(\varphi, t))$, $N(\mathscr{X}(\varphi, t), \mathscr{Y}(\varphi, t), \mathscr{Z}(\varphi, t))$, and $\mathscr{F}(\varphi, t)$ are linear, nonlinear, and known functions, respectively. Also D_t^{α} is in the Caputo-Fabrizio sense.

Further, we apply the variational iteration method on the above equation. Then, we found the following iterative form:

$$\mathscr{X}_{n+1}(\varphi, t) = \mathscr{X}_n(\varphi, t) + \lambda\{D_t^{\alpha} \mathscr{X}(\varphi, t) + L(\mathscr{X}(\varphi, t)) + N(\mathscr{X}(\varphi, t), \mathscr{Y}(\varphi, t), \mathscr{Z}(\varphi, t)) - \mathscr{F}(\varphi, t)\}. \tag{9}$$

Also, if we apply the Laplace transform on this equation, we transform the variable t, to the new variable s, such that

$$\mathscr{X}_{n+1}(\varphi, s) = \mathscr{X}_n(\varphi, s) + \lambda \mathscr{L}\left\{D_t^{\alpha} \mathscr{X}_n(\varphi, s) + L\tilde{\mathscr{X}}_n(\varphi, s) + N\left(\tilde{\mathscr{X}}_n(\varphi, s), \tilde{\mathscr{Y}}_n(\varphi, s), \tilde{\mathscr{Z}}_n(\varphi, s)\right) - \mathscr{F}(\varphi, s)\right\}, \tag{10}$$

where $\tilde{\mathscr{X}}_n(\varphi, t)$, etc. are restricted values, which means

$$\delta \tilde{\mathscr{X}}_n(\varphi, t) = 0. \tag{11}$$

Using the following relations:

$$\begin{aligned} \mathscr{L}\{D^{\alpha} \mathscr{X}_n(\varphi, t)\} &= s^{\alpha} \mathscr{X}_n(\varphi, s) - s^{\alpha-1} \mathscr{X}_n(\varphi, 0), \\ \mathscr{L}\{\delta D^{\alpha} \mathscr{X}_n(\varphi, t)\} &= s^{\alpha} \delta \mathscr{X}_n(\varphi, s) - s^{\alpha-1} \delta \mathscr{X}_n(\varphi, 0), \end{aligned} \tag{12}$$

where

$$\delta \mathscr{X}_n(\varphi, 0) = 0. \tag{13}$$

Then, we have

$$\mathscr{L}\{\delta D^{\alpha}\mathscr{X}_n(\varphi,t)\} = s^{\alpha}\delta\mathscr{X}_n(\varphi,s). \tag{14}$$

The optimization conditions,

$$\frac{\delta\tilde{\mathscr{X}}_{n+1}(\varphi,s)}{\delta\tilde{\mathscr{X}}_n(\varphi,s)} = 0, \quad \delta\widetilde{\mathscr{X}_n} = 0, \delta\widetilde{\mathscr{Y}_n} = 0, \delta\widetilde{\mathscr{Z}_n} = 0, \tag{15}$$

give

$$0 = 1 + \lambda\left\{\frac{s^{\alpha}\delta\tilde{\mathscr{X}}_n(\varphi,s)}{\delta\tilde{\mathscr{X}}_n(\varphi,s)}\right\}. \tag{16}$$

The above equation implies $\lambda = -1/s^{\alpha}$.

On substituting in equation (10), we obtain

$$\mathscr{X}_{n+1}(\varphi,s) = \mathscr{X}_n(\varphi,s) - \frac{1}{s^{\alpha}}\mathscr{L}\Big\{D_t^{\alpha}\mathscr{X}_n(\varphi,s) + L\tilde{\mathscr{X}}_n(\varphi,s) + N\Big(\tilde{\mathscr{X}}_n(\varphi,s), \tilde{\mathscr{Y}}_n(\varphi,s), \tilde{\mathscr{Z}}_n(\varphi,s)\Big) - \mathscr{F}(\varphi,t)\Big\}. \tag{17}$$

The inverse Laplace transform gives

$$\mathscr{X}_{n+1}(\varphi,t) = \mathscr{X}_n(\varphi,t) - \mathscr{L}^{-1}\left\{\frac{1}{s^{\alpha}}\mathscr{L}\{D_t^{\alpha}\mathscr{X}_n(\varphi,s) + L\mathscr{X}_n(\varphi,s) + N(\mathscr{X}_n(\varphi,s), \mathscr{Y}_n(\varphi,s), \mathscr{Z}_n(\varphi,s)) - \mathscr{F}(\varphi,s)\}\right\}. \tag{18}$$

Substituting $n = 0, 1, 2, \cdots$, we find the following successive approximations:

$$\mathscr{X}_1, \mathscr{X}_2, \mathscr{X}_3 \cdots. \tag{19}$$

3. Applications of VITM on Various FODE Types

In this section, we present and apply VITM on some important FODEs from related literature. Then, our first application takes into consideration the most general time fractional form of the Korteweg-de Vries equation (see [24]).

$$D_t^{\alpha}\mathscr{X} + \alpha_1\mathscr{X}\mathscr{X}_{\varphi} + \beta_1\mathscr{X}_{\varphi\varphi\varphi} = 0, \quad 0 < \alpha \le 1, \tag{20}$$

subject to

$$\mathscr{X}(\varphi,0) = \mathscr{X}_0 = \frac{a}{\cosh^2\beta_1\varphi}, \tag{21}$$

where

$$\alpha_1 = \frac{c_0}{2k^2}(\varepsilon c\lambda_3), \tag{22}$$

is the nonlinear parameter and

$$\beta_1 = \frac{c_0h^2}{6}, \tag{23}$$

is the dispersion parameter.

Applying the variational iteration and Laplace transform, we get

$$\mathscr{X}_{n+1}(\varphi,s) = \mathscr{X}_n(\varphi,s) + \mathscr{L}\left\{\lambda\left\{D_t^{\alpha}\tilde{\mathscr{X}}_n(\varphi,s) + \alpha_1\tilde{\mathscr{X}}_n(\varphi,s)\frac{\partial\tilde{\mathscr{X}}_n(\varphi,s)}{\partial\varphi} + \beta_1\frac{\partial^3\tilde{\mathscr{X}}_n(\varphi,s)}{\partial\varphi^3}\right\}\right\}. \tag{24}$$

Also, substituting the following relation

$$\mathscr{L}\{D_t^{\alpha}\mathscr{X}_n(\varphi,t)\} = s^{\alpha}\mathscr{X}_n(\varphi,s) - s^{\alpha-1}\mathscr{X}_n(\varphi,0), \tag{25}$$

and optimality conditions, etc., we get the next results:

$$\frac{\delta\tilde{\mathscr{X}}_{n+1}(\varphi,s)}{\delta\tilde{\mathscr{X}}_n(\varphi,s)} = 0,\ \delta\widetilde{\mathscr{X}_n} = 0, \lambda = \frac{-1}{s^{\alpha}}. \tag{26}$$

By substitution, we have

$$\mathscr{X}_{n+1}(\varphi,s) = \mathscr{X}_n(\varphi,s) - \frac{1}{s^{\alpha}}\mathscr{L}\left\{\alpha_1\mathscr{X}_n(\varphi,s)\frac{\partial\mathscr{X}_n(\varphi,s)}{\partial\varphi} + \beta_1\frac{\partial^3\mathscr{X}n(\varphi,s)}{\partial\varphi^3}\right\}. \tag{27}$$

Applying the inverse Laplace transform on the above equation and simplifying, we get

$$\mathscr{X}_{n+1}(\varphi,t) = \mathscr{X}_n(\varphi,t) - \mathscr{L}^{-1}\left\{\frac{1}{s^{\alpha}}\mathscr{L}\left\{D_t^{\alpha}\mathscr{X}_n(\varphi,s) + \alpha_1\mathscr{X}_n(\varphi,s)\frac{\partial\mathscr{X}_n(\varphi,s)}{\partial\varphi} + \beta_1\frac{\partial^3\mathscr{X}n(\varphi,s)}{\partial\varphi^3}\right\}\right\}. \tag{28}$$

For $n = 0, 1, 2, \cdots$, the following approximations are obtained: $\mathscr{X}_1, \mathscr{X}_2, \mathscr{X}_3 \cdots$, such as

$$\begin{aligned}\mathscr{X}_1(\varphi,t) = {} & a\operatorname{sech}^2\beta_1\varphi - \frac{t^{\alpha}}{\Gamma(1+\alpha)} \\ & \cdot\{-2a^2\alpha_1\beta_1 \sec h^4\beta_1\varphi \tan h\beta_1\varphi \\ & + \beta_1\{16a{\beta_1}^3 \sec h^4\beta_1\varphi \tan h\beta_1\varphi \\ & - 8a{\beta_1}^3 \sec h^2\beta_1\varphi \tan h^3\beta_1\varphi\}\},\end{aligned}$$

$$\mathcal{X}_2(\varphi,t)=a\sec h^2\beta_1\varphi-\frac{t^\alpha}{\Gamma(1+\alpha)}\cdot\left\{-2a^2\alpha_1\beta_1\sec h^4\beta_1\varphi\tan h\beta_1\varphi+\beta_1\left\{16a{\beta_1}^3\sec h^4\beta_1\varphi\tan h\beta_1\varphi-8a{\beta_1}^3\sec h^2\beta_1\varphi\tan h^3\beta_1\varphi\right\}\right\}-\frac{2a^3t^{2\alpha}{\alpha_1}^2{\beta_1}^2\sec h^8\beta_1\varphi}{\Gamma(1+2\alpha)}+\cdots, \tag{29}$$

and so on.

The solution $\mathcal{X}(\varphi,t)$ can be found as

$$\mathcal{X}(\varphi,t)=\lim_{i\longrightarrow\infty}\mathcal{X}_i. \tag{30}$$

As particular examples, let us consider further some versions of time fractional equations.

Example 1. The first particular example is a simple time fractional Korteweg-de Vries equation (see [18]).

$$D_t^\alpha\mathcal{X}-6\mathcal{X}\mathcal{X}_\varphi+\mathcal{X}_{\varphi\varphi\varphi}=0\,;\mathcal{X}(\varphi,0)=6\varphi. \tag{31}$$

Application of the proposed VITM step by step gives

$$\mathcal{X}_{n+1}(\varphi,s)=\mathcal{X}_n(\varphi,s)+\mathcal{L}\left\{\lambda\left\{D_t^\alpha\widetilde{\mathcal{X}_n}(\varphi,s)-6\widetilde{\mathcal{X}_n}(\varphi,s)\frac{\partial\widetilde{\mathcal{X}_n}(\varphi,s)}{\partial\varphi}+\frac{\partial^3\widetilde{\mathcal{X}_n}(\varphi,s)}{\partial\varphi^3}\right\}\right\}. \tag{32}$$

The optimality conditions, etc. give the following results:

$$\frac{\delta\tilde{\mathcal{X}}_{n+1}(\varphi,s)}{\delta\tilde{\mathcal{X}}_n(\varphi,s)}=0,\delta\widetilde{\mathcal{X}_n}=0,\ \lambda=\frac{-1}{s^\alpha} \tag{33}$$

Substitution and inverse Laplace transform implies

$$\mathcal{X}_{n+1}(\varphi,t)=\mathcal{X}_n(\varphi,t)-\mathcal{L}^{-1}\left\{\frac{1}{s^\alpha}\mathcal{L}\left\{D_t^\alpha\mathcal{X}_n(\varphi,s)-6\mathcal{X}_n(\varphi,s)\frac{\partial\mathcal{X}_n(\varphi,s)}{\partial\varphi}+\frac{\partial^3\mathcal{X}n(\varphi,s)}{\partial\varphi^3}\right\}\right\}. \tag{34}$$

For $n=0,1,2,\cdots$, we get the approximations $\mathcal{X}_1,\mathcal{X}_2,\mathcal{X}_3,\cdots$, such as

$$\begin{aligned}\mathcal{X}_1(\varphi,t)&=6\varphi+\frac{216\varphi t^\alpha}{\Gamma(1+\alpha)},\\ \mathcal{X}_2(\varphi,t)&=6\varphi+\frac{216\varphi t^\alpha}{\Gamma(1+\alpha)}-216\varphi t^\alpha\left\{\frac{-1+t^{-1+\alpha}\alpha\Gamma\alpha/\Gamma(2\alpha)}{\Gamma(1+\alpha)}-\frac{72t^\alpha}{\Gamma(1+2\alpha)}-\frac{1296t^{2\alpha}\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)}\right\}.\end{aligned} \tag{35}$$

The solution $\mathcal{X}(\varphi,t)$ can be found as

$$\mathcal{X}(\varphi,t)=\lim_{i\longrightarrow\infty}\mathcal{X}_i, \tag{36}$$

that is,

$$\mathcal{X}(\varphi,t)=6\varphi+\frac{216\varphi t^\alpha}{\Gamma(1+\alpha)}-216\varphi t^\alpha\left\{\frac{-1+t^{-1+\alpha}\alpha\Gamma\alpha/\Gamma(2\alpha)}{\Gamma(1+\alpha)}-\frac{72t^\alpha}{\Gamma(1+2\alpha)}-\frac{1296t^{2\alpha}\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)}\right\}+\cdots. \tag{37}$$

For $\alpha=1$, it turns out to be

$$\mathcal{X}(\varphi,t)=6\varphi+216\varphi t+7776\varphi t^2+\cdots, \tag{38}$$

which is the expansion of the exact solution $\mathcal{X}(\varphi,t)=6(\varphi)/(1-36t)$ that confirms the validity of the proposed VITM (see [18]).

Next, let us give a graphical representation of the approximated solution $\mathcal{X}(\varphi,t)$, for different values of α using Mathematica. Moreover, we will also give a graphical 3D representation for the exact solution $\mathcal{X}(\varphi,t)=6(\varphi)/(1-36t)$ (Figure 1(b)). In this way, we show how the proposed technique approaches the exact solution; see Figures 2(a), 2(b), and 1(a), which are the approximations of Figure 1(b). The scale for all the four figures is $-50\geq\varphi\leq 50$ and $-50\geq t\leq 50$.

Example 2. Let us consider another version of time fractional version of Korteweg-de Vries equation (see [18]).

$$D_t^\alpha\mathcal{X}-6\mathcal{X}\mathcal{X}_\varphi+\mathcal{X}_{\varphi\varphi\varphi}=0,\mathcal{X}(\varphi,0)=-2\sec h^2\varphi. \tag{39}$$

Applying VITM step by step, we obtain

$$\mathcal{X}_{n+1}(\varphi,s)=\mathcal{X}_n(\varphi,s)+\mathcal{L}\left\{\lambda\left\{D_t^\alpha\tilde{\mathcal{X}}_n(\varphi,s)-6\tilde{\mathcal{X}}_n(\varphi,s)\frac{\partial\tilde{\mathcal{X}}_n(\varphi,s)}{\partial\varphi}+\frac{\partial^3\tilde{\mathcal{X}}_n(\varphi,s)}{\partial\varphi^3}\right\}\right\}. \tag{40}$$

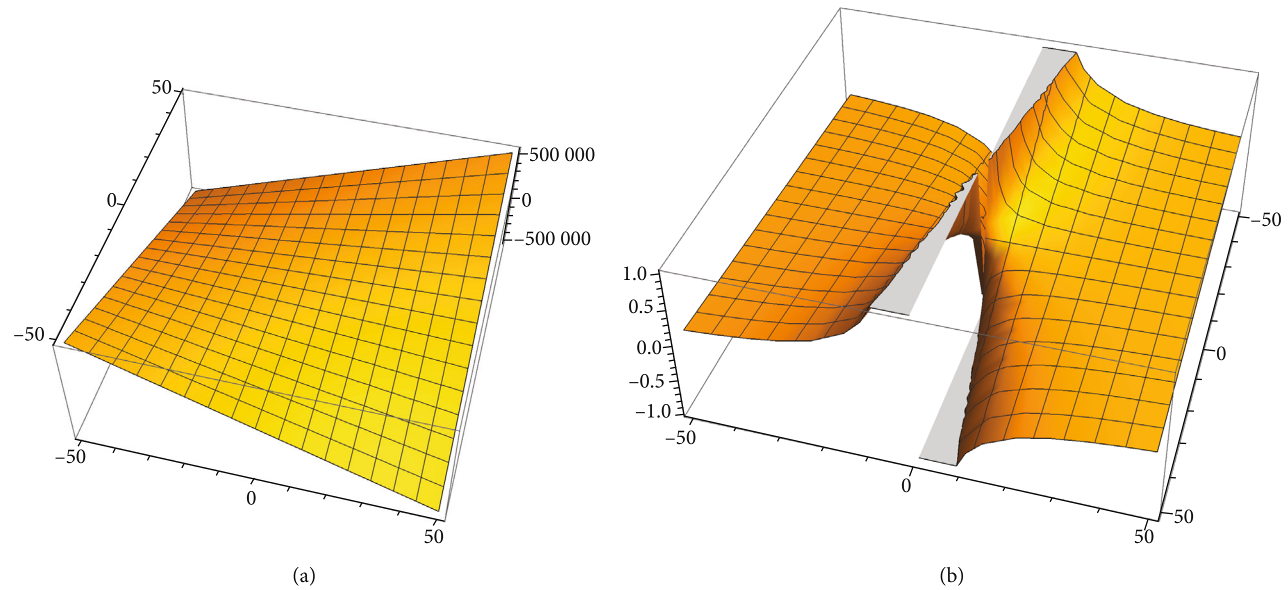


FIGURE 1: 3D representation of X for $\alpha = 1$ (a) and of the exact solution (b).

Using optimality conditions, etc., we get the following results:

$$\frac{\delta \tilde{\mathcal{X}}_{n+1}(\varphi, s)}{\delta \tilde{\mathcal{X}}_n(\varphi, s)} = 0,\ \delta \tilde{\mathcal{X}}_n = 0,\ \lambda = \frac{-1}{s^\alpha}. \tag{41}$$

Substitution and inverse Laplace transform implies

$$\mathcal{X}_{n+1}(\varphi, t) = \mathcal{X}_n(\varphi, t) - \mathcal{L}^{-1}\left\{\frac{1}{s^\alpha}\mathcal{L}\left\{D_t^\alpha \mathcal{X}_n(\varphi, s) - 6\mathcal{X}_n(\varphi, s)\frac{\partial \mathcal{X}_n(\varphi, s)}{\partial \varphi} + \frac{\partial^3 \mathcal{X} n(\varphi, s)}{\partial \varphi^3}\right\}\right\}. \tag{42}$$

For $n = 0, 1, 2, \cdots$, we get the approximations $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \cdots$, such as

$$\begin{aligned}
\mathcal{X}_1(\varphi, t) &= -2 \sec h^2\varphi - \frac{t^\alpha\left\{16 \sec h^4\varphi \tan h\varphi + 16 \sec h^2\varphi \tan h^3\varphi\right\}}{\Gamma(1+\alpha)},\\
\mathcal{X}_2(\varphi, t) &= -2 \sec h^2\varphi - \frac{t^\alpha\left\{16 \sec h^4\varphi \tan h\varphi + 16 \sec h^2\varphi \tan h^3\varphi\right\}}{\Gamma(1+\alpha)}\\
&\quad - \frac{16 t^{-1+\alpha} \sec h^2\varphi + \cdots}{\Gamma(1+\alpha)^2\Gamma(2\alpha)\Gamma(1+2\alpha)\Gamma(1+3\alpha)}.
\end{aligned} \tag{43}$$

The solution $\mathcal{X}(\varphi, t)$ can be found as

$$\mathcal{X}(\varphi, t) = \lim_{i\longrightarrow\infty} \mathcal{X}_i. \tag{44}$$

Then,

$$\begin{aligned}
\mathcal{X}(\varphi, t) &= -2 \sec h^2\varphi - \frac{t^\alpha\left\{16 \sec h^4\varphi \tan h\varphi + 16 \sec h^2\varphi \tan h^3\varphi\right\}}{\Gamma(1+\alpha)}\\
&\quad - \frac{16 t^{-1+\alpha} \sec h^2\varphi + \cdots}{\Gamma(1+\alpha)^2\Gamma(2\alpha)\Gamma(1+2\alpha)\Gamma(1+3\alpha)} + \cdots.
\end{aligned} \tag{45}$$

For $\alpha = 1$, we have

$$\begin{aligned}
\mathcal{X}(\varphi, t) &= -2 \sec h^2\varphi - t\left\{16 \sec h^4\varphi \tan h\varphi\right.\\
&\quad \left. + 16 \sec h^2\varphi \tan h^3\varphi\right\} + \cdots,
\end{aligned} \tag{46}$$

which is the expansion of the exact solution, $\mathcal{X}(\varphi, t) = -2 \sec h^2(\varphi - 4t)$ (see [18]).

Example 3. Consider the simple time fractional Burgers equation (see [18]).

$$D_t^\alpha \mathcal{X} + \mathcal{X}\mathcal{X}_\varphi + \mathcal{X}_{\varphi\varphi} = 0, \mathcal{X}(\varphi, 0) = \varphi. \tag{47}$$

Applying the proposed VITM step by step, we get

$$\mathcal{X}_{n+1}(\varphi, s) = \mathcal{X}_n(\varphi, s) + \mathcal{L}\left\{\lambda\left\{D_t^\alpha \widetilde{\mathcal{X}_n}(\varphi, s) + \widetilde{\mathcal{X}_n}(\varphi, s)\frac{\partial \widetilde{\mathcal{X}_n}(\varphi, s)}{\partial \varphi} + \frac{\partial^2 \widetilde{\mathcal{X}_n}(\varphi, s)}{\partial \varphi^2}\right\}\right\}. \tag{48}$$

The optimality conditions, etc. give the following results:

$$\frac{\delta \tilde{\mathcal{X}}_{n+1}(\varphi, s)}{\delta \widetilde{\mathcal{X}_n}(\varphi, s)} = 0,\ \delta \widetilde{\mathcal{X}_n} = 0, \lambda = \frac{-1}{s^\alpha}. \tag{49}$$

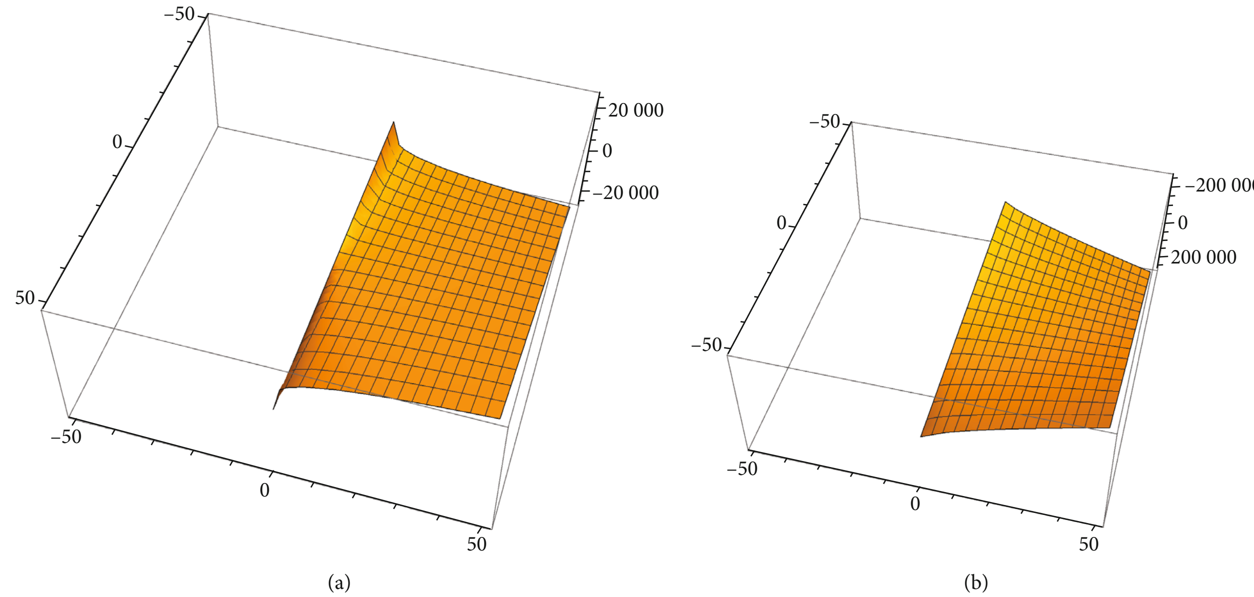


Figure 2: 3D representation of X for $\alpha = 0.2$ (a) and $\alpha = 0.8$ (b).

Substitution and inverse Laplace transform implies

$$\mathcal{X}_{n+1}(\varphi, t) = \mathcal{X}_n(\varphi, t) - \mathcal{L}^{-1}\left\{\frac{1}{s^\alpha}\mathcal{L}\left\{D_t^\alpha \mathcal{X}_n(\varphi, s) + \mathcal{X}_n(\varphi, s)\frac{\partial \mathcal{X}_n(\varphi, s)}{\partial \varphi} + \frac{\partial^2 \mathcal{X}n(\varphi, s)}{\partial \varphi^2}\right\}\right\}. \tag{50}$$

For $n = 0, 1, 2, \cdots$, we get the approximations $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \cdots$, such as

$$\begin{aligned}
\mathcal{X}_1(\varphi, t) &= \varphi - \frac{\varphi t^\alpha}{\Gamma(1+\alpha)}, \\
\mathcal{X}_2(\varphi, t) &= \varphi + \frac{\varphi t^\alpha}{\Gamma(1+\alpha)} - \varphi t^\alpha \left\{\frac{1 - t^{-1+\alpha}\alpha\Gamma\alpha/\Gamma(2\alpha)}{\Gamma(1+\alpha)} - \frac{2t^\alpha}{\Gamma(1+2\alpha)} + \frac{t^{2\alpha}\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)}\right\}.
\end{aligned} \tag{51}$$

The solution $\mathcal{X}(\varphi, t)$ can be found as

$$\mathcal{X}(\varphi, t) = \lim_{i \longrightarrow \infty} \mathcal{X}_i. \tag{52}$$

It means

$$\mathcal{X}(\varphi, t) = \varphi + \frac{\varphi t^\alpha}{\Gamma(1+\alpha)} - \varphi t^\alpha \left\{\frac{1 - t^{-1+\alpha}\alpha\Gamma\alpha/\Gamma(2\alpha)}{\Gamma(1+\alpha)} - \frac{2t^\alpha}{\Gamma(1+2\alpha)} + \frac{t^{2\alpha}\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)}\right\}. \tag{53}$$

For $\alpha = 1$, we have

$$\mathcal{X}(\varphi, t) = \varphi - \varphi t + \varphi t^2 + \cdots, \tag{54}$$

which resembles with the expansion of the exact solution $\mathcal{X}(\varphi, t) = \varphi/(1+t)$, confirming the validity of the proposed VITM (see [18]).

Further, let us draw some approximations of $\mathcal{X}(\varphi, t) = \varphi - \varphi t + \varphi t^2 + \cdots$, for different values of α. Then, see Figures 3(a), 3(b), 4(a), and 4(b), which are the approximations of Figure 5.

Example 4. Let us consider another time fractional version of Burgers equation (see [18]) as follows:

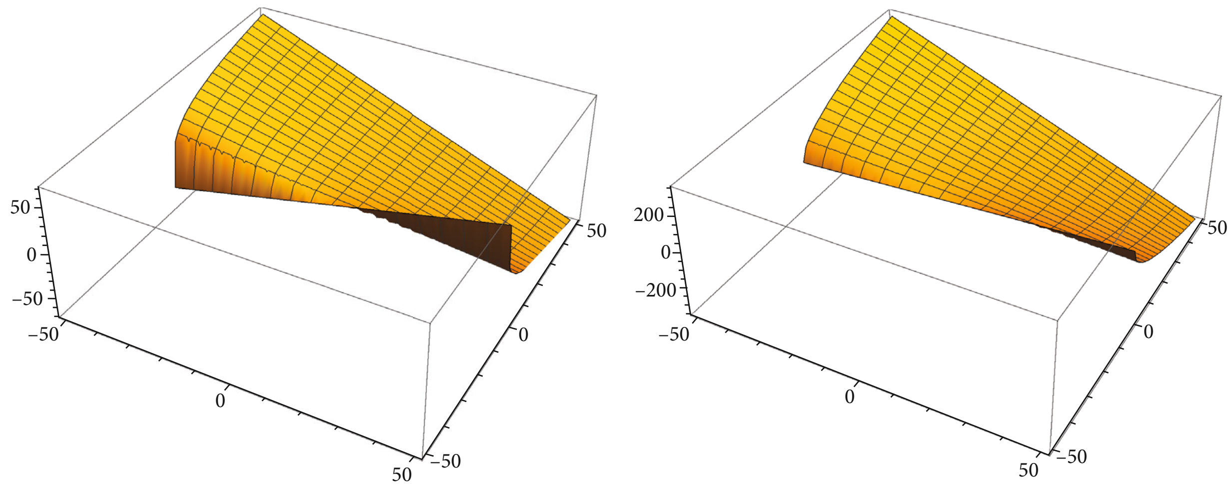
$$D_t^\alpha \mathcal{X} + \mathcal{X}\mathcal{X}_\varphi + \mathcal{X}_{\varphi\varphi} = 0\,;\, \mathcal{X}(\varphi, 0) = 2\tan\varphi. \tag{55}$$

Applying the variational iteration and Laplace transform, we get

$$\mathcal{X}_{n+1}(\varphi, s) = \mathcal{X}_n(\varphi, s) + \mathcal{L}\left\{\lambda\left\{D_t^\alpha \tilde{\mathcal{X}}_n(\varphi, s) + \tilde{\mathcal{X}}_n(\varphi, s)\frac{\partial \tilde{\mathcal{X}}_n(\varphi, s)}{\partial \varphi} + \frac{\partial^2 \tilde{\mathcal{X}}_n(\varphi, s)}{\partial \varphi^2}\right\}\right\}. \tag{56}$$

Using optimality conditions, etc., we obtain the following results:

$$\frac{\delta \tilde{\mathcal{X}}_{n+1}(\varphi, s)}{\delta \tilde{\mathcal{X}}_n(\varphi, s)} = 0,\ \delta \tilde{\mathcal{X}}_n = 0,\ \lambda = \frac{-1}{s^\alpha}. \tag{57}$$



(a) (b)

FIGURE 3: 3D representation of X for $\alpha = 0.2$ (a) and $\alpha = 0.5$ (b).

Substitution and inverse Laplace transform implies

$$\mathcal{X}_{n+1}(\varphi, t) = \mathcal{X}_n(\varphi, t) - \mathcal{L}^{-1}\left\{ \frac{1}{s^\alpha} \mathcal{L}\left\{ D_t^\alpha \mathcal{X}_n(\varphi, s) + \mathcal{X}_n(\varphi, s) \frac{\partial \mathcal{X}_n(\varphi, s)}{\partial \varphi} + \frac{\partial^2 \mathcal{X}_n(\varphi, s)}{\partial \varphi^2} \right\}\right\}. \tag{58}$$

For $n = 0, 1, 2, \cdots$, we get the approximations $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \cdots$, such as

$$\begin{aligned}
\mathcal{X}_1(\varphi, t) &= 2 \tan \varphi - \frac{8t^\alpha \sec^2\varphi \tan \varphi}{\Gamma(1+\alpha)}, \\
\mathcal{X}_2(\varphi, t) &= 2 \tan \varphi - \frac{8t^\alpha \sec^2\varphi \tan \varphi}{\Gamma(1+\alpha)} \\
&\quad - \frac{8t^{-1+\alpha} \sec^2\varphi \tan \varphi}{\Gamma(2\alpha)\Gamma(1+\alpha)^2\Gamma(1+2\alpha)\Gamma(1+3\alpha)} \\
&\quad \times \{-t^\alpha \alpha \Gamma(\alpha)\Gamma(1+\alpha)\Gamma(1+2\alpha)\Gamma(1+3\alpha) \\
&\quad + t\Gamma(2\alpha)\Gamma(1+\alpha)\Gamma(1+2\alpha)\Gamma(1+3\alpha) \\
&\quad + 4t^\alpha \sec^2\varphi(-4+\cos 2\varphi)\Gamma(1+\alpha)^2\Gamma(1+3\alpha) \\
&\quad - 8t^{2\alpha} \sec^4 x(-2+\cos 2x)\Gamma(1+2\alpha)^2\}.
\end{aligned} \tag{59}$$

The solution is in the series form, such as

$$\mathcal{X}(\varphi, t) = \lim_{i \longrightarrow \infty} \mathcal{X}_i, \tag{60}$$

which turns out to be the expansion of the exact solution $\mathcal{X}(\varphi, t) = -2 \tan \varphi$ for $\alpha = 1$, as discussed earlier (see [18]).

Example 5. Let us consider the time fractional version of the nonlinear simple Schrödinger equation [18] as follows

$$iD_t^\alpha \mathcal{X} + \mathcal{X}_{\varphi\varphi} - 2|\mathcal{X}|^2\mathcal{X} = 0, \mathcal{X}(\varphi, 0) = e^{i\varphi}. \tag{61}$$

Applying the variational iteration and Laplace transform, we obtain

$$\mathcal{X}_{n+1}(\varphi, s) = \mathcal{X}_n(\varphi, s) + \mathcal{L}\left\{ \lambda \left\{ iD_t^\alpha \tilde{\mathcal{X}}_n(\varphi, s) - 2 \,|\, \widetilde{\mathcal{X}^2}_n(\varphi, s) \,|\, \tilde{\mathcal{X}}_n(\varphi, s) + \frac{\partial^2 \tilde{\mathcal{X}}_n(\varphi, s)}{\partial \varphi^2} \right\}\right\}. \tag{62}$$

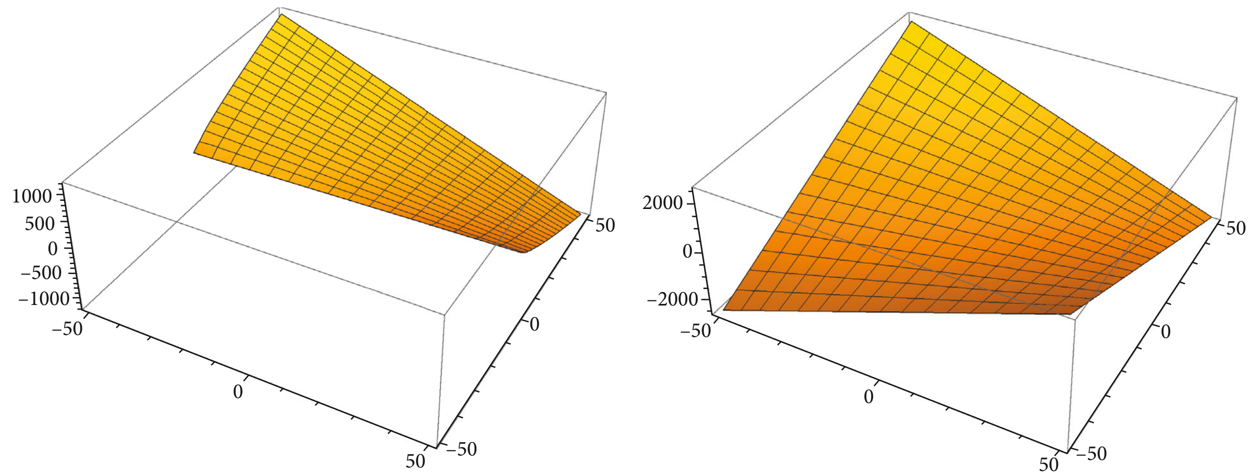
Using optimality conditions, etc., we get the following results:

$$\frac{\delta \tilde{\mathcal{X}}_{n+1}(\varphi, s)}{\delta \tilde{\mathcal{X}}_n(\varphi, s)} = 0, \ \delta \tilde{\mathcal{X}}_n = 0, \ \lambda = \frac{-1}{s^\alpha}. \tag{63}$$

Substitution and inverse Laplace transform implies

$$\mathcal{X}_{n+1}(\varphi, t) = \mathcal{X}_n(\varphi, t) - \mathcal{L}^{-1}\left\{ \frac{1}{s^\alpha} \mathcal{L}\left\{ iD_t^\alpha \mathcal{X}_n(\varphi, s) - 2|\mathcal{X}_n(\varphi, s)|^2 \mathcal{X}_n(\varphi, s) + \frac{\partial^2 \mathcal{X}_n(\varphi, s)}{\partial \varphi^2} \right\}\right\}. \tag{64}$$

For $n = 0, 1, 2, \cdots$, we get the approximations $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \cdots$, such as



(a) (b)

FIGURE 4: 3D representation of X for $\alpha = 0.8$ (a) and $\alpha = 1$ (b).

$$
\begin{aligned}
\mathcal{X}_1(\varphi, t) &= e^{i\varphi} + \frac{2e^{3i\varphi}t^{\alpha}}{\Gamma(1+\alpha)}, \\
\mathcal{X}_2(\varphi, t) &= e^{i\varphi} + \frac{2e^{3i\varphi}t^{\alpha}}{\Gamma(1+\alpha)} - 2e^{3i\varphi}t^{\alpha}\left\{\frac{-1 + it^{-1+\alpha}\alpha\Gamma(\alpha)/\Gamma(2\alpha)}{\Gamma(1+\alpha)}\right. \\
&\quad - \frac{6e^{2\alpha}t^{\alpha}}{\Gamma(1+2\alpha)} - \frac{12e^{4\alpha}t^{2\alpha}\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)} \\
&\quad \left. - \frac{8e^{6\alpha}t^{3\alpha}\Gamma(1+3\alpha)}{\Gamma(1+\alpha)^3\Gamma(1+4\alpha)}\right\}.
\end{aligned}
\tag{65}
$$

The solution is in the series form, such as

$$
\mathcal{X}(\varphi, t) = \lim_{i \longrightarrow \infty} \mathcal{X}_i, \tag{66}
$$

which turns out to be the exact solution $\mathcal{X}(\varphi, t) = e^{i(\varphi+t)}$ for $\alpha = 1$ (see [18]).

Open question: as an application of the VITM on nonlinear-time fractional differential equations towards fixed-point theorem. One can obtain some approximations of the solution $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3 \cdots$. Moreover, it can be asked whether these approximations of the solution are equivalent with the iterations of a sequence of successive approximations which are convergent to a fixed point or not? What are the minimum hypotheses imposed which lead us to the existence and the uniqueness of a fixed point in this case?

4. Discussions and Concluding Remarks

The proposed method VITM being the combination of two basic techniques, VIM and Laplace transform, is understandable by just having the formal knowledge of *advanced calculus*; indeed, it is understandable even for the reader who has no strong background and base in *calculus of variations*. It is simple and can be easily applied as compared to the more traditional VIM for fractional differential equations.

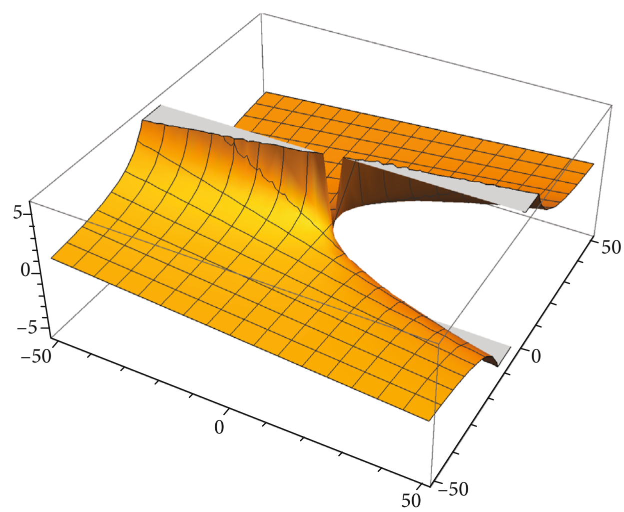


FIGURE 5: 3D representation of the exact solution.

Using the proposed method in the present paper, we study the convergence for some nonlinear fractional order partial differential equations as the Korteweg-de Vries equation, Schrödinger equation, and Burger equation. The rapid convergence of the method proves that it is a more reliable technique now more than ever than the existing ones to solve FPDEs, and it introduces a new significant improvement. The reliability and efficiency of this simple and newly introduced method is discussed by giving some numerical results and their graphical approximations.

Moreover, we propose an interesting connection between the approximations of the solution $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3 \cdots$ and the iterations of a sequence (convergent or not), from the area of fixed-point theory.

The main advantage of this proposed variational method together with Laplace transform helps to speed up the computational work and may easily be applied to nonlinear dynamical systems using software like Mathematica™, MATLAB™, and Maple™.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally and significantly in writing of this article. All authors have read and agreed to the last version of the manuscript.

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