Hindawi
Journal of Function Spaces
Volume 2021, Article ID 8513304, 31 pages
https://doi.org/10.1155/2021/8513304

Research Article

Some Reiteration Theorems for $\mathscr{R}$, $\mathscr{L}$, $\mathscr{R}\mathscr{R}$, $\mathscr{R}\mathscr{L}$, $\mathscr{L}\mathscr{R}$, and $\mathscr{L}\mathscr{L}$ Limiting Interpolation Spaces

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Received 10 May 2021; Accepted 29 June 2021; Published 2 August 2021

Academic Editor: Maria Alessandra Ragusa

We consider the K-interpolation methods involving slowly varying functions. We establish some reiteration formulae including so-called $\mathscr{L}$ or $\mathscr{R}$ limiting interpolation spaces as well as the $\mathscr{R}\mathscr{R}$, $\mathscr{R}\mathscr{L}$, $\mathscr{L}\mathscr{R}$, and $\mathscr{L}\mathscr{L}$ extremal interpolation spaces. These spaces arise in the limiting situations. The proofs of most reiteration formulae are based on Holmstedt-type formulae. Applications to grand and small Lorentz spaces in critical cases are given.

1. Introduction

Let $\bar{A} := (A_0, A_1)$ be a compatible couple of Banach or quasi-Banach spaces such that $A_0 \bigcap A_1 \neq \{0\}$. For $t > 0$, Peetre's K-functional is given by

$$K(t, f; \bar{A}) \equiv K(t, f) := \inf \left(\|f_0\|_{A_0} + t\|f_1\|_{A_1} : f = f_0 + f_1, f_i \in A_i \right). \tag{1}$$

We refer to, e.g., [1] for basic concepts on interpolation theory and properties of the K-functional. In recent years, the scale of interpolation spaces $\bar{A}_{\theta,q,a}$ has been intensively studied, which is defined via the (quasi-)norms

$$\|f\|_{\theta,q,a} := \left\| u^{-\theta-1/q} a(u) K(u, f) \right\|_{q,(0,\infty)}, \tag{2}$$

where $0 \leq \theta \leq 1$, $0 < q \leq \infty$, a is a slowly varying function (see Definition 5 below), and $\|*\|_{q,(0,\infty)}$ is the usual (quasi-)norm in the Lebesgue space L_q on the interval $(0, \infty)$.

An important property of this scale is that the reiteration spaces $(\bar{A}_{\theta_0,*}, \bar{A}_{\theta_1,*})_{\theta,r,a}$ with $0 \leq \theta_0 < \theta_1 \leq 1$ and $0 < \theta < 1$ belong to the same scale. (Here and below, "$*$" stands for further parameters.) The consideration of the limiting cases $\theta \in \{0, 1\}$ leads normally to some new limiting interpolation spaces $\bar{A}^{\mathscr{L}}_{\theta,*}$ and $\bar{A}^{\mathscr{R}}_{\theta,*}$. Following [2], we call them $\mathscr{R}$ and $\mathscr{L}$ spaces (see Definition 7 below). These limiting interpolation spaces occur naturally in many fields of analysis. For example, it is shown that the so-called grand and small Lorentz spaces can be described in terms of the $\mathscr{R}$ and $\mathscr{L}$ spaces [3, 4].

Naturally, the question arises about the description of the reiteration spaces $(*,*)_{\theta,r,a}$ for couples where one or both of the operands are $\mathscr{R}$ or $\mathscr{L}$ spaces. Such reiteration formulae under condition $0 < \theta < 1$ are established in the previous paper [5]. Some limiting cases $\theta \in \{0, 1\}$, but not all, are also considered in [5]. For example, if $0 < \theta_0 < \theta_1 < 1$, it has been shown in [5] that $(\bar{A}_{\theta_0,*}, \bar{A}^{\mathscr{L}}_{\theta_1,*})_{\theta,*} = \bar{A}_{\eta,*}$ if $0 < \theta < 1$ and $(\bar{A}_{\theta_0,*}, \bar{A}^{\mathscr{L}}_{\theta_1,*})_{0,*} = \bar{A}^{\mathscr{L}}_{\theta_0,*}$. However, the formula for $(\bar{A}_{\theta_0,*}, \bar{A}^{\mathscr{L}}_{\theta_1,*})_{1,*}$ has not been derived. The principal aim of this paper is to establish reiteration formulae for combinations of parameters, which are not included in [5].

The motivation for this work was the articles [6–8], where it has been shown that for some limiting combinations of parameters, new interpolation spaces are required. Following [6, 7], we call them $\mathscr{L}\mathscr{L}$, $\mathscr{L}\mathscr{R}$, $\mathscr{R}\mathscr{L}$, and $\mathscr{R}\mathscr{R}$ extremal interpolation spaces (see Definition 8 below). For example, $(\bar{A}_{\theta_0,*}, \bar{A}^{\mathscr{L}}_{\theta_1,*})_{1,*} = \bar{A}^{\mathscr{L},\mathscr{R}}_{\theta_1,*}$ (see Theorem 19 below).

Our paper [5] and the articles [6–8] appeared practically at the same time. The results of the papers [6–8] demonstrate—mathematically speaking—a large intersection and a symmetric difference with the ones in [5] and in this paper. Fernández-Martínez and Signes work with the scale $(A_0, A_1)_{\theta,E,b}$ of interpolation spaces based on the family of rearrangement invariant Banach function spaces E. They have introduced and studied this rich scale in [2, 9]. If $E = L_q$ and $1 \le q \le \infty$, the space $(A_0, A_1)_{\theta,E,b}$ coincides with the space $(A_0, A_1)_{\theta,q;b}$. However, the spaces $(A_0, A_1)_{\theta,q;b}$ with $0 < q < 1$ do not belong to the scale $(A_0, A_1)_{\theta,E,b}$. The same remark holds about $\mathscr{R}$ and $\mathscr{L}$ limiting interpolation spaces, too.

Note that in [5] and in this paper, the main parameters θ_0 and θ_1 for considered $\mathscr{R}$ und $\mathscr{L}$ spaces lie between 0 and 1. Limiting cases $\theta_0, \theta_1 \in \{0, 1\}$ will be the topic of a future paper.

The paper is organized as follows. Sections 2 and 3 contain necessary notations, definitions, and technical results. In Sections 4–6, we establish reiteration formulae for the couples, where one of the operands is either $\mathscr{L}$ or $\mathscr{R}$ space. Reiteration formulae for the couples, where both operands are the $\mathscr{L}$ or $\mathscr{R}$ space, are established in Section 7. In Section 8, we consider couples where one of the operands is $\mathscr{L}\mathscr{L}$, $\mathscr{L}\mathscr{R}$, $\mathscr{R}\mathscr{L}$, or $\mathscr{R}\mathscr{R}$ space. Note that latter combinations are not studied in [6–8]. Finally, in Section 9, we present interpolation results for the grand and small Lorentz spaces in critical cases as applications of our general reiteration theorems.

2. Preliminaries

Throughout the paper, we write $X \subset Y$ for two (quasi-)normed spaces X and Y to indicate that X is continuously embedded in Y. We write $X = Y$ if $X \subset Y$ and $Y \subset X$. For f and g being positive functions, we write $f \prec g$ if $f \le Cg$, where the constant C is independent of all significant quantities. Two functions f and g are considered equivalent ($f \approx g$) if $f \prec g$ and $g \prec f$. We adopt the conventions $1/\infty = 0$ and $1/0 = \infty$. The abbreviation LHS($*$) (RHS($*$)) will be used for the left- (right-) hand side of the relation ($*$). By $\chi_{(a,b)}$, we denote the characteristic function on an interval (a, b).

2.1. Slowly Varying Functions. In this section, we summarize some of the properties of slowly varying functions, which will be required later. For more details, we refer to, e.g., [2, 9–11].

Definition 1. We say that a positive Lebesgue measurable function b is slowly varying on $(0, \infty)$, notation $b \in SV$, if, for each $\varepsilon > 0$, the function $t^\varepsilon b(t)$ is equivalent to an increasing function while the function $t^{-\varepsilon} b(t)$ is equivalent to a decreasing function.

Lemma 2. *Let $b, b_1, b_2 \in SV$, $\alpha > 0$, $0 < q \le \infty$, $\lambda \in (-\infty, \infty)$, and $t \in (0,\infty)$.*

(i) Then, $b^\lambda \in SV$, $b(1/t) \in SV$, $b(t^\alpha b_1(t)) \in SV$, and $b_1 b_2 \in SV$

(ii) If $f \approx g$, then $b \circ f \approx b \circ g$

(iii) $\|u^{\alpha-1/q} b(u)\|_{q,(0,t)} \approx t^\alpha b(t)$ and $\|u^{-\alpha-1/q} b(u)\|_{q,(t,\infty)} \approx t^{-\alpha} b(t)$

(iv) The functions $\|u^{-1/q} b(u)\|_{q,(0,t)}$ and $\|u^{-1/q} b(u)\|_{q,(t,\infty)}$ (if exist) belong to SV and $b(t) \prec \|u^{-1/q} b(u)\|_{q,(0,t)}$ and $b(t) \prec \|u^{-1/q} b(u)\|_{q,(t,\infty)}$

(v) $\|u^{\lambda-1/q} b(u)\|_{q,(t,2t)} \approx t^\lambda b(t)$

We will often use these properties without referencing explicitly every time.

2.2. Hardy-Type Inequalities. The following Hardy-type inequalities will be applied later.

Lemma 3 (see [10], Lemma 2.7). *Let $1 \le P \le \infty$ and $b \in SV$.*

(i) The inequality

$$\left\| t^{\nu-1/P} b(t) \int_0^t g(u)\,du \right\|_{P,(0,\infty)} \prec \left\| t^{\nu+1-1/P} b(t) g(t) \right\|_{P,(0,\infty)}, \quad (3)$$

holds for all (Lebesgue) measurable nonnegative functions g on $(0, \infty)$ if and only if $\nu < 0$.

(ii) The inequality

$$\left\| t^{\nu-1/P} b(t) \int_t^\infty g(u)\,du \right\|_{P,(0,\infty)} \prec \left\| t^{\nu+1-1/P} b(t) g(t) \right\|_{P,(0,\infty)}, \quad (4)$$

holds for all (Lebesgue) measurable nonnegative functions g on $(0, \infty)$ if and only if $\nu > 0$.

These inequalities will be used in the form of the following corollary.

Corollary 4. *Let $0 < q \le r \le \infty$, $b \in SV$, and $\mu > 0$. Then, for all (Lebesgue) measurable nonnegative functions h on $(0, \infty)$,*

$$\left\| t^{-\mu-1/r} b(t) \left\| u^{-1/q} h(u) \right\|_{q,(0,t)} \right\|_{r,(0,\infty)} \prec \left\| t^{-\mu-1/r} b(t) h(t) \right\|_{r,(0,\infty)},$$
$$\left\| t^{\mu-1/r} b(t) \left\| u^{-1/q} h(u) \right\|_{q,(t,\infty)} \right\|_{r,(0,\infty)} \prec \left\| t^{\mu-1/r} b(t) h(t) \right\|_{r,(0,\infty)}. \quad (5)$$

Proof. We begin with the first estimate. If $0 < q < r < \infty$, using Lemma 3 with $g(u) = (u^{-1/q} h(u))^q$, $\nu = -q\mu$, and $P = r/q$, we get

$$\begin{aligned}
&\left\|t^{-\mu-1/r}b(t)\left\|u^{-1/q}h(u)\right\|_{q,(0,t)}\right\|_{r,(0,\infty)}\\
&=\left\{\int_0^\infty\left(t^{-q\mu-1/P}b(t)^q\int_0^t g(u)du\right)^P dt\right\}^{1/(Pq)}\\
&\prec\left\{\int_0^\infty\left(t^{-q\mu+1-1/P}b(t)^q g(t)\right)^P dt\right\}^{1/(Pq)}\\
&=\left\|t^{-\mu-1/r}b(t)h(t)\right\|_{r,(0,\infty)}.
\end{aligned} \tag{6}$$

The case $0<q<r=\infty$ can be considered similarly with $P=\infty$. The case $0<q=r\le\infty$ can be proved by Fubini's theorem or by the exchange of the essential suprema (if $p=\infty$).

The second estimate follows from the first one if we take $u=1/v$ and $t=1/s$. □

3. Interpolation Methods

Below is a collection of crucial definitions and statements building a base of the real interpolation methods involving slowly varying functions. In the following, let $\bar{A}\equiv(A_0,A_1)$ be a compatible couple of (quasi-)Banach spaces such that $A_0\bigcap A_1\ne\{0\}$. Peetre's K-functional on A_0+A_1 is given by (1).

3.1. Standard Interpolation Spaces

Definition 5 (see [10]). Let $0\le\theta\le1$, $0<q\le\infty$, and $b\in SV$. We put

$$\begin{aligned}
&\bar{A}_{\theta,q;b}\equiv(A_0,A_1)_{\theta,q;b}\\
&\quad:=\left\{f\in A_0+A_1:\|f\|_{\theta,q;b}=\left\|u^{-\theta-1/q}b(u)K(u,f)\right\|_{q,(0,\infty)}<\infty\right\}.
\end{aligned} \tag{7}$$

Lemma 6 (see [10], Proposition 2.5). *$A_0\bigcap A_1\subset\bar{A}_{\theta,q;b}\subset A_0+A_1$ if and only if one of the following conditions is satisfied:*

(i) $0<\theta<1$

(ii) $\theta=0$ and $\|u^{-1/q}b(u)\|_{q,(1,\infty)}<\infty$

(iii) $\theta=1$ and $\|u^{-1/q}b(u)\|_{q,(0,1)}<\infty$

Moreover, if none of these conditions holds, then $\bar{A}_{\theta,q;b}=\{0\}$

.

Spaces $\bar{A}_{\theta,q;b}$ are referred to as standard interpolation spaces.

3.2. $\mathscr{L}$ and $\mathscr{R}$ Spaces

Definition 7 (see [10]). Let $0<r,q\le\infty$, $0\le\theta\le1$, and $a,b\in SV$. We put

$$\begin{aligned}
&\bar{A}^{\mathscr{L}}_{\theta,r,b,q,a}\equiv(A_0,A_1)^{\mathscr{L}}_{\theta,r,b,q,a}\\
&\quad:=\left\{f\in A_0+A_1:\|f\|_{\mathscr{L};\theta,r,b,q,a}\right.\\
&\quad\left.:=\left\|t^{-1/r}b(t)\left\|u^{-\theta-1/q}a(u)K(u,f)\right\|_{q,(0,t)}\right\|_{r,(0,\infty)}<\infty\right\}.
\end{aligned} \tag{8}$$

Similarly,

$$\begin{aligned}
&\bar{A}^{\mathscr{R}}_{\theta,r,b,q,a}\equiv(A_0,A_1)^{\mathscr{R}}_{\theta,r,b,q,a}\\
&\quad:=\left\{f\in A_0+A_1:\|f\|_{\mathscr{R};\theta,r,b,q,a}\right.\\
&\quad\left.:=\left\|t^{-1/r}b(t)\left\|u^{-\theta-1/q}a(u)K(u,f)\right\|_{q,(t,\infty)}\right\|_{r,(0,\infty)}<\infty\right\}.
\end{aligned} \tag{9}$$

In the literature (e.g., [2, 5, 9, 12–14]), similar definitions are given alongside with properties of these spaces. As proposed in [2], we refer to the spaces $\bar{A}^{\mathscr{L}}_*$ and $\bar{A}^{\mathscr{R}}_*$ as $\mathscr{L}$ and $\mathscr{R}$ limiting interpolation spaces.

3.3. $\mathscr{L}\mathscr{L}$, $\mathscr{L}\mathscr{R}$, $\mathscr{R}\mathscr{L}$, and $\mathscr{R}\mathscr{R}$ Spaces. We introduce here four interpolation spaces. We follow [6–8] where it has been shown that they appear in relation to the extreme reiteration results.

Definition 8. Let $0<p,q,r\le\infty$, $0\le\theta\le1$, and $a,b,c\in SV$. The space $\bar{A}^{\mathscr{L},\mathscr{L}}_{\theta,p,c,r,b,q,a}\equiv(A_0,A_1)^{\mathscr{L},\mathscr{L}}_{\theta,p,c,r,b,q,a}$ is the set of all $f\in A_0+A_1$ for which

$$\|f\|_{\mathscr{L},\mathscr{L};\theta,p,c,r,b,q,a}:=\left\|t^{-1/p}c(t)\left\|s^{-1/r}b(s)\left\|u^{-\theta-1/q}a(u)K(u,f)\right\|_{q,(0,s)}\right\|_{r,(0,t)}\right\|_{p,(0,\infty)}<\infty. \tag{10}$$

The space $\bar{A}^{\mathscr{L},\mathscr{R}}_{\theta,p,c,r,b,q,a}\equiv(A_0,A_1)^{\mathscr{L},\mathscr{R}}_{\theta,p,c,r,b,q,a}$ is the set of all $f\in A_0+A_1$ for which

$$\|f\|_{\mathscr{L},\mathscr{R};\theta,p,c,r,b,q,a}:=\left\|t^{-1/p}c(t)\left\|s^{-1/r}b(s)\left\|u^{-\theta-1/q}a(u)K(u,f)\right\|_{q,(t,s)}\right\|_{r,(t,\infty)}\right\|_{p,(0,\infty)}<\infty. \tag{11}$$

The space $\bar{A}^{\mathscr{R},\mathscr{L}}_{\theta,p,c,r,b,q,a}\equiv(A_0,A_1)^{\mathscr{R},\mathscr{L}}_{\theta,p,c,r,b,q,a}$ is the set of all $f\in A_0+A_1$ for which

$$\|f\|_{\mathscr{R},\mathscr{L};\theta,p,c,r,b,q,a}:=\left\|t^{-1/p}c(t)\left\|s^{-1/r}b(s)\left\|u^{-\theta-1/q}a(u)K(u,f)\right\|_{q,(s,t)}\right\|_{r,(0,t)}\right\|_{p,(0,\infty)}<\infty. \tag{12}$$

The space $\bar{A}^{\mathscr{R},\mathscr{R}}_{\theta,p,c,r,b,q,a}\equiv(A_0,A_1)^{\mathscr{R},\mathscr{R}}_{\theta,p,c,r,b,q,a}$ is the set of all $f\in A_0+A_1$ for which

$$\|f\|_{\mathscr{R},\mathscr{R};\theta,p,c,r,b,q,a}:=\left\|t^{-1/p}c(t)\left\|s^{-1/r}b(s)\left\|u^{-\theta-1/q}a(u)K(u,f)\right\|_{q,(s,\infty)}\right\|_{r,(t,\infty)}\right\|_{p,(0,\infty)}<\infty. \tag{13}$$

We refer to these spaces as $\mathscr{L}\mathscr{L}$, $\mathscr{L}\mathscr{R}$, $\mathscr{R}\mathscr{L}$, and $\mathscr{R}\mathscr{R}$ extremal interpolation spaces.

3.4. Some Known Formulae, Auxiliary Lemmas, and Reiteration Theorems. Let $C\ge1$, $t>0$, and $\tilde{b}(t):=b(t^{-1})$. Under suitable conditions, the following formulae hold ([1], Chap. 5, Proposition 1.2) [6–8, 10]:

$$\begin{gathered} K(t,f) \le K(Ct,f) \le CK(t,f), \\ K(t,f;A_0,A_1) = tK\left(t^{-1},f;A_1,A_0\right), \\ (A_0,A_1)_{\theta,q,b} = (A_1,A_0)_{1-\theta,q,\tilde{b}}, \end{gathered} \tag{14}$$

$$(A_0,A_1)^{\mathscr{L}}_{\sigma,r,b,q,a} = (A_1,A_0)^{\mathscr{R}}_{1-\sigma,r,\tilde{b},q,\tilde{a}}, \tag{15}$$

$$\begin{gathered} (A_0,A_1)^{\mathscr{L},\mathscr{L}}_{\theta,p,c,r,b,q,a} = (A_1,A_0)^{\mathscr{R},\mathscr{R}}_{1-\theta,p,\tilde{c},r,\tilde{b},q,\tilde{a}}, \\ (A_0,A_1)^{\mathscr{L},\mathscr{R}}_{\theta,p,c,r,b,q,a} = (A_1,A_0)^{\mathscr{R},\mathscr{L}}_{1-\theta,p,\tilde{c},r,\tilde{b},q,\tilde{a}}. \end{gathered} \tag{16}$$

Lemma 9 (cf. [5]; [6], Remark 2.6). *Let $a \in SV$ and $0 < r < q \le \infty$. Then, for all $f \in A_0 + A_1$ and $t > 0$,*

$$\left\| u^{-\theta-1/q} a(u) K(u,f) \right\|_{q,(0,t)} \prec \left\| u^{-\theta-1/r} a(u) K(u,f) \right\|_{r,(0,t)}, \quad \theta < 1, \tag{17}$$

$$\left\| u^{-\theta-1/q} a(u) K(u,f) \right\|_{q,(t,\infty)} \prec \left\| u^{-\theta-1/r} a(u) K(u,f) \right\|_{r,(t,\infty)}, \quad 0 < \theta. \tag{18}$$

Moreover,

$$t^{-1} \left\| u^{-1/r} a(u) \right\|_{r,(0,t)} K(t,f) \le \left\| u^{-1-1/r} a(u) K(u,f) \right\|_{r,(0,t)}, \tag{19}$$

$$\left\| u^{-1/r} a(u) \right\|_{r,(t,\infty)} K(t,f) \le \left\| u^{-1/r} a(u) K(u,f) \right\|_{r,(t,\infty)}. \tag{20}$$

Proof. The estimates (19) and (20) follow immediately from the monotonicity of the K-functional. We prove the estimate (18). The estimate (17) can be proved similarly. For $u > t$, because $\theta > 0$ and $K(u,f)$ is nondecreasing, we have

$$\begin{aligned} u^{-\theta} a(u) K(u,f) &\approx K(u,f) \left\| s^{-\theta-1/r} a(s) \right\|_{r,(u,\infty)} \\ &\le \left\| s^{-\theta-1/r} a(s) K(s,f) \right\|_{r,(u,\infty)} \\ &\le \left\| s^{-\theta-1/r} a(s) K(s,f) \right\|_{r,(t,\infty)}. \end{aligned} \tag{21}$$

Thus, (18) is proved for $q = \infty$:

$$\sup_{t<u<\infty} u^{-\theta} a(u) K(u,f) \prec \left\| u^{-\theta-1/r} a(u) K(u,f) \right\|_{r,(t,\infty)}. \tag{22}$$

If $r < q < \infty$, using the last estimate, we get

$$\begin{aligned} &\left\| u^{-\theta-1/q} a(u) K(u,f) \right\|_{q,(t,\infty)} \\ &\le \left(\sup_{t<u<\infty} u^{-\theta} a(u) K(u,f) \right)^{(q-r)/q} \left\{ \int_t^{\infty} \left(u^{-\theta} a(u) K(u,f) \right)^r \frac{du}{u} \right\}^{1/q} \\ &\prec \left\| u^{-\theta-1/r} a(u) K(u,f) \right\|_{r,(t,\infty)}. \end{aligned} \tag{23}$$

This completes the proof. □

The next lemma can be proved by repeating the proofs of [6] (Lemma 2.12) and [7] (Lemma 2.10).

Lemma 10. *Let $0 \le \theta \le 1$, $0 < q, r \le \infty$, and $a, b \in SV$. Then, for all $f \in A_0 + A_1$ and $t > 0$,*

$$t^{-\theta} a(t) \left\| s^{-1/r} b(s) \right\|_{r,(0,t)} K(t,f) \prec \left\| s^{-1/r} b(s) \left\| u^{-\theta-1/q} a(u) K(u,f) \right\|_{q,(s,t)} \right\|_{r,(0,t)}, \tag{24}$$

$$t^{-\theta} a(t) \left\| s^{-1/r} b(s) \right\|_{r,(t,\infty)} K(t,f) \prec \left\| s^{-1/r} b(s) \left\| u^{-\theta-1/q} a(u) K(u,f) \right\|_{q,(t,s)} \right\|_{r,(t,\infty)}, \tag{25}$$

$$t^{-\theta} a(t) b(t) K(t,f) \prec \left\| s^{-1/r} b(s) \left\| u^{-\theta-1/q} a(u) K(u,f) \right\|_{q,(s,\infty)} \right\|_{r,(t,\infty)}, \tag{26}$$

$$t^{-\theta} a(t) b(t) K(t,f) \prec \left\| s^{-1/r} b(s) \left\| u^{-\theta-1/q} a(u) K(u,f) \right\|_{q,(0,s)} \right\|_{r,(0,t)}. \tag{27}$$

Lemma 11. *Let $0 < \theta < 1$, $a, b \in SV$, $0 < p \le \infty$, and $0 < r < q \le \infty$. Then, for all $f \in A_0 + A_1$ and $t > 0$,*

$$\begin{aligned} &\left\| s^{-1/q} b(s) \left\| u^{-\theta-1/p} a(u) K(u,f) \right\|_{p,(0,s)} \right\|_{q,(0,t)} \\ &\quad \prec \left\| s^{-1/r} b(s) \left\| u^{-\theta-1/p} a(u) K(u,f) \right\|_{p,(0,s)} \right\|_{r,(0,t)}, \\ &\left\| s^{-1/q} b(s) \left\| u^{-\theta-1/p} a(u) K(u,f) \right\|_{p,(s,\infty)} \right\|_{q,(t,\infty)} \\ &\quad \prec \left\| s^{-1/r} b(s) \left\| u^{-\theta-1/p} a(u) K(u,f) \right\|_{p,(s,\infty)} \right\|_{r,(t\infty)}. \end{aligned} \tag{28}$$

Proof. We prove the first estimate. The second one can be proved analogously. Because $0 < \theta < 1$, the function $s^{1-\theta} a(s)$ is equivalent to an increasing function. It is known ([10], Theorem 3.1) that for all $f \in A_0 + A_1$ and $t > 0$,

$$K\left(t^{1-\theta} a(t), f; \bar{A}_{\theta,p,\mathrm{a}}, A_1\right) \approx \left\| u^{-\theta-1/p} a(u) K(u,f) \right\|_{p,(0,t)}. \tag{29}$$

Therefore, the function $(1/t^{1-\theta}a(t))\|u^{-\theta-1/p}a(u)K(u,f)\|_{p,(0,t)}$ is equivalent to a decreasing function. Hence, for $v<t$, we get

$$\begin{aligned} b(v)\left\|u^{-\theta-1/p}a(u)K(u,f)\right\|_{p,(0,v)} &\approx \left\|s^{1-\theta-1/r}a(s)b(s)\right\|_{r,(0,v)}\frac{1}{v^{1-\theta}a(v)}\left\|u^{-\theta-1/p}a(u)K(u,f)\right\|_{p,(0,v)} \\ &\prec \left\|s^{-1/r}b(s)\left\|u^{-\theta-1/p}a(u)K(u,f)\right\|_{p,(0,s)}\right\|_{r,(0,v)} \\ &\le \left\|s^{-1/r}b(s)\left\|u^{-\theta-1/p}a(u)K(u,f)\right\|_{p,(0,s)}\right\|_{r,(0,t)}. \end{aligned} \tag{30}$$

Thus, (28) is proved for $q=\infty$:

$$\begin{aligned} &\sup_{0<s<t} b(s)\left\|u^{-\theta-1/p}a(u)K(u,f)\right\|_{p,(0,s)} \\ &\quad\prec \left\|s^{-1/r}b(s)\left\|u^{-\theta-1/p}a(u)K(u,f)\right\|_{p,(0,s)}\right\|_{r,(0,t)}. \end{aligned} \tag{31}$$

Denote $d(s)=b(s)\|u^{-\theta-1/p}a(u)K(u,f)\|_{p,(0,s)}$. If $r<q<\infty$, using the last estimate, we get

$$\begin{aligned} &\left\|s^{-1/q}b(s)\left\|u^{-\theta-1/p}a(u)K(u,f)\right\|_{p,(0,s)}\right\|_{q,(0,t)} \\ &\quad= \left\{\int_0^t (d(s))^q\frac{du}{u}\right\}^{1/q} \le \left(\sup_{0<u<t} d(s)\right)^{(q-r)/q} \\ &\qquad\times\left\{\int_0^t (d(s))^r\frac{du}{u}\right\}^{1/q} \prec \left\|s^{-1/r}b(s)\left\|u^{-\theta-1/p}a(u)K(u,f)\right\|_{p,(0,s)}\right\|_{r,(0,t)}. \end{aligned} \tag{32}$$

This completes the proof. □

It is worth remarking that the estimates from Lemmas 9–11 can be used to establish different embedding theorems.

Lemma 12 (cf. [8], Lemma 4.2). *Let $0<q,r\le\infty$, $a,b\in SV$, and $\beta>0$. Then, for all $f\in A_0+A_1$,*

$$\left\|t^{-\beta-1/r}b(t)\left\|u^{\alpha-1/q}a(u)K(u,f)\right\|_{q,(0,t)}\right\|_{r,(0,\infty)} \approx \left\|t^{\alpha-\beta-1/r}a(t)b(t)K(t,f)\right\|_{r,(0,\infty)}, \tag{33}$$

provided that $\alpha>-1$, and

$$\left\|t^{\beta-1/r}b(t)\left\|u^{\alpha-1/q}a(u)K(u,f)\right\|_{q,(t,\infty)}\right\|_{r,(0,\infty)} \approx \left\|t^{\alpha+\beta-1/r}a(t)b(t)K(t,f)\right\|_{r,(0,\infty)}, \tag{34}$$

provided that $\alpha<0$.

Proof. We prove the first equivalence. The second one can be proved analogously. We have

$$\begin{aligned} \left\|u^{\alpha-1/q}a(u)K(u,f)\right\|_{q,(0,t)} &\ge \left\|u^{\alpha-1/q}a(u)K(u,f)\right\|_{q,(t/2,t)} \\ &\ge \left\|u^{\alpha-1/q}a(u)\right\|_{q,(t/2,t)}K\left(\frac{t}{2},f\right) \\ &\approx t^{\alpha}a(t)K(t,f). \end{aligned} \tag{35}$$

Hence,

$$\begin{aligned} &\left\|t^{-\beta-1/r}b(t)\left\|u^{\alpha-1/q}a(u)K(u,f)\right\|_{q,(0,t)}\right\|_{r,(0,\infty)} \\ &\quad\succ \left\|t^{\alpha-\beta-1/r}a(t)b(t)K(t,f)\right\|_{r,(0,\infty)}. \end{aligned} \tag{36}$$

Let us prove the converse inequality.

Case $q\le r$. Because $\beta>0$, by Corollary 4 (taking $h(u)=u^{\alpha}a(u)K(u,f)$), we get

$$\begin{aligned} &\left\|t^{-\beta-1/r}b(t)\left\|u^{\alpha-1/q}a(u)K(u,f)\right\|_{q,(0,t)}\right\|_{r,(0,\infty)} \\ &\quad\prec \left\|t^{\alpha-\beta-1/r}a(t)b(t)K(t,f)\right\|_{r,(0,\infty)}. \end{aligned} \tag{37}$$

In particular,

$$\begin{aligned} &\left\|t^{-\beta-1/r}b(t)\left\|u^{\alpha-1/r}a(u)K(u,f)\right\|_{r,(0,t)}\right\|_{r,(0,\infty)} \\ &\quad\prec \left\|t^{\alpha-\beta-1/r}a(t)b(t)K(t,f)\right\|_{r,(0,\infty)}. \end{aligned} \tag{38}$$

Case $q>r$. Because $\alpha>-1$, using Lemma 9 and the last estimate, we get

$$\begin{aligned} &\left\|t^{-\beta-1/r}b(t)\left\|u^{\alpha-1/q}a(u)K(u,f)\right\|_{q,(0,t)}\right\|_{r,(0,\infty)} \\ &\quad\prec \left\|t^{-\beta-1/r}b(t)\left\|u^{\alpha-1/r}a(u)K(u,f)\right\|_{r,(0,t)}\right\|_{r,(0,\infty)} \\ &\quad\prec \left\|t^{\alpha-\beta-1/r}a(t)b(t)K(t,f)\right\|_{r,(0,\infty)}. \end{aligned} \tag{39}$$

This completes the proof. □

Lemma 13. *Let $0\le\theta_0<\theta_1\le 1$, $0<p,q,r\le\infty$, and $a,b,c\in SV$. Then, for all $f\in A_0+A_1$,*

$$\begin{aligned} &\left\|t^{\theta_0-\theta_1-1/p}c(t)\left\|s^{-1/r}b(s)\left\|u^{-\theta_0-1/q}a(u)K(u,f)\right\|_{q,(0,s)}\right\|_{r,(0,t)}\right\|_{p,(0,\infty)} \\ &\quad\approx \left\|t^{-\theta_1-1/p}a(t)b(t)c(t)K(t,f)\right\|_{p,(0,\infty)}, \end{aligned}$$

$$\left\| t^{\theta_1-\theta_0-1/p} c(t) \left\| s^{-1/r} b(s) \left\| u^{-\theta_1-1/q} a(u) K(u,f) \right\|_{q,(s,\infty)} \right\|_{r,(t,\infty)} \right\|_{p,(0,\infty)} \approx \left\| t^{-\theta_0-1/p} a(t) b(t) c(t) K(u,f) \right\|_{p,(0,\infty)}. \tag{40}$$

Proof. We prove the first equivalence. The second one can be proved analogously or using the first equivalence and symmetry argument. Denote $I(r) := \mathrm{LHS}(40)$.

Case $r \le p$. Because $\theta_0 - \theta_1 < 0$ and $\theta_0 < 1$, by Corollary 4 (taking $h(s) = b(s) \| u^{-\theta_0-1/q} a(u) K(u,f) \|_{q,(0,s)}$) and by Lemma 12, we get

$$\begin{aligned} I(r) &\prec \left\| t^{\theta_0-\theta_1-1/r} c(t) h(t) \right\|_{p,(0,\infty)} \\ &= \left\| t^{\theta_0-\theta_1-1/p} c(t) b(t) \left\| u^{-\theta_0-1/q} a(u) K(u,f) \right\|_{q,(0,t)} \right\|_{p,(0,\infty)} \\ &\approx \left\| t^{-\theta_1-1/p} a(t) c(t) b(t) K(t,f) \right\|_{p,(0,\infty)}. \end{aligned} \tag{41}$$

In particular, it was shown that

$$I(p) \prec \left\| t^{-\theta_1-1/p} a(t) c(t) b(t) K(t,f) \right\|_{p,(0,\infty)}. \tag{42}$$

Case $r > p$. By Lemma 11 and (42), we get

$$\begin{aligned} I(r) &\prec \left\| t^{\theta_0-\theta_1-1/p} c(t) \left\| s^{-1/p} b(s) \left\| u^{-\theta_0-1/q} a(u) K(u,f) \right\|_{q,(0,s)} \right\|_{p,(0,t)} \right\|_{p,(0,\infty)} \\ &= I(p) \prec \left\| t^{-\theta_1-1/p} a(t) c(t) b(t) K(t,f) \right\|_{p,(0,\infty)}. \end{aligned} \tag{43}$$

Let us prove the inverse estimate. By (27), we have

$$t^{-\theta_0} a(t) b(t) K(t,f) \prec \left\| s^{-1/r} b(s) \left\| u^{-\theta_0-1/q} a(u) K(u,f) \right\|_{q,(0,s)} \right\|_{r,(0,t)}. \tag{44}$$

Hence,

$$\mathrm{RHS}(40) \prec \left\| t^{\theta_0-\theta_1-1/p} c(t) \left\| s^{-1/r} b(s) \left\| u^{-\theta_0-1/q} a(u) K(u,f) \right\|_{q,(0,s)} \right\|_{r,(0,t)} \right\|_{p,(0,\infty)}. \tag{45}$$

This completes the proof. □

Lemma 14. *Let $0 \le \theta_0 < \theta_1 \le 1$, $0 < p, q, r \le \infty$, and $a, b \in SV$.*

(i) If $\| s^{-1/r} b(s) \|_{r,(0,1)} < \infty$, then for all $f \in A_0 + A_1$,

$$\left\| t^{\theta_0-\theta_1-1/p} c(t) \left\| s^{-1/r} b(s) \left\| u^{-\theta_0-1/q} a(u) K(u,f) \right\|_{q,(s,t)} \right\|_{r,(0,t)} \right\|_{p,(0,\infty)} \approx \left\| t^{-\theta_1-1/p} a(t) \| s^{-1/r} b(s) \|_{r,(0,t)} c(t) K(t,f) \right\|_{p,(0,\infty)}. \tag{46}$$

(ii) If $\| s^{-1/r} b(s) \|_{r,(1,\infty)} < \infty$, then for all $f \in A_0 + A_1$,

$$\left\| t^{\theta_1-\theta_0-1/p} c(t) \left\| s^{-1/r} b(s) \left\| u^{-\theta_1-1/q} a(u) K(u,f) \right\|_{q,(t,s)} \right\|_{r,(t,\infty)} \right\|_{p,(0,\infty)} \approx \left\| t^{-\theta_0-1/p} a(t) \| s^{-1/r} b(s) \|_{r,(t,\infty)} c(t) K(t,f) \right\|_{p,(0,\infty)}. \tag{47}$$

Proof. We prove the first equivalence. The second one can be proved analogously or using the symmetry argument. For $0 \le \theta_0 \le 1$, by (24), we have

$$t^{-\theta_0} a(t) \| s^{-1/r} b(s) \|_{r,(0,t)} K(t,f) \prec \left\| s^{-1/r} b(s) \left\| u^{-\theta_0-1/q} a(u) K(u,f) \right\|_{q,(s,t)} \right\|_{r,(0,t)}. \tag{48}$$

Hence, for each θ_1,

$$\mathrm{RHS}(46) \prec \left\| t^{\theta_0-\theta_1-1/p} c(s) \left\| s^{-1/r} b(s) \left\| u^{-\theta_0-1/q} a(u) K(u,f) \right\|_{q,(s,t)} \right\|_{r,(0,t)} \right\|_{p,(0,\infty)} = \mathrm{LHS}(46). \tag{49}$$

Let us prove the inverse estimate. Because $\theta_0 - \theta_1 < 0$ and $\theta_0 < 1$ by Lemma 12, we get

$$\begin{aligned} \mathrm{LHS}(46) &\le \left\| t^{\theta_0-\theta_1-1/p} c(t) \| s^{-1/r} b(s) \|_{r,(0,t)} \left\| u^{-\theta_0-1/q} a(u) K(u,f) \right\|_{q,(0,t)} \right\|_{p,(0,\infty)} \\ &\approx \left\| t^{-\theta_1-1/p} c(t) \| s^{-1/r} b(s) \|_{r,(0,t)} a(t) K(t,f) \right\|_{p,(0,\infty)} \\ &= \mathrm{RHS}(46). \end{aligned} \tag{50}$$

This completes the proof. □

The following lemma deals with the quasi-norms on the spaces $\bar{A}_{0,*;*}$ and $\bar{A}_{1,*;*}$.

Lemma 15. *Let $0 < r, p, q \le \infty$ and $a, b, c \in SV$.*

(i) If $\| s^{-1/r} c(s) \|_{r,(1,\infty)} < \infty$ and $\| s^{-1/p} b(s) \|_{p,(1,\infty)} < \infty$, then for all $f \in \bar{A}_{0,r;c}$,

$$\|f\|_{0,r;c} \approx \left\| t^{\theta-1/r} \frac{c(t)}{a(t)\|s^{-1/p}b(s)\|_{p,(t,\infty)}} \left\| s^{-1/p}b(s) \left\| u^{-\theta-1/q}a(u)K(u,f) \right\|_{q,(t,s)} \right\|_{p,(t,\infty)} \right\|_{r,(0,\infty)}, \quad (51)$$

provided that $0<\theta\le 1$. Furthermore,

$$\|f\|_{0,r;c} \prec \left\| t^{-1/r} \frac{c(t)}{a(t)\|s^{-1/p}b(s)\|_{p,(t,\infty)}} \left\| s^{-1/p}b(s) \|u^{-1/q}a(u)K(u,f)\|_{q,(t,s)} \right\|_{p,(t,\infty)} \right\|_{r,(0,\infty)}. \quad (52)$$

(ii) *If $\|s^{-1/r}c(s)\|_{r,(0,1)}<\infty$ and $\|s^{-1/p}b(s)\|_{p,(0,1)}<\infty$, then for all $f\in \bar{A}_{1,r;c}$,*

$$\|f\|_{1,r;c} \approx \left\| t^{\theta-1-1/r} \frac{c(t)}{a(t)\|s^{-1/p}b(s)\|_{p,(0,t)}} \left\| s^{-1/p}b(s) \left\| u^{-\theta-1/q}a(u)K(u,f) \right\|_{q,(s,t)} \right\|_{p,(0,t)} \right\|_{r,(0,\infty)}, \quad (53)$$

provided that $0\le\theta<1$. Furthermore,

$$\|f\|_{1,r;c} \prec \left\| t^{-1/r} \frac{c(t)}{a(t)\|s^{-1/r_0}b(s)\|_{r_0,(0,t)}} \left\| s^{-1/p}b(s) \|u^{-1-1/q}a(u)K(u,f)\|_{q,(s,t)} \right\|_{p,(0,t)} \right\|_{r,(0,\infty)}. \quad (54)$$

(iii) *If $\|s^{-1/r}c(s)\|_{r,(1,\infty)}<\infty$ and $0<\theta\le 1$, then for all $f \in \bar{A}_{0,r;c}$,*

$$\|f\|_{0,r,c} \approx \left\| t^{\theta-1/r} \frac{c(t)}{a(t)} \left\| u^{-\theta-1/q}a(u)K(u,f) \right\|_{q,(t,\infty)} \right\|_{r,(0,\infty)}. \quad (55)$$

(iv) *If $\|s^{-1/r}c(s)\|_{r,(0,1)}<\infty$ and $0\le\theta<1$, then for all $f\in \bar{A}_{1,r;c}$,*

$$\|f\|_{1,r,c} \approx \left\| t^{\theta-1-1/r} \frac{c(t)}{a(t)} \left\| u^{-\theta-1/q}a(u)K(u,f) \right\|_{q,(0,t)} \right\|_{r,(0,\infty)}. \quad (56)$$

Proof. We prove the first assertion. The second one can be proved analogously or using the symmetry argument. By Lemma 10, for $0\le\theta\le1$, we have

$$\begin{aligned} \|f\|_{0,r;c} &= \left\| t^{\theta-1/r} \frac{c(t)}{a(t)\|s^{-1/p}b(s)\|_{p,(t,\infty)}} t^{-\theta}a(t)\|s^{-1/p}b(s)\|_{p,(t,\infty)}K(t,f) \right\|_{r,(0,\infty)} \\ &\prec \left\| t^{\theta-1/r} \frac{c(t)}{a(t)\|s^{-1/p}b(s)\|_{p,(t,\infty)}} \left\| u^{-1/p}b(u) \|s^{-\theta-1/q}a(s)K(s,f)\|_{q,(t,u)} \right\|_{p,(t,\infty)} \right\|_{r,(0,\infty)}. \end{aligned} \quad (57)$$

Thus, LHS(51) $\prec$ RHS(51) and the estimate (52) is proved. Let $0<\theta\le1$. Note that

$$\begin{aligned} &\left\| s^{-1/r_1}b(s) \left\| u^{-\theta-1/q}a(u)K(u,f) \right\|_{q,(t,s)} \right\|_{r_1,(t,\infty)} \\ &\quad \le \left\| s^{-1/r_1}b(s) \right\|_{r_1,(t,\infty)} \left\| u^{-\theta-1/q}a(u)K(u,f) \right\|_{q,(t,\infty)}. \end{aligned} \quad (58)$$

Hence, by Lemma 12, we get

$$\begin{aligned} \text{RHS}(51) &\le \left\| t^{\theta-1/r} \frac{c(t)}{a(t)} \left\| u^{-\theta-1/q}a(u)K(u,f) \right\|_{q,(t,\infty)} \right\|_{r,(0,\infty)} \\ &\approx \left\| t^{-1/r}c(t)K(t,f) \right\|_{r,(0,\infty)} = \text{LHS}(51). \end{aligned} \quad (59)$$

The equivalences (iii) and (iv) follow immediately from Lemma 12. □

In Section 8, we will use the following reiteration theorems. See [9, 14].

Theorem 16. *Let $0<r,r_0\le\infty$, $a,b_0\in SV$, and $\|s^{-1/r_0}b_0(s)\|_{r_0,(1,\infty)}<\infty$. Put $\rho(t)=t\|s^{-1/r_0}b_0(s)\|_{r_0,(t,\infty)}$.*

(i) *If $0<\theta<1$, then*

$$\left(\bar{A}_{0,r_0,b_0}, A_1\right)_{\theta,r,a} = \bar{A}_{\theta,r,a^{\#}}, \quad (60)$$

where $a^{\#}(t)=(\|s^{-1/r_0}b_0(s)\|_{r_0,(t,\infty)})^{1-\theta}a(\rho(t))$.

(ii) *If $\|t^{-1/r}a(t)\|_{r,(1,\infty)}<\infty$, then*

$$\left(\bar{A}_{0,r_0,b_0}, A_1\right)_{0,r,a} = \bar{A}_{0,r,a^{\#}} \cap \bar{A}^{\mathscr{L}}_{0,r,a\circ\rho,r_0,b_0}, \quad (61)$$

where $a^{\#}(t)=\|s^{-1/r_0}b_0(s)\|_{r_0,(t,\infty)}a(\rho(t))$.

(iii) *If $\|t^{-1/r}a(t)\|_{r,(0,1)}<\infty$, then*

$$\left(\bar{A}_{0,r_0,b_0}, A_1\right)_{1,r,a} = \bar{A}_{1,r,a\circ\rho}. \quad (62)$$

Theorem 17. *Let $0<r,r_1\le\infty$, $a,b_1\in SV$, and $\|s^{-1/r_1}b_1(s)\|_{r_1,(0,1)}<\infty$. Put $\rho(t)=t/\|t^{-1/r_1}b_1(t)\|_{r_1,(0,t)}$.*

(i) *If $0<\theta<1$, then*

$$\left(A_0, \bar{A}_{1,r_1,b_1}\right)_{\theta,r,a} = \bar{A}_{\theta,r,a^{\#}}, \quad (63)$$

where $a^{\#}(t)=(\|s^{-1/r_1}b_1(s)\|_{r_1,(0,t)})^{\theta}a(\rho(t))$.

(ii) *If $\|t^{-1/r}a(t)\|_{r,(1,\infty)}<\infty$, then*

$$\left(A_0, \bar{A}_{1,r_1,b_1}\right)_{0,r,a} = \bar{A}_{0,r,a\circ\rho}. \quad (64)$$

(iii) If $\|t^{-1/r}a(t)\|_{r,(0,1)} < \infty$, *then*

$$\left(A_0, \bar{A}_{1,r_1,b_1}\right)_{1,r,a} = \bar{A}_{1,r,a^{\#}} \cap \bar{A}^{R}_{1,r,a\circ\rho,r_1,b_1}, \quad (65)$$

where $a^{\#}(t) = \|s^{-1/r_1}b_1(s)\|_{r_1,(0,t)}a(\rho(t))$.

4. Limiting Interpolation between the $\mathscr{L}$ Spaces on the Right and the Standard Interpolation Spaces

In this section, we establish some limiting reiteration formulae for the couples of the form $(\bar{A}_{\theta_0,*}, \bar{A}^{\mathscr{L}}_{\theta_1,*})$ $(0 \le \theta_0 < \theta_1 < 1)$. They amend Theorems 16, 17, and 19 from [5]. Here, we need the relevant Holmstedt-type formulae. The proofs of all Holmstedt-type formulae in this paper are based on the paper [15]. Therefore, we adopt the notation from it.

Theorem 18 (cf. [6], Theorem 3.4). *Let* $0 < \theta_0 < \theta_1 < 1$, $0 < r_0, q_1, r_1 \le \infty$, $b_0, a_1, b_1 \in SV$, *and* $\|s^{-1/r_1}b_1(s)\|_{r_1,(1,\infty)} < \infty$. *Put* $\chi_1(t) = a_1(t)\|s^{-1/r_1}b_1(s)\|_{r_1,(t,\infty)}$.

(i) Let $\rho(t) = t^{\theta_1-\theta_0}(b_0(t)/\chi_1(t))$. *Then, for all* $f \in \bar{A}_{\theta_0,r_0,b_0} + \bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}$ *and* $t > 0$,

$$\begin{aligned} K\left(\rho(t), f; \bar{A}_{\theta_0,r_0,b_0}, \bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}\right) &\approx \left\| u^{-\theta_0-1/r_0}b_0(u)K(u,f)\right\|_{r_0,(0,t)} \\ &+ \rho(t)\left\| s^{-1/r_1}b_1(s)\left\| u^{-\theta_1-1/q_1}a_1(u)K(u,f)\right\|_{q_1,(t,s)}\right\|_{r_1,(t,\infty)}. \end{aligned} \quad (66)$$

(ii) Let $\|s^{-1/r_0}b_0(s)\|_{r_0,(1,\infty)} < \infty$ *and* $\rho(t) = t^{\theta_1}(\|s^{-1/r_0}b_0(s)\|_{r_0,(t,\infty)}/\chi_1(t))$. *Then, for all* $f \in \bar{A}_{0,r_0,b_0} + \bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}$ *and* $t > 0$,

$$\begin{aligned} K\left(\rho(t), f; \bar{A}_{0,r_0,b_0}, \bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}\right) &\approx \left\| u^{-1/r_0}b_0(u)K(u,f)\right\|_{r_0,(0,t)} \\ &+ \rho(t)\left\| s^{-1/r_1}b_1(s)\left\| u^{-\theta_1-1/q_1}a_1(u)K(u,f)\right\|_{q_1,(t,s)}\right\|_{r_1,(t,\infty)}. \end{aligned} \quad (67)$$

(iii) Let $\rho(t) = t^{\theta_1}(1/\chi_1(t))$. *Then, for all* $f \in A_0 + \bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}$ *and* $t > 0$,

$$\begin{aligned} K&\left(\rho(t), f; A_0, \bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}\right) \\ &\approx \rho(t)\left\| s^{-1/r_1}b_1(s)\left\| u^{-\theta_1-1/q_1}a_1(u)K(u,f)\right\|_{q_1,(t,s)}\right\|_{r_1,(t,\infty)}. \end{aligned} \quad (68)$$

Proof. The formulae (i) and (iii) have been proved in [5]. Here, we prove the formula (ii). Let Φ_0 be the function space corresponding to $\bar{A}_{0,r_0,b_0}$:

$$\|F(*)\|_{\Phi_0} = \left\| u^{-1/r_0}b_0(u)F(u)\right\|_{r_0,(0,\infty)}. \quad (69)$$

Consider the functions $I(t,f)$, $g_0(t)$, and $h_0(t)$.

$$\begin{aligned} I(t,f) &:= \left\|\chi_{(0,t)}(*)K(*,f)\right\|_{\Phi_0} = \left\| u^{-1/r_0}b_0(u)K(u,f)\right\|_{r_0,(0,t)}, \\ g_0(t) &:= t\left\|\chi_{(t,\infty)}(*)\right\|_{\Phi_0} = t\left\| u^{-1/r_0}b_0(u)\right\|_{r_0,(t,\infty)}, \\ h_0(t) &:= \left\| *\chi_{(0,t)}(*)\right\|_{\Phi_0} = \left\| u^{1-1/r_0}b_0(u)\right\|_{r_0,(0,t)} \approx tb_0(t). \end{aligned} \quad (70)$$

Thus,

$$h_0(t) \prec g_0(t) \approx g_0(t) + h_0(t) \approx t\left\| u^{-1/r_0}b_0(u)\right\|_{r_0,(t,\infty)}. \quad (71)$$

Let Φ_1 be the function space corresponding to $\bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}$
:

$$\|F(*)\|_{\Phi_1} = \left\| s^{-1/r_1}b_1(s)\left\| u^{-\theta_1-1/q_1}a_1(u)F(u)\right\|_{q_1,(0,s)}\right\|_{r_1,(0,\infty)}. \quad (72)$$

Consider the functions $J(t,f)$, $g_1(t)$, and $h_1(t)$.

$$\begin{aligned} J(t,f) &:= \left\|\chi_{(t,\infty)}(*)K(*,f)\right\|_{\Phi_1} \\ &= \left\| s^{-1/r_1}b_1(s)\left\| u^{-\theta_1-1/q_1}a_1(u)K(u,f)\right\|_{q_1,(t,s)}\right\|_{r_1,(t,\infty)}, \end{aligned}$$

$$\begin{aligned} g_1(t) &:= t\left\|\chi_{(t,\infty)}(*)\right\|_{\Phi_1} \\ &= t\left\| s^{-1/r_1}b_1(s)\left\| u^{-\theta_1-1/q_1}a_1(u)\right\|_{q_1,(t,s)}\right\|_{r_1,(t,\infty)} \\ &\prec t^{1-\theta_1}\left\| s^{-1/r_1}b_1(s)\right\|_{r_1,(t,\infty)}a_1(t) = t^{1-\theta_1}\chi_1(t), \end{aligned}$$

$$
\begin{aligned}
h_1(t) &:= \left\| * \chi_{(0,t)}(*) \right\|_{\Phi_1} \\
&= \left\| s^{-1/r_1} b_1(s) \left\| u^{1-\theta_1-1/q_1} a_1(u) \right\|_{q_1,(0,\min(t,s))} \right\|_{r_1,(0,\infty)} \\
&\approx \left\| s^{-1/r_1} b_1(s) (\min(t,s))^{1-\theta_1} a_1(\min(t,s)) \right\|_{r_1,(0,\infty)} \\
&\approx \left\| s^{1-\theta_1-1/r_1} a_1(s) b_1(s) \right\|_{r_1,(0,t)} + t^{1-\theta_1} a_1(t) \left\| s^{-1/r_1} b_1(s) \right\|_{r_1,(t,\infty)} \\
&\approx t^{1-\theta_1} a_1(t) b_1(t) + t^{1-\theta_1} \chi_1(t).
\end{aligned} \tag{73}
$$

Because $a_1(t) b_1(t) \prec a_1(t) \| s^{-1/r_1} b_1(s) \|_{r_1,(t,\infty)} = \chi_1(t)$, we arrive at

$$
g_1(t) \prec h_1(t) \approx g_1(t) + h_1(t) \approx t^{1-\theta_1} \chi_1(t). \tag{74}
$$

Now, we estimate the functions $g(t)$ and $h(t)$.

$$
\begin{aligned}
\frac{1}{g(t)} &:= \left\| \chi_{(t,\infty)}(u) \frac{u}{g_0(u) + h_0(u)} \right\|_{\Phi_1} \\
&\approx \left\| s^{-1/r_1} b_1(s) \left\| u^{-\theta_1-1/q_1} \frac{a_1(u)}{\| v^{-1/r_0} b_0(v) \|_{r_0,(u,\infty)}} \right\|_{q_1,(t,s)} \right\|_{r_1,(t,\infty)} \\
&\le \left\| u^{-\theta_1-1/q_1} \frac{a_1(u)}{\| v^{-1/r_0} b_0(v) \|_{r_0,(u,\infty)}} \right\|_{q_1,(t,\infty)} \left\| s^{-1/r_1} b_1(s) \right\|_{r_1,(t,\infty)} \\
&\approx t^{-\theta_1} \frac{a_1(t)}{\| v^{-1/r_0} b_0(v) \|_{r_0,(t,\infty)}} \left\| s^{-1/r_1} b_1(s) \right\|_{r_1,(t,\infty)} = \frac{1}{\rho(t)},
\end{aligned}
$$

$$
\begin{aligned}
h(t) &:= \left\| \chi_{(0,t)}(u) \frac{u}{g_1(u) + h_1(u)} \right\|_{\Phi_0} \approx \left\| u^{\theta_1-1/r_0} \frac{b_0(u)}{\chi_1(u)} \right\|_{r_0,(0,t)} \\
&\approx t^{\theta_1} \frac{b_0(t)}{\chi_1(t)} \prec t^{\theta_1} \frac{\| s^{-1/r_0} b_0(s) \|_{r_0,(t,\infty)}}{\chi_1(t)} = \rho(t).
\end{aligned} \tag{75}
$$

We see that $h(t) \prec \rho(t) \prec g(t)$and $g_0(t)/(g_1(t) + h_1(t)) \approx (g_0(t) + h_0(t))/h_1(t) \approx \rho(t)$. Note that

$$
g_0(t) + \rho(t) h_1(t) \approx t \left\| s^{-1/r_0} b_0(s) \right\|_{r_0,(t,\infty)}. \tag{76}
$$

Hence, by [15] (Theorem 4, Case 1), we arrive at

$$
\begin{aligned}
K&\left(\rho(t), f; \bar{A}_{0,r_0,b_0}, \bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}\right) \\
&\approx \left\| u^{-1/r_0} b_0(u) K(u,f) \right\|_{r_0,(0,t)} \\
&\quad + \rho(t) \left\| s^{-1/r_1} b_1(s) \left\| u^{-\theta_1-1/q_1} a_1(u) K(u,f) \right\|_{q_1,(t,s)} \right\|_{r_1,(t,\infty)} \\
&\quad + \left\| s^{-1/r_0} b_0(s) \right\|_{r_0,(t,\infty)} K(t,f).
\end{aligned} \tag{77}
$$

Using Lemma 10, we get

$$
\| s^{-1/r_0} b_0(s) \|_{r_0,(t,\infty)} K(t,f) \prec \rho(t) \left\| s^{-1/r_1} b_1(s) \left\| u^{-\theta_1-1/q_1} a_1(u) K(u,f) \right\|_{q_1,(t,s)} \right\|_{r_1,(t,\infty)}. \tag{78}
$$

Therefore,

$$
\begin{aligned}
K&\left(\rho(t), f; \bar{A}_{0,r_0,b_0}, \bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}\right) \\
&\approx \left\| u^{-1/r_0} b_0(u) K(u,f) \right\|_{r_0,(0,t)} \\
&\quad + \rho(t) \left\| s^{-1/r_1} b_1(s) \left\| u^{-\theta_1-1/q_1} a_1(u) K(u,f) \right\|_{q_1,(t,s)} \right\|_{r_1,(t,\infty)}.
\end{aligned} \tag{79}
$$

This completes the proof. ☐

Theorem 19 (cf. [6], Theorem 4.7 (c)). *Let $0 < \theta_0 < \theta_1 < 1$, $0 < r, r_0, q_1, r_1 \le \infty$, $a, b_0, a_1, b_1 \in SV$, $\| s^{-1/r_1} b_1(s) \|_{r_1,(1,\infty)} < \infty$, and $\| s^{-1/r} a(s) \|_{r,(0,1)} < \infty$. Put $\chi_1(t) = a_1(t) \| s^{-1/r_1} b_1(s) \|_{r_1,(t,\infty)}$ and $\rho(t) = t^{\theta_1-\theta_0}(b_0(t)/\chi_1(t))$. Then,*

$$
\left(\bar{A}_{\theta_0,r_0,b_0}, \bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}\right)_{1,r,a} = \bar{A}^{\mathscr{L},\mathscr{R}}_{\theta_1,r,a\circ\rho,r_1,b_1,q_1,a_1}. \tag{80}
$$

Proof. Denote $X_0 = \bar{A}_{\theta_0,r_0,b_0}$, $X_1 = \bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}$, $\bar{K}(t,f) = K(t, f; \bar{X})$, and $Z = \bar{X}_{1,r,a}$. Using the change of variables $u = \rho(t)$ (see, e.g., [5], Remark 3) and Theorem 18 (i), we can write

$$
\begin{aligned}
\|f\|_Z &= \left\| u^{-1-1/r} a(u) \bar{K}(u,f) \right\|_{r,(0,\infty)} \\
&\approx \left\| \rho(t)^{-1} t^{-1/r} a(\rho(t)) \bar{K}(\rho(t),f) \right\|_{r,(0,\infty)} \approx I_1 + I_2,
\end{aligned} \tag{81}
$$

where

$$
\begin{aligned}
I_1 &:= \left\| t^{\theta_0-\theta_1-1/r} \frac{a(\rho(t))\chi_1(t)}{b_0(t)} \left\| u^{-\theta_0-1/r_0} b_0(u) K(u,f) \right\|_{r_0,(0,t)} \right\|_{r,(0,\infty)}, \\
I_2 &:= \left\| t^{-1/r} a(\rho(t)) \left\| s^{-1/r_1} b_1(s) \left\| u^{-\theta_1-1/q_1} a_1(u) K(u,f) \right\|_{q_1,(t,s)} \right\|_{r_1,(t,\infty)} \right\|_{r,(0,\infty)} \\
&= \|f\|_{\mathscr{L},\mathscr{R};\theta_1,r,a\circ\rho,r_1,b_1,q_1,a_1}.
\end{aligned} \tag{82}
$$

Lemmas 12 and 10 imply that

$$
I_1 \approx \left\| t^{-\theta_1-1/r} a(\rho(t)) a_1(t) \left\| s^{-1/r_1} b_1(s) \right\|_{r_1,(t,\infty)} K(t,f) \right\|_{r,(0,\infty)} \prec I_2. \tag{83}
$$

This completes the proof. ☐

Theorem 20 (cf. [6], Theorem 4.8 (b, c)). *Let $0 < \theta_1 < 1$, $0 < r, r_0, q_1, r_1 \le \infty$, $a, b_0, a_1, b_1 \in SV$, $\| s^{-1/r_0} b_0(s) \|_{r_0,(1,\infty)} < \infty$,* *and* $\|s^{-1/r_1}b_1(s)\|_{r_1,(1,\infty)} < \infty$. *Put* $\chi_1(t) = a_1(t)\|s^{-1/r_1}b_1(s)\|_{r_1,(t,\infty)}$ *and* $\rho(t) = t^{\theta_1}(\|s^{-1/r_0}b_0(s)\|_{r_0,(t,\infty)}/\chi_1(t))$.

(i) *If* $\|s^{-1/r}a(s)\|_{r_0,(1,\infty)} < \infty$, *then*

$$\left(\bar{A}_{0,r_0,b_0}, \bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}\right)_{0,r,a} = \bar{A}_{0,r,a^{\#}} \bigcap \bar{A}^{\mathscr{L}}_{0,r,a\circ\rho,r_0,b_0}, \tag{84}$$

where $a^{\#}(t) = \|s^{-1/r_0}b_0(s)\|_{r_0,(t,\infty)}a(\rho(t))$.

(ii) *If* $\|s^{-1/r}a(s)\|_{r_0,(0,1)} < \infty$, *then*

$$\left(\bar{A}_{0,r_0,b_0}, \bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}\right)_{1,r,a} = \bar{A}^{\mathscr{L},\mathscr{R}}_{\theta_1,r,a\circ\rho,r_1,b_1,q_1,a_1}. \tag{85}$$

Proof. Denote $X_0 = \bar{A}_{0,r_0,b_0}$, $X_1 = \bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}$, and $\bar{K}(t,f) = K(t,f;\bar{X})$. First, we prove (i). Let $Z = \bar{X}_{0,r,a}$. Using the change of variables $u = \rho(t)$ and Theorem 18 (ii), we can write

$$\|f\|_Z = \|u^{-1/r}a(u)\bar{K}(u,f)\|_{r,(0,\infty)} \approx \|t^{-1/r}a(\rho(t))\bar{K}(\rho(t),f)\|_{r,(0,\infty)} \approx I_1 + I_2, \tag{86}$$

where

$$I_1 := \left\| t^{-1/r}a(\rho(t))\|u^{-1/r_0}b_0(u)K(u,f)\|_{r_0,(0,t)} \right\|_{r,(0,\infty)} = \|f\|_{\mathscr{L};0,r,a\circ\rho,r_0,b_0},$$

$$\begin{aligned} I_2 &:= \left\| t^{-1/r}a(\rho(t))\rho(t) \left\| s^{-1/r_1}b_1(s) \|u^{-\theta_1-1/q_1}a_1(u)K(u,f)\|_{q_1,(t,s)} \right\|_{r_1,(t,\infty)} \right\|_{r,(0,\infty)} \\ &= \left\| t^{\theta_1-1/r}\frac{a^{\#}(t)}{\chi_1(t)} \left\| s^{-1/r_1}b_1(s) \|u^{-\theta_1-1/q_1}a_1(u)K(u,f)\|_{q_1,(t,s)} \right\|_{r_1,(t,\infty)} \right\|_{r,(0,\infty)}. \end{aligned} \tag{87}$$

By Lemma 15 (i), we get $I_2 \approx \|f\|_{0,r;a^{\#}}$. The formula (i) is proved.

Now, we prove (ii). Let $Z = \bar{X}_{1,r,a}$. Using the change of variables and Theorem 18 (ii), we can write

$$\begin{aligned} \|f\|_Z &= \|u^{-1-1/r}a(u)\bar{K}(u,f)\|_{r,(0,\infty)} \\ &\approx \|\rho(t)^{-1}t^{-1/r}a(\rho(t))\bar{K}(\rho(t),f)\|_{r,(0,\infty)} \approx I_1 + I_2, \end{aligned} \tag{88}$$

where

$$I_1 := \left\| t^{-\theta_1-1/r}\frac{\chi_1(t)}{\|s^{-1/r_0}b_0(s)\|_{r_0,(t,\infty)}}a(\rho(t))\|u^{-1/r_0}b_0(u)K(u,f)\|_{r_0,(0,t)} \right\|_{r,(0,\infty)},$$

$$\begin{aligned} I_2 &:= \left\| t^{-1/r}a(\rho(t)) \left\| s^{-1/r_1}b_1(s) \|u^{-\theta_1-1/q_1}a_1(u)K(u,f)\|_{q_1,(t,s)} \right\|_{r_1,(t,\infty)} \right\|_{r,(0,\infty)} \\ &= \|f\|_{\mathscr{L},\mathscr{R};\theta_1,r,a\circ\rho,r_1,b_1,q_1,a_1}. \end{aligned} \tag{89}$$

By Lemmas 12 and 10, we get

$$\begin{aligned} I_1 &\approx \left\| t^{-\theta_1-1/r}\frac{\chi_1(t)b_0(t)}{\|s^{-1/r_0}b_0(s)\|_{r_0,(t,\infty)}}a(\rho(t))K(t,f) \right\|_{r,(0,\infty)} \\ &\prec \left\| t^{-\theta_1-1/r}a_1(t)\|s^{-1/r_1}b_1(s)\|_{r_1,(t,\infty)}a(\rho(t))K(t,f) \right\|_{r,(0,\infty)} \\ &\prec \left\| t^{-1/r}a(\rho(t)) \left\| s^{-1/r_1}b_1(s) \|u^{-\theta_1-1/q_1}a_1(u)K(u,f)\|_{q_1,(t,s)} \right\|_{r_1,(t,\infty)} \right\|_{r,(0,\infty)} \\ &= I_2. \end{aligned} \tag{90}$$

This completes the proof. □

Theorem 21 (cf. [6], Theorem 4.9 (b)). *Let* $0 < \theta_1 < 1$, $0 < r, q_1, r_1 \le \infty$, $a, a_1, b_1 \in SV$, $\|s^{-1/r_1}b_1(s)\|_{r_1,(1,\infty)} < \infty$, *and* $\|s^{-1/r}a(s)\|_{r,(0,1)} < \infty$. *Put* $\chi_1(t) = a_1(t)\|s^{-1/r_1}b_1(s)\|_{r_1,(t,\infty)}$ *and* $\rho(t) = t^{\theta_1}(1/\chi_1(t))$. *Then,*

$$\left(A_0, \bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}\right)_{1,r,a} = \bar{A}^{\mathscr{L},\mathscr{R}}_{\theta_1,r,a\circ\rho,r_1,b_1,q_1,a_1}. \tag{91}$$

Proof. Denote $X_0 = A_0$, $X_1 = \bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}$, $\bar{K}(t,f) = K(t,f;\bar{X})$, and $Z = \bar{X}_{1,r,a}$. Using the change of variables and Theorem 18 (iii), we can write

$$\begin{aligned} \|f\|_Z &= \|u^{-1-1/r}a(u)\bar{K}(u,f)\|_{r,(0,\infty)} \approx \|\rho(t)^{-1}t^{-1/r}a(\rho(t))\bar{K}(\rho(t),f)\|_{r,(0,\infty)} \\ &\approx \left\| t^{-1/r}a(\rho(t)) \left\| s^{-1/r_1}b_1(s) \|u^{-\theta_1-1/q_1}a_1(u)K(u,f)\|_{q_1,(t,s)} \right\|_{r_1,(t,\infty)} \right\|_{r,(0,\infty)} \\ &= \|f\|_{\mathscr{L},\mathscr{R};\theta_1,r,a\circ\rho,r_1,b_1,q_1,a_1}. \end{aligned} \tag{92}$$

This completes the proof. □

5. Limiting Interpolation between the $\mathscr{L}$ Spaces on the Left and the Standard Interpolation Spaces

In this section, we establish some limiting reiteration formulae for the couples of the form $(\bar{A}^{\mathscr{L}}_{\theta_0,*}, \bar{A}_{\theta_1,*})(0 < \theta_0 < \theta_1 \le 1)$. They amend Theorems 11–13 from [5]. Here, we also need the relevant Holmstedt-type formulae.

Theorem 22 (cf. [7], Theorem 3.2). *Let* $0 < \theta_0 < \theta_1 < 1$, $0 < q_0, r_0, r_1 \le \infty$, $a_0, b_0, b_1 \in SV$, *and* $\|s^{-1/r_0}b_0(s)\|_{r_0,(1,\infty)} < \infty$. *Put* $\chi_0(t) = a_0(t)\|s^{-1/r_0}b_0(s)\|_{r_0,(t,\infty)}$.

(i) *Let* $\rho(t) = t^{\theta_1-\theta_0}(\chi_0(t)/b_1(t))$. *Then, for all* $f \in \bar{A}^{\mathscr{L}}_{\theta_0,r_0,b_0,q_0,a_0} + \bar{A}_{\theta_1,r_1,b_1}$ *and* $t > 0$,

$$K\left(\rho(t), f; \bar{A}^{\mathscr{L}}_{\theta_0,r_0,b_0,q_0,a_0}, \bar{A}_{\theta_1,r_1,b_1}\right)$$
$$\approx \left\| s^{-1/r_0} b_0(s) \left\| u^{-\theta_0-1/q_0} a_0(u) K(u,f) \right\|_{q_0,(0,s)} \right\|_{r_0,(0,t)}$$
$$+ \left\| s^{-1/r_0} b_0(s) \right\|_{r_0,(t,\infty)} \left\| u^{-\theta_0-1/q_0} a_0(u) K(u,f) \right\|_{q_0,(0,t)}$$
$$+ \rho(t) \left\| u^{-\theta_1-1/r_1} b_1(u) K(u,f) \right\|_{r_1,(t,\infty)}. \quad (93)$$

(ii) Let $\|s^{-1/r_1} b_1(s)\|_{r_1,(0,1)} < \infty$ *and* $\rho(t) = t^{1-\theta_0}(\chi_0(t) / \|s^{-1/r_1} b_1(s)\|_{r_1,(0,t)})$. *Then, for all* $f \in \bar{A}^{\mathscr{L}}_{\theta_0,r_0,b_0,q_0,a_0} + \bar{A}_{1,r_1,b_1}$ *and* $t > 0$,

$$K\left(\rho(t), f; \bar{A}^{\mathscr{L}}_{\theta_0,r_0,b_0,q_0,a_0}, \bar{A}_{1,r_1,b_1}\right)$$
$$\approx \left\| s^{-1/r_0} b_0(s) \left\| u^{-\theta_0-1/q_0} a_0(u) K(u,f) \right\|_{q_0,(0,s)} \right\|_{r_0,(0,t)}$$
$$+ \left\| s^{-1/r_0} b_0(s) \right\|_{r_0,(t,\infty)} \left\| u^{-\theta_0-1/q_0} a_0(u) K(u,f) \right\|_{q_0,(0,t)}$$
$$+ \rho(t) \left\| u^{-1-1/r_1} b_1(u) K(u,f) \right\|_{r_1,(t,\infty)}. \quad (94)$$

(iii) Let $\rho(t) = t^{1-\theta_0} \chi_0(t)$. *Then, for all* $f \in \bar{A}^{\mathscr{L}}_{\theta_0,r_0,b_0,q_0,a_0} + \bar{A}_1$ *and* $t > 0$,

$$K\left(\rho(t), f; \bar{A}^{\mathscr{L}}_{\theta_0,r_0,b_0,q_0,a_0}, A_1\right)$$
$$\approx \left\| s^{-1/r_0} b_0(s) \left\| u^{-\theta_0-1/q_0} a_0(u) K(u,f) \right\|_{q_0,(0,s)} \right\|_{r_0,(0,t)}$$
$$+ \left\| s^{-1/r_0} b_0(s) \right\|_{r_0,(t,\infty)} \left\| u^{-\theta_0-1/q_0} a_0(u) K(u,f) \right\|_{q_0,(0,t)}. \quad (95)$$

Proof. Recall that we adopt the notation from the paper [15]. Let Φ_0 be the function space corresponding to $\bar{A}^{\mathscr{L}}_{\theta_0,r_0,b_0,q_0,a_0}$:

$$\|F(*)\|_{\Phi_0} = \left\| s^{-1/r_0} b_0(s) \left\| u^{-\theta_0-1/q_0} a_0(u) F(u) \right\|_{q_0,(0,s)} \right\|_{r_0,(0,\infty)}. \quad (96)$$

Consider the functions $I(t,f)$, $g_0(t)$, and $h_0(t)$.

$$I(t,f) := \left\| \chi_{(0,t)}(*) K(*,f) \right\|_{\Phi_0}$$
$$= \left\| s^{-1/r_0} b_0(s) \left\| u^{-\theta_0-1/q_0} a_0(u) \chi_{(0,t)}(u) K(u,f) \right\|_{q_0,(0,s)} \right\|_{r_0,(0,\infty)}$$
$$\approx \left\| s^{-1/r_0} b_0(s) \left\| u^{-\theta_0-1/q_0} a_0(u) K(u,f) \right\|_{q_0,(0,s)} \right\|_{r_0,(0,t)}$$
$$+ \left\| s^{-1/r_0} b_0(s) \right\|_{r_0,(t,\infty)} \left\| u^{-\theta_0-1/q_0} a_0(u) K(u,f) \right\|_{q_0,(0,t)},$$

$$g_0(t) := t \left\| \chi_{(t,\infty)}(*) \right\|_{\Phi_0} = t \left\| s^{-1/r_0} b_0(s) \left\| u^{-\theta_0-1/q_0} a_0(u) \right\|_{q_0,(t,s)} \right\|_{r_0,(t,\infty)}$$
$$\le t \left\| u^{-\theta_0-1/q_0} a_0(u) \right\|_{q_0,(t,\infty)} \left\| s^{-1/r_0} b_0(s) \right\|_{r_0,(t,\infty)}$$
$$\approx t^{1-\theta_0} a_0(t) \left\| s^{-1/r_0} b_0(s) \right\|_{r_0,(t,\infty)} = t^{1-\theta_0} \chi_0(t),$$

$$h_0(t) := \left\| * \chi_{(0,t)}(*) \right\|_{\Phi_0} = \left\| s^{-1/r_0} b_0(s) \left\| u^{1-\theta_0-1/q_0} a_0(u) \right\|_{q_0,(0,\min(t,s))} \right\|_{r_0,(0,\infty)}$$
$$\approx \left\| s^{-1/r_0} b_0(s) \min(t,s)^{1-\theta_0} a_0(\min(t,s)) \right\|_{r_0,(0,\infty)}$$
$$\approx \left\| s^{1-\theta_0-1/r_0} b_0(s) a_0(s) \right\|_{r_0,(0,t)} + t^{1-\theta_0} a_0(t) \left\| s^{-1/r_0} b_0(s) \right\|_{r_0,(t,\infty)}$$
$$\approx t^{1-\theta_0} b_0(t) a_0(t) + t^{1-\theta_0} a_0(t) \left\| s^{-1/r_0} b_0(s) \right\|_{r_0,(t,\infty)}$$
$$\approx t^{1-\theta_0} a_0(t) \left\| s^{-1/r_0} b_0(s) \right\|_{r_0,(t,\infty)} = t^{1-\theta_0} \chi_0(t). \quad (97)$$

Thus,

$$g_0(t) \prec h_0(t) \approx g_0(t) + h_0(t) \approx t^{1-\theta_0} \chi_0(t). \quad (98)$$

First, we prove (i). Let Φ_1 be the function space corresponding to $\bar{A}_{\theta_1,r_1,b_1}$:

$$\|F(*)\|_{\Phi_1} = \left\| u^{-\theta_1-1/r_1} b_1(u) F(u) \right\|_{r_1,(0,\infty)}. \quad (99)$$

Consider the functions $J(t,f)$, $g_1(t)$, and $h_1(t)$.

$$J(t,f) := \left\| \chi_{(t,\infty)}(*) K(*,f) \right\|_{\Phi_1} = \left\| u^{-\theta_1-1/r_1} b_1(u) K(u,f) \right\|_{r_1,(t,\infty)},$$
$$g_1(t) := t \left\| \chi_{(t,\infty)}(*) \right\|_{\Phi_1} = t \left\| u^{-\theta_1-1/r_1} b_1(u) \right\|_{r_1,(t,\infty)} \approx t^{1-\theta_1} b_1(t),$$
$$h_1(t) := \left\| * \chi_{(0,t)}(*) \right\|_{\Phi_1} = \left\| u^{1-\theta_1-1/r_1} b_1(u) \right\|_{r_1,(0,t)} \approx t^{1-\theta_1} b_1(t). \quad (100)$$

Thus,

$$g_1(t) \approx h_1(t) \approx g_1(t) + h_1(t) \approx t^{1-\theta_1} b_1(t). \quad (101)$$

Now, we estimate the functions $g(t)$ and $h(t)$. Because

$\theta_0 - \theta_1 < 0$, we get

$$\frac{1}{g(t)} := \left\| \chi_{(t,\infty)}(u) \frac{u}{g_0(u)+h_0(u)} \right\|_{\Phi_1} \approx \left\| u^{\theta_0-\theta_1-1/r_1} \frac{b_1(u)}{\chi_0(u)} \right\|_{r_1,(t,\infty)} \approx t^{\theta_0-\theta_1} \frac{b_1(t)}{\chi_0(t)} = \frac{1}{\rho(t)},$$

$$\begin{aligned} h(t) &:= \left\| \chi_{(0,t)}(u) \frac{u}{g_1(u)+h_1(u)} \right\|_{\Phi_0} \\ &\approx \left\| s^{-1/r_0} b_0(s) \left\| u^{\theta_1-\theta_0-1/q_0} \frac{a_0(u)}{b_1(u)} \right\|_{q_0,(0,\min(t,s))} \right\|_{r_0,(0,\infty)} \\ &\approx \left\| s^{\theta_1-\theta_0-1/r_0} b_0(s) \frac{a_0(s)}{b_1(s)} \right\|_{r_0,(0,t)} + t^{\theta_1-\theta_0} \frac{a_0(t)}{b_1(t)} \left\| s^{-1/r_0} b_0(s) \right\|_{r_0,(t,\infty)} \\ &\approx t^{\theta_1-\theta_0} \frac{a_0(t) b_0(t)}{b_1(t)} + t^{\theta_1-\theta_0} \frac{a_0(t)}{b_1(t)} \left\| s^{-1/r_0} b_0(s) \right\|_{r_0,(t,\infty)} \\ &\approx t^{\theta_1-\theta_0} \frac{a_0(t)}{b_1(t)} \left\| s^{-1/r_0} b_0(s) \right\|_{r_0,(t,\infty)} = \rho(t). \end{aligned} \tag{102}$$

Thus,

$$h(t) \approx \rho(t) \approx g(t). \tag{103}$$

Using [15] (Theorem 4, Case 4), we arrive at (93).

Proof of statement (ii). Put $\chi_1(t) = \| s^{-1/r_1} b_1(s) \|_{r_1,(0,t)}$. Let Φ_1 be the function space corresponding to $\bar{A}_{1,r_1,b_1}$:

$$\| F(*) \|_{\Phi_1} = \| u^{-1-1/r_1} b_1(u) F(u) \|_{r_1,(0,\infty)}. \tag{104}$$

Consider the functions $J(t,f)$, $g_1(t)$, and $h_1(t)$.

$$\begin{aligned} J(t,f) &:= \left\| \chi_{(t,\infty)}(*) K(*,f) \right\|_{\Phi_1} = \| u^{-1-1/r_1} b_1(u) K(u,f) \|_{r_1,(t,\infty)}, \\ g_1(t) &:= t \left\| \chi_{(t,\infty)}(*) \right\|_{\Phi_1} = t \| u^{-1-1/r_1} b_1(u) \|_{r_1,(t,\infty)} \approx b_1(t), \\ h_1(t) &:= \left\| * \chi_{(0,t)}(*) \right\|_{\Phi_1} = \| u^{-1/r_1} b_1(u) \|_{r_1,(0,t)} = \chi_1(t). \end{aligned} \tag{105}$$

Thus,

$$g_1(t) \prec h_1(t) \approx g_1(t) + h_1(t) \approx \chi_1(t). \tag{106}$$

Now, we estimate the functions $g(t)$ and $h(t)$. Because $\theta_0 < 1$, we get

$$\frac{1}{g(t)} := \left\| \chi_{(t,\infty)}(u) \frac{u}{g_0(u)+h_0(u)} \right\|_{\Phi_1} \approx \left\| u^{\theta_0-1-1/r_1} \frac{b_1(u)}{\chi_0(u)} \right\|_{r_1,(t,\infty)} \approx t^{\theta_0-1} \frac{b_1(t)}{\chi_0(t)} \prec t^{\theta_0-1} \frac{\chi_1(t)}{\chi_0(t)} = \frac{1}{\rho(t)},$$

$$\begin{aligned} h(t) &:= \left\| \chi_{(0,t)}(u) \frac{u}{g_1(u)+h_1(u)} \right\|_{\Phi_0} \\ &\approx \left\| s^{-1/r_0} b_0(s) \left\| u^{1-\theta_0-1/q_0} \frac{a_0(u)}{\chi_1(t)} \right\|_{q_0,(0,\min(t,s))} \right\|_{r_0,(0,\infty)} \\ &\approx \left\| s^{1-\theta_0-1/r_0} b_0(s) \frac{a_0(s)}{\chi_1(s)} \right\|_{r_0,(0,t)} + t^{1-\theta_0} \frac{a_0(t)}{\chi_1(t)} \left\| s^{-1/r_0} b_0(s) \right\|_{r_0,(t,\infty)} \\ &\approx t^{1-\theta_0} b_0(t) \frac{a_0(t)}{\chi_1(t)} + t^{1-\theta_0} \frac{a_0(t)}{\chi_1(t)} \left\| s^{-1/r_0} b_0(s) \right\|_{r_0,(t,\infty)} \\ &\approx t^{1-\theta_0} \frac{a_0(t)}{\chi_1(t)} \left\| s^{-1/r_0} b_0(s) \right\|_{r_0,(t,\infty)} = \rho(t). \end{aligned} \tag{107}$$

Thus,

$$h(t) \approx \rho(t) \approx g(t). \tag{108}$$

Using [15] (Theorem 4, Case 4), we arrive at (94).

Proof of assertion (iii). Here, we use [15] (Theorem 1). Because $g_0(t) + h_0(t) \approx \rho(t)$, we get

$$\begin{aligned} K\left(\rho(t), f; \bar{A}^{\mathscr{L}}_{\theta_0,r_0,b_0,q_0,a_0}, A_1\right) &\approx \left\| s^{-1/r_0} b_0(s) \left\| u^{-\theta_0-1/q_0} a_0(u) K(u,f) \right\|_{q_0,(0,s)} \right\|_{r_0,(0,t)} \\ &+ \left\| s^{-1/r_0} b_0(s) \right\|_{r_0,(t,\infty)} \left\| u^{-\theta_0-1/q_0} a_0(u) K(u,f) \right\|_{q_0,(0,t)} \\ &+ t^{-\theta_0} \chi_0(t) K(t,f). \end{aligned} \tag{109}$$

By Lemma 9, we know that

$$\begin{aligned} t^{-\theta_0} \chi_0(t) K(t,f) &= \left\| s^{-1/r_0} b_0(s) \right\|_{r_0,(t,\infty)} t^{-\theta_0} a_0(t) K(t,f) \\ &\prec \left\| s^{-1/r_0} b_0(s) \right\|_{r_0,(t,\infty)} \left\| u^{-\theta_0-1/q_0} a_0(u) K(u,f) \right\|_{q_0,(0,t)}. \end{aligned} \tag{110}$$

So, we arrive at (95). ☐

Theorem 23 (cf. [7], Theorem 4.8). *Let $0 < \theta_0 < \theta_1 < 1$, $0 < r$, $q_0, r_0, r_1 \le \infty$, $a, a_0, b_0, b_1 \in SV$, $\| s^{-1/r_0} b_0(s) \|_{r_0,(1,\infty)} < \infty$, and $\| s^{-1/r} a(s) \|_{r,(1,\infty)} < \infty$. Put $\chi_0(t) = a_0(t) \| s^{-1/r_0} b_0(s) \|_{r_0,(t,\infty)}$ and $\rho(t) = t^{\theta_1-\theta_0}(\chi_0(t)/b_1(t))$. Then,*

$$\left(\bar{A}^{\mathscr{L}}_{\theta_0,r_0,b_0,q_0,a_0}, \bar{A}_{\theta_1,r_1,b_1} \right)_{0,r,a} = \bar{A}^{\mathscr{L}}_{\theta_0,r,a^{\#},q_0,a_0} \bigcap \bar{A}^{\mathscr{L},\mathscr{L}}_{\theta_0,r,a\circ\rho,r_0,b_0,q_0,a_0}, \tag{111}$$

where $a^{\#}(t) = \| s^{-1/r_0} b_0(s) \|_{r_0,(t,\infty)} a(\rho(t))$.

Proof. Denote $X_0 = \bar{A}^{\mathscr{L}}_{\theta_0,r_0,b_0,q_0,a_0}$, $X_1 = \bar{A}_{\theta_1,r_1,b_1}$, $\bar{K}(t,f) = K(t, f; \bar{X})$, and $Z = \bar{X}_{0,r,a}$. Using the change of variables and

Theorem 22 (i), we can write

$$\|f\|_Z = \|u^{-1/r}a(u)\bar{K}(u,f)\|_{r,(0,\infty)} \approx \|t^{-1/r}a(\rho(t))\bar{K}(\rho(t),f)\|_{r,(0,\infty)} \approx I_1 + I_2 + I_3, \tag{112}$$

where

$$I_1 := \left\| t^{-1/r}a(\rho(t)) \left\| s^{-1/r_0}b_0(s) \left\| u^{-\theta_0-1/q_0}a_0(u)K(u,f)\right\|_{q_0,(0,s)} \right\|_{r_0,(0,t)} \right\|_{r,(0,\infty)} = \|f\|_{\mathscr{L},\mathscr{L};\theta_0,r,a\circ\rho,r_0,b_0,q_0,a_0},$$

$$I_2 := \left\| t^{-1/r}a(\rho(t))\|s^{-1/r_0}b_0(s)\|_{r_0,(t,\infty)} \left\| u^{-\theta_0-1/q_0}a_0(u)K(u,f)\right\|_{q_0,(0,t)} \right\|_{r,(0,\infty)} = \|f\|_{\mathscr{L};\theta_0,r,a^{\#},q_0,a_0},$$

$$I_3 := \left\| t^{\theta_1-\theta_0-1/r}a(\rho(t))\frac{\chi_0(t)}{b_1(t)} \left\| u^{-\theta_1-1/r_1}b_1(u)K(u,f)\right\|_{r_1,(t,\infty)} \right\|_{r,(0,\infty)}. \tag{113}$$

It is enough to show that $I_3 \prec I_2$. By Lemmas 12 and 9, we get

$$I_3 \approx \left\| t^{-\theta_0-1/r}a(\rho(t))a_0(t)\|s^{-1/r_0}b_0(s)\|_{r_0,(t,\infty)}K(t,f)\right\|_{r,(0,\infty)} \prec \left\| t^{-1/r}a(\rho(t))\|s^{-1/r_0}b_0(s)\|_{r_0,(t,\infty)} \left\| u^{-\theta_0-1/q_0}a_0(u)K(u,f)\right\|_{q_0,(0,t)} \right\|_{r,(0,\infty)} = I_2. \tag{114}$$

This completes the proof. □

Theorem 24 (cf. [7], Theorem 4.9). *Let $0<\theta_0<1$, $0<r,q_0,r_0,r_1\le\infty$, $a,a_0,b_0,b_1\in SV$, $\|s^{-1/r_0}b_0(s)\|_{r_0,(1,\infty)}<\infty$, and $\|s^{-1/r_1}b_1(s)\|_{r_1,(0,1)}<\infty$. Put $\chi_0(t)=a_0(t)\|s^{-1/r_0}b_0(s)\|_{r_0,(t,\infty)}$ and $\rho(t)=t^{1-\theta_0}(\chi_0(t)/\|s^{-1/r_1}b_1(s)\|_{r_1,(0,t)})$.*

(i) If $\|s^{-1/r}a(s)\|_{r,(1,\infty)}<\infty$, then

$$\left(\bar{A}^{\mathscr{L}}_{\theta_0,r_0,b_0,q_0,a_0}, \bar{A}_{1,r_1,b_1}\right)_{0,r,a} = \bar{A}^{\mathscr{L}}_{\theta_0,r,a^{\#},q_0,a_0} \bigcap \bar{A}^{\mathscr{L},\mathscr{L}}_{\theta_0,r,a\circ\rho,r_0,b_0,q_0,a_0}, \tag{115}$$

where $a^{\#}(t)=\|s^{-1/r_0}b_0(s)\|_{r_0,(t,\infty)}a(\rho(t))$.

(ii) If $\|s^{-1/r}a(s)\|_{r,(0,1)}<\infty$, then

$$\left(\bar{A}^{\mathscr{L}}_{\theta_0,r_0,b_0,q_0,a_0}, \bar{A}_{1,r_1,b_1}\right)_{1,r,a} = \bar{A}_{1,r,a^{\#}} \bigcap \bar{A}^{\mathscr{R}}_{1,r,a\circ\rho,r_1,b_1}, \tag{116}$$

where $a^{\#}(t)=\|s^{-1/r_1}b_1(s)\|_{r_1,(0,t)}a(\rho(t))$.

Proof. Denote $X_0=\bar{A}^{\mathscr{L}}_{\theta_0,r_0,b_0,q_0,a_0}$, $X_1=\bar{A}_{1,r_1,b_1}$, and $\bar{K}(t,f)=K(t,f;\bar{X})$. First, we prove (i). Let $Z=\bar{X}_{0,r,a}$. Using the change of variables and Theorem 22 (ii), we can write

$$\|f\|_Z = \|u^{-1/r}a(u)\bar{K}(u,f)\|_{r,(0,\infty)} \approx \|t^{-1/r}a(\rho(t))\bar{K}(\rho(t),f)\|_{r,(0,\infty)} \approx I_1 + I_2 + I_3, \tag{117}$$

where

$$I_1 := \left\| t^{-1/r}a(\rho(t)) \left\| s^{-1/r_0}b_0(s) \left\| u^{-\theta_0-1/q_0}a_0(u)K(u,f)\right\|_{q_0,(0,s)} \right\|_{r_0,(0,t)} \right\|_{r,(0,\infty)} = \|f\|_{\mathscr{L},\mathscr{L};\theta_0,r,a\circ\rho,r_0,b_0,q_0,a_0},$$

$$I_2 := \left\| t^{-1/r}a(\rho(t))\|s^{-1/r_0}b_0(s)\|_{r_0,(t,\infty)} \left\| u^{-\theta_0-1/q_0}a_0(u)K(u,f)\right\|_{q_0,(0,t)} \right\|_{r,(0,\infty)} = \|f\|_{\mathscr{L};\theta_0,r,a^{\#},q_0,a_0},$$

$$I_3 := \left\| t^{1-\theta_0-1/r}a(\rho(t))\frac{\chi_0(t)}{\|s^{-1/r_1}b_1(s)\|_{r_1,(0,t)}} \|u^{-1-1/r_1}b_1(u)K(u,f)\|_{r_1,(t,\infty)} \right\|_{r,(0,\infty)}. \tag{118}$$

The estimate $I_3 \prec I_2$ can be proved by repeating the corresponding part of the proof of Theorem 23.

Now, we prove (ii). Let $Z=\bar{X}_{1,r,a}$. Using the change of variables and Theorem 22 (ii), we can write

$$\|f\|_Z = \|u^{-1-1/r}a(u)\bar{K}(u,f)\|_{r,(0,\infty)} \approx \|\rho(t)^{-1}t^{-1/r}a(\rho(t))\bar{K}(\rho(t),f)\|_{r,(0,\infty)} \approx I_1 + I_2 + I_3, \tag{119}$$

where

$$I_1 := \left\| t^{\theta_0-1-1/r}\frac{a^{\#}(t)}{\chi_0(t)} \left\| s^{-1/r_0}b_0(s) \left\| u^{-\theta_0-1/q_0}a_0(u)K(u,f)\right\|_{q_0,(0,s)} \right\|_{r_0,(0,t)} \right\|_{r,(0,\infty)},$$

$$I_2 := \left\| t^{\theta_0-1-1/r}\frac{a^{\#}(t)}{a_0(t)} \left\| u^{-\theta_0-1/q_0}a_0(u)K(u,f)\right\|_{q_0,(0,t)} \right\|_{r,(0,\infty)},$$

$$I_3 := \left\| t^{-1/r}a(\rho(t))\|u^{-1-1/r_1}b_1(u)K(u,f)\|_{r_1,(t,\infty)} \right\|_{r,(0,\infty)} = \|f\|_{\mathscr{R};1,r,a\circ\rho,r_1,b_1}. \tag{120}$$

Using Lemma 15 (iv), we get $I_2 \approx \|f\|_{1,r,a^{\#}}$. Hence, it is enough to show that $I_1 \prec \|f\|_{1,r,a^{\#}}$. By Lemma 13, we get

$$I_1 \approx \left\| t^{-1-1/r}\frac{a^{\#}(t)}{\|s^{-1/r_0}b_0(s)\|_{r_0,(t,\infty)}} b_0(s)K(t,f)\right\|_{r,(0,\infty)} \prec \|t^{-1-1/r}a^{\#}(t)K(t,f)\|_{r,(0,\infty)} = \|f\|_{1,r,a^{\#}}. \tag{121}$$

This completes the proof. □

Theorem 25 (cf. [7], Theorem 4.10). *Let $0<\theta_0<1$, $0<r,q_0,r_0\le\infty$, $a,a_0,b_0\in SV$, $\|s^{-1/r_0}b_0(s)\|_{r_0,(1,\infty)}<\infty$, and $\|s^{-1/r}a(s)\|_{r,(1,\infty)}<\infty$. Put $\chi_0(t)=a_0(t)\|s^{-1/r_0}b_0(s)\|_{r_0,(t,\infty)}$*

and $\rho(t)=t^{1-\theta_0}\chi_0(t)$. *Then,*

$$\left(\bar{A}^{\mathscr{L}}_{\theta_0,r_0,b_0,q_0,a_0},A_1\right)_{0,r,a}=\bar{A}^{\mathscr{L}}_{\theta_0,r,a^{\#},q_0,a_0}\bigcap\bar{A}^{\mathscr{L},\mathscr{L}}_{\theta_0,r,a\circ\rho,r_0,b_0,q_0,a_0},\tag{122}$$

where $a^{\#}(t)=\|s^{-1/r_0}b_0(s)\|_{r_0,(t,\infty)}a(\rho(t))$.

Proof. Denote $X_0=\bar{A}^{\mathscr{L}}_{\theta_0,r_0,b_0,q_0,a_0}$, $X_1=A_1$, $\bar{K}(t,f)=K(t,f;\bar{X})$, and $Z=\bar{X}_{0,r,a}$. Using the change of variables and Theorem 22 (iii), we can write

$$\|f\|_Z=\left\|u^{-1/r}a(u)\bar{K}(u,f)\right\|_{r,(0,\infty)}\approx\left\|t^{-1/r}a(\rho(t))\bar{K}(\rho(t),f)\right\|_{r,(0,\infty)}\approx I_1+I_2,\tag{123}$$

where

$$\begin{aligned}I_1&:=\left\|t^{-1/r}a(\rho(t))\left\|s^{-1/r_0}b_0(s)\left\|u^{-\theta_0-1/q_0}a_0(u)K(u,f)\right\|_{q_0,(0,s)}\right\|_{r_0,(0,t)}\right\|_{r,(0,\infty)}\\&=\|f\|_{\mathscr{L},\mathscr{L};\theta_0,r,a\circ\rho,r_0,b_0,q_0,a_0},\\ I_2&:=\left\|t^{-1/r}a(\rho(t))\|s^{-1/r_0}b_0(s)\|_{r_0,(t,\infty)}\left\|u^{-\theta_0-1/q_0}a_0(u)K(u,f)\right\|_{q_0,(0,t)}\right\|_{r,(0,\infty)}\\&=\|f\|_{\mathscr{L};\theta_0,r,a^{\#},q_0,a_0}.\end{aligned}\tag{124}$$

This completes the proof. ☐

6. Limiting Interpolation between the $\mathscr{R}$ Spaces and the Standard Interpolation Spaces

The theorems in this section amend theorems from [5] (Section 5). The proofs are based on the corresponding reiteration theorems from the previous sections and on the formulae (14)–(16). They are left to the reader. See, as an example, the proof of Theorem 34.

Theorem 26 (cf. [6], Theorem 4.4). *Let* $0<\theta_0<\theta_1<1$, $0<r,q_0,r_0,r_1\le\infty$, $a,b_0,a_1,b_1\in SV$, $\|s^{-1/r_0}b_0(s)\|_{r_0,(0,1)}<\infty$, *and* $\|s^{-1/r}a(s)\|_{r,(1,\infty)}<\infty$. *Put* $\chi_0(t)=a_0(t)\|s^{-1/r_0}b_0(s)\|_{r_0,(0,t)}$ *and* $\rho(t)=t^{\theta_1-\theta_0}(\chi_0(t)/b_1(t))$. *Then,*

$$\left(\bar{A}^{\mathscr{R}}_{\theta_0,r_0,b_0,q_0,a_0},\bar{A}_{\theta_1,r_1,b_1}\right)_{0,r,a}=\bar{A}^{\mathscr{R},\mathscr{L}}_{\theta_0,r,a\circ\rho,r_0,b_0,q_0,a_0}.\tag{125}$$

Theorem 27 (cf. [6], Theorem 4.5). *Let* $0<\theta_0<1$, $0<r,q_0,r_0,r_1\le\infty$, $a,a_0,b_0,b_1\in SV$, $\|s^{-1/r_0}b_0(s)\|_{r_0,(0,1)}<\infty$, *and* $\|s^{-1/r_1}b_1(s)\|_{r_1,(0,1)}<\infty$. *Put* $\chi_0(t)=a_0(t)\|s^{-1/r_0}b_0(s)\|_{r_0,(0,t)}$ *and* $\rho(t)=t^{1-\theta_0}(\chi_0(t)/\|s^{-1/r_1}b_1(s)\|_{r_1,(0,t)})$.

(i) If $\|s^{-1/r}a(s)\|_{r_0,(1,\infty)}<\infty$, *then*

$$\left(\bar{A}^{\mathscr{R}}_{\theta_0,r_0,b_0,q_0,a_0},\bar{A}_{1,r_1,b_1}\right)_{0,r,a}=\bar{A}^{\mathscr{R},\mathscr{L}}_{\theta_0,r,a\circ\rho,r_0,b_0,q_0,a_0}.\tag{126}$$

(ii) If $\|s^{-1/r}a(s)\|_{r_0,(0,1)}<\infty$, *then*

$$\left(\bar{A}^{\mathscr{R}}_{\theta_0,r_0,b_0,q_0,a_0},\bar{A}_{1,r_1,b_1}\right)_{1,r,a}=\bar{A}_{1,r,a^{\#}}\bigcap\bar{A}^{\mathscr{R}}_{1,r,a\circ\rho,r_1,b_1},\tag{127}$$

where $a^{\#}(t)=\|s^{-1/r_1}b_1(s)\|_{r_1,(0,t)}a(\rho(t))$.

Theorem 28 (cf. [6], Theorem 4.6). *Let* $0<\theta_0<1$, $0<r,q_0,r_0\le\infty$, $a,a_0,b_0\in SV$, $\|s^{-1/r_0}b_0(s)\|_{r_0,(0,1)}<\infty$, *and* $\|s^{-1/r}a(s)\|_{r,(1,\infty)}<\infty$. *Put* $\chi_0(t)=a_0(t)\|s^{-1/r_0}b_0(s)\|_{r_0,(0,t)}$ *and* $\rho(t)=t^{1-\theta_0}\chi_0(t)$. *Then,*

$$\left(\bar{A}^{\mathscr{R}}_{\theta_0,r_0,b_0,q_0,a_0},A_1\right)_{0,r,a}=\bar{A}^{\mathscr{R},\mathscr{L}}_{\theta_0,r,a\circ\rho,r_0,b_0,q_0,a_0}.\tag{128}$$

Theorem 29 (cf. [7], Theorem 4.5). *Let* $0<\theta_0<\theta_1<1$, $0<r,r_0,q_1,r_1\le\infty$, $a,b_0,a_1,b_1\in SV$, $\|s^{-1/r_1}b_1(s)\|_{r_1,(0,1)}<\infty$, *and* $\|s^{-1/r}a(s)\|_{r,(0,1)}<\infty$. *Put* $\chi_1(t)=a_1(t)\|s^{-1/r_1}b_1(s)\|_{r_1,(0,t)}$ *and* $\rho(t)=t^{\theta_1-\theta_0}(b_0(t)/\chi_1(t))$. *Then,*

$$\left(\bar{A}_{\theta_0,r_0,b_0},\bar{A}^{\mathscr{R}}_{\theta_1,r_1,b_1,q_1,a_1}\right)_{1,r,a}=\bar{A}^{\mathscr{R}}_{\theta_1,r,a^{\#},q_1,a_1}\bigcap\bar{A}^{\mathscr{R},\mathscr{R}}_{\theta_1,r,a\circ\rho,r_1,b_1,q_1,a_1},\tag{129}$$

where $a^{\#}(t)=\|s^{-1/r_1}b_1(s)\|_{r_1,(0,t)}a(\rho(t))$.

Theorem 30 (cf. [7], Theorem 4.6). *Let* $0<\theta_1<1$, $0<r,r_0,q_1,r_1\le\infty$, $a,b_0,a_1,b_1\in SV$, $\|s^{-1/r_0}b_0(s)\|_{r_0,(1,\infty)}<\infty$, *and* $\|s^{-1/r_1}b_1(s)\|_{r_1,(0,1)}<\infty$. *Put* $\chi_1(t)=a_1(t)\|s^{-1/r_1}b_1(s)\|_{r_1,(0,t)}$ *and* $\rho(t)=t^{\theta_1}(\|s^{-1/r_0}b_0(s)\|_{r_0,(t,\infty)}/\chi_1(t))$.

(i) If $\|s^{-1/r}a(s)\|_{r,(1,\infty)}<\infty$, *then*

$$\left(\bar{A}_{0,r_0,b_0},\bar{A}^{\mathscr{R}}_{\theta_1,r_1,b_1,q_1,a_1}\right)_{0,r,a}=\bar{A}_{0,r,a^{\#}}\bigcap\bar{A}^{\mathscr{L}}_{0,r,a\circ\rho,r_1,b_1},\tag{130}$$

where $a^{\#}(t)=\|s^{-1/r_0}b_0(s)\|_{r_0,(t,\infty)}a(\rho(t))$.

(ii) If $\|s^{-1/r}a(s)\|_{r,(0,1)}<\infty$, *then*

$$\left(\bar{A}_{0,r_0,b_0},\bar{A}^{\mathscr{R}}_{\theta_1,r_1,b_1,q_1,a_1}\right)_{1,r,a}=\bar{A}^{\mathscr{R}}_{\theta_1,r,a^{\#},q_1,a_1}\bigcap\bar{A}^{\mathscr{R},\mathscr{R}}_{\theta_1,r,a\circ\rho,r_1,b_1,q_1,a_1},\tag{131}$$

where $a^{\#}(t)=\|s^{-1/r_1}b_1(s)\|_{r_1,(0,t)}a(\rho(t))$.

Theorem 31 (cf. [7], Theorem 4.7). *Let* $0<\theta_1<1$, $0<r,q_1,r_1\le\infty$, $a,a_1,b_1\in SV$, $\|s^{-1/r_1}b_1(s)\|_{r_1,(0,1)}<\infty$, *and* $\|s^{-1/r}a(s)\|_{r,(0,1)}<\infty$. *Put* $\chi_1(t)=a_1(t)\|s^{-1/r_1}b_1(s)\|_{r_1,(0,t)}$ *and*

$\rho(t) = t^{\theta_1}(1/\chi_1(t))$. *Then,*

$$\left(A_0, \bar{A}^{\mathcal{R}}_{\theta_1,r_1,b_1,q_1,a_1}\right)_{1,r,a} = \bar{A}^{\mathcal{R}}_{\theta_1,r,a^{\#},q_1,a_1} \bigcap \bar{A}^{\mathcal{R},\mathcal{R}}_{\theta_1,r,a\circ\rho,r_1,b_1,q_1,a_1}, \tag{132}$$

where $a^{\#}(t) = \|s^{-1/r_1}b_1(s)\|_{r_1,(0,t)}a(\rho(t))$.

7. Limiting Reiteration Formulae for Couples Formed Only by $\mathcal{L}$ and $\mathcal{R}$ Spaces

The reiteration theorems in this section amend theorems from [5] (Section 6). Here, we also need the relevant Holmstedt-type formulae.

Theorem 32 (cf. [8], Theorem 3.2). *Let* $0 < \theta_0 < \theta_1 < 1$, $0 < q_0, r_0, q_1, r_1 \le \infty$, $a_0, b_0, a_1, b_1 \in SV$, $\|s^{-1/r_0}b_0(s)\|_{r_0,(1,\infty)} < \infty$, *and* $\|s^{-1/r_1}b_1(s)\|_{r_1,(1,\infty)} < \infty$. *Put* $\chi_0(t) = a_0(t)\|s^{-1/r_0}b_0(s)\|_{r_0,(t,\infty)}$, $\chi_1(t) = a_1(t)\|s^{-1/r_1}b_1(s)\|_{r_1,(t,\infty)}$, *and* $\rho(t) = t^{\theta_1-\theta_0}(\chi_0(t)/\chi_1(t))$. *Then, for all* $f \in \bar{A}^{\mathcal{L}}_{\theta_0,r_0,b_0,q_0,a_0} + \bar{A}^{\mathcal{L}}_{\theta_1,r_1,b_1,q_1,a_1}$ *and* $t > 0$,

$$\begin{aligned} &K\left(\rho(t), f; \bar{A}^{\mathcal{L}}_{\theta_0,r_0,b_0,q_0,a_0}, \bar{A}^{\mathcal{L}}_{\theta_1,r_1,b_1,q_1,a_1}\right) \\ &\quad\approx \left\|s^{-1/r_0}b_0(s)\left\|u^{-\theta_0-1/q_0}a_0(u)K(u,f)\right\|_{q_0,(0,s)}\right\|_{r_0,(0,t)} \\ &\qquad + \|s^{-1/r_0}b_0(s)\|_{r_0,(t,\infty)}\left\|u^{-\theta_0-1/q_0}a_0(u)K(u,f)\right\|_{q_0,(0,t)} \\ &\qquad + \rho(t)\left\|s^{-1/r_1}b_1(s)\left\|u^{-\theta_1-1/q_1}a_1(u)K(u,f)\right\|_{q_1,(t,s)}\right\|_{r_1,(t,\infty)}. \end{aligned} \tag{133}$$

Proof. Recall that we adopt the notation from the paper [15]. Let Φ_0 be the function space corresponding to $\bar{A}^{\mathcal{L}}_{\theta_0,r_0,b_0,q_0,a_0}$:

$$\|F(*)\|_{\Phi_0} = \left\|s^{-1/r_0}b_0(s)\left\|u^{-\theta_0-1/q_0}a_0(u)F(u)\right\|_{q_0,(0,s)}\right\|_{r_0,(0,\infty)}. \tag{134}$$

From the proof of Theorem 22, we know that

$$\begin{aligned} I(t,f) &\approx \left\|s^{-1/r_0}b_0(s)\left\|u^{-\theta_0-1/q_0}a_0(u)K(u,f)\right\|_{q_0,(0,s)}\right\|_{r_0,(0,t)} \\ &\quad + \|s^{-1/r_0}b_0(s)\|_{r_0,(t,\infty)}\left\|u^{-\theta_0-1/q_0}a_0(u)K(u,f)\right\|_{q_0,(0,t)}, \end{aligned}$$

$$g_0(t) \prec h_0(t) \approx g_0(t) + h_0(t) \approx t^{1-\theta_0}\chi_0(t). \tag{135}$$

Let Φ_1 be the function space corresponding to $\bar{A}^{\mathcal{L}}_{\theta_1,r_1,b_1,q_1,a_1}$:

$$\|F(*)\|_{\Phi_1} = \left\|s^{-1/r_1}b_1(s)\left\|u^{-\theta_1-1/q_1}a_1(u)F(u)\right\|_{q_1,(0,s)}\right\|_{r_1,(0,\infty)}. \tag{136}$$

From the proof of Theorem 18, we know that

$$\begin{aligned} J(t,f) &= \left\|s^{-1/r_1}b_1(s)\left\|u^{-\theta_1-1/q_1}a_1(u)K(u,f)\right\|_{q_1,(t,s)}\right\|_{r_1,(t,\infty)}, \\ &g_1(t) \prec h_1(t) \approx g_1(t) + h_1(t) \approx t^{1-\theta_1}\chi_1(t). \end{aligned} \tag{137}$$

Now, we estimate the functions $g(t)$ and $h(t)$. Because $\theta_0 - \theta_1 < 0$, we get

$$\begin{aligned} \frac{1}{g(t)} &:= \left\|\chi_{(t,\infty)}(u)\frac{u}{g_0(u)+h_0(u)}\right\|_{\Phi_1} \\ &\approx \left\|s^{-1/r_1}b_1(s)\left\|u^{\theta_0-\theta_1-1/q_1}\frac{a_1(u)}{\chi_0(u)}\right\|_{q_1,(t,s)}\right\|_{r_1,(t,\infty)} \\ &\le \|s^{-1/r_1}b_1(s)\|_{r_1,(t,\infty)}\left\|u^{\theta_0-\theta_1-1/q_1}\frac{a_1(u)}{\chi_0(u)}\right\|_{q_1,(t,\infty)} \\ &\approx \|s^{-1/r_1}b_1(s)\|_{r_1,(t,\infty)}t^{\theta_0-\theta_1}\frac{a_1(t)}{\chi_0(t)} = \frac{1}{\rho(t)}, \\ h(t) &:= \left\|\chi_{(0,t)}(u)\frac{u}{g_1(u)+h_1(u)}\right\|_{\Phi_0} \\ &\approx \left\|s^{-1/r_0}b_0(s)\left\|u^{\theta_1-\theta_0-1/q_0}\frac{a_0(u)}{\chi_1(u)}\right\|_{q_0,(0,\min(t,s))}\right\|_{r_0,(0,\infty)} \\ &\approx \left\|s^{\theta_1-\theta_0-1/r_0}b_0(s)\frac{a_0(s)}{\chi_1(s)}\right\|_{r_0,(0,t)} \\ &\quad + \|s^{-1/r_0}b_0(s)\|_{r_0,(t,\infty)}\left\|u^{\theta_1-\theta_0-1/q_0}\frac{a_0(u)}{\chi_1(u)}\right\|_{q_0,(0,t)} \\ &\approx t^{\theta_1-\theta_0}b_0(t)\frac{a_0(t)}{\chi_1(t)} + \|s^{-1/r_0}b_0(s)\|_{r_0,(t,\infty)}t^{\theta_1-\theta_0}\frac{a_0(t)}{\chi_1(t)} \\ &\approx \|s^{-1/r_0}b_0(s)\|_{r_0,(t,\infty)}t^{\theta_1-\theta_0}\frac{a_0(t)}{\chi_1(t)} = \rho(t). \end{aligned} \tag{138}$$

Thus,

$$h(t) \approx \rho(t) \prec g(t). \tag{139}$$

Therefore, using [15] (Theorem 4, Case 4), we arrive at (133). □

Theorem 33 (cf. [8], Theorem 4.4). *Let* $0 < \theta_0 < \theta_1 < 1$, $0 < r, q_0, r_0, q_1, r_1 \le \infty$, $a, a_0, b_0, a_1, b_1 \in SV$, $\|s^{-1/r_0}b_0(s)\|_{r_0,(1,\infty)} < \infty$, *and* $\|s^{-1/r_1}b_1(s)\|_{r_1,(1,\infty)} < \infty$. *Put* $\chi_0(t) = a_0(t)$

$\|s^{-1/r_0} b_0(s)\|_{r_0,(t,\infty)}$, $\chi_1(t) = a_1(t)\|s^{-1/r_1} b_1(s)\|_{r_1,(t,\infty)}$, *and* $\rho(t) = t^{\theta_1-\theta_0}(\chi_0(t)/\chi_1(t))$.

(i) *If* $\|s^{-1/r} a(s)\|_{r,(1,\infty)} < \infty$, *then*

$$\left(\bar{A}^{\mathscr{L}}_{\theta_0,r_0,b_0,q_0,a_0}, \bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}\right)_{0,r,a} = \bar{A}^{\mathscr{L}}_{\theta_0,r,a^{\#},q_0,a_0} \bigcap \bar{A}^{\mathscr{L},\mathscr{L}}_{\theta_0,r,a\circ\rho,r_0,b_0,q_0,a_0}, \tag{140}$$

where $a^{\#}(t) = \|s^{-1/r_0} b_0(s)\|_{r_0,(t,\infty)} a(\rho(t))$.

(ii) *If* $\|s^{-1/r} a(s)\|_{r,(0,1)} < \infty$, *then*

$$\left(\bar{A}^{\mathscr{L}}_{\theta_0,r_0,b_0,q_0,a_0}, \bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}\right)_{1,r,a} = \bar{A}^{\mathscr{L},\mathscr{R}}_{\theta_1,r,a\circ\rho,r_1,b_1,q_1,a_1}. \tag{141}$$

Proof. Denote $X_0 = \bar{A}^{\mathscr{L}}_{\theta_0,r_0,b_0,q_0,a_0}$, $X_1 = \bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}$, and $\bar{K}(t,f) = K(t,f;\bar{X})$. First, we prove (i). Let $Z = \bar{X}_{0,r,a}$. Using the change of variables and Theorem 32, we can write

$$\begin{aligned} \|f\|_Z &= \left\|u^{-1/r} a(u)\bar{K}(u,f)\right\|_{r,(0,\infty)} \\ &\approx \left\|t^{-1/r} a(\rho(t))\bar{K}(\rho(t),f)\right\|_{r,(0,\infty)} \approx I_1 + I_2 + I_3, \end{aligned} \tag{142}$$

where

$$\begin{aligned} I_1 &:= \left\|t^{-1/r} a(\rho(t)) \left\|s^{-1/r_0} b_0(s) \left\|u^{-\theta_0-1/q_0} a_0(u) K(u,f)\right\|_{q_0,(0,s)}\right\|_{r_0,(0,t)}\right\|_{r,(0,\infty)} \\ &= \|f\|_{\mathscr{L},\mathscr{L};\theta_0,r,a\circ\rho,r_0,b_0,q_0,a_0}, \end{aligned}$$

$$\begin{aligned} I_2 &:= \left\|t^{-1/r} a(\rho(t)) \|s^{-1/r_0} b_0(s)\|_{r_0,(t,\infty)} \left\|u^{-\theta_0-1/q_0} a_0(u) K(u,f)\right\|_{q_0,(0,t)}\right\|_{r,(0,\infty)} \\ &= \|f\|_{\mathscr{L};\theta_0,r,a^{\#},q_0,a_0}, \end{aligned}$$

$$I_3 := \left\|t^{\theta_1-\theta_0-1/r} a(\rho(t)) \frac{\chi_0(t)}{\chi_1(t)} \left\|s^{-1/r_1} b_1(s) \left\|u^{-\theta_1-1/q_1} a_1(u) K(u,f)\right\|_{q_1,(t,s)}\right\|_{r_1,(t,\infty)}\right\|_{r,(0,\infty)}. \tag{143}$$

It is enough to show that $I_3 \prec I_2$. Using Lemmas 12 and 9, we get

$$\begin{aligned} I_3 &\le \left\|t^{\theta_1-\theta_0-1/r} a(\rho(t)) \frac{\chi_0(t)}{\chi_1(t)} \|s^{-1/r_1} b_1(s)\|_{r_1,(t,\infty)} \left\|u^{-\theta_1-1/q_1} a_1(u) K(u,f)\right\|_{q_1,(t,\infty)}\right\|_{r,(0,\infty)} \\ &\approx \left\|t^{-\theta_0-1/r} a(\rho(t)) a_0(t) \|s^{-1/r_0} b_0(s)\|_{r_0,(t,\infty)} K(t,f)\right\|_{r,(0,\infty)} \\ &\prec \left\|t^{-1/r} a(\rho(t)) \|s^{-1/r_0} b_0(s)\|_{r_0,(t,\infty)} \left\|u^{-\theta_0-1/q_0} a_0(u) K(u,f)\right\|_{q_0,(0,t)}\right\|_{r,(0,\infty)} = I_2. \end{aligned} \tag{144}$$

Statement (i) is proved.

Now, we prove (ii). Let $Z = \bar{X}_{1,r,a}$. Using the change of variables and Theorem 32, we can write

$$\begin{aligned} \|f\|_Z &= \left\|u^{-1-1/r} a(u)\bar{K}(u,f)\right\|_{r,(0,\infty)} \\ &\approx \left\|\rho(t)^{-1} t^{-1/r} a(\rho(t))\bar{K}(\rho(t),f)\right\|_{r,(0,\infty)} \approx I_1 + I_2 + I_3, \end{aligned} \tag{145}$$

where

$$\begin{aligned} I_1 &:= \left\|t^{\theta_0-\theta_1-1/r} a(\rho(t)) \frac{\chi_1(t)}{\chi_0(t)} \left\|s^{-1/r_0} b_0(s) \left\|u^{-\theta_0-1/q_0} a_0(u) K(u,f)\right\|_{q_0,(0,s)}\right\|_{r_0,(0,t)}\right\|_{r,(0,\infty)}, \\ I_2 &:= \left\|t^{\theta_0-\theta_1-1/r} a(\rho(t)) \frac{\chi_1(t)}{a_0(t)} \left\|u^{-\theta_0-1/q_0} a_0(u) K(u,f)\right\|_{q_0,(0,t)}\right\|_{r,(0,\infty)}, \\ I_3 &:= \left\|t^{-1/r} a(\rho(t)) \left\|s^{-1/r_1} b_1(s) \left\|u^{-\theta_1-1/q_1} a_1(u) K(u,f)\right\|_{q_1,(t,s)}\right\|_{r_1,(t,\infty)}\right\|_{r,(0,\infty)} \\ &= \|f\|_{\mathscr{L},\mathscr{R};\theta_1,r,a\circ\rho,r_1,b_1,q_1,a_1}. \end{aligned} \tag{146}$$

Note that due to Lemma 10, we have

$$I_3 \succ \left\|t^{-\theta_1-1/r} a(\rho(t)) a_1(t) \|s^{-1/r_1} b_1(s)\|_{r_1,(t,\infty)} K(t,f)\right\|_{r,(0,\infty)} := I_4. \tag{147}$$

Hence, it is enough to show that $I_1 \prec I_4$ and $I_2 \prec I_4$. Consider I_1. By Lemma 13, we have

$$\begin{aligned} I_1 &\approx \left\|t^{-\theta_1-1/r} a(\rho(t)) \frac{\chi_1(t)}{\chi_0(t)} b_0(t) a_0(t) K(t,f)\right\|_{r,(0,\infty)} \\ &= \left\|t^{-\theta_1-1/r} a(\rho(t)) \frac{a_1(t)\|s^{-1/r_1} b_1(s)\|_{r_1,(t,\infty)}}{\|s^{-1/r_0} b_0(s)\|_{r_0,(t,\infty)}} b_0(t) K(t,f)\right\|_{r,(0,\infty)} \\ &\prec \left\|t^{-\theta_1-1/r} a(\rho(t)) a_1(t) \|s^{-1/r_1} b_1(s)\|_{r_1,(t,\infty)} K(t,f)\right\|_{r,(0,\infty)} = I_4. \end{aligned} \tag{148}$$

Consider I_2. By Lemma 12, we get

$$I_2 \approx \left\|t^{-\theta_1-1/r} a(\rho(t)) \chi_1(t) K(t,f)\right\|_{r,(0,\infty)} = I_4. \tag{149}$$

This completes the proof. □

The next assertion is a symmetric counterpart of Theorem 33.

Theorem 34 (cf. [8], Theorem 4.3). *Let* $0 < \theta_0 < \theta_1 < 1$, $0 < r, q_0, r_0, q_1, r_1 \le \infty$, $a, a_0, b_0, a_1, b_1 \in SV$, $\|s^{-1/r_0} b_0(s)\|_{r_0,(0,1)} < \infty$, *and* $\|s^{-1/r_1} b_1(s)\|_{r_1,(0,1)} < \infty$. *Put* $\chi_0(t) = a_0(t) \|s^{-1/r_0} b_0(s)\|_{r_0,(0,t)}$, $\chi_1(t) = a_1(t)\|s^{-1/r_1} b_1(s)\|_{r_1,(0,t)}$, *and* $\rho(t) = t^{\theta_1-\theta_0}(\chi_0(t)/\chi_1(t))$.

(i) *If* $\|s^{-1/r} a(s)\|_{r,(1,\infty)} < \infty$, *then*

$$\left(\bar{A}^{\mathcal{R}}_{\theta_0,r_0,b_0,q_0,a_0}, \bar{A}^{\mathcal{R}}_{\theta_1,r_1,b_1,q_1,a_1}\right)_{0,r,a} = \bar{A}^{\mathcal{R},\mathcal{L}}_{\theta_0,r,a\circ\rho,r_0,b_0,q_0,a_0}. \quad (150)$$

(ii) If $\|s^{-1/r}a(s)\|_{r,(0,1)} < \infty$, *then*

$$\left(\bar{A}^{\mathcal{R}}_{\theta_0,r_0,b_0,q_0,a_0}, \bar{A}^{\mathcal{R}}_{\theta_1,r_1,b_1,q_1,a_1}\right)_{1,r,a} = \bar{A}^{\mathcal{R}}_{\theta_1,r,a^{\#},q_1,a_1} \bigcap \bar{A}^{\mathcal{R},\mathcal{R}}_{\theta_1,r,a\circ\rho,r_1,b_1,q_1,a_1}, \quad (151)$$

where $a^{\#}(t) = \|s^{-1/r_1}b_1(s)\|_{r_1,(0,t)}a(\rho(t))$.

Proof. We prove the second formula. The first one can be proved similarly. Recall that $\bar{A} = (A_0, A_1)$, and we use the denotation $\tilde{b}(t) = b(t^{-1})$. Note that

$$\left\|s^{-1/r_i}\tilde{b}_i(s)\right\|_{r_i,(1,\infty)} = \left\|s^{-1/r_i}b_i(s)\right\|_{r_i,(0,1)} < \infty (i = 0, 1),$$
$$\left\|s^{-1/r}\tilde{a}(s)\right\|_{r,(1,\infty)} = \left\|s^{-1/r}a(s)\right\|_{r,(0,1)} < \infty. \quad (152)$$

Using Theorem 33 (i) and formulae (14) and (15), we get

$$\begin{aligned}\left(\bar{A}^{\mathcal{R}}_{\theta_0,r_0,b_0,q_0,a_0}, \bar{A}^{\mathcal{R}}_{\theta_1,r_1,b_1,q_1,a_1}\right)_{1,r,a} &= \left((A_1, A_0)^{\mathcal{L}}_{1-\theta_0,r_0,\tilde{b}_0,q_0,\tilde{a}_0}, (A_1, A_0)^{\mathcal{L}}_{1-\theta_1,r_1,\tilde{b}_1,q_1,\tilde{a}_1}\right)_{1,r,a} \\ &= \left((A_1, A_0)^{\mathcal{L}}_{1-\theta_1,r_1,\tilde{b}_1,q_1,\tilde{a}_1}, (A_1, A_0)^{\mathcal{L}}_{1-\theta_1,r_0,\tilde{b}_0,q_0,\tilde{a}_0}\right)_{0,r,\tilde{a}} \\ &= (A_1, A_0)^{\mathcal{L}}_{1-\theta_1,r,c,q_1,\tilde{a}_1} \bigcap (A_1, A_0)^{\mathcal{L},\mathcal{L}}_{1-\theta_1,r,\tilde{a}\circ\rho_1,r_0,\tilde{b}_0,q_0,\tilde{a}_0},\end{aligned} \quad (153)$$

where

$$\begin{aligned}\rho_1(t) &= t^{\theta_1-\theta_0}\frac{\tilde{a}_1(t)\left\|s^{-1/r_1}\tilde{b}_1(s)\right\|_{r_1,(t,\infty)}}{\tilde{a}_0(t)\left\|s^{-1/r_0}\tilde{b}_0(s)\right\|_{r_0,(t,\infty)}} \\ &= t^{\theta_1-\theta_0}\frac{a_1(t^{-1})\|s^{-1/r_1}b_1(s)\|_{r_1,(0,t^{-1})}}{a_0(t^{-1})\|s^{-1/r_0}b_0(s)\|_{r_0,(0,t^{-1})}} = \frac{1}{\rho(t^{-1})} = \frac{1}{\tilde{\rho}(t)},\end{aligned}$$

$$\begin{aligned}c(t) &= \left\|s^{-1/r_0}\tilde{b}_1(s)\right\|_{r_0,(t,\infty)}\tilde{a}(\rho_1(t)) = \|s^{-1/r_0}b_1(s)\|_{r_0,(0,t^{-1})}\tilde{a}\left(\frac{1}{\tilde{\rho}(t)}\right) \\ &= \|s^{-1/r_0}b_1(s)\|_{r_0,(0,t^{-1})}a(\tilde{\rho}(t)) = \|s^{-1/r_0}b_1(s)\|_{r_0,(0,t^{-1})}a\left(\rho(t^{-1})\right) \\ &= a^{\#}\left(t^{-1}\right) = \widetilde{a^{\#}}(t).\end{aligned} \quad (154)$$

Additionally, we have

$$(\tilde{a}\circ\rho_1)(t) = a\left(\frac{1}{\rho_1(t)}\right) = a(\tilde{\rho}(t)) = (a\circ\rho)\left(t^{-1}\right). \quad (155)$$

Hence, $\widetilde{\tilde{a}\circ\rho_1} = a\circ\rho$. Therefore,

$$(A_1, A_0)^{\mathcal{L}}_{1-\theta_1,r,c,q_1,\tilde{a}_1} = \bar{A}^{\mathcal{R}}_{\theta_1,r,a^{\#},q_1,a_1}, \quad (156)$$

and by (16),

$$(A_1, A_0)^{\mathcal{L},\mathcal{L}}_{1-\theta_1,r,\tilde{a}\circ\rho_1,r_0,\tilde{b}_0,q_0,\tilde{a}_0} = \bar{A}^{\mathcal{R},\mathcal{R}}_{\theta_1,r,\widetilde{\tilde{a}\circ\rho_1},r_0,b_0,q_0,a_0} = \bar{A}^{\mathcal{R},\mathcal{R}}_{\theta_1,r,a\circ\rho,r_0,b_0,q_0,a_0}. \quad (157)$$

This completes the proof. □

Theorem 35 (cf. [8], Theorem 3.4). *Let* $0 < \theta_0 < \theta_1 < 1$, $0 < q_0, r_0, q_1, r_1 \leq \infty$, $a_0, b_0, a_1, b_1 \in SV$, $\|s^{-1/r_0}b_0(s)\|_{r_0,(1,\infty)} < \infty$, *and* $\|s^{-1/r_1}b_1(s)\|_{r_1,(0,1)} < \infty$. *Put* $\chi_0(t) = a_0(t)\|s^{-1/r_0}b_0(s)\|_{r_0,(t,\infty)}$, $\chi_1(t) = a_1(t)\|s^{-1/r_1}b_1(s)\|_{r_1,(0,t)}$, *and* $\rho(t) = t^{\theta_1-\theta_0}(\chi_0(t)/\chi_1(t))$. *Then, for all* $f \in \bar{A}^{\mathcal{L}}_{\theta_0,r_0,b_0,q_0,a_0} + \bar{A}^{\mathcal{R}}_{\theta_1,r_1,b_1,q_1,a_1}$ *and* $t > 0$,

$$\begin{aligned}K&\left(\rho(t), f; \bar{A}^{\mathcal{L}}_{\theta_0,r_0,b_0,q_0,a_0}, \bar{A}^{\mathcal{R}}_{\theta_1,r_1,b_1,q_1,a_1}\right) \\ &\approx \left\|s^{-1/r_0}b_0(s)\left\|u^{-\theta_0-1/q_0}a_0(u)K(u,f)\right\|_{q_0,(0,s)}\right\|_{r_0,(0,t)} \\ &\quad + \left\|s^{-1/r_0}b_0(s)\right\|_{r_0,(t,\infty)}\left\|u^{-\theta_0-1/q_0}a_0(u)K(u,f)\right\|_{q_0,(0,t)} \\ &\quad + \rho(t)\left\|s^{-1/r_1}b_1(s)\right\|_{r_1,(0,t)}\left\|u^{-\theta_1-1/q_1}a_1(u)K(u,f)\right\|_{q_1,(t,\infty)} \\ &\quad + \rho(t)\left\|s^{-1/r_1}b_1(s)\left\|u^{-\theta_1-1/q_1}a_1(u)K(u,f)\right\|_{q_1,(s,\infty)}\right\|_{r_1,(t,\infty)}.\end{aligned} \quad (158)$$

Proof. As usual, we use the notation from the paper [15]. Let Φ_0 be the function space corresponding to $\bar{A}^{\mathcal{L}}_{\theta_0,r_0,b_0,q_0,a_0}$:

$$\|F(*)\|_{\Phi_0} = \left\|s^{-1/r_0}b_0(s)\left\|u^{-\theta_0-1/q_0}a_0(u)F(u)\right\|_{q_0,(0,s)}\right\|_{r_0,(0,\infty)}. \quad (159)$$

From the proof of Theorem 22, we know that

$$\begin{aligned}I(t,f) &\approx \left\|s^{-1/r_0}b_0(s)\left\|u^{-\theta_0-1/q_0}a_0(u)K(u,f)\right\|_{q_0,(0,s)}\right\|_{r_0,(0,t)} \\ &\quad + \left\|s^{-1/r_0}b_0(s)\right\|_{r_0,(t,\infty)}\left\|u^{-\theta_0-1/q_0}a_0(u)K(u,f)\right\|_{q_0,(0,t)},\end{aligned}$$

$$g_0(t) \prec h_0(t) \approx g_0(t) + h_0(t) \approx t^{1-\theta_0}\chi_0(t). \quad (160)$$

Let Φ_1 be the function space corresponding to $\bar{A}^{\mathcal{R}}_{\theta_1,r_1,b_1,q_1,a_1}$:

$$\|F(*)\|_{\Phi_1} = \left\|s^{-1/r_1}b_1(s)\left\|u^{-\theta_1-1/q_1}a_1(u)F(u)\right\|_{q_1,(s,\infty)}\right\|_{r_1,(0,\infty)}. \quad (161)$$

Consider the functions $J(t,f)$, $g_1(t)$, and $h_1(t)$.

$$\begin{aligned} J(t,f) &:= \left\|\chi_{(t,\infty)}(*)K(*,f)\right\|_{\Phi_1} \\ &= \left\|s^{-1/r_1}b_1(s)\left\|u^{-\theta_1-1/q_1}a_1(u)K(u,f)\right\|_{q_1,(\max(s,t),\infty)}\right\|_{r_1,(0,\infty)} \\ &\approx \left\|s^{-1/r_1}b_1(s)\right\|_{r_1,(0,t)}\left\|u^{-\theta_1-1/q_1}a_1(u)K(u,f)\right\|_{q_1,(t,\infty)} \\ &\quad + \left\|s^{-1/r_1}b_1(s)\left\|u^{-\theta_1-1/q_1}a_1(u)K(u,f)\right\|_{q_1,(s,\infty)}\right\|_{r_1,(t,\infty)}, \end{aligned}$$

$$\begin{aligned} g_1(t) &:= t\left\|\chi_{(t,\infty)}(*)\right\|_{\Phi_1} \\ &= t\left\|s^{-1/r_1}b_1(s)\left\|u^{-\theta_1-1/q_1}a_1(u)\right\|_{q_1,(\max(s,t),\infty)}\right\|_{r_1,(0,\infty)} \\ &\approx t\Big(\left\|s^{-1/r_1}b_1(s)\right\|_{r_1,(0,t)}\left\|u^{-\theta_1-1/q_1}a_1(u)\right\|_{q_1,(t,\infty)} \\ &\quad + \left\|s^{-1/r_1}b_1(s)\left\|u^{-\theta_1-1/q_1}a_1(u)\right\|_{q_1,(s,\infty)}\right\|_{r_1,(t,\infty)}\Big) \\ &\approx t\Big(\left\|s^{-1/r_1}b_1(s)\right\|_{r_1,(0,t)}t^{-\theta_1}a_1(t) + \left\|s^{-\theta_1-1/r_1}b_1(s)a_1(s)\right\|_{r_1,(t,\infty)}\Big) \\ &\approx t^{1-\theta_1}a_1(t)\Big(\left\|s^{-1/r_1}b_1(s)\right\|_{r_1,(0,t)} + b_1(t)\Big) \approx t^{1-\theta_1}a_1(t)\left\|s^{-1/r_1}b_1(s)\right\|_{r_1,(0,t)} \\ &= t^{1-\theta_1}\chi_1(t), \end{aligned}$$

$$\begin{aligned} h_1(t) &:= \left\|*\chi_{(0,t)}(*)\right\|_{\Phi_1} \\ &= \left\|s^{-1/r_1}b_1(s)\left\|u^{1-\theta_1-1/q_1}a_1(u)\right\|_{q_1,(s,t)}\right\|_{r_1,(0,t)} \\ &\le \left\|s^{-1/r_1}b_1(s)\right\|_{r_1,(0,t)}\left\|u^{1-\theta_1-1/q_1}a_1(u)\right\|_{q_1,(0,t)} \\ &\approx \left\|s^{-1/r_1}b_1(s)\right\|_{r_1,(0,t)}t^{1-\theta_1}a_1(t) = t^{1-\theta_1}\chi_1(t). \end{aligned} \tag{162}$$

Thus,

$$h_1(t) \prec g_1(t) \approx g_1(t)+h_1(t) \approx t^{1-\theta_1}\chi_1(t). \tag{163}$$

Now, we estimate the functions $g(t)$ and $h(t)$. Because $\theta_0-\theta_1<0$, we get

$$\begin{aligned} \frac{1}{g(t)} &:= \left\|\chi_{(t,\infty)}(u)\frac{u}{g_0(u)+h_0(u)}\right\|_{\Phi_1} \\ &\approx \left\|s^{-1/r_1}b_1(s)\left\|u^{\theta_0-\theta_1-1/q_1}\frac{a_1(u)}{\chi_0(u)}\right\|_{q_1,(\max(s,t),\infty)}\right\|_{r_1,(0,\infty)} \\ &\approx \left\|s^{-1/r_1}b_1(s)\right\|_{r_1,(0,t)}\left\|u^{\theta_0-\theta_1-1/q_1}\frac{a_1(u)}{\chi_0(u)}\right\|_{q_1,(t,\infty)} \\ &\quad + \left\|s^{-1/r_1}b_1(s)\left\|u^{\theta_0-\theta_1-1/q_1}\frac{a_1(u)}{\chi_0(u)}\right\|_{q_1,(s,\infty)}\right\|_{r_1,(t,\infty)} \\ &\approx \left\|s^{-1/r_1}b_1(s)\right\|_{r_1,(0,t)}t^{\theta_0-\theta_1}\frac{a_1(t)}{\chi_0(t)} + \left\|s^{\theta_0-\theta_1-1/r_1}b_1(s)\frac{a_1(s)}{\chi_0(s)}\right\|_{r_1,(t,\infty)} \\ &\approx t^{\theta_0-\theta_1}\frac{a_1(t)}{\chi_0(t)}\Big(\left\|s^{-1/r_1}b_1(s)\right\|_{r_1,(0,t)} + b_1(t)\Big) \approx t^{\theta_0-\theta_1}\frac{a_1(t)}{\chi_0(t)}\left\|s^{-1/r_1}b_1(s)\right\|_{r_1,(0,t)} \\ &= \frac{1}{\rho(t)}, \end{aligned}$$

$$\begin{aligned} h(t) &:= \left\|\chi_{(0,t)}(u)\frac{u}{g_1(u)+h_1(u)}\right\|_{\Phi_0} \\ &\approx \left\|s^{-1/r_0}b_0(s)\left\|u^{\theta_1-\theta_0-1/q_0}\frac{a_0(u)}{\chi_1(u)}\right\|_{q_0,(0,\min(s,t))}\right\|_{r_0,(0,\infty)} \\ &\approx \left\|s^{-1/r_0}b_0(s)\left\|u^{\theta_1-\theta_0-1/q_0}\frac{a_0(u)}{\chi_1(u)}\right\|_{q_0,(0,s)}\right\|_{r_0,(0,t)} \\ &\quad + \left\|s^{-1/r_0}b_0(s)\right\|_{r_0,(t,\infty)}\left\|u^{\theta_1-\theta_0-1/q_0}\frac{a_0(u)}{\chi_1(u)}\right\|_{q_0,(0,t)} \\ &\approx \left\|s^{\theta_1-\theta_0-1/r_0}b_0(s)\frac{a_0(s)}{\chi_1(s)}\right\|_{r_0,(0,t)} \\ &\quad + \left\|s^{-1/r_0}b_0(s)\right\|_{r_0,(t,\infty)}t^{\theta_1-\theta_0}\frac{a_0(t)}{\chi_1(t)} \\ &\approx t^{\theta_1-\theta_0}\frac{a_0(t)}{\chi_1(t)}\Big(b_0(t) + \left\|s^{-1/r_0}b_0(s)\right\|_{r_0,(t,\infty)}\Big) \\ &\approx t^{\theta_1-\theta_0}\frac{a_0(t)}{\chi_1(t)}\left\|s^{-1/r_0}b_0(s)\right\|_{r_0,(t,\infty)} = \rho(t). \end{aligned} \tag{164}$$

Thus, $h(t)\approx\rho(t)\approx g(t)$. Therefore, using [15] (Theorem 4, Case 4), we arrive at (158). ☐

Theorem 36 (cf. [8], Theorem 4.6). *Let $0<\theta_0<\theta_1<1$, $0<r, q_0, r_0, q_1, r_1\le\infty$, $a, a_0, b_0, a_1, b_1\in SV$, $\|s^{-1/r_0}b_0(s)\|_{r_0,(1,\infty)}<\infty$, and $\|s^{-1/r_1}b_1(s)\|_{r_1,(0,1)}<\infty$. Put $\chi_0(t)=a_0(t)\|s^{-1/r_0}b_0(s)\|_{r_0,(t,\infty)}$, $\chi_1(t)=a_1(t)\|s^{-1/r_1}b_1(s)\|_{r_1,(0,t)}$, and $\rho(t)=t^{\theta_1-\theta_0}(\chi_0(t)/\chi_1(t))$.*

(i) If $\|s^{-1/r}a(s)\|_{r,(1,\infty)}<\infty$, then

$$\left(\bar{A}^{\mathscr{L}}_{\theta_0,r_0,b_0,q_0,a_0}, \bar{A}^{\mathscr{R}}_{\theta_1,r_1,b_1,q_1,a_1}\right)_{0,r,a} = \bar{A}^{\mathscr{L}}_{\theta_0,r,a^{\#},q_0,a_0}\bigcap\bar{A}^{\mathscr{L},\mathscr{L}}_{\theta_0,r,a\circ\rho,r_0,b_0,q_0,a_0}, \tag{165}$$

where $a^{\#}(t)=\|s^{-1/r_0}b_0(s)\|_{r_0,(t,\infty)}a(\rho(t))$.

(ii) If $\|s^{-1/r}a(s)\|_{r,(0,1)}<\infty$, then

$$\left(\bar{A}^{\mathscr{L}}_{\theta_0,r_0,b_0,q_0,a_0}, \bar{A}^{\mathscr{R}}_{\theta_1,r_1,b_1,q_1,a_1}\right)_{1,r,a} = \bar{A}^{\mathscr{R}}_{\theta_1,r,a^{\#},q_1,a_1}\bigcap\bar{A}^{\mathscr{R},\mathscr{R}}_{\theta_1,r,a\circ\rho,r_1,b_1,q_1,a_1}, \tag{166}$$

where $a^{\#}(t)=\|s^{-1/r_1}b_1(s)\|_{r_1,(0,t)}a(\rho(t))$.

Proof. Denote $X_0=\bar{A}^{\mathscr{L}}_{\theta_0,r_0,b_0,q_0,a_0}$, $X_1=\bar{A}^{\mathscr{R}}_{\theta_1,r_1,b_1,q_1,a_1}$, and $\bar{K}(t,f)=K(t,f;\bar{X})$. We prove (i). Statement (ii) can be proved analogously. Let $Z=\bar{X}_{0,r,a}$. Using the change of variables and Theorem 35, we can write

$$\begin{aligned}\|f\|_Z &= \left\|u^{-1/r}a(u)\bar{K}(u,f)\right\|_{r,(0,\infty)}\\ &\approx \left\|t^{-1/r}a(\rho(t))\bar{K}(\rho(t),f)\right\|_{r,(0,\infty)} \approx I_1+I_2+I_3+I_4,\end{aligned} \tag{167}$$

where

$$\begin{aligned}I_1 &:= \left\|t^{-1/r}a(\rho(t))\left\|s^{-1/r_0}b_0(s)\left\|u^{-\theta_0-1/q_0}a_0(u)K(u,f)\right\|_{q_0,(0,s)}\right\|_{r_0,(0,t)}\right\|_{r,(0,\infty)}\\ &= \|f\|_{\mathscr{L},\mathscr{L};\theta_0,r,a\circ\rho,r_0,b_0,q_0,a_0},\\ I_2 &:= \left\|t^{-1/r}a(\rho(t))\left\|s^{-1/r_0}b_0(s)\right\|_{r_0,(t,\infty)}\left\|u^{-\theta_0-1/q_0}a_0(u)K(u,f)\right\|_{q_0,(0,t)}\right\|_{r,(0,\infty)}\\ &= \|f\|_{\mathscr{L};\theta_0,r,a^{\#},q_0,a_0},\\ I_3 &:= \left\|t^{\theta_1-\theta_0-1/r}a(\rho(t))\frac{\chi_0(t)}{a_1(t)}\left\|u^{-\theta_1-1/q_1}a_1(u)K(u,f)\right\|_{q_1,(t,\infty)}\right\|_{r,(0,\infty)},\\ I_4 &:= \left\|t^{\theta_1-\theta_0-1/r}a(\rho(t))\frac{\chi_0(t)}{\chi_1(t)}\left\|s^{-1/r_1}b_1(s)\left\|u^{-\theta_1-1/q_1}a_1(u)K(u,f)\right\|_{q_1,(s,\infty)}\right\|_{r_1,(t,\infty)}\right\|_{r,(0,\infty)}.\end{aligned} \tag{168}$$

It is enough to show that $I_3 \prec I_2$ and $I_4 \prec I_1$. By Lemmas 12 and 9, we get

$$\begin{aligned}I_3 &\approx \left\|t^{-\theta_0-1/r}a(\rho(t))a_0(t)\left\|s^{-1/r_0}b_0(s)\right\|_{r_0,(t,\infty)}K(t,f)\right\|_{r,(0,\infty)}\\ &\prec \left\|t^{-1/r}a(\rho(t))\left\|s^{-1/r_0}b_0(s)\right\|_{r_0,(t,\infty)}\left\|u^{-\theta_0-1/q_0}a_0(u)K(u,f)\right\|_{q_0,(0,t)}\right\|_{r,(0,\infty)}\\ &= I_2.\end{aligned} \tag{169}$$

Using Lemmas 13 and 10, we arrive at

$$\begin{aligned}I_4 &\approx \left\|t^{-\theta_0-1/r}a(\rho(t))\frac{\chi_0(t)}{\chi_1(t)}b_1(t)a_1(t)K(t,f)\right\|_{r,(0,\infty)}\\ &= \left\|t^{-\theta_0-1/r}a(\rho(t))\frac{\chi_0(t)b_1(t)}{\left\|s^{-1/r_1}b_1(s)\right\|_{r_1,(0,t)}}K(t,f)\right\|_{r,(0,\infty)}\\ &\prec \left\|t^{-\theta_0-1/r}a(\rho(t))a_0(t)\left\|s^{-1/r_0}b_0(s)\right\|_{r_0,(t,\infty)}K(u,f)\right\|_{r,(0,\infty)}\\ &\prec \left\|t^{-1/r}a(\rho(t))\left\|s^{-1/r_0}b_0(s)\left\|u^{-\theta_0-1/q}a_0(u)K(u,f)\right\|_{q,(t,s)}\right\|_{r_0,(t,\infty)}\right\|_{r,(0,\infty)}\\ &\leq \left\|t^{-1/r}a(\rho(t))\left\|s^{-1/r_0}b_0(s)\left\|u^{-\theta_0-1/q}a_0(u)K(u,f)\right\|_{q,(0,s)}\right\|_{r_0,(t,\infty)}\right\|_{r,(0,\infty)} = I_1.\end{aligned} \tag{170}$$

This completes the proof. □

Theorem 37 (cf. [8], Theorem 3.3). *Let $0<\theta_0<\theta_1<1$, $0<q_0,r_0,q_1,r_1\leq\infty$, $a_0,b_0,a_1,b_1\in SV$, $\|s^{-1/r_0}b_0(s)\|_{r_0,(0,1)}<\infty$, and $\|s^{-1/r_1}b_1(s)\|_{r_1,(1,\infty)}<\infty$. Put $\chi_0(t)=a_0(t)\|s^{-1/r_0}b_0(s)\|_{r_0,(0,t)}$, $\chi_1(t)=a_1(t)\|s^{-1/r_1}b_1(s)\|_{r_1,(t,\infty)}$, and $\rho(t)=t^{\theta_1-\theta_0}(\chi_0(t)/\chi_1(t))$. Then, for all $f\in\bar{A}^{\mathscr{R}}_{\theta_0,r_0,b_0,q_0,a_0}+\bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}$ and $t>0$,*

$$\begin{aligned}&K\left(\rho(t),f;\bar{A}^{\mathscr{R}}_{\theta_0,r_0,b_0,q_0,a_0},\bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}\right)\\ &\quad\approx \left\|s^{-1/r_0}b_0(s)\left\|u^{-\theta_0-1/q_0}a_0(u)K(u,f)\right\|_{q_0,(s,t)}\right\|_{r_0,(0,t)}\\ &\qquad+\rho(t)\left\|s^{-1/r_1}b_1(s)\left\|u^{-\theta_1-1/q_1}a_1(u)K(u,f)\right\|_{q_1,(t,s)}\right\|_{r_1,(t,\infty)}.\end{aligned} \tag{171}$$

Proof. Let Φ_0 be the function space corresponding to $\bar{A}^{\mathscr{R}}_{\theta_0,r_0,b_0,q_0,a_0}$:

$$\|F(*)\|_{\Phi_0} = \left\|s^{-1/r_0}b_0(s)\left\|u^{-\theta_0-1/q_0}a_0(u)F(u)\right\|_{q_0,(s,\infty)}\right\|_{r_0,(0,\infty)}. \tag{172}$$

Consider the functions $I(t,f)$, $g_0(t)$, and $h_0(t)$.

$$\begin{aligned}I(t,f) &:= \left\|\chi_{(0,t)}(*)K(*,f)\right\|_{\Phi_0}\\ &= \left\|s^{-1/r_0}b_0(s)\left\|u^{-\theta_0-1/q_0}a_0(u)K(u,f)\right\|_{q_0,(s,t)}\right\|_{r_0,(0,t)},\end{aligned}$$

$$\begin{aligned}g_0(t) &:= t\left\|\chi_{(t,\infty)}(*)\right\|_{\Phi_0}\\ &= t\left\|s^{-1/r_0}b_0(s)\left\|u^{-\theta_0-1/q_0}a_0(u)\right\|_{q_0,(\max(t,s),\infty)}\right\|_{r_0,(0,\infty)}\\ &\approx t\Bigg(\left\|s^{-1/r_0}b_0(s)\right\|_{r_0,(0,t)}\left\|u^{-\theta_0-1/q_0}a_0(u)\right\|_{q_0,(t,\infty)}\\ &\qquad+\left\|s^{-1/r_0}b_0(s)\left\|u^{-\theta_0-1/q_0}a_0(u)\right\|_{q_0,(s,\infty)}\right\|_{r_0,(t,\infty)}\Bigg)\\ &\approx t\left(\left\|s^{-1/r_0}b_0(s)\right\|_{r_0,(0,t)}t^{1-\theta_0}a_0(t)+\left\|s^{-\theta_0-1/r_0}a_0(s)b_0(s)\right\|_{r_0,(t,\infty)}\right)\\ &\approx t^{1-\theta_0}a_0(t)\left(\left\|s^{-1/r_0}b_0(s)\right\|_{r_0,(0,t)}+b_0(t)\right)\approx t^{1-\theta_0}a_0(t)\left\|s^{-1/r_0}b_0(s)\right\|_{r_0,(0,t)}\\ &= t^{1-\theta_0}\chi_0(t),\end{aligned} \tag{173}$$

$$\begin{aligned}h_0(t) &:= \left\|*\chi_{(0,t)}(*)\right\|_{\Phi_0} = \left\|s^{-1/r_0}b_0(s)\left\|u^{1-\theta_0-1/q_0}a_0(u)\right\|_{q_0,(s,t)}\right\|_{r_0,(0,\infty)}\\ &\leq \left\|u^{1-\theta_0-1/q_0}a_0(u)\right\|_{q_0,(0,t)}\left\|s^{-1/r_0}b_0(s)\right\|_{r_0,(0,\infty)}\\ &\approx t^{1-\theta_0}a_0(t)\left\|s^{-1/r_0}b_0(s)\right\|_{r_0,(0,\infty)} = t^{1-\theta_0}\chi_0(t).\end{aligned} \tag{174}$$

Thus,

$$h_0(t)\prec g_0(t)\approx g_0(t)+h_0(t)\approx t^{1-\theta_0}\chi_0(t). \tag{175}$$

Let Φ_1 be the function space corresponding to $\bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}$

:

$$\|F(*)\|_{\Phi_1} = \left\| s^{-1/r_1} b_1(s) \left\| u^{-\theta_1-1/q_1} a_1(u) F(u) \right\|_{q_1,(0,s)} \right\|_{r_1,(0,\infty)}. \tag{176}$$

From the proof of Theorem 18, we know that

$$J(t,f) = \left\| s^{-1/r_1} b_1(s) \left\| u^{-\theta_1-1/q_1} a_1(u) K(u,f) \right\|_{q_1,(t,s)} \right\|_{r_1,(t,\infty)},$$
$$g_1(t) \prec h_1(t) \approx g_1(t) + h_1(t) \approx t^{1-\theta_1} \chi_1(t). \tag{177}$$

Now, we estimate the functions $g(t)$ and $h(t)$.

$$\begin{aligned}\frac{1}{g(t)} &:= \left\| \chi_{(t,\infty)}(u) \frac{u}{g_0(u)+h_0(u)} \right\|_{\Phi_1} \\ &\approx \left\| s^{-1/r_1} b_1(s) \left\| u^{\theta_0-\theta_1-1/q_1} \frac{a_1(u)}{\chi_0(u)} \right\|_{q_1,(t,s)} \right\|_{r_1,(t,\infty)} \\ &\leq \left\| u^{\theta_0-\theta_1-1/q_1} \frac{a_1(u)}{\chi_0(u)} \right\|_{q_1,(t,\infty)} \left\| s^{-1/r_1} b_1(s) \right\|_{r_1,(t,\infty)} \\ &\approx t^{\theta_0-\theta_1} \frac{a_1(t)}{\chi_0(t)} \left\| s^{-1/r_1} b_1(s) \right\|_{r_1,(t,\infty)} = \frac{1}{\rho(t)},\end{aligned}$$

$$\begin{aligned}h(t) &:= \left\| \chi_{(0,t)}(u) \frac{u}{g_1(u)+h_1(u)} \right\|_{\Phi_0} \\ &\approx \left\| s^{-1/r_0} b_0(s) \left\| u^{\theta_1-\theta_0-1/q_0} \frac{a_0(u)}{\chi_1(u)} \right\|_{q_0,(s,t)} \right\|_{r_0,(0,\infty)} \\ &\leq \left\| u^{\theta_1-\theta_0-1/q_0} \frac{a_0(u)}{\chi_1(u)} \right\|_{q_0,(0,t)} \left\| s^{-1/r_0} b_0(s) \right\|_{r_0,(0,\infty)} \\ &\approx t^{\theta_1-\theta_0} \frac{a_0(t)}{\chi_1(t)} \left\| s^{-1/r_0} b_0(s) \right\|_{r_0,(0,\infty)} = \rho(t).\end{aligned} \tag{178}$$

Because $h(t) \prec \rho(t) \prec g(t)$, using [15] (Theorem 4, Case 2), we arrive at

$$\begin{aligned}&K\left(\rho(t), f; \bar{A}^{\mathscr{R}}_{\theta_0,r_0,b_0,q_0,a_0}, \bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}\right) \\ &\quad\approx \left\| s^{-1/r_0} b_0(s) \left\| u^{-\theta_0-1/q_0} a_0(u) K(u,f) \right\|_{q_0,(s,t)} \right\|_{r_0,(0,t)} \\ &\qquad + \rho(t) \left\| s^{-1/r_1} b_1(s) \left\| u^{-\theta_1-1/q_1} a_1(u) K(u,f) \right\|_{q_1,(t,s)} \right\|_{r_1,(t,\infty)} \\ &\qquad + t^{-\theta_0} \chi_0(t) K(t,f).\end{aligned} \tag{179}$$

By Lemma 10, we get

$$t^{-\theta_0} \chi_0(t) K(t,f) \prec \left\| s^{-1/r_0} b_0(s) \left\| u^{-\theta_0-1/q_0} a_0(u) K(u,f) \right\|_{q_0,(s,t)} \right\|_{r_0,(0,t)}. \tag{180}$$

Thus, (171) is proved. □

Theorem 38 (cf. [8], Theorem 4.5). *Let $0 < \theta_0 < \theta_1 < 1$, $0 < r, q_0, r_0, q_1, r_1 \leq \infty$, $a, a_0, b_0, a_1, b_1 \in SV$, $\|s^{-1/r_0} b_0(s)\|_{r_0,(0,1)} < \infty$, and $\|s^{-1/r_1} b_1(s)\|_{r_1,(1,\infty)} < \infty$. Put $\chi_0(t) = a_0(t) \|s^{-1/r_0} b_0(s)\|_{r_0,(0,t)}$, $\chi_1(t) = a_1(t) \|s^{-1/r_1} b_1(s)\|_{r_1,(t,\infty)}$, and $\rho(t) = t^{\theta_1-\theta_0}(\chi_0(t)/\chi_1(t))$.*

(i) If $0 < \theta < 1$, then

$$\left(\bar{A}^{\mathscr{R}}_{\theta_0,r_0,b_0,q_0,a_0}, \bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}\right)_{\theta,r,a} = \bar{A}_{\eta,r,a^{\#}}, \tag{181}$$

where $\eta = (1-\theta)\theta_0 + \theta\theta_1$ and $a^{\#}(t) = \chi_0(t)^{1-\theta} \chi_1(t)^{\theta} a(\rho(t))$.

(ii) If $\|s^{-1/r} a(s)\|_{r,(1,\infty)} < \infty$, then

$$\left(\bar{A}^{\mathscr{R}}_{\theta_0,r_0,b_0,q_0,a_0}, \bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}\right)_{0,r,a} = \bar{A}^{\mathscr{R},\mathscr{L}}_{\theta_0,r,a\circ\rho,r_0,b_0,q_0,a_0}. \tag{182}$$

(iii) If $\|s^{-1/r} a(s)\|_{r,(0,1)} < \infty$, then

$$\left(\bar{A}^{\mathscr{R}}_{\theta_0,r_0,b_0,q_0,a_0}, \bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}\right)_{1,r,a} = \bar{A}^{\mathscr{L},\mathscr{R}}_{\theta_1,r,a\circ\rho,r_1,b_1,q_1,a_1}. \tag{183}$$

Proof. Denote $X_0 = \bar{A}^{\mathscr{R}}_{\theta_0,r_0,b_0,q_0,a_0}$, $X_1 = \bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}$, $\bar{K}(t,f) = K(t,f;\bar{X})$, and $Z = \bar{X}_{\theta,r,a}$ $(0 \leq \theta \leq 1)$. Using the change of variables and Theorem 37, we can write

$$\begin{aligned}\|f\|_Z &= \left\| u^{-\theta-1/r} a(u) \bar{K}(u,f) \right\|_{r,(0,\infty)} \\ &\approx \left\| t^{-1/r} \rho(t)^{-\theta} a(\rho(t)) \bar{K}(\rho(t),f) \right\|_{r,(0,\infty)} \approx I_1 + I_2,\end{aligned} \tag{184}$$

where

$$\begin{aligned}I_1 &:= \left\| t^{-\theta(\theta_1-\theta_0)-1/r} \left(\frac{\chi_1(t)}{\chi_0(t)}\right)^{\theta} a(\rho(t)) \right\| s^{-1/r_0} b_0(s) \\ &\quad \cdot \left\| u^{-\theta_0-1/q_0} a_0(u) K(u,f) \right\|_{q_0,(s,t)} \Big\|_{r_0,(0,t)} \Big\|_{r,(0,\infty)},\end{aligned}$$

$$I_2 := \left\| t^{(1-\theta)(\theta_1-\theta_0)-1/r} \left(\frac{\chi_1(t)}{\chi_0(t)}\right)^{\theta-1} a(\rho(t)) * \left\| s^{-1/r_1} b_1(s) \left\| u^{-\theta_1-1/q_1} a_1(u) K(u,f) \right\|_{q_1,(t,s)} \right\|_{r_1,(t,\infty)} \right\|_{r,(0,\infty)}. \tag{185}$$

First, let $0<\theta<1$. In this case, $-\theta(\theta_1-\theta_0)=\theta_0-\eta$ and $(1-\theta)(\theta_1-\theta_0)=\theta_1-\eta$. By Lemma 14 (i), we get

$$\begin{aligned} I_1 &\approx \left\| t^{-\eta-1/r} \left(\frac{\chi_1(t)}{\chi_0(t)}\right)^{\theta} a(\rho(t)) a_0(t) \left\| s^{-1/r_0} b_0(s) \right\|_{r_0,(0,t)} K(t,f) \right\|_{r,(0,\infty)} \\ &= \left\| t^{-\eta-1/r} a^{\#}(t) K(t,f) \right\|_{r,(0,\infty)} = \|f\|_{\eta,r,a^{\#}}. \end{aligned} \tag{186}$$

So, it is enough to show that $I_2 \prec \|f\|_{\eta,r,a^{\#}}$. Note that

$$\begin{aligned} &\left\| s^{-1/r_1} b_1(s) \left\| u^{-\theta_1-1/q_1} a_1(u) K(u,f) \right\|_{q_1,(t,s)} \right\|_{r_1,(t,\infty)} \\ &\quad \le \left\| s^{-1/r_1} b_1(s) \right\|_{r_1,(t,\infty)} \left\| u^{-\theta_1-1/q_1} a_1(u) K(u,f) \right\|_{q_1,(t,\infty)}. \end{aligned} \tag{187}$$

Thus, by Lemma 12, we get

$$\begin{aligned} I_2 &\le \left\| t^{\theta_1-\eta-1/r} \left(\frac{\chi_1(t)}{\chi_0(t)}\right)^{\theta-1} a(\rho(t)) \left\| s^{-1/r_1} b_1(s) \right\|_{r_1,(t,\infty)} \cdot \left\| u^{-\theta_1-1/q_1} a_1(u) K(u,f) \right\|_{q_1,(t,\infty)} \right\|_{r,(0,\infty)} \\ &\approx \left\| t^{-\eta-1/r} \left(\frac{\chi_1(t)}{\chi_0(t)}\right)^{\theta-1} a(\rho(t)) a_1(t) \left\| s^{-1/r_1} b_1(s) \right\|_{r_1,(t,\infty)} K(t,f) \right\|_{r,(0,\infty)} \\ &= \left\| t^{-\eta-1/r} a^{\#}(t) K(t,f) \right\|_{r,(0,\infty)} = \|f\|_{\eta,r,a^{\#}}. \end{aligned} \tag{188}$$

Now, we prove (ii). In this case,

$$\begin{aligned} I_1 &= \left\| t^{-1/r} a(\rho(t)) \left\| s^{-1/r_0} b_0(s) \left\| u^{-\theta_0-1/q_0} a_0(u) K(u,f) \right\|_{q_0,(s,t)} \right\|_{r_0,(0,t)} \right\|_{r,(0,\infty)} \\ &= \|f\|_{\mathscr{R},\mathscr{L};\theta_0,r,a\circ\rho,r_0,b_0,q_0,a_0}, \end{aligned}$$

$$I_2 = \left\| t^{\theta_1-\theta_0-1/r} a(\rho(t)) \frac{\chi_0(t)}{\chi_1(t)} \left\| s^{-1/r_1} b_1(s) \left\| u^{-\theta_1-1/q_1} a_1(u) K(u,f) \right\|_{q_1,(t,s)} \right\|_{r_1,(t,\infty)} \right\|_{r,(0,\infty)}. \tag{189}$$

It is enough to show that $I_2 \prec I_1$. Using Lemmas 12 and 10, we have

$$\begin{aligned} I_2 &\le \left\| t^{\theta_1-\theta_0-1/r} a(\rho(t)) \frac{\chi_0(t)}{\chi_1(t)} \left\| s^{-1/r_1} b_1(s) \right\|_{r_1,(t,\infty)} \left\| u^{-\theta_1-1/q_1} a_1(u) K(u,f) \right\|_{q_1,(t,\infty)} \right\|_{r,(0,\infty)} \\ &\approx \left\| t^{-\theta_0-1/r} a(\rho(t)) a_0(t) \left\| s^{-1/r_0} b_0(s) \right\|_{r_0,(0,t)} K(t,f) \right\|_{r,(0,\infty)} \\ &\prec \left\| t^{-1/r} a(\rho(t)) \left\| s^{-1/r_0} b_0(s) \left\| u^{-\theta_0-1/q_0} a_0(u) K(u,f) \right\|_{q_0,(s,t)} \right\|_{r_0,(0,t)} \right\|_{r,(0,\infty)} = I_1. \end{aligned} \tag{190}$$

The proof of (iii) follows similar steps. □

Remark 39. Assertion (i) of Theorem 38 coincides with [5] (Theorem 29). But there it has been wrongly stated that it can be proved based on symmetry arguments. The correct proof is presented above.

8. Reiteration Formulae for Couples Where One of the Operands Is $\mathscr{L}\mathscr{L}$, $\mathscr{L}\mathscr{R}$, $\mathscr{R}\mathscr{L}$, or $\mathscr{R}\mathscr{R}$ Space

In this section, we establish reiteration formulae for the couples in which one of the operands is $\mathscr{L}\mathscr{L}$, $\mathscr{L}\mathscr{R}$, $\mathscr{R}\mathscr{L}$, or $\mathscr{R}\mathscr{R}$ space. Here, we have chosen an indirect method of the proofs. Compared to the direct method, i.e., calculation of the corresponding Holmstedt-type formulae, we subsequently apply known formulae but unfortunately often arrive at weaker results (embeddings instead of isomorphisms).

Theorem 40. *Let $0<\theta_0<\theta_1<1$, $0<r,r_0,p_1,q_1,r_1\le\infty$, a, $b_0,a_1,b_1,c_1\in SV$, $\|s^{-1/r_1}b_1(s)\|_{r_1,(1,\infty)}<\infty$, and $\|u^{-1/p_1}c_1(u)\|_{p_1,(0,1)}<\infty$. Put $\chi_1(t)=a_1(t)\|s^{-1/r_1}b_1(s)\|_{r_1,(t,\infty)}\|u^{-1/p_1}c_1(u)\|_{p_1,(0,t)}$ and $\rho(t)=t^{\theta_1-\theta_0}(b_0(t)/\chi_1(t))$.*

(i) If $0<\theta<1$, then

$$\left(\bar{A}_{\theta_0,r_0,b_0}, \bar{A}^{\mathscr{L},\mathscr{R}}_{\theta_1,p_1,c_1,r_1,b_1,q_1,a_1}\right)_{\theta,r,a} = \bar{A}_{\eta,r,a^{\#}}, \tag{191}$$

where $\eta=(1-\theta)\theta_0+\theta\theta_1$ and $a^{\#}(t)=b_0(t)^{1-\theta}\chi_1(t)^{\theta}a(\rho(t))$.

(ii) If $\|t^{-1/r}a(t)\|_{r,(1,\infty)}<\infty$, then

$$\left(\bar{A}_{\theta_0,r_0,b_0}, \bar{A}^{\mathscr{L},\mathscr{R}}_{\theta_1,p_1,c_1,r_1,b_1,q_1,a_1}\right)_{0,r,a} = \bar{A}^{\mathscr{L}}_{\theta_0,r,a\circ\rho,r_0,b_0}. \tag{192}$$

(iii) If $\|t^{-1/r}a(t)\|_{r,(0,1)}<\infty$, then

$$\left(\bar{A}_{\theta_0,r_0,b_0}, \bar{A}^{\mathscr{L},\mathscr{R}}_{\theta_1,p_1,c_1,r_1,b_1,q_1,a_1}\right)_{1,r,a} \subset \bar{A}^{\mathscr{L},\mathscr{R}}_{\theta_1,r,a\circ\rho,r_1,b_1,q_1,a_1}. \tag{193}$$

Proof. Denote $X_0=\bar{A}_{\theta_0,r_0,b_0}$, $X_1=\bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}$, and $\psi(t)=a_1(t)\|s^{-1/r_1}b_1(s)\|_{r_1,(t,\infty)}$. Let σ be a strongly increasing, differentiable function such that $\sigma(t) \approx t^{\theta_1-\theta_0}(b_0(t)/\psi(t))$ and $\sigma(1)=1$. Additionally, denote $x=c_1\circ\sigma^{(-1)}$, $Z=(\bar{A}_{\theta_0,r_0,b_0},\bar{A}^{\mathscr{L},\mathscr{R}}_{\theta_1,p_1,c_1,r_1,b_1,q_1,a_1})_{\theta,r,a}$, and $y(t)=(\|s^{-1/p_1}x(s)\|_{p_1,(0,t)})^{\theta} a(t/\|s^{-1/p_1}x(s)\|_{p_1,(0,t)})$ $(0\le\theta\le1)$. By the change of variables $u=\sigma^{(-1)}(s)$, we get $\|s^{-1/p_1}x(s)\|_{p_1,(0,\sigma(t))}\approx\|u^{-1/p_1}c_1(u)\|_{p_1,(0,t)}$. Hence, $\|s^{-1/p_1}x(s)\|_{p_1,(0,1)}<\infty$ and

$$y(\sigma(t))\approx\left(\|u^{-1/p_1}c_1(u)\|_{p_1,(0,t)}\right)^{\theta}a(\rho(t)). \tag{194}$$

By Theorem 19, we obtain

$$\bar{A}^{\mathscr{L},\mathscr{R}}_{\theta_1,p_1,c_1,r_1,b_1,q_1,a_1}=\left(\bar{A}_{\theta_0,r_0,b_0},\bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}\right)_{1,p_1,x}. \tag{195}$$

So, $Z=(X_0,\bar{X}_{1,p_1,x})_{\theta,r,a}$. Using Theorem 17, we conclude that for $0\le\theta<1$, it holds

$$Z=\bar{X}_{\theta,r,y}=\left(\bar{A}_{\theta_0,r_0,b_0},\bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}\right)_{\theta,r,y}. \tag{196}$$

If $0<\theta<1$, by [5] (Theorem 19) (i) and (194), we get $Z=\bar{A}_{\eta,r,a^{\#}}$, where

$$\begin{aligned}a^{\#}(t)&=b_0(t)^{1-\theta}\left(a_1(t)\|s^{-1/r_1}b_1(s)\|_{r_1,(t,\infty)}\right)^{\theta}y(\sigma(t))\\&\approx b_0(t)^{1-\theta}\left(a_1(t)\|s^{-1/r_1}b_1(s)\|_{r_1,(t,\infty)}\|u^{-1/p_1}c_1(u)\|_{p_1,(0,t)}\right)^{\theta}a(\rho(t))\\&=b_0(t)^{1-\theta}\chi_1(t)^{\theta}a(\rho(t)).\end{aligned} \tag{197}$$

If $\theta=0$, by [5] (Theorem 19) (ii), we get

$$Z=\left(\bar{A}_{\theta_0,r_0,b_0},\bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}\right)_{0,r,y}=\bar{A}^{\mathscr{L}}_{\theta_0,r,a\circ\rho,r_0,b_0}, \tag{198}$$

while due to (194), $y\circ\sigma\approx a\circ\rho$. Note that $\|t^{-1/r}y(t)\|_{r,(1,\infty)}<\infty$.

If $\theta=1$, using Theorems 17 and 19, we conclude that

$$\begin{aligned}Z&=\left(X_0,\bar{X}_{1,p_1,x}\right)_{1,r,a}\subset\bar{X}_{1,r,y}=\left(\bar{A}_{\theta_0,r_0,b_0},\bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}\right)_{1,r,y}\\&=\bar{A}^{\mathscr{L},\mathscr{R}}_{\theta_1,r,y\circ\sigma,r_1,b_1,q_1,a_1}=\bar{A}^{\mathscr{L},\mathscr{R}}_{\theta_1,r,a\circ\rho,r_1,b_1,q_1,a_1}.\end{aligned} \tag{199}$$

This completes the proof. □

Theorem 41. *Let $0<\theta_1<1$, $0<r,r_0,p_1,q_1,r_1\le\infty$, $a,b_0,a_1,b_1,c_1\in SV$, $\|s^{-1/r_0}b_0(s)\|_{r_0,(1,\infty)}<\infty$, $\|s^{-1/r_1}b_1(s)\|_{r_1,(1,\infty)}<\infty$, and $\|u^{-1/p_1}c_1(u)\|_{p_1,(0,1)}<\infty$. Put $\chi_0(t)=\|s^{-1/r_0}b_0(s)\|_{r_0,(t,\infty)}$, $\chi_1(t)=a_1(t)\|s^{-1/r_1}b_1(s)\|_{r_1,(t,\infty)}\|u^{-1/p_1}c_1(u)\|_{p_1,(0,t)}$, and $\rho(t)=t^{\theta_1}(\chi_0(t)/\chi_1(t))$.*

(i) If $0<\theta<1$, then

$$\left(\bar{A}_{0,r_0,b_0},\bar{A}^{\mathscr{L},\mathscr{R}}_{\theta_1,p_1,c_1,r_1,b_1,q_1,a_1}\right)_{\theta,r,a}=\bar{A}_{\eta,r,a^{\#}}, \tag{200}$$

where $\eta=\theta\theta_1$ and $a^{\#}(t)=\chi_0(t)^{1-\theta}\chi_1(t)^{\theta}a(\rho(t))$.

(ii) If $\|t^{-1/r}a(t)\|_{r,(1,\infty)}<\infty$, then

$$\left(\bar{A}_{0,r_0,b_0},\bar{A}^{\mathscr{L},\mathscr{R}}_{\theta_1,p_1,c_1,r_1,b_1,q_1,a_1}\right)_{0,r,a}=\bar{A}_{0,r,a^{\#}}\bigcap\bar{A}^{\mathscr{L}}_{0,r,a\circ\rho,r_0,b_0}, \tag{201}$$

where $a^{\#}(t)=\chi_0(t)a(\rho(t))$.

(iii) If $\|t^{-1/r}a(t)\|_{r,(0,1)}<\infty$, then

$$\left(\bar{A}_{0,r_0,b_0},\bar{A}^{\mathscr{L},\mathscr{R}}_{\theta_1,p_1,c_1,r_1,b_1,q_1,a_1}\right)_{1,r,a}\subset\bar{A}^{\mathscr{L},\mathscr{R}}_{\theta_1,r,a\circ\rho,r_1,b_1,q_1,a_1}. \tag{202}$$

Proof. Denote $X_0=\bar{A}_{0,r_0,b_0}$, $X_1=\bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}$, and $\psi(t)=a_1(t)\|s^{-1/r_1}b_1(s)\|_{r_1,(t,\infty)}$. Let σ be a strongly increasing, differentiable function such that $\sigma(t)\approx t^{\theta_1}(\chi_0(t)/\psi(t))$ and $\sigma(1)=1$. Additionally, denote $x=c_1\circ\sigma^{(-1)}$, $Z=(\bar{A}_{0,r_0,b_0},\bar{A}^{\mathscr{L},\mathscr{R}}_{\theta_1,p_1,c_1,r_1,b_1,q_1,a_1})_{\theta,r,a}$, and $y(t)=(\|s^{-1/p_1}x(s)\|_{p_1,(0,t)})^{\theta}a(t/\|s^{-1/p_1}x(s)\|_{p_1,(0,t)})$ $(0\le\theta\le1)$. By the change of variables $u=\sigma^{(-1)}(s)$, we get $\|s^{-1/p_1}x(s)\|_{p_1,(0,\sigma(t))}\approx\|u^{-1/p_1}c_1(u)\|_{p_1,(0,t)}$ and

$$y(\sigma(t))\approx\left(\|u^{-1/p_1}c_1(u)\|_{p_1,(0,t)}\right)^{\theta}a(\rho(t)). \tag{203}$$

By Theorem 20 (ii), we obtain

$$\bar{A}^{\mathscr{L},\mathscr{R}}_{\theta_1,p_1,c_1,r_1,b_1,q_1,a_1}\cong\left(\bar{A}_{0,r_0,b_0},\bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}\right)_{1,p_1,x}. \tag{204}$$

Hence, $Z=(X_0,\bar{X}_{1,p_1,x})_{\theta,r,a}$. Using Theorem 17, we conclude that for $0\le\theta<1$, it holds

$$Z=\bar{X}_{\theta,r,y}=\left(\bar{A}_{0,r_0,b_0},\bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}\right)_{\theta,r,y}. \tag{205}$$

If $0<\theta<1$, by [5] (Theorem 17) and (203), we get

$$Z=\left(\bar{A}_{0,r_0,b_0},\bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}\right)_{\theta,r,y}\cong\bar{A}_{\eta,r,a^{\#}}, \tag{206}$$

where $\eta=\theta\theta_1$ and $a^{\#}(t)=\chi_0(t)^{1-\theta}\psi(t)^{\theta}y(\sigma(t))\approx\chi_0(t)^{1-\theta}\chi_1(t)^{\theta}a(\rho(t))$.

If $\theta=0$, using Theorem 20 (i), we get

$$Z=\left(\bar{A}_{0,r_0,b_0},\bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}\right)_{0,r,y}=\bar{A}_{0,r,a^{\#}}\bigcap\bar{A}^{\mathscr{L}}_{0,r,a\circ\rho,r_0,b_0}, \tag{207}$$

where $a^{\#}(t)=\chi_0(t)y(\sigma(t))\approx\chi_0(t)a(\rho(t))$.

If $\theta = 1$, using Theorems 17 and 20, we conclude that

$$Z = \left(X_0, \bar{X}_{1,p_1,x}\right)_{1,r,a} \subset \bar{X}_{1,r,y} = \left(\bar{A}_{0,r_0,b_0}, \bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}\right)_{1,r,y} = \bar{A}^{\mathscr{L},\mathscr{R}}_{\theta_1,r,y\circ\sigma,r_1,b_1,q_1,a_1} = \bar{A}^{\mathscr{L},\mathscr{R}}_{\theta_1,r,a\circ\rho,r_1,b_1,q_1,a_1}. \tag{208}$$

This completes the proof. □

Theorem 42. *Let $0 < \theta_1 < 1$, $0 < r, p_1, q_1, r_1 \le \infty$, $a, a_1, b_1, c_1 \in SV$, $\|s^{-1/r_1} b_1(s)\|_{r_1,(1,\infty)} < \infty$, and $\|u^{-1/p_1} c_1(u)\|_{p_1,(0,1)} < \infty$. Put $\chi_1(t) = a_1(t)\|s^{-1/r_1} b_1(s)\|_{r_1,(t,\infty)} \|u^{-1/p_1} c_1(u)\|_{p_1,(0,t)}$ and $\rho(t) = t^{\theta_1}(1/\chi_1(t))$.*

(i) If $0 < \theta < 1$, then

$$\left(A_0, \bar{A}^{\mathscr{L},\mathscr{R}}_{\theta_1,p_1,c_1,r_1,b_1,q_1,a_1}\right)_{\theta,r,a} = \bar{A}_{\eta,r,a^\#}, \tag{209}$$

where $\eta = \theta\theta_1$ and $a^\#(t) = \chi_1(t)^\theta a(\rho(t))$.

(ii) If $\|t^{-1/r} a(t)\|_{r,(1,\infty)} < \infty$, then

$$\left(A_0, \bar{A}^{\mathscr{L},R}_{\theta_1,p_1,c_1,r_1,b_1,q_1,a_1}\right)_{0,r,a} = \bar{A}_{0,r,a\circ\rho}. \tag{210}$$

(iii) If $\|t^{-1/r} a(t)\|_{r,(0,1)} < \infty$, then

$$\left(A_0, \bar{A}^{\mathscr{L},\mathscr{R}}_{\theta_1,p_1,c_1,r_1,b_1,q_1,a_1}\right)_{1,r,a} \subset \bar{A}^{\mathscr{L},\mathscr{R}}_{\theta_1,r,a\circ\rho,r_1,b_1,q_1,a_1}. \tag{211}$$

Proof. Denote $X_0 = A_0$, $X_1 = \bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}$, and $\psi(t) = a_1(t)\|s^{-1/r_1} b_1(s)\|_{r_1,(t,\infty)}$. Let σ be a strongly increasing, differentiable function such that $\sigma(t) \approx t^{\theta_1}/\psi(t)$ and$\sigma(1) = 1$. Additionally, denote $x = c_1 \circ \sigma^{(-1)}$, $Z = (A_0, \bar{A}^{\mathscr{L},\mathscr{R}}_{\theta_1,p_1,c_1,r_1,b_1,q_1,a_1})_{\theta,r,a}$, and $y(t) = (\|s^{-1/p_1} x(s)\|_{p_1,(0,t)})^\theta a(t/\|s^{-1/p_1} x(s)\|_{p_1,(0,t)})$ $(0 \le \theta < 1)$. By the change of variables $u = \sigma^{(-1)}(s)$, we get $\|s^{-1/p_1} x(s)\|_{p_1,(0,\sigma(t))} \approx \|u^{-1/p_1} c_1(u)\|_{p_1,(0,t)}$ and

$$y(\sigma(t)) \approx \left(\|u^{-1/p_1} c_1(u)\|_{p_1,(0,t)}\right)^\theta a(\rho(t)). \tag{212}$$

By Theorem 21, we obtain

$$\bar{A}^{\mathscr{L},\mathscr{R}}_{\theta_1,p_1,c_1,r_1,b_1,q_1,a_1} = \left(A_0, \bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}\right)_{1,p_1,x}. \tag{213}$$

Hence, $Z = (X_0, \bar{X}_{1,p_1,x})_{\theta,r,a}$.

If $0 \le \theta < 1$, by Theorem 17, [5] (Theorem 16), and (212), we get

$$Z = \bar{X}_{\theta,r,y} = \left(A_0, \bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}\right)_{\theta,r,y} = \bar{A}_{\eta,r,a^\#}, \tag{214}$$

where $\eta = \theta\theta_1$ and $a^\#(t) = \psi(t)^\theta y(\sigma(t)) \approx \chi_1(t)^\theta a(\rho(t))$.

If $\theta = 1$, using Theorems 17 and 21, we conclude that

$$Z = \left(X_0, \bar{X}_{1,p_1,x}\right)_{1,r,a} \subset \bar{X}_{1,r,y} = \left(A_0, \bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}\right)_{1,r,y} = \bar{A}^{\mathscr{L},\mathscr{R}}_{\theta_1,r,y\circ\sigma,r_1,b_1,q_1,a_1} = \bar{A}^{\mathscr{L},\mathscr{R}}_{\theta_1,r,a\circ\rho,r_1,b_1,q_1,a_1}. \tag{215}$$

This completes the proof. □

Theorem 43. *Let $0 < \theta_0 < \theta_1 < 1$, $0 < r, q_0, r_0, p_1, q_1, r_1 \le \infty$, $a, a_0, b_0, a_1, b_1, c_1 \in SV$, $\|s^{-1/r_0} b_0(s)\|_{r_0,(1,\infty)} < \infty$, $\|s^{-1/r_1} b_1(s)\|_{r_1,(1,\infty)} < \infty$, and $\|u^{-1/p_1} c_1(u)\|_{p_1,(0,1)} < \infty$. Put $\chi_0(t) = a_0(t)\|s^{-1/r_0} b_0(s)\|_{r_0,(t,\infty)}$, $\chi_1(t) = a_1(t)\|s^{-1/r_1} b_1(s)\|_{r_1,(t,\infty)} \|u^{-1/p_1} c_1(u)\|_{p_1,(0,t)}$, and $\rho(t) = t^{\theta_1-\theta_0}(\chi_0(t)/\chi_1(t))$.*

(i) If $0 < \theta < 1$, then

$$\left(\bar{A}^{\mathscr{L}}_{\theta_0,r_0,b_0,q_0,a_0}, \bar{A}^{\mathscr{L},\mathscr{R}}_{\theta_1,p_1,c_1,r_1,b_1,q_1,a_1}\right)_{\theta,r,a} = \bar{A}_{\eta,r,a^\#}, \tag{216}$$

where $\eta = (1-\theta)\theta_0 + \theta\theta_1$ and $a^\#(t) = \chi_0(t)^{1-\theta} \chi_1(t)^\theta a(\rho(t))$.

(ii) If $\|t^{-1/r} a(t)\|_{r,(1,\infty)} < \infty$, then

$$\left(\bar{A}^{\mathscr{L}}_{\theta_0,r_0,b_0,q_0,a_0}, \bar{A}^{\mathscr{L},\mathscr{R}}_{\theta_1,p_1,c_1,r_1,b_1,q_1,a_1}\right)_{0,r,a} = \bar{A}^{\mathscr{L}}_{\theta_0,r,a^\#,q_0,a_0} \bigcap \bar{A}^{\mathscr{L},\mathscr{L}}_{\theta_0,r,a\circ\rho,r_0,b_0,q_0,a_0}, \tag{217}$$

where $a^\#(t) = \|s^{-1/r_0} b_0(s)\|_{r_0,(t,\infty)} a(\rho(t))$.

(iii) If $\|t^{-1/r} a(t)\|_{r,(0,1)} < \infty$, then

$$\left(\bar{A}^{\mathscr{L}}_{\theta_0,r_0,b_0,q_0,a_0}, \bar{A}^{\mathscr{L},\mathscr{R}}_{\theta_1,p_1,c_1,r_1,b_1,q_1,a_1}\right)_{1,r,a} \subset \bar{A}^{\mathscr{L},\mathscr{R}}_{\theta_1,r,a\circ\rho,r_1,b_1,q_1,a_1}. \tag{218}$$

Proof. Denote $X_0 = \bar{A}^{\mathscr{L}}_{\theta_0,r_0,b_0,q_0,a_0}$, $X_1 = \bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}$, and $\psi(t) = a_1(t)\|s^{-1/r_1} b_1(s)\|_{r_1,(t,\infty)}$. Let σ be a strongly increasing, differentiable function such that $\sigma(t) \approx t^{\theta_1-\theta_0}(\chi_0(t)/\psi(t))$ and $\sigma(1) = 1$. Additionally, denote $x = c_1 \circ \sigma^{(-1)}$, $Z = (\bar{A}^{\mathscr{L}}_{\theta_0,r_0,b_0,q_0,a_0}, \bar{A}^{\mathscr{L},\mathscr{R}}_{\theta_1,p_1,c_1,r_1,b_1,q_1,a_1})_{\theta,r,a}$, and $y(t) = (\|s^{-1/p_1} x(s)\|_{p_1,(0,t)})^\theta a(t/\|s^{-1/p_1} x(s)\|_{p_1,(0,t)})$. By the change of variables $u = \sigma^{(-1)}(s)$, we get $\|s^{-1/p_1} x(s)\|_{p_1,(0,\sigma(t))} \approx$

$\|u^{-1/p_1}c_1(u)\|_{p_1,(0,t)}$ and

$$y(\sigma(t)) \approx \left(\|u^{-1/p_1}c_1(u)\|_{p_1,(0,t)}\right)^{\theta} a(\rho(t)). \quad (219)$$

By Theorem 33 (ii), we obtain

$$\bar{A}^{\mathscr{L},\mathscr{R}}_{\theta_1,p_1,c_1,r_1,b_1,q_1,a_1} = \left(\bar{A}^{\mathscr{L}}_{\theta_0,r_0,b_0,q_0,a_0}, \bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}\right)_{1,p_1,x}. \quad (220)$$

Hence, $Z = (X_0, \bar{X}_{1,p_1,x})_{\theta,r,a}$. Using Theorem 17, we conclude that for $0 \le \theta < 1$, it holds

$$Z = \bar{X}_{\theta,r,y} = \left(\bar{A}^{\mathscr{L}}_{\theta_0,r_0,b_0,q_0,a_0}, \bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}\right)_{\theta,r,y}. \quad (221)$$

If $0 < \theta < 1$, by [5] (Theorem 26) (i) and (219), we get $Z = \bar{A}_{\eta,r,a^\#}$, where $a^\#(t) = \chi_0(t)^{1-\theta}\psi(t)^{\theta}y(\sigma(t)) \approx \chi_0(t)^{1-\theta}\chi_1(t)^{\theta}a(\rho(t))$.

If $\theta = 0$, by Theorem 33 (i), we get

$$Z = \left(\bar{A}^{\mathscr{L}}_{\theta_0,r_0,b_0,q_0,a_0}, \bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}\right)_{0,r,y} = \bar{A}^{\mathscr{L}}_{\theta_0,r,a^\#,q_0,a_0} \bigcap \bar{A}^{\mathscr{L},\mathscr{L}}_{\theta_0,r,a\circ\rho,r_0,b_0,q_0,a_0}, \quad (222)$$

where $a^\#(t) = \|s^{-1/r_0}b_0(s)\|_{r_0,(t,\infty)}y(\sigma(t)) \approx \|s^{-1/r_0}b_0(s)\|_{r_0,(t,\infty)}a(\rho(t))$.

If $\theta = 1$, using Theorems 17 and 33 (ii), we conclude that

$$Z = \left(X_0, \bar{X}_{1,p_1,x}\right)_{1,r,a} \subset \bar{X}_{1,r,y} = \left(\bar{A}^{\mathscr{L}}_{\theta_0,r_0,b_0,q_0,a_0}, \bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}\right)_{1,r,y} = \bar{A}^{\mathscr{L},\mathscr{R}}_{\theta_1,r,y\circ\sigma,r_1,b_1,q_1,a_1} = \bar{A}^{\mathscr{L},\mathscr{R}}_{\theta_1,r,a\circ\rho,r_1,b_1,q_1,a_1}. \quad (223)$$

This completes the proof. □

Theorem 44. *Let $0 < \theta_0 < \theta_1 < 1$, $0 < r, r_0, q_0, p_1, q_1, r_1 \le \infty$, $a, a_0, a_1, b_0, b_1 \in SV$, $\|s^{-1/r_0}b_0(s)\|_{r_0,(0,1)} < \infty$, $\|s^{-1/r_1}b_1(s)\|_{r_1,(1,\infty)} < \infty$, and $\|u^{-1/p_1}c_1(u)\|_{p_1,(0,1)} < \infty$. Put $\chi_0(t) = a_0(t)\|s^{-1/r_0}b_0(s)\|_{r_0,(0,t)}$, $\chi_1(t) = a_1(t)\|s^{-1/r_1}b_1(s)\|_{r_1,(t,\infty)}\|u^{-1/p_1}c_1(u)\|_{p_1,(0,t)}$, and $\rho(t) = t^{\theta_1-\theta_0}(\chi_0(t)/\chi_1(t))$.*

(i) If $0 < \theta < 1$, then

$$\left(\bar{A}^{\mathscr{R}}_{\theta_0,r_0,b_0,q_0,a_0}, \bar{A}^{\mathscr{L},\mathscr{R}}_{\theta_1,p_1,c_1,r_1,b_1,q_1,a_1}\right)_{\theta,r,a} = \bar{A}_{\eta,r,a^\#}, \quad (224)$$

where $\eta = (1-\theta)\theta_0 + \theta\theta_1$ and $a^\#(t) = \chi_0(t)^{1-\theta}\chi_1(t)^{\theta}a(\rho(t))$.

(ii) If $\|t^{-1/r}a(t)\|_{r,(1,\infty)} < \infty$, then

$$\left(\bar{A}^{\mathscr{R}}_{\theta_0,r_0,b_0,q_0,a_0}, \bar{A}^{\mathscr{L},\mathscr{R}}_{\theta_1,p_1,c_1,r_1,b_1,q_1,a_1}\right)_{0,r,a} = \bar{A}^{\mathscr{R},\mathscr{L}}_{\theta_0,r,a\circ\rho,r_0,b_0,q_0,a_0}. \quad (225)$$

(iii) If $\|t^{-1/r}a(t)\|_{r,(0,1)} < \infty$, then

$$\left(\bar{A}^{\mathscr{R}}_{\theta_0,r_0,b_0,q_0,a_0}, \bar{A}^{\mathscr{L},\mathscr{R}}_{\theta_1,p_1,c_1,r_1,b_1,q_1,a_1}\right)_{1,r,a} \subset \bar{A}^{\mathscr{L},R}_{\theta_1,r,a\circ\rho,r_1,b_1,q_1,a_1}. \quad (226)$$

Proof. Denote $X_0 = \bar{A}^{\mathscr{R}}_{\theta_0,r_0,b_0,q_0,a_0}$, $X_1 = \bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}$, and $\psi(t) = a_1(t)\|s^{-1/r_1}b_1(s)\|_{r_1,(t,\infty)}$. Let σ be a strongly increasing, differentiable function such that $\sigma(t) \approx t^{\theta_1-\theta_0}(\chi_0(t)/\psi(t))$ and $\sigma(1) = 1$. Additionally, denote $x = c_1 \circ \sigma^{(-1)}$, $Z = (\bar{A}^{\mathscr{R}}_{\theta_0,r_0,b_0,q_0,a_0}, \bar{A}^{\mathscr{L},\mathscr{R}}_{\theta_1,p_1,c_1,r_1,b_1,q_1,a_1})_{\theta,r,a}$, and $y(t) = (\|s^{-1/p_1}x(s)\|_{p_1,(0,t)})^{\theta}a(t/\|s^{-1/p_1}x(s)\|_{p_1,(0,t)})$. By the change of variables $u = \sigma^{(-1)}(s)$, we get $\|s^{-1/p_1}x(s)\|_{p_1,(0,\sigma(t))} \approx \|u^{-1/p_1}c_1(u)\|_{p_1,(0,t)}$ and

$$y(\sigma(t)) \approx \left(\|u^{-1/p_1}c_1(u)\|_{p_1,(0,t)}\right)^{\theta} a(\rho(t)). \quad (227)$$

By Theorem 38 (iii), we obtain

$$\left(\bar{A}^{\mathscr{R}}_{\theta_0,r_0,b_0,q_0,a_0}, \bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}\right)_{1,p_1,x} = \bar{A}^{\mathscr{L},\mathscr{R}}_{\theta_1,p_1,c_1,r_1,b_1,q_1,a_1}. \quad (228)$$

Hence, $Z = (X_0, \bar{X}_{1,p_1,x})_{\theta,r,a}$. Using Theorem 17, we conclude that for $0 \le \theta < 1$, it holds

$$Z = \bar{X}_{\theta,r,y} = \left(\bar{A}^{\mathscr{R}}_{\theta_0,r_0,b_0,q_0,a_0}, \bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}\right)_{\theta,r,y}. \quad (229)$$

If $0 < \theta < 1$, by Theorem 38 (i) and (227), we get $Z = \bar{A}_{\eta,r,a^\#}$, where $a^\#(t) = \chi_0(t)^{1-\theta}\psi(t)^{\theta}y(\sigma(t)) \approx \chi_0(t)^{1-\theta}\chi_1(t)^{\theta}a(\rho(t))$.

If $\theta = 0$, by Theorem 38 (ii), we get

$$Z = \left(\bar{A}^{\mathscr{R}}_{\theta_0,r_0,b_0,q_0,a_0}, \bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}\right)_{0,r,y} = \bar{A}^{\mathscr{R},\mathscr{L}}_{\theta_0,r,y\circ\sigma,r_0,b_0,q_0,a_0} = \bar{A}^{\mathscr{R},\mathscr{L}}_{\theta_0,r,a^\#,r_0,b_0,q_0,a_0}, \quad (230)$$

where $a^\#(t) = y(\sigma(t)) \approx a(\rho(t))$.

If $\theta = 1$, using Theorems 17 and 38 (iii), we conclude that

$$Z = \left(X_0, \bar{X}_{1,p_1,x}\right)_{1,r,a} \subset \bar{X}_{1,r,y} = \left(\bar{A}^{\mathscr{R}}_{\theta_0,r_0,b_0,q_0,a_0}, \bar{A}^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,a_1}\right)_{1,r,y} = \bar{A}^{\mathscr{L},\mathscr{R}}_{\theta_1,r,y\circ\sigma,r_1,b_1,q_1,a_1} = \bar{A}^{\mathscr{L},\mathscr{R}}_{\theta_1,r,a\circ\rho,r_1,b_1,q_1,a_1}. \quad (231)$$

This completes the proof. □

From the previous theorems in this section, using the symmetry arguments given by the formulae (14)–(16), the following five theorems can be proved.

Theorem 45. *Let* $0<\theta_0<\theta_1<1$, $0<r,p_0,q_0,r_0,r_1\le\infty$, a, $b_0,a_0,c_0,b_1\in SV$, $\|s^{-1/r_0}b_0(s)\|_{r_0,(0,1)}<\infty$, *and* $\|u^{-1/p_0}c_0(u)\|_{p_0,(1,\infty)}<\infty$. *Put* $\chi_0(t)=a_0(t)\|s^{-1/r_0}b_0(s)\|_{r_0,(0,t)}\|u^{-1/p_0}c_0(u)\|_{p_0,(t,\infty)}$ *and* $\rho(t)=t^{\theta_1-\theta_0}(\chi_0(t)/b_1(t))$.

(i) *If* $0<\theta<1$, *then*

$$\left(\bar{A}^{\mathcal{R},\mathcal{L}}_{\theta_0,p_0,c_0,r_0,b_0,q_0,a_0},\bar{A}_{\theta_1,r_1,b_1}\right)_{\theta,r,a}=\bar{A}_{\eta,r,a^{\#}},\tag{232}$$

where $\eta=(1-\theta)\theta_0+\theta\theta_1$ *and* $a^{\#}(t)=\chi_0(t)^{1-\theta}b_1(t)^{\theta}a(\rho(t))$.

(ii) *If* $\|t^{-1/r}a(t)\|_{r,(0,1)}<\infty$, *then*

$$\left(\bar{A}^{\mathcal{R},\mathcal{L}}_{\theta_0,p_0,c_0,r_0,b_0,q_0,a_0},\bar{A}_{\theta_1,r_1,b_1}\right)_{0,r,a}\subset\bar{A}^{\mathcal{R},\mathcal{L}}_{\theta_0,r,a\circ\rho,r_0,b_0,q_0,a_0}.\tag{233}$$

(iii) *If* $\|t^{-1/r}a(t)\|_{r,(0,1)}<\infty$, *then*

$$\left(\bar{A}^{\mathcal{R},\mathcal{L}}_{\theta_0,p_0,c_0,r_0,b_0,q_0,a_0},\bar{A}_{\theta_1,r_1,b_1}\right)_{1,r,a}=\bar{A}^{\mathcal{R}}_{\theta_1,r,a\circ\rho,r_1,b_1}.\tag{234}$$

Theorem 46. *Let* $0<\theta_0<1$, $0<r,p_0,q_0,r_0,r_1\le\infty$, $a,a_0,b_0,b_1,c_0\in SV$, $\|s^{-1/r_0}b_0(s)\|_{r_0,(0,1)}<\infty$, $\|u^{-1/p_0}c_0(u)\|_{p_0,(1,\infty)}<\infty$, *and* $\|s^{-1/r_1}b_1(s)\|_{r_1,(0,1)}<\infty$. *Put* $\chi_0(t)=a_0(t)\|s^{-1/r_0}b_0(s)\|_{r_0,(0,t)}\|u^{-1/p_0}c_0(u)\|_{p_0,(t,\infty)}$, $\chi_1(t)=\|s^{-1/r_1}b_1(s)\|_{r_1,(0,t)}$, *and* $\rho(t)=t^{1-\theta_0}(\chi_0(t)/\chi_1(t))$.

(i) *If* $0<\theta<1$, *then*

$$\left(\bar{A}^{\mathcal{R},\mathcal{L}}_{\theta_0,p_0,c_0,r_0,b_0,q_0,a_0},\bar{A}_{1,r_1,b_1}\right)_{\theta,r,a}=\bar{A}_{\eta,r,a^{\#}},\tag{235}$$

where $\eta=(1-\theta)\theta_0+\theta$ *and* $a^{\#}(t)=\chi_0(t)^{1-\theta}\chi_1(t)^{\theta}a(\rho(t))$.

(ii) *If* $\|t^{-1/r}a(t)\|_{r,(1,\infty)}<\infty$, *then*

$$\left(\bar{A}^{\mathcal{R},\mathcal{L}}_{\theta_0,p_0,c_0,r_0,b_0,q_0,a_0},\bar{A}_{1,r_1,b_1}\right)_{0,r,a}\subset\bar{A}^{\mathcal{R},\mathcal{L}}_{\theta_0,r,a\circ\rho,r_0,b_0,q_0,a_0}.\tag{236}$$

(iii) *If* $\|t^{-1/r}a(t)\|_{r,(0,1)}<\infty$, *then*

$$\left(\bar{A}^{\mathcal{R},\mathcal{L}}_{\theta_0,p_0,c_0,r_0,b_0,q_0,a_0},\bar{A}_{1,r_1,b_1}\right)_{1,r,a}=\bar{A}_{1,r,a^{\#}}\bigcap\bar{A}^{\mathcal{R}}_{1,r,a\circ\rho,r_1,b_1},\tag{237}$$

where $a^{\#}(t)=\chi_1(t)a(\rho(t))$.

Theorem 47. *Let* $0<\theta_0<1$, $0<r,p_0,q_0,r_0\le\infty$, $a,a_0,b_0,c_0\in SV$, $\|s^{-1/r_0}b_0(s)\|_{r_0,(0,1)}<\infty$, *and* $\|u^{-1/p_0}c_0(u)\|_{p_0,(1,\infty)}<\infty$. *Put* $\chi_0(t)=a_0(t)\|s^{-1/r_0}b_0(s)\|_{r_0,(0,t)}\|u^{-1/p_0}c_0(u)\|_{p_0,(t,\infty)}$ *and* $\rho(t)=t^{1-\theta_0}\chi_0(t)$.

(i) *If* $0<\theta<1$, *then*

$$\left(\bar{A}^{\mathcal{R},\mathcal{L}}_{\theta_0,p_0,c_0,r_0,b_0,q_0,a_0},A_1\right)_{\theta,r,a}=\bar{A}_{\eta,r,a^{\#}},\tag{238}$$

where $\eta=(1-\theta)\theta_0+\theta$ *and* $a^{\#}(t)=\chi_0(t)^{1-\theta}a(\rho(t))$.

(ii) *If* $\|t^{-1/r}a(t)\|_{r,(1,\infty)}<\infty$, *then*

$$\left(\bar{A}^{\mathcal{R},\mathcal{L}}_{\theta_0,p_0,c_0,r_0,b_0,q_0,a_0},A_1\right)_{0,r,a}\subset\bar{A}^{\mathcal{R},\mathcal{L}}_{\theta_0,r,a\circ\rho,r_0,b_0,q_0,a_0}.\tag{239}$$

(iii) *If* $\|t^{-1/r}a(t)\|_{r,(0,1)}<\infty$, *then*

$$\left(\bar{A}^{\mathcal{R},\mathcal{L}}_{\theta_0,p_0,c_0,r_0,b_0,q_0,a_0},A_1\right)_{1,r,a}=\bar{A}_{1,r,a\circ\rho}.\tag{240}$$

Theorem 48. *Let* $0<\theta_0<\theta_1<1$, $0<r,p_0,q_0,r_0,r_1,q_1\le\infty$, $a,a_0,b_0,a_1,b_1,c_0\in SV$, $\|s^{-1/r_0}b_0(s)\|_{r_0,(0,1)}<\infty$, $\|u^{-1/p_0}c_0(u)\|_{p_0,(1,\infty)}<\infty$, *and* $\|s^{-1/r_1}b_1(s)\|_{r_1,(0,1)}<\infty$. *Put* $\chi_0(t)=a_0(t)\|s^{-1/r_0}b_0(s)\|_{r_0,(0,t)}\|u^{-1/p_0}c_0(u)\|_{p_0,(t,\infty)}$, $\chi_1(t)=a_1(t)\|s^{-1/r_1}b_1(s)\|_{r_1,(0,t)}$, *and* $\rho(t)=t^{\theta_1-\theta_0}(\chi_0(t)/\chi_1(t))$.

(i) *If* $0<\theta<1$, *then*

$$\left(\bar{A}^{\mathcal{R},\mathcal{L}}_{\theta_0,p_0,c_0,r_0,b_0,q_0,a_0},\bar{A}^{\mathcal{R}}_{\theta_1,r_1,b_1,q_1,a_1}\right)_{\theta,r,a}=\bar{A}_{\eta,r,a^{\#}},\tag{241}$$

where $\eta=(1-\theta)\theta_0+\theta\theta_1$ *and* $a^{\#}(t)=\chi_0(t)^{1-\theta}\chi_1(t)^{\theta}a(\rho(t))$.

(ii) *If* $\|t^{-1/r}a(t)\|_{r,(1,\infty)}<\infty$, *then*

$$\left(\bar{A}^{\mathcal{R},\mathcal{L}}_{\theta_0,p_0,c_0,r_0,b_0,q_0,a_0},\bar{A}^{\mathcal{R}}_{\theta_1,r_1,b_1,q_1,a_1}\right)_{0,r,a}\subset\bar{A}^{\mathcal{R},\mathcal{L}}_{\theta_0,r,a\circ\rho,r_0,b_0,q_0,a_0}.\tag{242}$$

(iii) *If* $\|t^{-1/r}a(t)\|_{r,(0,1)}<\infty$, *then*

$$\left(\bar{A}^{\mathcal{R},\mathcal{L}}_{\theta_0,p_0,c_0,r_0,b_0,q_0,a_0}, \bar{A}^{\mathcal{R}}_{\theta_1,r_1,b_1,q_1,a_1}\right)_{1,r,a} = \bar{A}^{\mathcal{R}}_{\theta_1,r,a^\#,q_1,a_1} \bigcap \bar{A}^{\mathcal{R},\mathcal{R}}_{\theta_1,r,a\circ\rho,r_1,b_1,q_1,a_1}, \quad (243)$$

where $a^\#(t) = \|s^{-1/r_1} b_1(s)\|_{r_1,(0,t)} a(\rho(t))$.

Theorem 49. *Let* $0 < \theta_0 < \theta_1 < 1$, $0 < r, p_0, q_0, r_0, r_1, q_1 \le \infty$, $a_0, a_1, b_0, b_1 \in SV$, $\|s^{-1/r_0} b_0(s)\|_{r_0,(0,1)} < \infty$, $\|u^{-1/p_0} c_0(u)\|_{p_0,(1,\infty)} < \infty$, *and* $\|s^{-1/r_1} b_1(s)\|_{r_1,(1,\infty)} < \infty$. *Put* $\chi_0(t) = a_0(t)\|s^{-1/r_0} b_0(s)\|_{r_0,(0,t)} \|u^{-1/p_0} c_0(u)\|_{p_0,(t,\infty)}$, $\chi_1(t) = a_1(t)\|s^{-1/r_1} b_1(s)\|_{r_1,(t,\infty)}$, *and* $\rho(t) = t^{\theta_1-\theta_0}(\chi_0(t)/\chi_1(t))$.

(i) *If* $0 < \theta < 1$, *then*

$$\left(\bar{A}^{\mathcal{R},\mathcal{L}}_{\theta_0,p_0,c_0,r_0,b_0,q_0,a_0}, \bar{A}^{\mathcal{L}}_{\theta_1,r_1,b_1,q_1,a_1}\right)_{\theta,r,a} = \bar{A}_{\eta,r,a^\#}, \quad (244)$$

where $\eta = (1-\theta)\theta_0 + \theta\theta_1$ *and* $a^\#(t) = \chi_0(t)^{1-\theta}\chi_1(t)^\theta a(\rho(t))$.

(ii) *If* $\|t^{-1/r} a(t)\|_{r,(1,\infty)} < \infty$, *then*

$$\left(\bar{A}^{\mathcal{R},\mathcal{L}}_{\theta_0,p_0,c_0,r_0,b_0,q_0,a_0}, \bar{A}^{\mathcal{L}}_{\theta_1,r_1,b_1,q_1,a_1}\right)_{0,r,a} \subset \bar{A}^{\mathcal{R},\mathcal{L}}_{\theta_0,r,a\circ\rho,r_0,b_0,q_0,a_0}. \quad (245)$$

(iii) *If* $\|t^{-1/r} a(t)\|_{r,(0,1)} < \infty$, *then*

$$\left(\bar{A}^{\mathcal{R},\mathcal{L}}_{\theta_0,p_0,c_0,r_0,b_0,q_0,a_0}, \bar{A}^{\mathcal{L}}_{\theta_1,r_1,b_1,q_1,a_1}\right)_{1,r,a} = \bar{A}^{\mathcal{L},\mathcal{R}}_{\theta_1,r,a\circ\rho,r_1,b_1,q_1,a_1}. \quad (246)$$

The technique used in the section allows considering also some couples, where one of the operands is $\mathcal{L}\mathcal{L}$ or $\mathcal{R}\mathcal{R}$ space. In these cases, though, only "reiteration embedding" can be proved.

Theorem 50. *Let* $0 < \theta_0 < \theta_1 < 1$, $0 < r, p_0, q_0, r_0, r_1 \le \infty$, $a, a_0, b_0, c_0, b_1 \in SV$, $\|s^{-1/r_0} b_0(s)\|_{r_0,(1,\infty)} < \infty$, $\|u^{-1/p_0} c_0(u)\|_{p_0,(1,\infty)} < \infty$, *and* $\|s^{-1/r} a(s)\|_{r,(1,\infty)} < \infty$. *Put* $\chi_0(t) = a_0(t)\|s^{-1/r_0} b_0(s)\|_{r_0,(t,\infty)} \|u^{-1/p_0} c_0(u)\|_{p_0,(t,\infty)}$ *and* $\rho(t) = t^{\theta_1-\theta_0}(\chi_0(t)/b_1(t))$.

(i) *If* $0 < \theta < 1$, *then*

$$\left(\bar{A}^{\mathcal{L},\mathcal{L}}_{\theta_0,p_0,c_0,r_0,b_0,q_0,a_0}, \bar{A}_{\theta_1,r_1,b_1}\right)_{\theta,r,a} \supset \bar{A}_{\eta,r,a^\#}, \quad (247)$$

where $\eta = (1-\theta)\theta_0 + \theta\theta_1$ *and* $a^\#(t) = \chi_0(t)^{1-\theta} b_1(t)^\theta a(\rho(t))$.

(ii) *If* $\|t^{-1/r} a(t)\|_{r,(0,1)} < \infty$, *then*

$$\left(\bar{A}^{\mathcal{L},\mathcal{L}}_{\theta_0,p_0,c_0,r_0,b_0,q_0,a_0}, \bar{A}_{\theta_1,r_1,b_1}\right)_{1,r,a} \supset \bar{A}^{\mathcal{R}}_{\theta_1,r,a\circ\rho,r_1,b_1}. \quad (248)$$

Proof. Denote $X_0 = \bar{A}^{\mathcal{L}}_{\theta_0,r_0,b_0,q_0,a_0}$, $X_1 = \bar{A}_{\theta_1,r_1,b_1}$, and $\psi(t) = a_0(t)\|s^{-1/r_0} b_0(s)\|_{r_0,(t,\infty)}$. Let σ be a strongly increasing, differentiable function such that $\sigma(t) \approx t^{\theta_1-\theta_0}(\psi(t)/b_1(t))$ and $\sigma(1) = 1$. Additionally, denote $x = c_0 \circ \sigma^{(-1)}$, $Z = (\bar{A}^{\mathcal{L},\mathcal{L}}_{\theta_0,p_0,c_0,r_0,b_0,q_0,a_0}, \bar{A}_{\theta_1,r_1,b_1})_{\theta,r,a}$, and $y(t) = (\|s^{-1/p_0} x(s)\|_{p_0,(t,\infty)})^{1-\theta} a(t\|s^{-1/p_0} x(s)\|_{p_0,(t,\infty)})$. By the change of variables $u = \sigma^{(-1)}(s)$, we get $\|s^{-1/p_0} x(s)\|_{p_0,(\sigma(t),\infty)} \approx \|u^{-1/p_0} c_0(u)\|_{p_0,(t,\infty)}$ and

$$y(\sigma(t)) \approx \left(\|u^{-1/p_0} c_0(u)\|_{p_0,(t,\infty)}\right)^{1-\theta} a(\rho(t)). \quad (249)$$

By Theorem 23, we obtain

$$\bar{A}^{\mathcal{L},\mathcal{L}}_{\theta_0,p_0,c_0,r_0,b_0,q_0,a_0} \supset \left(\bar{A}^{\mathcal{L}}_{\theta_0,r_0,b_0,q_0,a_0}, \bar{A}_{\theta_1,r_1,b_1}\right)_{0,p_0,x}. \quad (250)$$

Thus, $Z \supset (\bar{X}_{0,p_0,x}, X_1)_{\theta,r,a}$. Using Theorem 16, we conclude that for $0 < \theta \le 1$, it holds

$$\left(\bar{X}_{0,p_0,x}, X_1\right)_{\theta,r,a} = \bar{X}_{\theta,r,y} = \left(\bar{A}^{\mathcal{L}}_{\theta_0,r_0,b_0,q_0,a_0}, \bar{A}_{\theta_1,r_1,b_1}\right)_{\theta,r,y}. \quad (251)$$

If $0 < \theta < 1$, by [5] (Theorem 19) (i) and (249), we get $Z \supset \bar{A}_{\eta,r,a^\#}$, where $a^\#(t) = \psi(t)^{1-\theta} b_1(t)^\theta y(\sigma(t)) \approx \chi_0(t)^{1-\theta} b_1(t)^\theta a(\rho(t))$.

Let $\theta = 1$. Using [5] (Theorem 11) (ii), we arrive at

$$Z \supset \left(\bar{A}^{\mathcal{L}}_{\theta_0,r_0,b_0,q_0,a_0}, \bar{A}_{\theta_1,r_1,b_1}\right)_{1,r,y} = \bar{A}^{\mathcal{R}}_{\theta_1,r,y\circ\sigma,r_1,b_1} = \bar{A}^{\mathcal{R}}_{\theta_1,r,a\circ\rho,r_1,b_1}. \quad (252)$$

□

Similar theorems can be formulated and proved also for the couples $(\bar{A}^{\mathcal{L},\mathcal{L}}_{\theta_0,p_0,c_0,r_0,b_0,q_0,a_0}, \bar{A}_{1,r_1,b_1})$, $(\bar{A}^{\mathcal{L},\mathcal{L}}_{\theta_0,p_0,c_0,r_0,b_0,q_0,a_0}, A_1)$, $(\bar{A}^{\mathcal{L},\mathcal{L}}_{\theta_0,p_0,c_0,r_0,b_0,q_0,a_0}, \bar{A}^{\mathcal{L}}_{\theta_1,r_1,b_1,q_1,a_1})$, $(\bar{A}^{\mathcal{L},\mathcal{L}}_{\theta_0,p_0,c_0,r_0,b_0,q_0,a_0}, \bar{A}^{\mathcal{R}}_{\theta_1,r_1,b_1,q_1,a_1})$, $(\bar{A}_{\theta_0,r_0,b_0}, \bar{A}^{\mathcal{R},\mathcal{R}}_{\theta_1,p_1,c_1,r_1,b_1,q_1,a_1})_{\theta,r,a}$, $(\bar{A}_{0,r_0,b_0}, \bar{A}^{\mathcal{R},\mathcal{R}}_{\theta_1,p_1,c_1,r_1,b_1,q_1,a_1})$, $(A_0, \bar{A}^{\mathcal{R},\mathcal{R}}_{\theta_1,p_1,c_1,r_1,b_1,q_1,a_1})$, $(\bar{A}^{\mathcal{R}}_{\theta_0,r_0,b_0,q_0,a_0}, \bar{A}^{\mathcal{R},\mathcal{R}}_{\theta_1,p_1,c_1,r_1,b_1,q_1,a_1})$, and $(\bar{A}^{\mathcal{L}}_{\theta_0,r_0,b_0,q_0,a_0}, \bar{A}^{\mathcal{R},\mathcal{R}}_{\theta_1,p_1,c_1,r_1,b_1,q_1,a_1})$. We leave this to the reader.

9. Applications

Here, we demonstrate how our general reiteration theorems can be used to establish limiting interpolation results for the grand and small Lorentz spaces. For the sake of shortness, we present only some possible results.

Let (Ω, μ) denote a σ-finite measure space with a non-atomic measure μ. We consider functions f from the set $\mathfrak{M}(\Omega, \mu)$ of all μ-measurable functions on Ω. As conventional (see, e.g., [1]), $f^*(t)$ $(t > 0)$ denotes the nonincreasing rearrangement of f and the maximal function of f^* is defined by

$$f^{**}(t) := \frac{1}{t}\int_0^t f^*(u)du. \tag{253}$$

For simplicity, we consider below only interpolation spaces between L^1 and L^∞. For functions from these spaces, Peetre has shown that ([1], Theorem V.1.6)

$$K(t, f; L_1, L_\infty) = \int_0^t f^*(s)ds = tf^{**}(t). \tag{254}$$

The following assertion is a modification of [10] (Lemma 5.2) and can be proved similarly.

Lemma 51. *Let $0 < \theta \le 1$, $0 < q \le \infty$, $b \in SV$, and $p = 1/(1-\theta)$. If $\theta = 1$, we additionally suppose that $\|u^{-1/q}b(u)\|_{q,(0,1)} < \infty$. Then, for all $f \in L_1 + L_\infty$ and $t > 0$,*

$$\begin{aligned}&\left\|u^{-\theta-(1/q)}b(u)K(u,f;L_1,L_\infty)\right\|_{q,(0,t)}\\&\quad=\left\|u^{(1/p)-(1/q)}b(u)f^{**}(u)\right\|_{q,(0,t)}\approx\left\|u^{(1/p)-(1/q)}b(u)f^{*}(u)\right\|_{q,(0,t)}.\end{aligned} \tag{255}$$

9.1. Function Spaces. First, we define function spaces under consideration.

Definition 52. Let $0 < p, q \le \infty$ and $b \in SV$. The Lorentz–Karamata space $L_{p,q;b}$ is the set of all $f \in \mathfrak{M}(\Omega, \mu)$ such that

$$\|f\|_{p,q;\alpha} := \left\|t^{(1/p)-(1/q)}b(t)f^*(t)\right\|_{q,(0,\infty)} < \infty. \tag{256}$$

The Lorentz–Karamata spaces form an important scale of spaces. It contains, e.g., the Lebesgue space L_p, Lorentz space $L_{p,q}$, Lorentz–Zygmund, and the generalized Lorentz–Zygmund spaces. For further information about Lorentz–Karamata spaces, we refer to, e.g., [10, 11, 16]. They have found many different important applications in analysis (see, e.g., [1, 4, 10–12, 14, 16–19] and the references therein).

Lemma 51 implies the following interpolation result.

Lemma 53 (cf. [10], Corollary 5.3; [9]). *Let $0 < \theta \le 1$, $0 < q \le \infty$, $b \in SV$, and $p = 1/(1-\theta)$. Then, $(L_1, L_\infty)_{\theta,q,b} = L_{p,q,b}$.*

Definition 54 (cf. [10], (5.21), (5.33)). Let $1 < p < \infty$, $0 < q, r \le \infty$, and $a, b \in SV$. The spaces $L^{\mathscr{L}}_{p,r,b,q,a}$ and $L^{\mathscr{R}}_{p,r,b,q,a}$ are the sets of all $f \in \mathfrak{M}(\Omega, \mu)$ such that

$$\|f\|_{L^{\mathscr{L}}_{p,r,b,q,a}} := \left\|t^{-1/r}b(t)\left\|u^{(1/p)-(1/q)}a(u)f^*(u)\right\|_{q,(0,t)}\right\|_{r,(0,\infty)} < \infty, \tag{257}$$

or

$$\|f\|_{L^{\mathscr{R}}_{p,r,b,q,a}} := \left\|t^{-1/r}b(t)\left\|u^{(1/p)-(1/q)}a(u)f^{**}(u)\right\|_{q,(t,\infty)}\right\|_{r,(0,\infty)} < \infty, \tag{258}$$

correspondingly.

We will require that $\|t^{-1/r}b(t)\|_{r,(1,\infty)} < \infty$ for $L^{\mathscr{L}}_{p,r,b,q,a}$ and $\|t^{-1/r}b(t)\|_{r,(0,1)} < \infty$ for $L^{\mathscr{R}}_{p,r,b,q,a}$. Otherwise, the corresponding space consists only of the null element. Similar definitions can be found in [8, 9, 17]. We refer to the spaces $L^{\mathscr{L}}_*$ and $L^{\mathscr{R}}_*$ as $L^{\mathscr{L}}$ and $L^{\mathscr{R}}$ spaces, respectively. Note that the spaces $L^{\mathscr{L}}_*$ are a special case of the generalized gamma space with double weights [3].

In order to be able to compare our results with the results from [3, 4, 7–9], we introduce the generalized grand and small Lorentz spaces. Note that the interpolation results in [3, 4] do not contain limiting cases $\theta \in \{0, 1\}$.

Definition 55. Let $1 < p < \infty$, $0 < q, r \le \infty$, and $b \in SV$. We define the small Lorentz space $L^{(p,q,r}_b$ as the set of all $f \in \mathfrak{M}(\Omega, \mu)$ such that

$$\|f\|_{L^{(p,q,r}_b} := \left\|t^{-1/r}b(t)\left\|u^{(1/p)-(1/q)}f^*(u)\right\|_{q,(0,t)}\right\|_{r,(0,\infty)} < \infty. \tag{259}$$

We define the grand Lorentz space $L^{p),q,r}_b$ as the set of all $f \in \mathfrak{M}(\Omega, \mu)$ such that

$$\|f\|_{L^{p),q,r}_b} := \left\|t^{-1/r}b(t)\left\|u^{(1/p)-(1/q)}f^{**}(u)\right\|_{q,(t,\infty)}\right\|_{r,(0,\infty)} < \infty. \tag{260}$$

It is clear that $L^{(p,q,r}_b = L^{\mathscr{L}}_{p,r,b,q,1}$ and $L^{p),q,r}_b = L^{\mathscr{R}}_{p,r,b,q,1}$. Grand and small Lebesgue and Lorentz spaces find many different important applications and have been intensively studied by many authors (see [3, 4, 6–9] and the references therein). Normally, they are defined on a bounded domain Ω in R^n with measure 1. In [4], it is required that the functions are real-valued. We do not require that $\mu(\Omega) = 1$ and neither that the functions are real-valued.

The following lemma characterizes the $L^{\mathscr{L}}$ and $L^{\mathscr{R}}$ spaces as appropriate $\mathscr{L}$ and $\mathscr{R}$ limiting interpolation spaces. It follows from the corresponding definitions, formula (254), and Lemma 51.

Lemma 56 (see [10], Lemmas 5.4 and 5.9). *Let $1 < p < \infty$, $0 < q, r \le \infty$, $a, b \in SV$, and $\theta = 1 - (1/p)$. Then, $L^{\mathscr{L}}_{p,r,b,q,a} = (L^1, L^\infty)^{\mathscr{L}}_{\theta,r,b,q,a}$ and $L^{\mathscr{R}}_{p,r,b,q,a} = (L^1, L^\infty)^{\mathscr{R}}_{\theta,r,b,q,a}$.*

In particular, $L^{(p,q,r}_b = (L^1, L^\infty)^{\mathscr{L}}_{\theta,r,b,q,1}$ and $L^{p),q,r}_b =$

$(L^1, L^\infty)^{\mathscr{R}}_{\theta,r,b,q,1}$.

Definition 57. Let $1 < p < \infty$, $0 < q, r, s \le \infty$, and $a, b, c \in SV$. The spaces $L^{\mathscr{L},\mathscr{L}}_{p,(s,c,r,b,q,a)}$, $L^{\mathscr{L},\mathscr{R}}_{p,(s,c,r,b,q,a)}$, $L^{\mathscr{R},\mathscr{L}}_{p,(s,c,r,b,q,a)}$, and $L^{\mathscr{R},\mathscr{R}}_{p,(s,c,r,b,q,a)}$ are the sets of all $f \in \mathfrak{M}(\Omega,\mu)$ such that

$$\|f\|_{L^{\mathscr{L},\mathscr{L}}_{p,(s,c,r,b,q,a)}} := \left\| u^{-1/s}c(u) \left\| t^{-1/r} b(t) \left\| v^{(1/p)-(1/q)} a(v) f^*(v)\right\|_{q,(0,t)}\right\|_{r,(0,u)}\right\|_{s,(0,\infty)} < \infty, \tag{261}$$

$$\|f\|_{L^{\mathscr{L},\mathscr{R}}_{p,(s,c,r,b,q,a)}} := \Big\| u^{-1/s}c(u) \Big\| t^{-1/r} b(t) \cdot \left\| v^{(1/p)-(1/q)} a(v) f^{**}(v)\right\|_{q,(u,t)}\Big\|_{r,(u,\infty)}\Big\|_{s,(0,\infty)} < \infty, \tag{262}$$

$$\|f\|_{L^{\mathscr{R},\mathscr{L}}_{p,(s,c,r,b,q,a)}} := \left\| u^{-1/s}c(u) \left\| t^{-1/r} b(t) \left\| v^{(1/p)-(1/q)} a(v) f^{**}(v)\right\|_{q,(t,u)}\right\|_{r,(0,u)}\right\|_{s,(0,\infty)} < \infty, \tag{263}$$

or

$$\|f\|_{L^{\mathscr{R},\mathscr{R}}_{p,(s,c,r,b,q,a)}} := \left\| u^{-1/s}c(u) \left\| t^{-1/r} b(t) \left\| v^{(1/p)-(1/q)} a(v) f^{**}(v)\right\|_{q,(t,\infty)}\right\|_{r,(u,\infty)}\right\|_{s,(0,\infty)} < \infty, \tag{264}$$

correspondingly.

Remark 58. Because of Lemma 51, if in the formulae (257), (259), and (261) f^* will be replaced with f^{**}, one gets equivalent (quasi-)norms for corresponding spaces.

In view of Lemma 51, Remark 58, and (254), the following analogue of Lemma 56 holds.

Lemma 59. *Let $1 < p < \infty$, $0 < q, r, s \le \infty$, $a, b \in SV$, and $\theta = 1 - (1/p)$. Then,*

$$\begin{aligned} L^{\mathscr{L},\mathscr{L}}_{p,(s,c,r,b,q,a)} &= \left(L^1, L^\infty\right)^{\mathscr{L},\mathscr{L}}_{\theta,s,c,r,b,q,a}, \\ L^{\mathscr{L},\mathscr{R}}_{p,(s,c,r,b,q,a)} &= \left(L^1, L^\infty\right)^{\mathscr{L},\mathscr{R}}_{\theta,s,c,r,b,q,a}, \\ L^{\mathscr{R},\mathscr{L}}_{p,(s,c,r,b,q,a)} &= \left(L^1, L^\infty\right)^{\mathscr{R},\mathscr{L}}_{\theta,s,c,r,b,q,a}, \\ L^{\mathscr{R},\mathscr{R}}_{p,(s,c,r,b,q,a)} &= \left(L^1, L^\infty\right)^{\mathscr{R},\mathscr{R}}_{\theta,s,c,r,b,q,a}. \end{aligned} \tag{265}$$

9.2. Limiting Interpolation between the Small or Great Lorentz Spaces and the Lorentz–Karamata Spaces. Using results from Sections 4–6 and Lemmas 56 and 59, we are able to characterize some limiting interpolation spaces lying between $L^{\mathscr{L}}$ or $L^{\mathscr{R}}$ spaces and Lorentz–Karamata spaces. In the corollaries below, we restrict ourselves to the grand and small Lorentz spaces.

Corollary 60 (cf. [9], Corollary 7.8; [7], Corollary 5.12). *Let $1 < p_0 < \infty$, $0 < r, r_0, q_0 \le \infty$, $a, b_0 \in SV$, and $\|s^{-1/r_0} b_0(s)\|_{r_0,(1,\infty)} < \infty$. Put $\chi_0(t) = \|s^{-1/r_0} b_0(s)\|_{r_0,(t,\infty)}$ and $\rho(t) = t^{1/p_0}\chi_0(t)$.*

(i) If $\|s^{-1/r} a(s)\|_{r,(1,\infty)} < \infty$, then

$$\left(L^{(p_0,q_0,r_0}_{b_0}, L_\infty\right)_{0,r,a} = L^{(p_0,q_0,r}_{a^\#} \bigcap L^{\mathscr{L},\mathscr{L}}_{p_0,(r,a\circ\rho,r_0,b_0,q_0,1)}, \tag{266}$$

where $a^\#(t) = \chi_0(t) a(\rho(t))$.

(ii) If $\|s^{-1/r} a(s)\|_{r,(0,1)} < \infty$, then

$$\left(L^{(p_0,q_0,r_0}_{b_0}, L_\infty\right)_{1,r,a} \cong L_{\infty,r,a\circ\rho}. \tag{267}$$

Proof. Let $\theta_0 = 1 - (1/p_0)$. By Lemmas 56 and 59 and Theorem 25, we have

$$\begin{aligned} \left(L^{(p_0,q_0,r_0}_{b_0}, L_\infty\right)_{0,r,a} &= \left(\left(L^1, L^\infty\right)^{\mathscr{L}}_{\theta_0,r_0,b_0,q_0,1}, L_\infty\right)_{0,r,a} \\ &= \left(L^1, L^\infty\right)^{\mathscr{L}}_{\theta_0,r,a^\#,q_0,1} \bigcap \left(L^1, L^\infty\right)^{\mathscr{L},\mathscr{L}}_{\theta_0,r,a\circ\rho,r_0,b_0,q_0,1} \\ &= L^{(p_0,q_0,r}_{a^\#} \bigcap L^{\mathscr{L},\mathscr{L}}_{p_0,(r,a\circ\rho,r_0,b_0,q_0,1)}. \end{aligned} \tag{268}$$

Similarly, by [5] (Theorem 13) and Lemma 53, we get

$$\begin{aligned} \left(L^{(p_0,q_0,r_0}_{b_0}, L_\infty\right)_{1,r,a} &= \left(\left(L^1, L^\infty\right)^{\mathscr{L}}_{\theta_0,r_0,b_0,q_0,1}, L_\infty\right)_{1,r,a} = \left(L_1, L_\infty\right)_{\infty,r,a\circ\rho} \\ &= L_{\infty,r,a\circ\rho}. \end{aligned} \tag{269}$$

This completes the proof. □

Corollary 61 (cf. [7], Corollary 5.10). *Let $1 < p_0 < p_1 < \infty$, $0 < r, r_0, r_1, q_0 \le \infty$, $a, b_0, b_1 \in SV$, and $\|s^{-1/r_0} b_0(s)\|_{r_0,(1,\infty)} < \infty$. Put $\chi_0(t) = \|s^{-1/r_0} b_0(s)\|_{r_0,(t,\infty)}$ and $\rho(t) = t^{(1/p_0)-(1/p_1)}(\chi_0(t)/b_1(t))$. Then,*

(i) If $\|s^{-1/r} a(s)\|_{r,(1,\infty)} < \infty$, then

$$\left(L^{(p_0,q_0,r_0}_{b_0}, L_{p_1,r_1,b_1}\right)_{0,r,a} = L^{(p_0,q_0,r}_{a^\#} \bigcap L^{\mathscr{L},\mathscr{L}}_{p_0,(r,a\circ\rho,r_0,b_0,q_0,1)}, \tag{270}$$

where $a^\#(t) = \chi_0(t) a(\rho(t))$.

(ii) If $\|s^{-1/r} a(s)\|_{r,(0,1)} < \infty$, then

$$\left(L^{(p_0,q_0,r_0}_{b_0}, L_{p_1,r_1,b_1}\right)_{1,r,a} \cong L^{p_1),r_1,r}_{a\circ\rho}. \tag{271}$$

Proof. Let $\theta_0 = 1 - (1/p_0)$ and $\theta_1 = 1 - (1/p_1)$. By Lemmas 56, 53, and 59 and Theorem 23, we have

$$\begin{aligned}\left(L_{b_0}^{(p_0,q_0,r_0},L_{p_1,r_1,b_1}\right)_{0,r,a}&=\left(\left(L^1,L^\infty\right)^{\mathscr{L}}_{\theta_0,r_0,b_0,q_0,1},\left(L_1,L_\infty\right)_{\theta_1,r_1,b_1}\right)_{0,r,a}\\&=\left(L_1,L_\infty\right)^{\mathscr{L}}_{\theta_0,r,a^\#,q_0,1}\bigcap\left(L_1,L_\infty\right)^{\mathscr{L},\mathscr{L}}_{\theta_0,r,a\circ\rho,r_0,b_0,q_0,1}\\&=L_{a^\#}^{(p_0,q_0,r}\bigcap L^{\mathscr{L},\mathscr{L}}_{p_0,(r,a\circ\rho,r_0,b_0,q_0,1)}.\end{aligned}\tag{272}$$

Similarly, by [5] (Theorem 11), we get

$$\begin{aligned}\left(L_{b_0}^{(p_0,q_0,r_0},L_{p_1,r_1,b_1}\right)_{1,r,a}&=\left(\left(L^1,L^\infty\right)^{\mathscr{L}}_{\theta_0,r_0,b_0,q_0,1},\left(L_1,L_\infty\right)_{\theta_1,r_1,b_1}\right)_{1,r,a}\\&=\left(L_1,L_\infty\right)^{\mathscr{R}}_{\theta_1,r,a\circ\rho,r_1,1}=L_{a\circ\rho}^{p_1),r_1,r}.\end{aligned}\tag{273}$$

This completes the proof. ☐

Analogously, by using Theorems 28 and 26 and [5] (Theorem 23), the next two corollaries can be proved.

Corollary 62. (cf. [6], Corollary 5.9). *Let $1<p_0<\infty$, $0<r$, $r_0,q_0\le\infty$, $a,b_0\in SV$, and $\|s^{-1/r_0}b_0(s)\|_{r_0,(0,1)}<\infty$. Put $\rho(t)=t^{1/p_0}\|s^{-1/r_0}b_0(s)\|_{r_0,(0,t)}$. Then,*

(i) If $\|s^{-1/r}a(s)\|_{r,(1,\infty)}<\infty$, then

$$\left(L_{b_0}^{p_0),q_0,r_0},L_\infty\right)_{0,r,a}=L^{\mathscr{R},\mathscr{L}}_{p_0,(r,a\circ\rho,r_0,b_0,q_0,1)}.\tag{274}$$

(ii) If $\|s^{-1/r}a(s)\|_{r,(0,1)}<\infty$, then

$$\left(L_{b_0}^{p_0),q_0,r_0},L_\infty\right)_{1,r,a}\cong L_{\infty,r,a\circ\rho}.\tag{275}$$

Corollary 63 (cf. [6], Corollary 5.5). *Let $1<p_0<p_1<\infty$, $0<r,r_0,r_1,q_0\le\infty$, $a,b_0,b_1\in SV$, $\|s^{-1/r_0}b_0(s)\|_{r_0,(0,1)}<\infty$, and $\|s^{-1/r}a(s)\|_{r,(1,\infty)}<\infty$. Put $\rho(t)=t^{(1/p_0)-(1/p_1)}(\|s^{-1/r_0}b_0(s)\|_{r_0,(0,t)}/b_1(t))$. Then,*

$$\left(L_{b_0}^{p_0),q_0,r_0},L_{p_1,r_1,b_1}\right)_{0,r,a}=L^{\mathscr{R},\mathscr{L}}_{p_0,(r,a\circ\rho,r_0,b_0,q_0,1)}.\tag{276}$$

Four corollaries above amend Corollaries 38–40 and 47–49 from [5]. Other corollaries from Sections 8.2 and 8.3 of [5] can be similarly amended.

9.3. Limiting Interpolation Formulae for Couples Where Both Operands Are Small or Grand Lorentz Spaces. Four corollaries below amend corollaries from Section 8.4 of [5]. We prove only the first one. Three further corollaries can be proved analogously using theorems from Section 7.

Corollary 64 (cf. [6], Corollary 5.12; [8], Corollary 5.7). *Let $1<p_0<p_1<\infty$, $0<r,r_0,r_1,q_0,q_1\le\infty$, $b_0,b_1\in SV$, $\|s^{-1/r_0}b_0(s)\|_{r_0,(1,\infty)}<\infty$, and $\|s^{-1/r_1}b_1(s)\|_{r_1,(1,\infty)}<\infty$. Put $\chi_0(t)=\|s^{-1/r_0}b_0(s)\|_{r_0,(t,\infty)}$, $\chi_1(t)=\|s^{-1/r_1}b_1(s)\|_{r_1,(t,\infty)}$, and $\rho(t)=t^{(1/p_0)-(1/p_1)}(\chi_0(t)/\chi_1(t))$.*

(i) If $\|s^{-1/r}a(s)\|_{r,(1,\infty)}<\infty$, then

$$\left(L_{b_0}^{(p_0,q_0,r_0},L_{b_1}^{(p_1,q_1,r_1}\right)_{0,r,a}=L_{a^\#}^{(p_0,q_0,r}\bigcap L^{\mathscr{L},\mathscr{L}}_{p_0,(r,a\circ\rho,r_0,b_0,q_0,1)},\tag{277}$$

where $a^\#(t)=\chi_0(t)a(\rho(t))$.

(ii) If $\|s^{-1/r}a(s)\|_{r,(0,1)}<\infty$, then

$$\left(L_{b_0}^{(p_0,q_0,r_0},L_{b_1}^{(p_1,q_1,r_1}\right)_{1,r,a}=L^{\mathscr{L},\mathscr{R}}_{p_1,(r,a\circ\rho,r_1,b_1,q_1,1)}.\tag{278}$$

Proof. Let $\theta_0=1-(1/p_0)$ and $\theta_1=1-(1/p_1)$. Using Lemmas 56 and 59 and Theorem 33, we get

$$\begin{aligned}\left(L_{b_0}^{(p_0,q_0,r_0},L_{b_1}^{(p_1,q_1,r_1}\right)_{0,r,a}&=\left(\left(L^1,L^\infty\right)^{\mathscr{L}}_{\theta_0,r_0,b_0,q_0,1},\left(L^1,L^\infty\right)^{\mathscr{L}}_{\theta_1,r_1,b_1,q_1,1}\right)_{0,r,a}\\&=\left(L_1,L_\infty\right)^{\mathscr{L}}_{\theta_0,r,a^\#,q_0,1}\bigcap\left(L_1,L_\infty\right)^{\mathscr{L},\mathscr{L}}_{\theta_0,r,a\circ\rho,r_0,b_0,q_0,1}\\&=L_{a^\#}^{(p_0,q_0,r}\bigcap L^{\mathscr{L},\mathscr{L}}_{p_0,(r,a\circ\rho,r_0,b_0,q_0,1)}.\end{aligned}\tag{279}$$

Similarly,

$$\left(L_{b_0}^{(p_0,q_0,r_0},L_{b_1}^{(p_1,q_1,r_1}\right)_{1,r,a}=\left(L_1,L_\infty\right)^{\mathscr{L},\mathscr{R}}_{\theta_1,r,a\circ\rho,r_1,b_1,q_1,1}=L^{\mathscr{L},\mathscr{R}}_{p_1,(r,a\circ\rho,r_1,b_1,q_1,1)}.\tag{280}$$

This completes the proof. ☐

Corollary 65 (cf. [6], Corollary 5.7; [8], Corollary 5.5). *Let $1<p_0<p_1<\infty$, $0<r,r_0,r_1,q_0,q_1\le\infty$, $b_0,b_1\in SV$, $\|s^{-1/r_0}b_0(s)\|_{r_0,(0,1)}<\infty$, and $\|s^{-1/r_1}b_1(s)\|_{r_1,(0,1)}<\infty$. Put $\chi_0(t)=\|s^{-1/r_0}b_0(s)\|_{r_0,(0,t)}$, $\chi_1(t)=\|s^{-1/r_1}b_1(s)\|_{r_1,(0,t)}$, and $\rho(t)=t^{(1/p_0)-(1/p_1)}(\chi_0(t)/\chi_1(t))$.*

(i) If $\|s^{-1/r}a(s)\|_{r,(1,\infty)}<\infty$, then

$$\left(L_{b_0}^{p_0),q_0,r_0},L_{b_1}^{p_1),q_1,r_1}\right)_{0,r,a}=L^{\mathscr{R},\mathscr{L}}_{p_0,(r,a\circ\rho,r_0,b_0,q_0,1)}.\tag{281}$$

(ii) If $\|s^{-1/r}a(s)\|_{r,(0,1)}<\infty$, then

$$\left(L_{b_0}^{p_0),q_0,r_0},L_{b_1}^{p_1),q_1,r_1}\right)_{1,r,a}=L_{a^\#}^{p_1),q_1,r_1}\bigcap L^{\mathscr{R},\mathscr{R}}_{p_1,(r,a\circ\rho,r_1,b_1,q_1,1)},\tag{282}$$

where $a^\#(t)=\chi_1(t)a(\rho(t))$.

Corollary 66 (cf. [7], Theorem 5.7; [8], Corollary 5.10). *Let* $1 < p_0 < p_1 < \infty$, $0 < r, r_0, r_1, q_0, q_1 \le \infty$, $b_0, b_1 \in SV$, $\|s^{-1/r_0} b_0(s)\|_{r_0,(1,\infty)} < \infty$, *and* $\|s^{-1/r_1} b_1(s)\|_{r_1,(0,1)} < \infty$. *Put* $\chi_0(t) = \|s^{-1/r_0} b_0(s)\|_{r_0,(t,\infty)}$, $\chi_1(t) = \|s^{-1/r_1} b_1(s)\|_{r_1,(0,t)}$, *and* $\rho(t) = t^{(1/p_0)-(1/p_1)}(\chi_0(t)/\chi_1(t))$.

(i) *If* $\|s^{-1/r} a(s)\|_{r,(1,\infty)} < \infty$, *then*

$$\left(L_{b_0}^{(p_0,q_0,r_0}, L_{b_1}^{p_1),q_1,r_1}\right)_{0,r,a} = L_{a^{\#}}^{(p_0,q_0,r} \bigcap L_{p_0,(r,a\circ\rho,r_0,b_0,q_0,1)}^{\mathscr{L},\mathscr{L}}, \quad (283)$$

where $a^{\#}(t) = \chi_0(t) a(\rho(t))$.

(ii) *If* $\|s^{-1/r} a(s)\|_{r,(0,1)} < \infty$, *then*

$$\left(L_{b_0}^{(p_0,q_0,r_0}, L_{b_1}^{p_1),q_1,r_1}\right)_{1,r,a} = L_{a^{\#}}^{p_1),q_1,r} \bigcap L_{p_1,(r,a\circ\rho,r_1,b_1,q_1,1)}^{\mathscr{R},\mathscr{R}}, \quad (284)$$

where $a^{\#}(t) = \chi_1(t) a(\rho(t))$.

Corollary 67 (cf. [8], Corollary 5.9). *Let* $1 < p_0 < p_1 < \infty$, $0 < r, r_0, r_1, q_0, q_1 \le \infty$, $b_0, b_1 \in SV$, $\|s^{-1/r_0} b_0(s)\|_{r_0,(0,1)} < \infty$, *and* $\|s^{-1/r_1} b_1(s)\|_{r_1,(1,\infty)} < \infty$. *Put* $\chi_0(t) = \|s^{-1/r_0} b_0(s)\|_{r_0,(0,t)}$, $\chi_1(t) = \|s^{-1/r_1} b_1(s)\|_{r_1,(t,\infty)}$, *and* $\rho(t) = t^{(1/p_0)-(1/p_1)}(\chi_0(t)/\chi_1(t))$.

(i) *If* $\|s^{-1/r} a(s)\|_{r,(1,\infty)} < \infty$, *then*

$$\left(L_{b_0}^{p_0),q_0,r_0}, L_{b_1}^{(p_1,q_1,r_1}\right)_{0,r,a} = L_{p_0,(r,a\circ\rho,r_0,b_0,q_0,1)}^{\mathscr{R},\mathscr{L}}. \quad (285)$$

(ii) *If* $\|s^{-1/r} a(s)\|_{r,(0,1)} < \infty$, *then*

$$\left(L_{b_0}^{p_0),q_0,r_0}, L_{b_1}^{(p_1,q_1,r_1}\right)_{1,r,a} = L_{p_1,(r,a\circ\rho,r_1,b_1,q_1,1)}^{\mathscr{L},\mathscr{R}}. \quad (286)$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there is no conflict of interest regarding the publication of this paper.

Acknowledgments

The author is grateful to Pedro Fernández-Martínez and Teresa Signes for the possibility to read the paper [8] before it was posted in arXiv and for the productive discussion about our papers. We address a grateful thanks to Teresa Signes for pointing out a wrong statement in [5]. See Remark 39.

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