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Research Article

Fixed Points of Monotone Total Asymptotically Nonexpansive Mapping in Hyperbolic Space via New Algorithm

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In this article, we consider an extensive class of monotone nonexpansive mappings and introduce a new iteration algorithm to approximate the fixed point for monotone total asymptotically nonexpansive mappings in the framework of hyperbolic space. Faster convergence and stability results are proved for that iteration; also, fixed point is approximated numerically in a nontrivial example by using MATLAB.

1. Introduction

The concept of hyperbolic space was given by Reich and Shafrir [1] in 1990, which is defined as a metric space (Σ, σ) that has a family *Fourier* of metric lines; for any two unique endpoints $l, m \in \Sigma$, there is a unique metric line in *Fourier*. This metric line works out a unique metric segment symbolize by $[l, m]$, which is an isometric image of $[0, \sigma(l, m)]$. A unique point $z \in [l.m]$ is denoted by $\alpha l \oplus (1-\alpha)m$ which satisfies

$$\begin{aligned} \sigma(l, z) &= (1-\alpha)\sigma(l, m), \\ \sigma(z, m) &= \alpha\sigma(l, m), \end{aligned} \tag{1}$$

where $\alpha \in [0, 1]$. Such a metric space is called a *convex metric space*, such that

$$\sigma(\alpha l \oplus (1-\alpha)m, \alpha p \oplus (1-\alpha)q) \le \alpha\sigma(l, p) + (1-\alpha)\sigma(m, q), \tag{2}$$

for all $l, m, p, q \in \Sigma$, then Σ is called *hyperbolic metric space* (abbreviated as H.M.S).

The class of hyperbolic spaces contains the normed spaces, CAT(0) spaces, and many others. There are many examples in literature which show that hyperbolic spaces are more general than Banach spaces; for details, see [2] and Example 1.1 of [3].

Recently, a new direction has been discovered dealing with the extension of the Banach Contraction Principle [4] to partially ordered metric spaces. The case of monotone nonexpansive mappings was first considered in [5]. After that, Dehaish and Khamsi [6] gave an analogue to Browder [7] and Göhde [8] fixed point theorems for monotone nonexpansive mappings. In 2018, Alfuraidan and Khamsi [9] extended Goebel and Kirk's fixed point theorem [10] for asymptotically nonexpansive mappings to the case of monotone mappings. Multiple articles [11–14] can be found in the literature on fixed point of asymptotically nonexpansive mapping using multistep iterations and strong convergence analysis.

In 2016, Alber et al. [15] introduced the concept of total asymptotically nonexpansive mappings that generalizes the family of mapping that are the extension of asymptotically nonexpansive mappings. Example 1 of [16] and Example 3.1 of [4] show that total asymptotically nonexpansive mappings properly contain the asymptotically nonexpansive mappings.

In this article, we define monotone total asymptotically nonexpansive mappings (abbreviated as M.T.A.N.M) and also extend Alber's fixed point theorem [10] for the respective class. In addition, this result also generalizes the results of Alfuraidan and Khamsi in hyperbolic space [9].

In Section 4, we introduce a new iteration scheme, prove fast convergence and stability results, and also provide comparison with some famous iterations listed as Banach [17], Mann [18], Ishikawa [19], Agarwal et al. [20], Noor [21], Abbas and Nazir [22], Vatan Two-step [23], an accelerated iteration [24] (by Chen and Wen [24]), and Thakur New [25]. Numerically, we compare the convergence of new iteration with these iterations in a nontrivial example.

2. Preliminaries

Let $(\Sigma, \sigma, \preceq)$ be a partially ordered (abbreviated as P.O) metric space, any two points $a, b \in \Sigma$ are comparable whenever $a \preceq b$ or $b \preceq a$.

Definition 1. Consider $(\Sigma, \sigma, \preceq)$ be a partially ordered space and Y be a self map of Σ, which is said to be

(i) monotone or order preserving [5] if

$$a \preceq b \Rightarrow Ya \preceq Yb \tag{3}$$

(ii) monotone Lipschitzian mapping [5] if Y is order preserving and there exist $L \geq 0$ such that

$$\sigma(Y(a), Y(b)) \leq H\sigma(a, b). \tag{4}$$

If $H = 1$, the mapping Y is said to be *order preserving nonexpansive mapping*

(iii) monotone asymptotically nonexpansive mapping [9] if there exists a sequence $\{H_\upsilon\}$ for $\upsilon \in \mathbb{N}$ such that

$$\begin{gathered} \lim_{\upsilon \longrightarrow \infty} H_\upsilon = 1, \\ \sigma(Y^\upsilon(a), Y^\upsilon(b)) \leq H_\upsilon \sigma(a, b), \end{gathered} \tag{5}$$

for every $a, b \in \Sigma$, such that a and b are comparable

Now, we will define M.T.A.N.M in hyperbolic space.

Definition 2. Let Σ be a hyperbolic metric space having a nonempty subset K. A self map Y is monotone total asymptotically nonexpansive mapping on K if there exist nonnegative sequences $\{\mu_\upsilon\}$ and $\{\xi_\upsilon\}$ with $\mu_\upsilon \longrightarrow 0, \xi_\upsilon \longrightarrow 0$, as $\upsilon \longrightarrow \infty$, a strictly increasing continuous function

$$\vartheta : [0,\infty) \longrightarrow [0,\infty) \text{ with } \vartheta(0) = 0, \tag{6}$$

such that

$$\sigma(Y^\upsilon s, Y^\upsilon t) \leq \sigma(s, t) + \mu_\upsilon \vartheta(\sigma(s, t)) + \xi_\upsilon \text{ for all } \upsilon \geq 1, \tag{7}$$

and there exists a constant $R^* > 0$ such that $\vartheta(\lambda) \leq R^* \lambda$ for $\lambda > 0$, then

$$\sigma(Y^\upsilon s, Y^\upsilon t) \leq (1 + R^* \mu_\upsilon)\sigma(s, t) + \xi_\upsilon, \tag{8}$$

for every comparable elements $s, t \in K$.

Example 1. Consider the real line $\mathbb{R}$ as a hyperbolic metric space and K be the subset of $\mathbb{R}$, $K = [0, \pi/2]$, and $T : K \longrightarrow K$ be a mapping defined by $Tx = \sin x$. Suppose that there exist two nonnegative sequences $\{x_n\}$ and $\{\xi_n\}$ with $x_n \longrightarrow 0$ and $\xi_n \longrightarrow 0$ as $n \longrightarrow 0$ and a strictly increasing continuous function $\theta : [0,\infty) \longrightarrow [0,\infty)$ with $\theta(0) = 0, \theta(\lambda) = \lambda + 1, \lambda \in [0,\infty)$. Then, T is *M.T.A.N.M.*

Example 2. Consider the real plane $\mathbb{R}^2$ as a hyperbolic metric space. Let $\sigma : \Sigma \times \Sigma \longrightarrow \mathbb{R}$ be defined as

$$\sigma(w, h) = |w_1 - h_1| + |w_1 w_2 - h_1 h_2| \text{ where } w = (w_1, w_2), h = (h_1, h_2). \tag{9}$$

Let $K = [0, 1] \times [0, 1] \subset \Sigma$ and $Y : K \longrightarrow K$ be a mapping defined by

$$Y(w, h) = \begin{cases} \left(\dfrac{w}{10}, \dfrac{h}{10}\right) & \text{if } (w, h) \neq \left(\dfrac{1}{2}, \dfrac{1}{2}\right); \\ (0, 0) & \text{if } (w, h) = \left(\dfrac{1}{2}, \dfrac{1}{2}\right). \end{cases} \tag{10}$$

Thus, $(0, 0)$ is the fixed point of Y. Suppose $\exists$ two nonnegative sequences $\{\mu_\upsilon\}$ and $\{\xi_\upsilon\}$ with $\mu_\upsilon \longrightarrow 0, \xi_\upsilon \longrightarrow 0$, as $\upsilon \longrightarrow \infty$, and a strictly increasing continuous function

$$\vartheta : [0,\infty) \longrightarrow [0,\infty) \text{ with } \vartheta(0) = 0. \tag{11}$$

Now, we consider the following cases with assumptions $(w_1, w_2) \preceq (h_1, h_2)$:

Case 1. If $(w_1, w_2) = (1/2, 1/2) = (h_1, h_2)$, then $Y(w_1, w_2) = (0, 0) = Y(h_1, h_2)$. In this case, Y is monotone and satisfies all the conditions of *total asymptotically nonexpansive mapping.*

Case 2. If $(w_1, w_2) = (1/2, 1/2), (h_1, h_2) \neq (1/2, 1/2)$, then $Y(w_1, w_2) = (0, 0)$, $Y(h_1, h_2) = (h_1/10, h_2/10)$, and $Y^{\upsilon}(w_1, w_2) = (0, 0), Y^{\upsilon}(h_1, h_2) = (h_1/10^{\upsilon}, h_2/10^{\upsilon})$. Also,

$$\begin{aligned}\sigma(Y^{\upsilon}(w_1, w_2), Y^{\upsilon}(h_1, h_2)) &= \frac{h_1}{10^{\upsilon}} + \frac{h_1 h_2}{10^{2\upsilon}} \leq \left|\frac{1}{2} - h_1\right| \\ &+ \left|\frac{1}{2^2} - h_1 h_2\right| \leq \sigma((w_1, w_2), (h_1, h_2)) \\ &+ \mu_{\upsilon} \vartheta(\sigma((w_1, w_2), (h_1, h_2))) + \xi_{\upsilon}\end{aligned} \tag{12}$$

implies that Y is a M.T.A.N.M.

Case 3. If $(w_1, w_2) \neq (1/2, 1/2) \neq (h_1, h_2)$, then $Y(w_1, w_2) = (w_1/10, w_2/10), Y(h_1, h_2) = (h_1/10, h_2/10)$, and $Y^{\upsilon}(w_1, w_2) = (w_1/10^{\upsilon}, w_2/10^{\upsilon}), Y^{\upsilon}(h_1, h_2) = (h_1/10^{\upsilon}, h_2/10^{\upsilon})$. Now,

$$\begin{aligned}\sigma(Y^{\upsilon}(w_1, w_2), Y^{\upsilon}(h_1, h_2)) &= \left|\frac{w_1 - h_1}{10^{\upsilon}}\right| + \left|\frac{w_1 w_2 - h_1 h_2}{10^{2\upsilon}}\right| \\ &\leq |w_1 - h_1| + |w_1 w_2 - h_1 h_2| \leq \sigma((w_1, w_2), (h_1, h_2)) \\ &+ \mu_{\upsilon} \vartheta(\sigma((w_1, w_2), (h_1, h_2))) + \xi_{\upsilon}.\end{aligned} \tag{13}$$

Hence, Y is M.T.A.N.M.

Next, we have some definitions and lemmas that will be useful in the proof of the main result.

Definition 3 (see [26]). A hyperbolic space Σ with metric σ is said to be uniformly convex if for any $w \in \Sigma$, for every $z > 0$, and for each $\varepsilon > 0$

$$\begin{aligned}\delta(z, \varepsilon) = \inf &\left\{1 - \frac{1}{z}\sigma\left(\frac{1}{2}s \oplus \frac{1}{2}t, w\right); \sigma(s, w)\right. \\ &\left.\leq z, \sigma(t, w) \leq z, \sigma(s, t) \geq z\varepsilon\right\} > 0.\end{aligned} \tag{14}$$

The function δ is called the modulus of uniform convexity of Σ.

A hyperbolic space (Σ, σ) satisfies the *property* (R) [26]. If $\{K_{\upsilon}\}$ is nonincreasing sequence of nonempty, bounded, closed, and convex subset of Σ, $\cap_{\upsilon=1}^{\infty} K_{\upsilon} \neq \vartheta$.

Definition 4 (see [27]). A bounded sequence $\{r_{\upsilon}\} \in \Sigma$ is *Δ-convege to* $r \in \Sigma$, if r is the unique asymptotic centre of every subsequence $\{r_{\upsilon k}\}$ *of* $\{r_{\upsilon}\}$.

Definition 5 (see [9]). A partially ordered hyperbolic metric space Σ satisfies the monotone weak Opial condition if any sequence in Σ which is monotone and weakly converges to s, then the following

$$\limsup_{\upsilon \longrightarrow \infty} \sigma(s_{\upsilon}, s) \leq \limsup_{\upsilon \longrightarrow \infty} \sigma(s_{\upsilon}, t), \tag{15}$$

for every $t \in \Sigma$ such that $s \preceq t$ or $t \preceq s$.

Throughout in article, the order intervals are assumed to be closed and convex and any of the subsets

$$\begin{aligned}[s, \longrightarrow) &= \{y \in \Sigma; s \preceq y\}, \\ (\longleftarrow, t] &= \{y \in \Sigma; y \preceq t\}.\end{aligned} \tag{16}$$

for every $s, t \in \Sigma$.

Lemma 6 [9]. *Suppose (Σ, σ) be uniformly convex H.M.S, and K be a subset of Σ which is closed nonempty and convex. Let $\tau : K \longrightarrow [0, \infty)$ be a type function if $\exists$ a bounded sequence $\{s_{\upsilon}\} \in \Sigma$ such that*

$$\tau(s) = \limsup_{\upsilon \longrightarrow \infty} \sigma(s_{\upsilon}, s), \tag{17}$$

for any $s \in K$. Since Σ is hyperbolic space, τ is convex and continuous with distinctive minimum point $u \in K$ such that

$$\tau(u) = \inf \{\tau(s); s \in K\} = \tau_0. \tag{18}$$

Moreover, if $\{Y^{\upsilon}(u)\}$ in K is the minimizing sequence of τ, i.e.,

$$\lim_{\upsilon \longrightarrow \infty} \tau(Y^{\upsilon}(u)) = \tau_0, \tag{19}$$

then, $\{Y^{\upsilon}(u)\}$ strongly converges to u.

3. Main Result

Theorem 7. *Let a uniformly convex P.O H.M.S be $(\Sigma, \sigma, \preccurlyeq)$ with nonempty convex closed bounded subset K. Let Y be a continuous M.T.A.N.M on K. Assume $\exists s_0 \in K$, such that $s_0 \preccurlyeq Y s_0$. Then, Y has a fixed point.*

Proof. Let $s_0 \in K$ be such that

$$s_0 \preccurlyeq Y s_0. \tag{20}$$

By the monotonicity of Y, we get

$$Y^{\upsilon} s_0 \preccurlyeq Y^{\upsilon+1} s_0, \tag{21}$$

for each $\upsilon \in \mathbb{N}$, and $\{Y^{\upsilon} s_0\}$ is a monotone increasing sequence. Also, the order intervals are closed and convex. So, we have

$$K_{\infty} = \bigcap_{\upsilon \geq 0} [Y^{\upsilon} s_0, \longrightarrow) \cap K = \cap \{s \in K; Y^{\upsilon} s_0 \preccurlyeq s\} \neq \phi. \tag{22}$$

Let $s \in K_\infty$, then

$$Y^v s_0 \preccurlyeq s, \tag{23}$$

and the monotonicity of Y implies

$$Y(Y^v s_0) = Y^{v+1} s_0 \preccurlyeq Ys, \tag{24}$$

for every $v \in \mathbb{N}$, i.e., $Y(K_\infty) \subset K_\infty$. Consider the type function $\tau : K_\infty \longrightarrow [0,+\infty)$ produced by $\{Y^v s_0\}$ given as $\tau(s) = \limsup_{v\longrightarrow\infty} \sigma(Y^v s_0, s)$ for any $s \in K_\infty$. Above lemma shows the occurrence of a unique $a \in K_\infty$ such that $\tau(a) = \inf \{\tau(s) ; s \in K_\infty\} = \tau_0$. Since $a \in K_\infty$, we have $Y^d(a) \in K_\infty$, for every $d \in \mathbb{N}$, which implies

$$\begin{aligned}\tau\left(Y^d(a)\right) &= \lim \sup_{v\longrightarrow\infty} \sigma\left(Y^v s_0, Y^d a\right)\\ &\le \lim \sup_{v\longrightarrow\infty} \left[\sigma(Y^v s_0, a) + \mu_p \vartheta(\sigma(Y^v s_0, a)) + \xi_d\right]\\ &= \tau(a) + \mu_d \lim \sup_{v\longrightarrow\infty} (\vartheta(\sigma(Y^v s_0, a))) + \xi_d.\end{aligned} \tag{25}$$

As Y is total asymptotically nonexpansive mapping, so $\mu_d \longrightarrow 0$, $\xi_d \longrightarrow 0$, when $d \longrightarrow \infty$. Hence,

$$\lim_{d\longrightarrow\infty} \tau\left(Y^d(a)\right) = \tau_0, \tag{26}$$

hence, $\{Y^d(a)\}$ is a minimizing sequence of τ. Using Lemma 6, $Y^d(a)$ converges to a. Since Y is continous, we have

$$\lim_{v\longrightarrow\infty} Y(Y^v a) = Y(a) = a, \tag{27}$$

i.e., a is a fixed point of Y. □

The following corollary is the conclusion of Theorem 3.3 of [9].

Corollary 8. *Let a uniformly convex P.O H.M.S be $(\Sigma, \sigma, \preccurlyeq)$ with nonempty convex closed bounded subset K. Let Y be a continuous M.T.A.N.M on K. Assume $\exists s_0 \in K$, such that $s_0 \preccurlyeq Ys_0$. Then, Y has a fixed point.*

The corollary given below is the consequence of Theorem 7 by replacing the continuity condition with weak Opial condition.

Corollary 9. *Let (Σ, σ') be a uniformly convex P.O H.M.S, satisfying monotone weak Opial condition with a nonempty convex closed bounded subset K. Let Y be a M.T.A.N.M on K. Assume $\exists s_0 \in K$, such that $s_0 \preccurlyeq Ys_0$. Then, Y has a fixed point.*

4. Convergence Theorem and Stability Results

We introduce the new iteration scheme given below, let K be a nonempty convex subset of a hyperbolic space Σ, for $v \ge 0$, where $\{\alpha_v\}$, $\{\beta_v\}$, and $\{\gamma_v\}$ are sequences in $[0, 1]$, such that

$$\begin{cases} s_0 \in K,\\ \eta_v = (1-\gamma_v)s_v \oplus \gamma_v Y^v s_v,\\ \theta_v = (1-\beta_v)\eta_v \oplus \beta_v Y^v \eta_v,\\ s_{v+1} = Y^v((1-\alpha_v)Y^v s_v \oplus \alpha_v Y^v \theta_v).\end{cases} \tag{28}$$

Fastness and stability play an important role for an iteration process to be preferred on another iteration process, so now, we prove that new iteration is stable and has good speed of convergence than others. For faster convergence and new class of mapping on metric space (Σ, σ) introduced by Berinde [28], satisfying

$$\sigma(Ys, Yt) \le a\sigma(s, t) + L\sigma(s, Ys), \tag{29}$$

here, we will modify this mapping as

$$\sigma(Y^v s, Y^v t) \le a\sigma(s, t) + L\sigma(s, Y^v s), \tag{30}$$

for all $s, t \in \Sigma$, where $a \in [0, 1)$ and $L \ge 0$.

The following definitions and lemma will be helpful for faster convergence results given in [29].

Definition 10. Let $\{a_v\}$ and $\{b_v\}$ be two sequences, having convergent points a and b, respectively, then $\{a_v\}$ converges faster than $\{b_v\}$ if

$$\lim_{v\longrightarrow\infty} \frac{\sigma(a_v, a)}{\sigma(b_v, b)} = 0. \tag{31}$$

Definition 11. Let $\{k_v\}$ and $\{l_v\}$ be two fixed point schemes that converge to the same fixed point s. If

$$\sigma(k_v, s) \le s_v \text{ and } \sigma(l_v, s) \le t_v \text{ for all } v \ge 0, \tag{32}$$

where s_v and t_v are two sequences that converge to 0. If s_v converges faster than t_v, then $\{k_v\}$ converges faster than $\{l_v\}$ to s.

Lemma 12. *If b is a real number such that $0 \le b < 1$ and $\{a_v\}_{v=0}^{\infty}$ be a sequence such that*

$$\lim_{v\longrightarrow\infty} a_v = 0, \tag{33}$$

then for any positive sequence $\{s_v\}_{v=0}^{\infty}$ satisfying

$$s_{v+1} \le bs_v + a_v \Rightarrow \lim_{v\longrightarrow\infty} s_v = 0. \tag{34}$$

Lemma 13 (see [4]). *Suppose $\{l_v\}$, $\{m_v\}$, and $\{\delta_v\}$ be* *sequences of nonnegative satisfying*

$$l_{v+1} \leq (1+\delta_v) l_v + m_v \quad \forall v \geq 1. \tag{35}$$

If $\Sigma\delta_v < \infty$ and $\Sigma m_v < \infty$, then $\lim_{v\longrightarrow\infty} l_{v+1}$ exists.

Lemma 14 (see [4]). *Suppose Σ be a uniformly convex H.M.S. Let $P \in [0,\infty)$ be such that*

$$\begin{gathered} \limsup_{v\longrightarrow\infty} \sigma(s_v, a) \leq P, \\ \limsup_{v\longrightarrow\infty} \sigma(\theta_v, a) \leq P, \\ \lim_{v\longrightarrow\infty} \sigma(a, \alpha_v s_v \oplus (1-\alpha_v)\theta_v) = P, \end{gathered} \tag{36}$$

where $\alpha_v \in [a, b]$ with $0 < a \leq b < 1$. Then, we get

$$\lim_{v\longrightarrow\infty} \sigma(s_v, \theta_v) = 0. \tag{37}$$

Lemma 15 (see [26]). *Let (Σ, σ') be a P.O hyperbolic space. Let K be a nonempty convex and closed subset of Σ. Let $Y : K \longrightarrow K$ be a monotone mapping. Let $s_1 \in K$, such that $s_1 \preceq Ys_1$ or $(Yt_1 \preceq t_1)$. Then, the sequence $\{s_v\}$ in (1) then*

(a) $s_v \preceq Ys_v \preceq s_{v+1}$ or $(s_{v+1} \preceq Ys_v \preceq s_v)$

(b) $s_v \preceq s$ or $(s \preceq s_v)$, *provided that $\{s_v\}\Delta$-converge to $s \in K$, $\forall v \in \mathbb{N}$*

Lemma 16. *Let a uniformly convex P.O H.M.S be $(\Sigma, \sigma, \preceq)$ with nonempty convex closed bounded subset K. Let $Y : K \longrightarrow K$ be a M.T.A.N.M with $F(Y) \neq \vartheta$. If the sequence $\{s_v\}$ is defined by (1) with $s_1 \preceq Ys_1$ or $(Ys_1 \preceq s_1)$. Then, the following holds*

$$\begin{gathered} (a) \lim_{v\longrightarrow\infty} \sigma(s_v, s) \text{ exist for } s \in F(Y), \\ (b) \lim_{v\longrightarrow\infty} \sigma(Y^v s_v, s_v) = 0. \end{gathered} \tag{38}$$

Proof. Let $s \in F(Y) \Rightarrow Ys = s$. By the above lemma $s \preceq s_v$, as Y is monotone

$$Ys \preceq Ys_v \Rightarrow s \preceq Y^v s_v. \tag{39}$$

Now, using Definition 2 and after simplification, we get

$$\sigma(\eta_v, s) \leq (1 + R^* \mu_v)\sigma(s_v, s) + \xi_v. \tag{40}$$

Again using Definition 2 and (40), we get

$$\sigma(\theta_v, s) \leq (1 + R^* \mu_v)^2 \sigma(s_v, s) + (2 + R^* \mu_v)\xi_v. \tag{41}$$

Consider

$$\sigma(s_{v+1}, s) = \sigma(Y^v((1-\alpha_v)Y^v \eta_v \oplus \alpha_v Y^v \theta_v), s). \tag{42}$$

Using Definition 2, (40) and (41), we get

$$\sigma(s_{v+1}, s) \leq (1 + \delta_v)\sigma(s_v, s) + b_v, \tag{43}$$

where $\delta_v = (4 + 6R^*\mu_v + 4(R^*\mu_v)^2 + (R^*\mu_v)^3)R^*\mu_v$ and $b_v = (4 + 6R^*\mu_v + 4(R^*\mu_v)^2 + (R^*\mu_v)^3)\xi_v$. Using Lemma 13, $\lim_{v\longrightarrow\infty} \sigma(s_v, s)$ exist for $s \in F(Y)$.

For part (b), we have to show that

$$\lim_{v\longrightarrow\infty} \sigma(Y^v s_v, s_v) = 0, \tag{44}$$

the proof resembles to Theorem 2.1 of [4]. □ □

Theorem 17. *Let a uniformly convex P.O H.M.S be $(\Sigma, \sigma, \preceq)$ with convex closed bounded and nonempty subset Y. Let Y be a continuous M.T.A.N.M on Y, with $F(Y) \neq \vartheta$. If $\{s_v\}$ is defined by (1) with $s_1 \preceq Ys_1$ or $(Ys_1 \preceq s_1)$. If $s \preceq s_1$ or $(s_1 \preceq s)$ for $s \in F(Y)$, then $\{s_v\}\Delta$ -converges to a fixed point of Y.*

Proof. By Lemma 16,

$$\lim_{v\longrightarrow\infty} \sigma(s_v, s) \text{ exist} \forall s \in F(Y), \tag{45}$$

sequence is bounded and $\lim_{v\longrightarrow\infty} \sigma(s_v, Y^v s_v) = 0$. Let $\{s_{vk}\}$ be any subsequence of $\{s_v\}$ for $k \in \mathbb{N}$, such that $\{s_{vk}\}\Delta$-converges to $p \in Y$. By Lemma 15, we have

$$s_1 \preceq s_{vk} \preceq p. \tag{46}$$

Now, we have to show that every Δ-convergent subsequence of $\{s_v\}$ has a unique Δ-limit in $F(Y)$. Let $\{s_{vk}\}$ and $\{s_{vr}\}$ be two subsequences of $\{s_v\}\Delta$-converging to w and h, respectively. By Lemma 16, $\{s_{vk}\}$ is bounded and

$$\lim_{v\longrightarrow\infty} \sigma(s_{vk}, Y^v s_{vk}) = 0. \tag{47}$$

We claim that $w \in F(Y)$, and τ produced by $\{s_{vk}\}$ is

$$\tau(w) = \limsup_{k\longrightarrow\infty} \sigma(s_{vk}, w). \tag{48}$$

From Theorem 7, $Y(w) = w$, same for $Y(h) = h$. By the definition of Δ-convergence and Lemma 6, we have

$$\begin{aligned} \limsup_{k\longrightarrow\infty} \sigma(s_v, w) = \limsup_{k\longrightarrow\infty} \sigma(s_{vk}, w) &< \limsup_{k\longrightarrow\infty} \sigma(s_{vk}, h) \\ &= \limsup_{v\longrightarrow\infty} \sigma(s_v, h) = \limsup_{r\longrightarrow\infty} \sigma(s_{vr}, h) \\ &< \limsup_{r\longrightarrow\infty} \sigma(s_{vr}, w) = \limsup_{v\longrightarrow\infty} \sigma(s_v, w), \end{aligned} \tag{49}$$

which is contradiction, unless $w = h$. □

Theorem 18. *Let a uniformly convex P.O H.M.S be $(\Sigma, \sigma, \preceq)$ with nonempty convex closed bounded subset K. Let Y be a continuous M.T.A.N.M on K with $F(Y) \neq \vartheta$. If $\{s_v\}$ is defined by (1) with $s_1 \preceq Ys_1$ or $(Ys_1 \preceq s_1)$. If $s \preceq s_1$ or $(s_1 \preceq s)$ for $s \in F(Y)$,* *then $\{s_\upsilon\}$ converges to a fixed point of Y if and only if*

$$\liminf_{\upsilon\longrightarrow\infty}\sigma(s_\upsilon, F(Y)) = 0. \quad (50)$$

Proof. If $\{s_\upsilon\}$ converges to a fixed point of Y, then

$$\liminf_{\upsilon\longrightarrow\infty}\sigma(s_\upsilon, F(Y)) = 0 \quad \text{for all } \upsilon\in\mathbb{N}. \quad (51)$$

Conversely, consider $\liminf_{\upsilon\longrightarrow\infty}\sigma(s_\upsilon, F(Y)) = 0$. From Lemma 16

$$\lim_{\upsilon\longrightarrow\infty}\sigma(s_\upsilon, s) \text{ exist for each } s\in F(Y), \quad (52)$$

therefore, $\lim_{\upsilon\longrightarrow\infty}\sigma(s_\upsilon, F(Y)$ exists. As $\liminf_{\upsilon\longrightarrow\infty}\sigma(s_\upsilon, F(Y)) = 0$, so we get

$$\lim_{\upsilon\longrightarrow\infty}\sigma(s_\upsilon, F(Y)) = 0. \quad (53)$$

Now, we prove that $\{s_\upsilon\}$ is a Cauchy sequence in K. For $\varepsilon>0, \exists\upsilon_0\in\mathbb{N}$, such that for all $\upsilon\geq\upsilon_0$

$$\sigma(s_\upsilon, F(Y)) < \frac{\varepsilon}{2}, \quad (54)$$

in particular inf $\{\sigma(s_{\upsilon_0}, s): s\in F(Y)\} < \varepsilon/2$, so $\exists$ a fixed point $p\in F(Y)$ such that

$$\sigma\left(s_{\upsilon_0}, s\right) < \frac{\varepsilon}{2}. \quad (55)$$

For $m, \upsilon\geq\upsilon_0$,

$$\sigma(s_{\upsilon+m}, s_\upsilon) \leq \sigma(s_{\upsilon+m}, s) + \sigma(s, s_\upsilon) < 2\sigma\left(s_{\upsilon_0}, s\right) < \varepsilon. \quad (56)$$

Hence, $\{s_\upsilon\}$ is a Cauchy sequence in closed subset K of Σ; therefore, it converges in K such that

$$\lim_{\upsilon\longrightarrow\infty} s_\upsilon = q \text{ for } q\in K. \quad (57)$$

As we have $\lim_{\upsilon\longrightarrow\infty}\sigma(s_\upsilon, F(Y)) = 0 \Rightarrow \sigma(q, F(Y)) = 0$, since $F(Y)$ is closed so $q\in F(Y)$.

Now, we prove that newly proposed iteration scheme (1) is faster than Thakur New [25] for a mapping defined in (2) in hyperbolic metric space. □

Theorem 19. *Let $(\Sigma, \sigma, \preccurlyeq)$ be a P.O H.M.S. Let K be a nonempty convex closed bounded subset of Σ, and Y be a mapping satisfying (30) with $F(Y)\neq\vartheta$. Let $\{s_\upsilon\}$ be defined by (28), and $\{u_\upsilon\}$ defined in [25], then $\{s_\upsilon\}$ converges faster than $\{u_\upsilon\}$.*

Proof. Let $s\in F(Y)\Rightarrow Y^\upsilon s = s$. Now, using (28) and (30), we have

$$\begin{aligned}\sigma(\eta_\upsilon, s) &\leq (1-\gamma_\upsilon)\sigma(s_\upsilon, s) + \gamma_\upsilon[a\sigma(s_\upsilon, s) + L\sigma(s, Y^\upsilon s)] \\ &= (1-\gamma_\upsilon(1-a))\sigma(s_\upsilon, s).\end{aligned} \quad (58)$$

Again, using (28), (30), and (58), we have

$$\sigma(\theta_\upsilon, s) \leq (1-\beta_\upsilon(1-a))(1-\gamma_\upsilon(1-a))\sigma(s_\upsilon, s). \quad (59)$$

Further, consider

$$\sigma(s_{\upsilon+1}, s) = \sigma(Y^\upsilon((1-\alpha_\upsilon)Y^\upsilon\eta_\upsilon \oplus \alpha_\upsilon Y^\upsilon\theta_\upsilon), s). \quad (60)$$

Now, using (30) and then (28), we get

$$\sigma(s_{\upsilon+1}, s) \leq a[a(1-\gamma_\upsilon(1-a))(1-(1-a)\alpha_\upsilon\beta_\upsilon)]\sigma(s_\upsilon, s). \quad (61)$$

Let

$$K_\upsilon = a^\upsilon[a[(1-\gamma(1-a))(1-(1-a)\alpha\beta)]^\upsilon]\sigma(s_1, s), \quad (62)$$

and $\sigma_\upsilon = a^\upsilon[(1-\gamma(1-a))(1-(1-a)\alpha\beta)]^\upsilon\sigma(u_1, s)$ calculated in Theorem 3.1 of Thakur New [25]. Then,

$$\begin{aligned}\frac{K_\upsilon}{\sigma_\upsilon} &= \frac{a^\upsilon[a[(1-\gamma(1-a))(1-(1-a)\alpha\beta)]^\upsilon]\sigma(s_1, s)}{a^\upsilon[(1-\gamma(1-a))(1-(1-a)\alpha\beta)]^\upsilon\sigma(u_1, s)} \\ &= \frac{a[(1-\gamma(1-a))(1-(1-a)\alpha\beta)]^\upsilon\sigma(s_1, s)}{[(1-\gamma(1-a))(1-(1-a)\alpha\beta)]^\upsilon\sigma(u_1, s)} \longrightarrow 0 \text{ as } \upsilon\longrightarrow\infty.\end{aligned} \quad (63)$$

Hence, $\{s_\upsilon\}$ converges faster than $\{u_\upsilon\}$. □

Now, we will prove the stability result for this we have the following definition by [30].

Definition 20. Let $\{p_\upsilon\}_{\upsilon=0}^\infty\subset K$ be any arbitrary sequence, then iteration sequence $s_{\upsilon+1}$ converging to unique fixed point s is said to be Y -stable if for $\varepsilon_\upsilon = \sigma(p_\upsilon, s_{\upsilon+1})\upsilon\geq 0$, we have

$$\lim_{\upsilon\longrightarrow\infty}\varepsilon_\upsilon = 0 \Leftrightarrow \lim_{\upsilon\longrightarrow\infty} p_\upsilon = s. \quad (64)$$

Theorem 21. *Let (Σ, σ') be a P.O H.M.S. Let K be a nonempty convex closed bounded subset of Σ, and Y be a mapping satisfying (2) with $F(Y)\neq\vartheta$. Let $\{s_\upsilon\}$ be defined by (1), satisfying $\Sigma\alpha_\upsilon = \infty$, then the iteration (1) is Y -stable.*

Proof. Let $\{p_\upsilon\}_{\upsilon=0}^\infty\subset K$ be any arbitrary sequence, the sequence defined by (28) converging to unique fixed point s, and

$$\varepsilon_\upsilon = \sigma(p_\upsilon, s_{\upsilon+1}). \quad (65)$$

We have to prove that

$$\lim_{\upsilon\longrightarrow\infty}\varepsilon_\upsilon = 0 \Leftrightarrow \lim_{\upsilon\longrightarrow\infty} p_\upsilon = s. \quad (66)$$

Table 1: The convergence behavior of Mann, Ishikawa, Agarwal, and Noor iterations with new iteration for the parameters $\alpha = 0:6$, $\beta = 0:3$, and $\gamma = 0:5$, with the initial values $x_0 = 0$, $y_0 = 3$, and tolerance $= 10^{-6}$.

Steps	Mann	Ishikawa	Agarwal	Noor	New
0	(0, 3)	(0, 3)	(0, 3)	(0, 3)	(0, 3)
1	(0, 2.244506)	(0, 2.013105)	(0, 1.509443)	(0, 1.916515)	(0, 0.311990)
2	(0, 1.519333)	(0, 1.196547)	(0, 0.479098)	(0, 1.097796)	(0, 0.011353)
3	(0, 0.949002)	(0, 0.663545)	(0, 0.116180)	(0, 0.594503)	(0, 0.000384)
4	(0, 0.561764)	(0, 0.355178)	(0, 0.025817)	(0, 0.313100)	(0, 0.000012)
5	(0, 0.321995)	(0, 0.186769)	(0, 0.005616)	(0, 0.162620)	(0, 0.000000)
6	(0, 0.181202)	(0, 0.097333)	(0, 0.001216)	(0, 0.083873)	(0, 0.000000)
7	(0, 0.100930)	(0, 0.050492)	(0, 0.000263)	(0, 0.043105)	(0, 0.000000)
8	(0, 0.055900)	(0, 0.026132)	(0, 0.000056)	(0, 0.022113)	(0, 0.000000)
9	(0, 0.030863)	(0, 0.013508)	(0, 0.000012)	(0, 0.011333)	(0, 0.000000)
10	(0, 0.017010)	(0, 0.006978)	(0, 0.000002)	(0, 0.005806)	(0, 0.000000)
11	(0, 0.009366)	(0, 0.003603)	(0, 0.000000)	(0, 0.002973)	(0, 0.000000)
12	(0, 0.005155)	(0, 0.001860)	(0, 0.000000)	(0, 0.001522)	(0, 0.000000)
13	(0, 0.002836)	(0, 0.000960)	(0, 0.000000)	(0, 0.000779)	(0, 0.000000)
14	(0, 0.001560)	(0, 0.000495)	(0, 0.000000)	(0, 0.000399)	(0, 0.000000)
15	(0, 0.000858)	(0, 0.000256)	(0, 0.000000)	(0, 0.000204)	(0, 0.000000)
16	(0, 0.000472)	(0, 0.000132)	(0, 0.000000)	(0, 0.000104)	(0, 0.000000)
17	(0, 0.000259)	(0, 0.000068)	(0, 0.000000)	(0, 0.000053)	(0, 0.000000)
18	(0, 0.000142)	(0, 0.000035)	(0, 0.000000)	(0, 0.000027)	(0, 0.000000)
19	(0, 0.000078)	(0, 0.000018)	(0, 0.000000)	(0, 0.000014)	(0, 0.000000)
20	(0, 0.000043)	(0, 0.000009)	(0, 0.000000)	(0, 0.000007)	(0, 0.000000)
21	(0, 0.000023)	(0, 0.000004)	(0, 0.000000)	(0, 0.000003)	(0, 0.000000)
22	(0, 0.000013)	(0, 0.000002)	(0, 0.000000)	(0, 0.000001)	(0, 0.000000)
23	(0, 0.000007)	(0, 0.000001)	(0, 0.000000)	(0, 0.000000)	(0, 0.000000)
24	(0, 0.000003)	(0, 0.000000)	(0, 0.000000)	(0, 0.000000)	(0, 0.000000)
25	(0, 0.000002)	(0, 0.000000)	(0, 0.000000)	(0, 0.000000)	(0, 0.000000)
26	(0, 0.000001)	(0, 0.000000)	(0, 0.000000)	(0, 0.000000)	(0, 0.000000)
27	(0, 0.000000)	(0, 0.000000)	(0, 0.000000)	(0, 0.000000)	(0, 0.000000)

Let $\lim_{\upsilon \longrightarrow \infty} \varepsilon_\upsilon = 0$, then

$$\begin{aligned} \sigma(p_\upsilon, s) &\leq \sigma(p_\upsilon, s_{\upsilon+1}) + \sigma(s_{\upsilon+1}, s) \\ &= \varepsilon_\upsilon + \sigma(Y^\upsilon((1-\alpha_\upsilon)Y^\upsilon s_\upsilon \oplus \alpha_\upsilon Y^\upsilon \theta_\upsilon), s) \\ &\leq \varepsilon_\upsilon + a\sigma((1-\alpha_\upsilon)Y^\upsilon \eta_\upsilon \oplus \alpha_\upsilon Y^\upsilon \theta_\upsilon, s) \\ &\quad + L\sigma(s, Y^\upsilon s) = \varepsilon_\upsilon + a\sigma((1-\alpha_\upsilon) \\ &\quad \cdot Y^\upsilon \eta_\upsilon \oplus \alpha_\upsilon Y^\upsilon \theta_\upsilon, s) \leq \varepsilon_\upsilon + a[a(1-\gamma_\upsilon \\ &\quad \cdot (1-a))(1-(1-a)\alpha_\upsilon \beta_\upsilon)]\sigma(s_\upsilon, s), \end{aligned} \tag{67}$$

$0 \leq a[a(1-\gamma_\upsilon(1-a))(1-(1-a)\alpha_\upsilon\beta_\upsilon)] < 1$, applying $\lim_{\upsilon \longrightarrow \infty}$, we get

$$\lim_{\upsilon \longrightarrow \infty} \sigma(p_\upsilon, s) = 0 \Rightarrow \lim_{\upsilon \longrightarrow \infty} p_\upsilon = s. \tag{68}$$

Conversely, let $\lim_{\upsilon \longrightarrow \infty} p_\upsilon = s$, we have

Table 2: The convergence of Abbas, Thakur, and accelerated iteration with new iteration for the same initial values, parameters, and tolerance 10^{-6}.

Steps	Abbas	An accelerated	Thakur	New
0	(0, 3)	(0, 3)	(0, 3)	(0, 3)
1	(0, 0.917554)	(0, 0.890744)	(0, 0.969760)	(0, 0.311990)
2	(0, 0.133217)	(0, 0.126269)	(0, 0.161770)	(0, 0.011353)
3	(0, 0.015917)	(0, 0.015024)	(0, 0.022578)	(0, 0.000384)
4	(0, 0.001852)	(0, 0.001747)	(0, 0.003065)	(0, 0.000012)
5	(0, 0.000214)	(0, 0.000202)	(0, 0.000414)	(0, 0.000000)
6	(0, 0.000024)	(0, 0.000023)	(0, 0.000056)	(0, 0.000000)
7	(0, 0.000002)	(0, 0.000002)	(0, 0.000007)	(0, 0.000000)
8	(0, 0.000000)	(0, 0.000000)	(0, 0.000001)	(0, 0.000000)
9	(0, 0.000000)	(0, 0.000000)	(0, 0.000000)	(0, 0.000000)

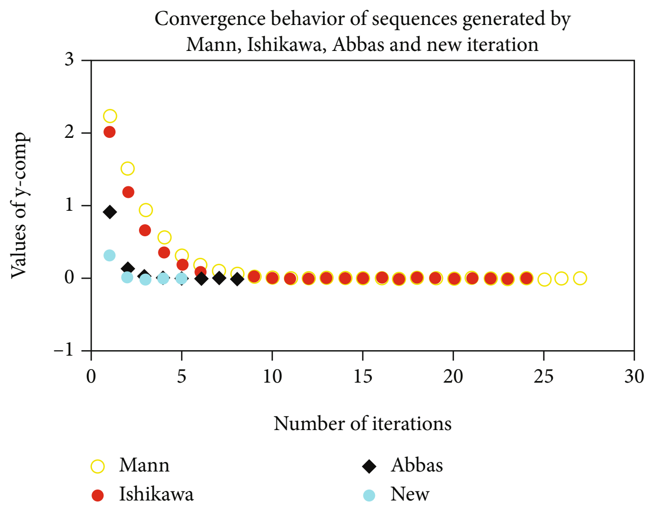

FIGURE 1: Convergence behavior of Mann, Ishikawa, and Abbas with new iteration for the initial values (0,3) with parameters $\alpha = 0.6, \beta = 0.3, \gamma = 0.5$.

$$\begin{aligned}\varepsilon_v = \sigma(p_v, s_{v+1}) &\leq \sigma(p_v, s) + \sigma(s_{v+1}, s) \leq \sigma(p_v, s)\\ &+ a[a(1-\gamma_v(1-a))(1-(1-a)\alpha_v\beta_v)]\sigma(s_v, s)\\ &\Rightarrow \lim_{v\longrightarrow\infty} \varepsilon_v = 0.\end{aligned} \quad (69)$$

Hence, it is Y-stable. □

We have a nontrivial example for M.T.A.N.M, and fixed point is numerically approximated by using MATLAB.

Example 3. Let $\Sigma = \mathbb{R}^2$ be a hyperbolic space. Define a relation as

$$(r_1, t_1)'(r_2, t_2) \Leftrightarrow r_1 \leq r_2 \text{and} t_1 \leq t_2. \quad (70)$$

Let $\sigma : \Sigma \times \Sigma \longrightarrow \mathbb{R}$ be defined as

$$\sigma(r, t) = |r_1 - r_2| + |r_1 t_1 - r_2 t_2| \text{ where } r = (r_1, t_1), y = (r_2, t_2). \quad (71)$$

Let $K = [0, 3] \times [0, 3] \subset \Sigma$ and $Y : K \longrightarrow K$ be a mapping defined by

$$Y(r, t) = \left\{ \left(\frac{(1 - \cos r)}{2}, \frac{\exp(t/2) - 1}{2} \right); (r, t) \in K. \right. \quad (72)$$

As $(0, 0)$ is the fixed point of Y, we have to show that Y is monotone for this consider

$$(r_1, t_1)'(r_2, t_2) \Leftrightarrow r_1 \leq r_2 \text{ and } t_1 \leq t_2, \quad (73)$$

then

$$\frac{1 - \cos r_1}{2} \leq \frac{1 - \cos r_2}{2} \text{and} \frac{\exp(t_1/2) - 1}{2} \leq \frac{\exp(t_2/2) - 1}{2}, \quad (74)$$

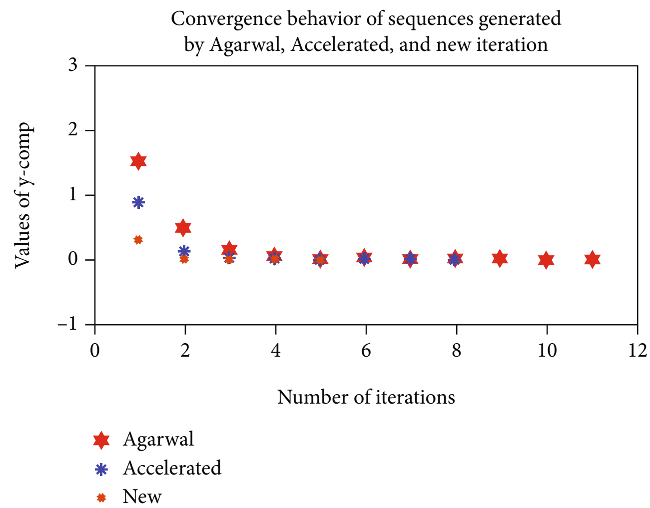

FIGURE 2: Convergence behavior of Agarwal, accelerated with new iteration for the initial values (0,3) with parameters $\alpha = 0.6, \beta = 0.3, \gamma = 0.5$.

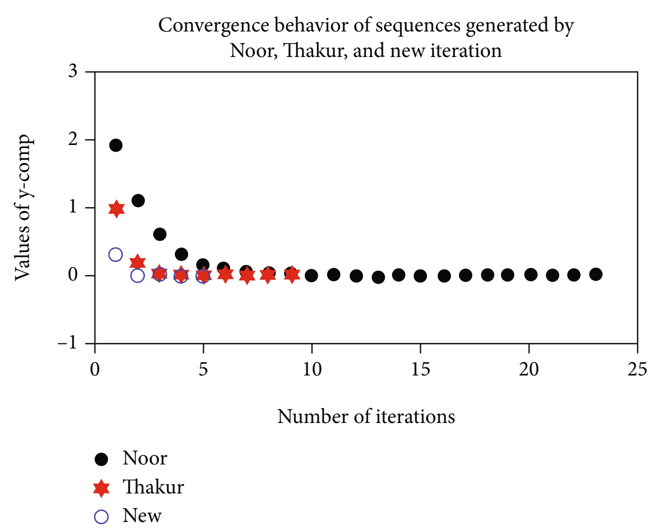

FIGURE 3: Convergence behavior of Noor and Thakur with new iteration for the initial values (0,3) with parameters $\alpha = 0.6, \beta = 0.3, \gamma = 0.5$.

that is

$$Y(r_1, t_1)'Y(r_2, t_2). \quad (75)$$

Next,

$$\begin{aligned} Y^v(r_1, t_1) = (0, 0), Y^v(r_2, t_2) &= (0, 0), \text{ for large } v,\\ &\Rightarrow \sigma(Y^v(r_1, t_1), Y^v(r_2, t_2))\\ &= 0 \leq \sigma((r_1, t_1), (r_2, t_2)).\end{aligned} \quad (76)$$

Hence, Y is M.T.A.N.M with $\mu_v = \xi_v = 0$.

The rate of convergence of Mann, Ishikawa, Agarwal, Noor, Abbas, Thakur, and accelerated and new iterations

Table 3: Influence of parameters and initial values by setting stopping parameter at 10^{-15}.

For $\alpha = 0.7$, $\beta = \gamma = 0.1$ with $(x_0, y_0) = (0, 3)$								
Iteration process	Mann	Ishikawa	Agarwal	Noor	Abbas	An accelerated	Thakur	New
Number of iterations	50	48	26	48	22	22	25	13
For $\alpha = 0.2 = \beta$, $\gamma = 0.3$ with $(x_0, y_0) = (0,2.5)$								
Iteration process	Mann	Ishikawa	Agarwal	Noor	Abbas	An accelerated	Thakur	New
Number of iterations	221	202	26	203	15	15	22	12
For $\alpha = 0.3 = \beta = 0.7$, $\gamma = 0.2$ with $(x_0, y_0) = (0, 2)$								
Iteration process	Mann	Ishikawa	Agarwal	Noor	Abbas	An accelerated	Thakur	New
Number of iterations	140	116	24	115	16	16	21	12
For $\alpha = 0.5 = \beta = 0.9$, $\gamma = 0.2$ with $(x_0, y_0) = (0, 1)$								
Iteration process	Mann	Ishikawa	Agarwal	Noor	Abbas	An accelerated	Thakur	New
Number of iterations	74	57	20	56	18	18	18	11
For $\alpha = 0.3 = \beta = 0.9$, $\gamma = 0.5$ with $(x_0, y_0) = (0,1.5)$								
Iteration process	Mann	Ishikawa	Agarwal	Noor	Abbas	An accelerated	Thakur	New
Number of iterations	138	109	23	106	14	14	17	11

for the mapping defined in Example 2 is given below. Table 1 shows the convergence behavior of Mann, Ishikawa, Agarwal, and Noor iterations with new iteration for the parameters $\alpha = 0.6, \beta = 0.3$, and $\gamma = 0.5$, with the initial values $x_0 = 0, y_0 = 3$, and tolerance $= 10^{-6}$. New iteration requires less number of iterations for convergence.

Table 2 shows the convergence of Abbas, Thakur, and accelerated iteration with new iteration for the same initial values, parameters, and tolerance.

The following figures show the convergence behavior of different iterations with new iteration in Figures 1–3.

All iterations converges to $(x, y) = (0, 0)$. Comparison shows that new iteration requires the least number of iterations for convergence. Table 3 shows that different parameters have an effect on iterations and by changing the initial values, new iteration not only converges faster but also stable than other iterations

5. Conclusions

In the present article, the concept of monotone asymptotically nonexpansive mapping has been generalized to monotone total asymptotically nonexpansive mapping in the framework of hyperbolic space. New iteration has been introduced to approximate the fixed point for that mapping. We proved the existence of fixed point, faster convergence, and stability results for new iteration. We also constructed a nontrivial example to approximate the fixed point numerically and compare the convergence result of new iteration with some well-known iterations by using MATLAB.

By relaxing the condition of monotonicity, we can also achieve some similar results presented in recent articles [31, 32] by using the proposed iteration.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors have no conflict of interests regarding the publication of this paper.

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