Research Article

Discrete Fractional Inequalities Pertaining a Fractional Sum Operator with Some Applications on Time Scales

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This content replicates some discrete nonlinear fractional inequalities by virtue of the fractional sum operator $\bar{\Psi}$ on time scales. Through the recognition of the principle of discrete fractional calculus, we are able to recover the precise estimates for unknown functions of inequalities of the Gronwall type. The resultant inequalities are of unique structure comparative with the latest reviewing disclosures and can be described as a complementary tool for numerically testing the solutions of discrete partial differential equations. The foremost consequences are probably confirmed via handling of assessment procedure and technique of mean value speculation. We display few examples of the proposed inequalities to represent the incentives of our effort.

1. Introduction and Essentials

Fractional calculus and its conceptual applications have acquired a huge amount of potential in terms of the reality that fractional operators are becoming a valuable asset with more specificity and success in demonstrating a few complicated discoveries in numerous seemingly diverse and wide fields of science and in many areas, such as fluid flow, physics, chaos, image analysis, virology, and financial economy [1, 2]. A few years earlier, fractional differential equations and dynamic systems have been validated as being significant gadgets in showing a few marvels in different parts of applied and pure sciences. They draw enormous importance in research-oriented fields (see the basic monograph and the interesting paper [3, 4]). The set of implications that encouraged the formation of a discrete fractional theory is established [5].

The aim of the paper is to impose discrete fractional sum equations in order to build up a procedure to comprehend certain equations and to extract corresponding Gronwall sort of inequality. Especially, Gronwall's inequality is illustrated to be among the primary inequalities for the foundation of differential equations. From now on and into the near future, various assumptions and development of such inequalities

have ended up being a major component. Discrete Gronwall inequality was suggested by Sugiyama [6] in 1969. He carried out the related framework of reliable and discrete type of Gronwall inequality:

Theorem 1. Let $\mathcal{L}(\bar{\Omega})$ and $\mathfrak{f}(\bar{\Omega})$ be real-valued functions defined for $\bar{\Omega} \in \mathbb{N}_0$ and $\mathfrak{f}(\bar{\Omega}) > 0$ for every $\bar{\Omega} \in \mathbb{N}_0$. If

$$\mathcal{L}(\bar{\Omega}) \leq \hat{\omega}_0 + \sum_{\bar{p}=0}^{\bar{\Omega}-1} \mathfrak{f}(\bar{p}) \mathcal{L}(\bar{p}), \quad \bar{\Omega} \in \mathbb{N}_0, \quad (1)$$

where $\hat{\omega}_0$ is a nonnegative constant, and then

$$\mathcal{L}(\bar{\Omega}) \leq \hat{\omega}_0 \prod_{\bar{p}=0}^{\bar{\Omega}-1} [1 + \mathfrak{f}(\bar{p})]. \quad (2)$$

Theorem 1 is often used differential and integral equations that possess the unification of discrete factor models.

It is interesting that discretization cycle is among the most demanded tools for researchers who are captivated in multiplication and computational assessment. Keep in mind

that not all discrete operators have identical characteristics to continuous ones, and the formation of discrete fractional calculus is becoming an essential prerequisite. Several other authors have dedicated their resources to the quest for arbitrary new operators. Definitely, the range of such methods provides analysts with more chances to adapt them in multiple models.

Fractional calculus that consists of derivative and integral of noninteger order is normal augmentations of the standard integer order calculus. Fractional calculus is by all accounts universal in light of its fascinating applications with regards to different aspects of science, for example, viscoelastic materials, dispersion, central nervous biology, regulation hypothesis, and statistical data [7–9].

Despite the existence of a rigorous scientific standard for the continuity of fractional calculus, the possibility of improving a discrete fractional calculus has been insufficient. Although we all realize, discovering fractional difference equations requires a thorough understanding of system identification. Recently, surprising achievement has been produced as a result of arduous attempts in fractional difference structures by Du and Jia [10]. The existence and uniqueness of solutions are the foundation for examining the stability problem that has been exploited using fractional Gronwall and Bihari inequality, for example [11, 12].

The essentially identical to discrete hypothesis by a fractional sum of order $\bar{\Psi} > 0$ was identified due to Miller and Ross [13] through solution based on linear differential equation, and many key aspects of proposed operator were tested. Moreover, Atici and Eloe [14] implemented a discrete method for Laplace transformation containing a fractional class of finite difference equations. Atici and Eloe [15] identified the causes of the initial value in the discrete fractional analysis. Atici and Eloe [16] investigated the structure of a discrete fractional calculus with the nabla operator. They created exponential laws and the item rule to the forward fractional calculus. Atici and Sengul [17] built up the Leibniz rule and summation by part equation in discrete fractional theory. Bastos and Torres [18] introduced the more broad discrete fractional operator which was specified by delta and nabla fractional sums. Holm [19] presented operators with fractional sums and applied one such hypothesis to tackle fractional initial value problems. Anastassiou [20] determined the privilege discrete nabla fractional of Taylor equation. The innovation that made look like a consequence of this depiction was charming to several readers and now subjected to outrageous review, in numerous approaches: discrete nature and precision of fractional equations, tumor formation simulating [21], consistency of tumor cell solutions related order of Legendre's derivative $\bar{\Psi}$ [22], and Euler-Lagrange equation and Legendre's optimality condition for the calculus of variations problems [23]. The idea of a discrete version of fractional calculus is adopted just as late, usually because of the impact of exploration in fractional analysis (see [24–26]).

Inequalities of finite difference that demonstrate distinct bounds of undefined functions suggest a highly useful and beneficial way to enhance understanding of finite difference equations. As a consequence, difference equations tend to

be a realistic instrument that correctly represents real-life scenarios like question queueing, power systems, and financial measurements, and to attempt such kind of mechanism, this protection is mandated. Probably the least impossible enormously difference equations right now have begun to achieve the attention [27].

In the stage when we have to examine many features of a differential equation, there are multiple interpretations for certain categories of inequalities. Essentially, based on capability of the aforementioned inquiry, we formulate in this material some generalizations of discrete fractional nonlinear inequalities linked to the fractional sum operator $\bar{\Psi}$ that assemble to describe fractional inequalities and incorporate some proven publication tests. To reflect theoretical hypotheses, it was shown that the transmitted inequalities may be used to evaluate certain classifications of discrete fractional equations. In order to explore the usefulness and drawbacks of the usage of fractional sum difference equations, the completion of this paper secures a few instances.

Definitive portions of the document are classified as such. We address relevant actual considerations and basic assumptions in Section 2. Section 3 is committed to the theoretical experiences of nonlinear discrete fractional inequalities with some remarks. The remaining section is considered in accomplishing the theoretical examination specifications.

2. Preliminaries

And with that initiative, without the absence of a specific argument, let $\bar{M}$ be a constant, $\mathbb{N}_{\bar{\rho}} = \{\bar{\rho}, \bar{\rho} + 1, \bar{\rho} + 2, \dots\}$, $H_{\bar{\rho}} = [\bar{\rho}, \bar{M}] \cap \mathbb{N}_{\bar{\rho}}$, where $\bar{M}, \bar{\rho} \in \mathbb{N}_{\bar{\rho}}$, $\sum_{\bar{\rho}=\bar{c}}^{\bar{\rho}} \mathfrak{f}(\bar{\rho}) = 0$, $\mathbb{R}_+ = [0, \infty)$, and difference operator of a be assigned as $\Delta a(\bar{\vartheta}) = a(\bar{\vartheta}) - a(\bar{\vartheta} - 1)$, $\bar{\vartheta} \in \mathbb{N}_{\bar{\rho}}$.

A part of primitive specifications and theorems of discrete fractional measurement is represented as

Definition 2 (see [17]). Let $\bar{\Psi}$ be any positive real number, $\mathfrak{g}$ be any real number and $\sigma(\bar{\rho}) = \bar{\rho} + 1$, and then $\bar{\Psi}$ -th fractional sum of $\mathfrak{f}$ is defined for $\bar{\Omega} = \mathfrak{g} \pmod{1}$ by

$$\Delta_{\mathfrak{g}}^{-\bar{\Psi}} \mathfrak{f}(\bar{\rho}) = \frac{1}{\Gamma(\bar{\Psi})} \sum_{\bar{\Omega}=\mathfrak{g}}^{\bar{\rho}-\bar{\Psi}} (\bar{\rho} - \sigma(\bar{\Omega}))^{\bar{\Psi}-1} \mathfrak{f}(\bar{\Omega}), \quad (3)$$

such that $(\bar{\rho})^{\bar{\Psi}} = \Gamma(\bar{\rho} + 1)/\Gamma(\bar{\rho} - \bar{\Psi} + 1)$, and $\Delta_{\mathfrak{g}}^{-\bar{\Psi}} \mathfrak{f}$ is defined for $\bar{\rho} = \mathfrak{g} + \bar{\Psi} \pmod{1}$ and $\Delta_{\mathfrak{g}}^{-\bar{\Psi}} : \mathbb{N}_{\mathfrak{g}} \longrightarrow \mathbb{N}_{\mathfrak{g}+\bar{\Psi}}$.

Definition 3 (see [17]). Let $\hat{\nu} > 0$, and $\hat{\Lambda} - 1 < \hat{\nu} < \hat{\Lambda}$. Then, $\hat{\nu}$ -th fractional difference of $\mathfrak{f}$ is classified as

$$\Delta^{\hat{\nu}} \mathfrak{f}(\bar{\rho}) = \Delta^{\hat{\Lambda}-\bar{\Psi}} \mathfrak{f}(\bar{\rho}) = \Delta^{\hat{\Lambda}} \left(\Delta^{-\bar{\Psi}} \mathfrak{f}(\bar{\rho}) \right), \quad (4)$$

where $\hat{\Lambda}$ is a positive integer and $-\bar{\Psi} = \hat{\nu} - \hat{\Lambda}$.

Theorem 4 (see [15]). If a real-valued function $\mathfrak{k}$ be prescribed on $\mathbb{N}_g$, such that $\widehat{\nu}, \bar{\Psi} > 0$, so

$$\Delta^{-\bar{\Psi}}(\Delta^{-\nu \wedge} \mathfrak{k}(\bar{\rho})) = \Delta^{-(\bar{\Psi} + \nu \wedge)} \mathfrak{k}(\bar{\rho}) = \Delta^{-\nu \wedge}(\Delta^{-\bar{\Psi}} \mathfrak{k}(\bar{\rho})). \quad (5)$$

Theorem 5 (see [15]). Let $\bar{\Psi} > 0$ and $\mathfrak{k}$ be a function which is real valued on $\mathbb{N}_g$, and then

$$\Delta^{-\bar{\Psi}} \Delta \mathfrak{k}(\bar{\rho}) = \Delta \Delta^{-\bar{\Psi}} \mathfrak{k}(\bar{\rho}) = \frac{(\bar{\rho} - 1)^{\bar{\Psi} - 1}}{\Gamma(\bar{\Psi})} \mathfrak{k}(g). \quad (6)$$

The reader may bring attention to [15, 17] for further desirable characteristics on a discrete fractional proposition.

3. Result Declaration

Presently, we will adjust the basic tests.

Theorem 6. Suppose that $\mathcal{L} \in \mathbb{N}_{\bar{\Psi}-1} \longrightarrow \mathbb{R}_+$, $\mathfrak{F}, \mathfrak{U} : \mathbb{N}_{\bar{\Psi}} \longrightarrow \mathbb{R}_+$ are functions, $0 < \bar{\Psi} \leq 1$ and $\widehat{\omega}_0 \geq 0$ are constants, and $\widehat{\Pi} : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a nondecreasing continuous function with $\widehat{\Pi}(t) > 0$ for $t > 0$. If

$$\begin{aligned} \mathcal{L}(\bar{\rho}) &\leq \widehat{\omega}_0 + \Delta_0^{-\bar{\Psi}} \left[\mathfrak{F}(\bar{\rho}) \widehat{\Pi}(\mathcal{L}(\bar{\rho} + \bar{\Psi} - 1)) \right] \\ &+ \frac{1}{\Gamma(\bar{\Psi})} \sum_{\bar{\Omega}=0}^{\bar{\rho}-\bar{\Psi}} (\bar{\rho} - \bar{\Omega} - 1)^{\bar{\Psi}-1} \mathfrak{U}(\bar{\Omega}) \mathcal{L}(\bar{\Omega} + \bar{\Psi} - 1) \widehat{\Pi} \\ &\cdot (\mathcal{L}(\bar{\Omega} + \bar{\Psi} - 1)), \bar{\rho} \in H_{\bar{\Psi}-1}, \end{aligned} \quad (7)$$

is satisfied, then

$$\begin{aligned} \mathcal{L}(\bar{\rho}) &\leq u \wedge^{-1} \left[\mathfrak{N} \wedge^{-1} \left\{ \widehat{\mathfrak{N}} \left(\widehat{u}(\widehat{\omega}_0) + \sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}} \mathfrak{j}(\bar{\Omega} - \bar{\Psi}) \right) \right. \right. \\ &\left. \left. + \sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}} \mathfrak{u}(\bar{\Omega} - \bar{\Psi}) \right\} \right], \bar{\rho} \in H_{\bar{\Psi}-1}, \end{aligned} \quad (8)$$

for $0 \leq \bar{\rho} \leq \bar{\rho}_1$, provided with

$$\widehat{u}(\mathfrak{n}) = \int_{\mathfrak{n}_0}^{\mathfrak{n}} \frac{1}{\widehat{\Pi}(\mathfrak{a})} d\mathfrak{a}, \mathfrak{n} \geq \mathfrak{n}_0 > 0, \lim_{\mathfrak{n} \rightarrow \infty} \widehat{u}(\mathfrak{n}) = \infty, \quad (9)$$

$$\widehat{\mathfrak{N}}(\mathfrak{l}) = \int_{\mathfrak{l}_0}^{\mathfrak{l}} \frac{1}{u \wedge^{-1}(\mathfrak{q})} d\mathfrak{q}, \mathfrak{l} \geq \mathfrak{l}_0 > 0, \lim_{\mathfrak{l} \rightarrow \infty} \widehat{\mathfrak{N}}(\mathfrak{l}) = \infty. \quad (10)$$

$u \wedge^{-1}$, $\mathfrak{N} \wedge^{-1}$ is the inverses of $\widehat{u}$, $\widehat{\mathfrak{N}}$, and $\bar{\rho}_1 \in H_{\bar{\Psi}-1}$ is chosen so that

$$\widehat{\mathfrak{N}} \left(\widehat{u}(\widehat{\omega}_0) + \sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}} \mathfrak{j}(\bar{\Omega} - \bar{\Psi}) \right) + \sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}} \mathfrak{u}(\bar{\Omega} - \bar{\Psi}) \in \text{Dom}(\mathfrak{N} \wedge^{-1}), \quad (11)$$

$$\mathfrak{F}(\bar{\Omega}, \bar{\rho}) = \frac{1}{\Gamma(\bar{\Psi})} (\bar{\rho} - \bar{\Omega} - 1)^{\bar{\Psi}-1} \mathfrak{F}(\bar{\Omega}), \quad (12)$$

$$\mathfrak{U}(\bar{\Omega}, \bar{\rho}) = \frac{1}{\Gamma(\bar{\Psi})} (\bar{\rho} - \bar{\Omega} - 1)^{\bar{\Psi}-1} \mathfrak{U}(\bar{\Omega}). \quad (13)$$

Proof. Let $\widehat{\omega}_0 > 0$. Defining

$$\begin{aligned} \mathfrak{y}(\bar{\rho}) &= \widehat{\omega}_0 + \Delta_0^{-\bar{\Psi}} \left[\mathfrak{F}(\bar{\rho}) \widehat{\Pi}(\mathcal{L}(\bar{\rho} + \bar{\Psi} - 1)) \right] \\ &+ \frac{1}{\Gamma(\bar{\Psi})} \sum_{\bar{\Omega}=0}^{\bar{\rho}-\bar{\Psi}} (\bar{\rho} - \bar{\Omega} - 1)^{\bar{\Psi}-1} \mathfrak{U}(\bar{\Omega}) \mathcal{L}(\bar{\Omega} + \bar{\Psi} - 1) \widehat{\Pi} \\ &\cdot (\mathcal{L}(\bar{\Omega} + \bar{\Psi} - 1)), \end{aligned} \quad (14)$$

then one has

$$\mathcal{L}(\bar{\rho}) \leq \mathfrak{y}(\bar{\rho}), \bar{\rho} \in H_{\bar{\Psi}-1}. \quad (15)$$

From Definition 2 and (14), we have

$$\begin{aligned} \mathfrak{y}(\bar{\rho}) &= \widehat{\omega}_0 + \frac{1}{\Gamma(\bar{\Psi})} \sum_{\bar{\Omega}=0}^{\bar{\rho}-\bar{\Psi}} (\bar{\rho} - \bar{\Omega} - 1)^{\bar{\Psi}-1} \mathfrak{F}(\bar{\Omega}) \widehat{\Pi}(\mathcal{L}(\bar{\Omega} + \bar{\Psi} - 1)) \\ &+ \frac{1}{\Gamma(\bar{\Psi})} \sum_{\bar{\Omega}=0}^{\bar{\rho}-\bar{\Psi}} (\bar{\rho} - \bar{\Omega} - 1)^{\bar{\Psi}-1} \mathfrak{U}(\bar{\Omega}) \mathcal{L}(\bar{\Omega} + \bar{\Psi} - 1) \widehat{\Pi} \\ &\cdot (\mathcal{L}(\bar{\Omega} + \bar{\Psi} - 1)), \bar{\rho} \in H_{\bar{\Psi}-1} = \widehat{\omega}_0 + \sum_{\bar{\Omega}=0}^{\bar{\rho}-\bar{\Psi}} \mathfrak{F}(\bar{\Omega}, \bar{\rho}) \widehat{\Pi} \\ &\cdot (\mathcal{L}(\bar{\Omega} + \bar{\Psi} - 1)) + \sum_{\bar{\Omega}=0}^{\bar{\rho}-\bar{\Psi}} \mathfrak{U}(\bar{\Omega}, \bar{\rho}) \mathcal{L}(\bar{\Omega} + \bar{\Psi} - 1) \widehat{\Pi} \\ &\cdot (\mathcal{L}(\bar{\Omega} + \bar{\Psi} - 1)), \end{aligned} \quad (16)$$

where $\mathfrak{F}(\bar{\Omega}, \bar{\rho})$ and $\mathfrak{U}(\bar{\Omega}, \bar{\rho})$ are defined as in (11) and (13). Hence, $\mathfrak{y}(\bar{\rho}) \geq 0$ is nondecreasing. Now, $\mathfrak{F}(\bar{\Omega}, \bar{\rho})$, $\mathfrak{U}(\bar{\Omega}, \bar{\rho})$, and $(\bar{\rho})^{\bar{\Psi}}$ by their definition and $\mathfrak{F}(\bar{\Omega}, \bar{\rho})$, $\mathfrak{U}(\bar{\Omega}, \bar{\rho})$ is decreasing in $\bar{\rho}$ for each $\bar{\Omega} \in \mathbb{N}_0$. In the equation (14) using straightforward computation for $\bar{\rho} \in H_{\bar{\Psi}}$ and (8), we get

$$\begin{aligned} \mathfrak{y}(\bar{\rho}) - \mathfrak{y}(\bar{\rho} - 1) &= \mathfrak{F}(\bar{\rho} - \bar{\Psi}, \bar{\rho}) \widehat{\Pi}(\mathcal{L}(\bar{\rho} - 1)) + \sum_{\bar{\Omega}=0}^{\bar{\rho}-\bar{\Psi}-1} [\mathfrak{F}(\bar{\Omega}, \bar{\rho}) \\ &- \mathfrak{F}(\bar{\Omega}, \bar{\rho} - 1)] \widehat{\Pi}(\mathcal{L}(\bar{\Omega} + \bar{\Psi} - 1)) \\ &+ \mathfrak{U}(\bar{\rho} - \bar{\Psi}, \bar{\rho}) \mathcal{L}(\bar{\rho} - 1) \widehat{\Pi}(\mathcal{L}(\bar{\rho} - 1)) \\ &+ \sum_{\bar{\Omega}=0}^{\bar{\rho}-\bar{\Psi}-1} [\mathfrak{U}(\bar{\Omega}, \bar{\rho}) - \mathfrak{U}(\bar{\Omega}, \bar{\rho} - 1)] \\ &\times \mathcal{L}(\bar{\Omega} + \bar{\Psi} - 1) \widehat{\Pi}(\mathcal{L}(\bar{\Omega} + \bar{\Psi} - 1)) \\ &= \mathfrak{j}(\bar{\rho} - \bar{\Psi}) \widehat{\Pi}(\mathfrak{y}(\bar{\rho} - 1)) + \mathfrak{u}(\bar{\rho} - \bar{\Psi}) \mathfrak{y}(\bar{\rho} - 1) \widehat{\Pi} \\ &\cdot (\mathfrak{y}(\bar{\rho} - 1)), \end{aligned} \quad (17)$$

monotonicity of $\widehat{\Pi}$, and $\mathfrak{y}$ produces

$$\widehat{\Pi}(\mathfrak{y}(\bar{\rho} - 1)) > \widehat{\Pi}(\mathfrak{y}(\bar{\Psi} - 1)) = \widehat{\Pi}(\widehat{\omega}_0) > 0, \bar{\rho} \in H_{\bar{\Psi}}, \quad (18)$$

from (17) and (18), and one has

$$\frac{\Delta \mathfrak{y}(\bar{\rho} - 1)}{\widehat{\Pi}(\mathfrak{y}(\bar{\rho} - 1))} \leq \mathfrak{j}(\bar{\rho} - \bar{\Psi}) + \mathfrak{u}(\bar{\rho} - \bar{\Psi}) \mathfrak{y}(\bar{\rho} - 1), \bar{\rho} \in H_{\bar{\Psi}}. \quad (19)$$

By mean value theorem, it can be seen that

$$\begin{aligned} \Delta \widehat{\mathfrak{u}}(\mathfrak{y}(\bar{\rho} - 1)) &= \widehat{\mathfrak{u}}(\mathfrak{y}(\bar{\rho})) - \widehat{\mathfrak{u}}(\mathfrak{y}(\bar{\rho} - 1)) = \mathfrak{u} \wedge'(\bar{\rho}) \Delta \mathfrak{y}(\bar{\rho} - 1) \\ &= \frac{\Delta \mathfrak{y}(\bar{\rho} - 1)}{\widehat{\Pi}(\bar{\rho})} \leq \frac{\Delta \mathfrak{y}(\bar{\rho} - 1)}{\widehat{\Pi}(\mathfrak{y}(\bar{\rho} - 1))}; \bar{\rho} \in [\mathfrak{y}(\bar{\rho} - 1), \mathfrak{y}(\bar{\rho})], \end{aligned} \quad (20)$$

relations (19) and (20) summarize into

$$\Delta \widehat{\mathfrak{u}}(\mathfrak{y}(\bar{\rho} - 1)) \leq \mathfrak{j}(\bar{\rho} - \bar{\Psi}) + \mathfrak{u}(\bar{\rho} - \bar{\Psi}) \mathfrak{y}(\bar{\rho} - 1), \bar{\rho} \in H_{\bar{\Psi}}, \quad (21)$$

summation (21) from $\bar{\Psi}$ to $\bar{\rho} - 1$ and from (9), we obtain

$$\sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}-1} \Delta \widehat{\mathfrak{u}}(\mathfrak{y}(\bar{\Omega} - 1)) \leq \sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}-1} \mathfrak{j}(\bar{\Omega} - \bar{\Psi}) + \sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}-1} \mathfrak{u}(\bar{\Omega} - \bar{\Psi}) \mathfrak{y}(\bar{\Omega} - 1), \quad (22)$$

particularly

$$\widehat{\mathfrak{u}}(\mathfrak{y}(\bar{\rho} - 1)) \leq \widehat{\mathfrak{u}}(\widehat{\omega}_0) + \sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}-1} \mathfrak{j}(\bar{\Omega} - \bar{\Psi}) + \sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}-1} \mathfrak{u}(\bar{\Omega} - \bar{\Psi}) \mathfrak{y}(\bar{\Omega} - 1), \quad (23)$$

$$\mathfrak{y}(\bar{\rho} - 1) \leq \mathfrak{u} \wedge^{-1}(\mathfrak{v}(\bar{\rho})), \quad (24)$$

where

$$\mathfrak{v}(\bar{\rho}) = \widehat{\mathfrak{u}}(\widehat{\omega}_0) + \sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}-1} \mathfrak{j}(\bar{\Omega} - \bar{\Psi}) + \sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}-1} \mathfrak{u}(\bar{\Omega} - \bar{\Psi}) \mathfrak{y}(\bar{\Omega} - 1), \quad (25)$$

for $0 \leq \bar{\rho} \leq \bar{\rho}_1$, and we get

$$\mathfrak{v}(\bar{\Psi} - 1) = \widehat{\mathfrak{u}}(\widehat{\omega}_0) + \sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}_1-1} \mathfrak{j}(\bar{\Omega} - \bar{\Psi}), \quad (26)$$

and furthermore with (24) and (25), we proceed to

$$\mathfrak{v}(\bar{\rho} - 1) - \mathfrak{v}(\bar{\rho}) = \mathfrak{u}(\bar{\rho} - \bar{\Psi} - 1) \mathfrak{y}(\bar{\rho} - 2) \leq \mathfrak{u}(\bar{\rho} - \bar{\Psi} - 1) \mathfrak{u} \wedge^{-1}(\mathfrak{v}(\bar{\rho} - 1)), \quad (27)$$

related moves from (18)-(20) with acceptable improvements to the above inequality yields

$$\frac{\Delta \mathfrak{v}(\bar{\rho} - 1)}{\mathfrak{u} \wedge^{-1}(\mathfrak{v}(\bar{\rho} - 1))} \leq \mathfrak{u}(\bar{\rho} - \bar{\Psi} - 1), \quad (28)$$

sum over $[\bar{\Psi}, \bar{\rho} - 1]$ in (28) and from (10), (26) with $\bar{\rho}_1$ is chosen arbitrary, and we acquire

$$\begin{aligned} \widehat{\mathfrak{N}}(\mathfrak{v}(\bar{\rho} - 1)) &\leq \widehat{\mathfrak{N}}\left(\widehat{\mathfrak{u}}(\widehat{\omega}_0) + \sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}-1} \mathfrak{j}(\bar{\Omega} - \bar{\Psi})\right) \\ &\quad + \sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}-1} \mathfrak{u}(\bar{\Omega} - \bar{\Psi}), \bar{\rho} \in H_{\bar{\Psi}}, \end{aligned} \quad (29)$$

and thus

$$\begin{aligned} \widehat{\mathfrak{N}}(\mathfrak{v}(\bar{\rho})) &\leq \widehat{\mathfrak{N}}\left(\widehat{\mathfrak{u}}(\widehat{\omega}_0) + \sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}} \mathfrak{j}(\bar{\Omega} - \bar{\Psi})\right) \\ &\quad + \sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}} \mathfrak{u}(\bar{\Omega} - \bar{\Psi}), \bar{\rho} \in H_{\bar{\Psi}-1}, \end{aligned} \quad (30)$$

and the conclusion of (8) can be followed by substituting (30) in (15) and (24) for $\bar{\rho} \in H_{\bar{\Psi}-1}$. $\square$

Remark 7. By inserting $\bar{\Psi} = 1$, $\Gamma(1) = 1$ (gamma function property), $\widehat{\Pi}(\mathcal{L}) = \mathcal{L}$, and $\mathfrak{U} = 0$ in (7), hence Theorem 6 shifts to Theorem 1 [6].

Remark 8. Theorem 6 converted into Theorem 7 by Du and Jia [24] if $\mathfrak{F} = 1$ and $\mathfrak{U} = 0$ in (7).

Theorem 9. Under the same suppositions of $\mathcal{L}$, $\mathfrak{F}$, $\widehat{\Pi}$, $\bar{\Psi}$, $\widehat{\omega}_0$, and $\mathfrak{F}(\bar{\Omega}, \bar{\rho})$ of Theorem 6, if the inequality

$$\mathcal{L}(\bar{\rho}) \leq \widehat{\omega}_0 + \Delta_0^{-\bar{\Psi}} \left[\mathfrak{F}(\bar{\rho}) \mathcal{L}^{\mathfrak{r}}(\bar{\rho} + \bar{\Psi} - 1) \widehat{\Pi}(\mathcal{L}(\bar{\rho} + \bar{\Psi} - 1)) \right], \bar{\rho} \in H_{\bar{\Psi}-1}, \quad (31)$$

satisfies for $\mathfrak{r} > 0$, $\mathfrak{r} \neq 1$, $\mathfrak{r}$ is a constant, then

$$\mathcal{L}(\bar{\rho}) \leq \left\{ \mathfrak{N} \wedge_I^{-1} \left[\mathfrak{N} \wedge_I \left(\omega \wedge_0^{\frac{1}{1-\mathfrak{r}}} \right) + (1 - \mathfrak{r}) \sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}} \mathfrak{j}(\bar{\Omega} - \bar{\Psi}) \right] \right\}^{\frac{1}{1-\mathfrak{r}}}, \bar{\rho} \in H_{\bar{\Psi}-1}, \quad (32)$$

where

$$\widehat{\mathfrak{N}}_I(\mathfrak{f}) = \int_{\mathfrak{f}_0}^{\mathfrak{f}} \frac{1}{\widehat{\Pi}(\mathfrak{q}^{1/\mathfrak{r}})} d\mathfrak{q}, \mathfrak{f} \geq \mathfrak{f}_0 > 0, \lim_{\mathfrak{f} \rightarrow \infty} \widehat{\mathfrak{N}}_I(\mathfrak{f}) = \infty. \quad (33)$$

$\widehat{\mathfrak{N}}_I^{-1}$ is the inverse of $\widehat{\mathfrak{N}}_I$, so that

$$\widehat{\mathfrak{N}}_I(\widehat{\omega}_0) + (1 - \mathfrak{r}) \sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}} \mathfrak{j}(\bar{\Omega} - \bar{\Psi}) \in \text{Dom}\left(\widehat{\mathfrak{N}}_I^{-1}\right). \quad (34)$$

Proof. Infer $\widehat{\omega}_0 > 0$ and denoting

$$\mathfrak{p}_1(\bar{\rho}) = \widehat{\omega}_0 + \Delta_0^{-\bar{\Psi}} \left[\mathfrak{F}(\bar{\rho}) \mathcal{L}^r(\bar{\rho} + \bar{\Psi} - 1) \widehat{\Pi}(\mathcal{L}(\bar{\rho} + \bar{\Psi} - 1)) \right], \quad (35)$$

therefore, one has

$$\mathcal{L}(\bar{\rho}) \leq \mathfrak{p}_1(\bar{\rho}), \quad (36)$$

and employing Definition 2 to (35), we deduce

$$\begin{aligned} \mathfrak{p}_1(\bar{\rho}) &= \widehat{\omega}_0 + \frac{1}{\Gamma(\bar{\Psi})} \sum_{\bar{\Omega}=0}^{\bar{\rho}-\bar{\Psi}} (\bar{\rho} - \bar{\Omega} - 1)^{\bar{\Psi}-1} \mathfrak{F}(\bar{\Omega}) \mathcal{L}^r \\ &\quad \cdot (\bar{\Omega} + \bar{\Psi} - 1) \widehat{\Pi}(\mathcal{L}(\bar{\Omega} + \bar{\Psi} - 1)), \bar{\rho} \in H_{\bar{\Psi}-1}, \end{aligned} \quad (37)$$

where $J(\bar{\Omega}, \bar{\rho})$ is defined in (11), and $\mathfrak{p}_1(\bar{\rho}) \geq 0$ is nondecreasing. With the assistance of direct calculation for $\bar{\rho} \in H_{\bar{\Psi}}$, decreasing property of $\mathfrak{F}(\bar{\Omega}, \bar{\rho})$ for $\bar{\Omega} \in \mathbb{N}_0$, the definition of $\mathfrak{F}(\bar{\Omega}, \bar{\rho})$, $(\bar{\rho})^{\bar{\Psi}}$, (35), and (36), we conclude

$$\begin{aligned} \mathfrak{p}_1(\bar{\rho}) - \mathfrak{p}_1(\bar{\rho} - 1) &= \mathfrak{F}(\bar{\rho} - \bar{\Psi}, \bar{\rho}) \mathcal{L}^r(\bar{\rho} - 1) \widehat{\Pi}(\mathcal{L}(\bar{\rho} - 1)) \\ &\quad + \sum_{\bar{\Omega}=0}^{\bar{\rho}-\bar{\Psi}-1} [\mathfrak{F}(\bar{\Omega}, \bar{\rho}) - \mathfrak{F}(\bar{\Omega}, \bar{\rho} - 1)] \mathcal{L}^r \\ &\quad \cdot (\bar{\Omega} + \bar{\Psi} - 1) \widehat{\Pi}(\mathcal{L}(\bar{\Omega} + \bar{\Psi} - 1)) \\ &\leq \mathfrak{j}(\bar{\rho} - \bar{\Psi}) \mathfrak{p}_1^r(\bar{\rho} - 1) \widehat{\Pi}(\mathfrak{p}_1(\bar{\rho} - 1)). \end{aligned} \quad (38)$$

By mean value axiom, we accomplish

$$\frac{1}{1-r} [\mathfrak{p}_1^{1-r}(\bar{\rho}) - \mathfrak{p}_1^{1-r}(\bar{\rho} - 1)] = \frac{\mathfrak{p}_1^{1-r}(\bar{\rho}) - \mathfrak{p}_1^{1-r}(\bar{\rho} - 1)}{[\rho \wedge(\bar{\rho})]^r}, \quad (39)$$

for some $\widehat{\rho}(\bar{\rho}) \in [\mathfrak{p}_1(\bar{\rho} - 1), \mathfrak{p}_1(\bar{\rho})]$. Therefore,

$$\begin{aligned} \mathfrak{p}_1^{1-r}(\bar{\rho}) - \mathfrak{p}_1^{1-r}(\bar{\rho} - 1) &\leq (1-r) \left[\frac{\mathfrak{p}_1(\bar{\rho}) - \mathfrak{p}_1(\bar{\rho} - 1)}{[\mathfrak{p}_1(\bar{\rho} - 1)]^r} \right] \\ &\leq (1-r) \mathfrak{j}(\bar{\rho} - \bar{\Psi}) \widehat{\Pi}(\mathfrak{p}_1(\bar{\rho} - 1)), \bar{\rho} \in H_{\bar{\Psi}}, \end{aligned} \quad (40)$$

summing prior inequality from $\bar{\Psi}$ to $\bar{\rho} - 1$ and taking into account $\mathfrak{p}_1(\bar{\Psi} - 1) = \widehat{\omega}_0$ implies

$$\mathfrak{p}_1^{1-r}(\bar{\rho} - 1) \leq \widehat{\omega}_0^{1-r} + (1-r) \sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}-1} \mathfrak{j}(\bar{\Omega} - \bar{\Psi}) \widehat{\Pi}(\mathfrak{p}_1(\bar{\Omega} - 1)). \quad (41)$$

Let

$$\mathfrak{p}_2(\bar{\rho}) = \widehat{\omega}_0^{1-r} + (1-r) \sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}-1} \mathfrak{j}(\bar{\Omega} - \bar{\Psi}) \widehat{\Pi}(\mathfrak{p}_1(\bar{\Omega} - 1)), \quad (42)$$

from (41) and (42), and we have

$$\mathfrak{p}_1^{1-r}(\bar{\rho} - 1) \leq \mathfrak{p}_2(\bar{\rho}) \Rightarrow \mathfrak{p}_1(\bar{\rho} - 1) \leq \mathfrak{p}_2^{\frac{1}{1-r}}(\bar{\rho}), \quad (43)$$

in addition from (42); we see

$$\begin{aligned} \mathfrak{p}_2(\bar{\rho}) - \mathfrak{p}_2(\bar{\rho} - 1) &= (1-r) \mathfrak{j}(\bar{\rho} - \bar{\Psi} - 1) \widehat{\Pi}(\mathfrak{p}_1(\bar{\rho} - 2)) \\ &\leq (1-r) \mathfrak{j}(\bar{\rho} - \bar{\Psi} - 1) \widehat{\Pi}(\mathfrak{p}_2^{\frac{1}{1-r}}(\bar{\rho} - 1)), \end{aligned} \quad (44)$$

monotonicity of $\widehat{\Pi}$, and $\mathfrak{p}_2$ gives

$$\widehat{\Pi}(\mathfrak{p}_2^{\frac{1}{1-r}}(\bar{\rho} - 1)) > \widehat{\Pi}(\mathfrak{p}_2^{\frac{1}{1-r}}(\bar{\Psi} - 1)) = \widehat{\Pi}(\widehat{\omega}_0^{\frac{1}{1-r}}) > 0, \bar{\rho} \in H_{\bar{\Psi}}, \quad (45)$$

equation (44) with inequality (45) becomes

$$\frac{\Delta \mathfrak{p}_2(\bar{\rho} - 1)}{\widehat{\Pi}(\mathfrak{p}_2^{1/(1-r)}(\bar{\rho} - 1))} \leq (1-r) \mathfrak{j}(\bar{\rho} - \bar{\Psi} - 1), \bar{\rho} \in H_{\bar{\Psi}}. \quad (46)$$

analysis of mean value theorem approaches to

$$\begin{aligned} \Delta \widehat{\mathfrak{N}}_1(\mathfrak{p}_2(\bar{\rho} - 1)) &= \widehat{\mathfrak{N}}_1(\mathfrak{p}_2(\bar{\rho})) - \widehat{\mathfrak{N}}_1(\mathfrak{p}_2(\bar{\rho} - 1)) \\ &= \widehat{\mathfrak{N}}_1'(\widehat{\rho}) \Delta \mathfrak{p}_2(\bar{\rho} - 1) = \frac{\Delta \mathfrak{p}_2(\bar{\rho} - 1)}{\widehat{\Pi}(\rho \wedge)^{1/(1-r)}} \\ &\leq \frac{\Delta \mathfrak{p}_2(\bar{\rho} - 1)}{\widehat{\Pi}(\mathfrak{p}_2^{1/(1-r)}(\bar{\rho} - 1))}; \widehat{\rho} \in [\mathfrak{p}_2(\bar{\rho} - 1), \mathfrak{p}_2(\bar{\rho})], \end{aligned} \quad (47)$$

inequalities (46) and (47) that offer

$$\Delta \widehat{\mathfrak{N}}_1(\mathfrak{p}_2(\bar{\rho} - 1)) \leq (1-r) \mathfrak{j}(\bar{\rho} - \bar{\Psi} - 1), \bar{\rho} \in H_{\bar{\Psi}}, \quad (48)$$

inequality (48) by summing from $\bar{\Psi}$ to $\bar{\rho} - 1$ and utilizing $\mathfrak{p}_2(\bar{\Psi} - 1) = \widehat{\omega}_0^{1/(1-r)}$ equals to

$$\sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}-1} \Delta \widehat{\mathfrak{N}}_1(\mathfrak{p}_2(\bar{\Omega} - 1)) \leq (1-r) \sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}-1} \mathfrak{j}(\bar{\Omega} - \bar{\Psi}), \quad (49)$$

that is,

$$\widehat{\mathfrak{N}}_1(\mathfrak{p}_2(\bar{\rho} - 1)) \leq \widehat{\mathfrak{N}}_1(\widehat{\omega}_0^{\frac{1}{1-r}}) + (1-r) \sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}-1} \mathfrak{j}(\bar{\Omega} - \bar{\Psi}), \bar{\rho} \in H_{\bar{\Psi}}, \quad (50)$$

$$\mathfrak{p}_2(\bar{\rho}) \leq \widehat{\mathfrak{N}}_1^{-1} \left[\widehat{\mathfrak{N}}_1(\widehat{\omega}_0^{\frac{1}{1-r}}) + (1-r) \sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}} \mathfrak{j}(\bar{\Omega} - \bar{\Psi}) \right], \bar{\rho} \in H_{\bar{\Psi}-1}, \quad (51)$$

substitute the resulting inequality in (43) and (36) to get the acquired bound in (32) with $\bar{\rho} \in H_{\bar{\Psi}-1}$. $\square$

Remark 10. By taking Theorem 9 alters into [28], Lemma 5 (β_1), by letting $\mathfrak{r} = 1$, $\bar{\Psi} = 1$, $\Gamma(1) = 1$, $\hat{\omega}_0 = \hat{\omega}(\bar{\rho})$, and $\widehat{\Pi}(\mathcal{L}(\bar{\rho})) = 1$.

Remark 11. Theorem 9 changes to Theorem 7 by taking $\mathfrak{r} = 0$ and $\mathfrak{J} = 1$ due to Du and Jia [24].

4. Boundedness and Uniqueness

This segment is related to a valid procedure of Theorem 6 to determine boundedness and uniqueness of discrete fractional inequalities. Consider the following pattern of fractional difference equation:

$$\mathcal{L}(\bar{\rho}) = \hat{\omega}_0 + \Delta_0^{-\bar{\Psi}} [\mathfrak{S}(\bar{\rho}, \mathcal{L}(\bar{\rho} + \bar{\Psi} - 1))], \bar{\rho} \in H_{\bar{\Psi}-1}, \quad (52)$$

where $\mathfrak{S} : \mathbb{N}_0 \times \mathbb{R} \longrightarrow \mathbb{R}$ and $\bar{\rho}$, $\bar{\Psi}$, $\hat{\omega}_0$, $\mathcal{L}$, and $\widehat{\Pi}$ be the same as in Theorem 6.

The accompanying example can describe the boundedness on the solutions of (52).

Example 12. Suppose that

$$|\mathfrak{S}(\bar{\rho}, \mathcal{L})| \leq \mathfrak{J}(\bar{\rho}) \left| \widehat{\Pi}(\mathcal{L}) \right| + \mathfrak{U}(\bar{\rho}) |\mathcal{L}| \left| \widehat{\Pi}(\mathcal{L}) \right|, \quad (53)$$

for $\bar{\rho} \in \mathbb{N}_0$, $\mathcal{L} \in \mathbb{R}$. If $\mathcal{L}(\bar{\rho})$ is a solution of (52), then

$$|\mathcal{L}(\bar{\rho})| \leq \mathfrak{u} \wedge^{-1} \left[\mathfrak{N} \wedge^{-1} \left\{ \hat{\mathfrak{N}} \left(\hat{\mathfrak{u}}(\hat{\omega}_0) + \sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}} \mathfrak{i}(\bar{\Omega} - \bar{\Psi}) \right) + \sum_{\bar{\Omega}=\bar{\Psi}}^{\bar{\rho}} \mathfrak{u}(\bar{\Omega} - \bar{\Psi}) \right\} \right], \bar{\rho} \in H_{\bar{\Psi}-1}, \quad (54)$$

Proof. Equation (52) with the blend of Definition 2 is encoded into

$$\mathcal{L}(\bar{\rho}) = \hat{\omega}_0 + \frac{1}{\Gamma(\bar{\Psi})} \sum_{\bar{\Omega}=0}^{\bar{\rho}-\bar{\Psi}} (\bar{\rho} - \bar{\Omega} - 1)^{\bar{\Psi}-1} \mathfrak{S}(\bar{\Omega}, \mathcal{L}(\bar{\Omega} + \bar{\Psi} - 1)). \quad (55)$$

Evidently, equation (55) with the utilization of (53) takes the form

$$\begin{aligned} |\mathcal{L}(\bar{\rho})| &= \left| \hat{\omega}_0 + \frac{1}{\Gamma(\bar{\Psi})} \sum_{\bar{\Omega}=0}^{\bar{\rho}-\bar{\Psi}} (\bar{\rho} - \bar{\Omega} - 1)^{\bar{\Psi}-1} \mathfrak{S}(\bar{\Omega}, \mathcal{L}(\bar{\Omega} + \bar{\Psi} - 1)) \right| \\ &\leq |\hat{\omega}_0| + \frac{1}{\Gamma(\bar{\Psi})} \sum_{\bar{\Omega}=0}^{\bar{\rho}-\bar{\Psi}} (\bar{\rho} - \bar{\Omega} - 1)^{\bar{\Psi}-1} \times \left[\mathfrak{J}(\bar{\Omega}) \left| \widehat{\Pi}(\mathcal{L}(\bar{\Omega} + \bar{\Psi} - 1)) \right| \right. \\ &\quad \left. + \mathfrak{U}(\bar{\Omega}) |\mathcal{L}(\bar{\Omega} + \bar{\Psi} - 1)| \left| \widehat{\Pi}(\mathcal{L}(\bar{\Omega} + \bar{\Psi} - 1)) \right| \right] \\ &\leq |\hat{\omega}_0| + \frac{1}{\Gamma(\bar{\Psi})} \sum_{\bar{\Omega}=0}^{\bar{\rho}-\bar{\Psi}} (\bar{\rho} - \bar{\Omega} - 1)^{\bar{\Psi}-1} \mathfrak{J}(\bar{\Omega}) \left| \widehat{\Pi}(\mathcal{L}(\bar{\Omega} + \bar{\Psi} - 1)) \right| \\ &\quad + \frac{1}{\Gamma(\bar{\Psi})} \sum_{\bar{\Omega}=0}^{\bar{\rho}-\bar{\Psi}} (\bar{\rho} - \bar{\Omega} - 1)^{\bar{\Psi}-1} \mathfrak{U}(\bar{\Omega}) |\mathcal{L}(\bar{\Omega} + \bar{\Psi} - 1)| \left| \widehat{\Pi}(\mathcal{L}(\bar{\Omega} + \bar{\Psi} - 1)) \right|. \end{aligned} \quad (56)$$

The rest of the calculations can be performed by assuming the right composition of Theorem 6 in order to gather the necessary inequality (54). $\square$

The uniqueness of (52) solutions can be defined from an illustration below.

Example 13. Let

$$\begin{aligned} |\mathfrak{S}(\bar{\rho}, \mathcal{L}_1) - \mathfrak{S}(\bar{\rho}, \mathcal{L}_2)| &\leq \mathfrak{J}(\bar{\rho}) \left| \widehat{\Pi}(\mathcal{L}_1) - \widehat{\Pi}(\mathcal{L}_2) \right| \\ &\quad + \mathfrak{U}(\bar{\rho}) |\mathcal{L}_1 - \mathcal{L}_2| \left| \widehat{\Pi}(\mathcal{L}_1) - \widehat{\Pi}(\mathcal{L}_2) \right|, \end{aligned} \quad (57)$$

and then (52) has at most one solution.

Proof. Equation (52) with solutions $\mathcal{L}_1(\bar{\rho})$ and $\mathcal{L}_2(\bar{\rho})$ can be represented as

$$\begin{aligned} \mathcal{L}_1(\bar{\rho}) - \mathcal{L}_2(\bar{\rho}) &= \frac{1}{\Gamma(\bar{\Psi})} \sum_{\bar{\Omega}=0}^{\bar{\rho}-\bar{\Psi}} (\bar{\rho} - \bar{\Omega} - 1)^{\bar{\Psi}-1} \mathfrak{S}(\bar{\Omega}, \mathcal{L}_1(\bar{\Omega} + \bar{\Psi} - 1)) \\ &\quad - \frac{1}{\Gamma(\bar{\Psi})} \sum_{\bar{\Omega}=0}^{\bar{\rho}-\bar{\Psi}} (\bar{\rho} - \bar{\Omega} - 1)^{\bar{\Psi}-1} \mathfrak{S}(\bar{\Omega}, \mathcal{L}_2(\bar{\Omega} + \bar{\Psi} - 1)). \end{aligned} \quad (58)$$

Assertion (57) with the prior inequality generates

$$\begin{aligned} |\mathcal{L}_1(\bar{\rho}) - \mathcal{L}_2(\bar{\rho})| &= \left| \frac{1}{\Gamma(\bar{\Psi})} \sum_{\bar{\Omega}=0}^{\bar{\rho}-\bar{\Psi}} (\bar{\rho} - \bar{\Omega} - 1)^{\bar{\Psi}-1} \mathfrak{S}(\bar{\Omega}, \mathcal{L}_1(\bar{\Omega} + \bar{\Psi} - 1)) \right. \\ &\quad \left. - \frac{1}{\Gamma(\bar{\Psi})} \sum_{\bar{\Omega}=0}^{\bar{\rho}-\bar{\Psi}} (\bar{\rho} - \bar{\Omega} - 1)^{\bar{\Psi}-1} \mathfrak{S}(\bar{\Omega}, \mathcal{L}_2(\bar{\Omega} + \bar{\Psi} - 1)) \right| \\ &\leq \frac{1}{\Gamma(\bar{\Psi})} \sum_{\bar{\Omega}=0}^{\bar{\rho}-\bar{\Psi}} (\bar{\rho} - \bar{\Omega} - 1)^{\bar{\Psi}-1} \mathfrak{J}(\bar{\Omega}) \left| \widehat{\Pi}(\mathcal{L}_1(\bar{\Omega} + \bar{\Psi} - 1)) \right. \\ &\quad \left. - \widehat{\Pi}(\mathcal{L}_2(\bar{\Omega} + \bar{\Psi} - 1)) \right| \\ &\quad + \frac{1}{\Gamma(\bar{\Psi})} \sum_{\bar{\Omega}=0}^{\bar{\rho}-\bar{\Psi}} (\bar{\rho} - \bar{\Omega} - 1)^{\bar{\Psi}-1} \mathfrak{U}(\bar{\Omega}) |\mathcal{L}_1(\bar{\Omega} + \bar{\Psi} - 1) \\ &\quad - \mathcal{L}_2(\bar{\Omega} + \bar{\Psi} - 1)| \times \left| \widehat{\Pi}(\mathcal{L}_1(\bar{\Omega} + \bar{\Psi} - 1)) \right. \\ &\quad \left. - \widehat{\Pi}(\mathcal{L}_2(\bar{\Omega} + \bar{\Psi} - 1)) \right|. \end{aligned} \quad (59)$$

The previous inequality by having a few amendments to $|\mathcal{L}_1(\bar{\rho}) - \mathcal{L}_2(\bar{\rho})|$ in the process of Theorem 6 introduces

$$|\mathcal{L}_1(\bar{\rho}) - \mathcal{L}_2(\bar{\rho})| \leq 0. \quad (60)$$

Subsequently, $\mathcal{L}_1(\bar{\rho}) = \mathcal{L}_2(\bar{\rho})$ and at least one solution of fractional difference equation (52) exist. $\square$

5. Concluding Remarks

Discrete fractional calculus has made great progress of real-world phenomena, like fractional chaotic maps, image coding, and more discrete time modeling. One of the pre-eminent crucial issues in investigation of difference equations is to explore the subjective attributes of solutions of these previously mentioned fields. Discrete fractional variants are notable pathways that speed disabling. In this article, fixed on the framework of discrete fractional analytics and with the aid of fractional sum inequalities, we proposed new kinds of discrete fractional Gronwall inequalities. We also extracted the expansion of the decreasing feature sequences in the time-scale domain frame. Such inequalities can be shown not only to recall explicit estimates for solutions of fractional difference equations of a discrete form but also to the uniqueness and continuous dependency on initial value of the solutions in the literature.

Data Availability

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that there are no competing interests.

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