Research Article

On Fixed Point Results in Partial b -Metric Spaces

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Partial b -metric spaces are characterised by a modified triangular inequality and that the self-distance of any point of space may not be zero and the symmetry is preserved. The spaces with a symmetric property have interesting topological properties. This manuscript paper deals with the existence and uniqueness of fixed point points for contraction mappings using triangular weak α -admissibility with regard to η and C -class functions in the class of partial b -metric spaces. We also introduce an example to demonstrate the obtained results.

1. Introduction and Preliminaries

The contribution of fixed point theorems has been appeared as a really effective and successful strategy for understanding many mathematical issues, which rise from actual applications. Fixed point theory is an essential tool to solve several equations in diverse fields, such as integral and differential (fractional) equations appeared in economics, game theory, engineering, physics, and chemistry. For instance, see [1–8]. As the significance and accessibility of Banach contraction mapping (BCM), various authors have amplified and generalized this contraction and set up various interesting supplements and alterations. In this continuation, Czerwik [9] and Bakhtin [10] originated the idea of a b -metric space as a suitable generalization of metric spaces. They demonstrated the truth of new weak

contraction mapping standards in b -metric spaces that generalize the celebrated Banach contraction. Subsequently, several authors have dealt with related fixed point results in this setting (see [11–17]). Furthermore, Matthews [18] presented the idea of a partial metric space by considering the distance between any point to itself that requires not be equal to zero. For more related works in this setting, see [19–21]. Recently, Shukla [22] studied the connection, as topological properties, between a partial metric and a b -metric space to introduce an unification of the concept of a partial b -metric space.

Definition 1 ([22]). Given a nonempty set W . A function $\rho_{pb} : W \times W \longrightarrow [0, \infty)$ is denominated as a b -partial metric if for all $v, \kappa, z \in W$

$$\begin{aligned}
(\rho_{pb1}) v = \kappa \text{ if and only if } \rho_{pb}(v, v) = \rho_{pb}(v, \kappa) = \rho_{pb}b(\kappa, \kappa), \\
(\rho_{pb2}) \rho_{pb}(v, v) \leq \rho_{pb}(v, \kappa), \\
(\rho_{pb3}) \rho_{pb}(v, \kappa) = \rho_{pb}(\kappa, v), \\
(\rho_{pb4}) \text{ there is a real number } s \geq 1 \text{ such that } \rho_{pb}(v, \kappa) \\
\leq s [\rho_{pb}(v, z) + \rho_{pb}(z, \kappa)] - \rho_{pb}(z, z).
\end{aligned} \tag{1}$$

If $\rho_{pb}(v, \kappa) = 0$, then $v = \kappa$. The converse does not hold in general. Now, consider $W = [0, \infty)$ and $p > 1$. Take $\rho_{pb} : W \times W \longrightarrow \mathbb{R}^+$ as

$$\rho_{pb}(v, \kappa) = [\max \{v, \kappa\}]^p - |v - \kappa|^p, \tag{2}$$

for all $v, \kappa \in W$. Then, (X, ρ_{pb}) is a partial b -metric space (Here, $s = 2p > 1$). It is neither a partial metric, nor a b -metric space. Let ρ_p be a partial metric and ρ_b be a b -metric (with $s > 1$) on a nonempty set W . Then, $\rho_{pb} : W \times W \longrightarrow [0, \infty)$ defined by $\rho_{pb}(v, \kappa) = \rho_p(v, \kappa) + \rho_b(v, \kappa)$ for all $v, \kappa \in W$, is a partial b -metric on W .

Definition 2 ([22, 23]). Let (W, ρ_{pb}) be a partial b -metric space with $s \geq 1$. Let $\{v_n\}$ be a sequence in W and $v \in W$. Then,

- (i) $\{v_n\} \subseteq W$ converges to $v \in W$ if $\lim_{n \rightarrow \infty} \rho_{pb}(v_n, v) = \rho_{pb}(v, v)$
- (ii) $\{v_n\} \subseteq W$ is Cauchy in (W, ρ_{pb}) if $\lim_{n, m \rightarrow \infty} \rho_{pb}(v_n, v_m)$ exists and is finite
- (iii) (W, ρ_{pb}) is complete if for every Cauchy sequence $\{v_n\} \subseteq W$, there is $v \in W$ so that

$$\lim_{n, m \rightarrow \infty} \rho_{pb}(v_n, v_m) = \lim_{n \rightarrow \infty} \rho_{pb}(v_n, v) = \rho_{pb}(v, v). \tag{3}$$

Given a nonempty set W . Given $f : W \longrightarrow W$ and $\alpha : W \times W \longrightarrow [0, \infty)$. Then, f is called α -admissible [24] if for all $v, \kappa \in W$ with $\alpha(v, \kappa) \geq 1$, we have $\alpha(fv, f\kappa) \geq 1$. The conception of triangular α -admissibility was initiated in [25]. This concept has been weakened as follows.

Definition 3 ([26]). Given a nonempty set W . Given $\alpha, \eta : W \times W \longrightarrow [0, \infty)$. $S : W \longrightarrow W$ is triangular weak α -admissible with regard to η if

- (1) $\alpha(v, S^n v) \geq \eta(v, S^n v)$, implies that $\alpha(S^n v, S^{n+1} v) \geq \eta(S^n v, S^{n+1} v)$, for all $n \in \mathbb{N}$
- (2) $\alpha(v, z) \geq \eta(v, z)$ and $\alpha(z, \kappa) \geq \eta(z, \kappa)$, imply that $\alpha(v, \kappa) \geq \eta(v, \kappa)$.

Let $W = [0, \infty)$. Given $S : W \longrightarrow W$ as $Sv = v^2$. Take $\alpha, \eta : W \times W \longrightarrow [0, \infty)$ as $\alpha(v, \kappa) = e^{v+\kappa}$ and $\eta(v, \kappa) = e^{\kappa-v}$. Here, S is triangular weak α -admissible with regard to η .

Lemma 4 ([26]). Given $\alpha, \eta : W \times W \longrightarrow \mathbb{R}$. Let $S : W \longrightarrow W$ be triangular weak α -admissible with regard to η . Assume that there is $v_0 \in W$ such that $\alpha(v_0, Sv_0) \geq \eta(v_0, Sv_0)$. Define $\{v_n\}$ in W by $Sv_n = v_{n+1}$. Then, for all integers $n < m$

$$\alpha(v_n, v_m) \geq \eta(v_n, v_m). \tag{4}$$

Let (W, σ) be a metric space. Given $\alpha, \eta : W \times W \longrightarrow [0, \infty)$. Following [27], the mapping $T : W \longrightarrow W$ is said to be α - η -continuous if every sequence $\{v_n\}$ in W with $\alpha(v_n, v_{n+1}) \geq \eta(v_n, v_{n+1})$ for all $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} \sigma(v_n, v) = 0$, we have $\lim_{n \rightarrow \infty} \sigma(Tv_n, Tv) = 0$. This concept could be extended to b -partial metric spaces as follows.

Definition 5. Let (W, ρ_{pb}) be a b -partial metric space. Given $\alpha, \eta : W \times W \longrightarrow [0, \infty)$. The mapping $T : W \longrightarrow W$ is said to be α - η -continuous if every sequence $\{v_n\}$ in W with $\alpha(v_n, v_{n+1}) \geq \eta(v_n, v_{n+1})$ for all $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} \rho_{pb}(v_n, v) = \rho_{pb}(v, v)$, we have $\lim_{n \rightarrow \infty} d(Tv_n, Tv) = \rho_{pb}(Tv, Tv)$.

The conception of C -class functions is posed as follows.

Definition 6 ([28]). We say that $\mathcal{F} : [0, \infty) \times [0, \infty) \longrightarrow \mathbb{R}$ is a C -class function if it is continuous and for $v, \kappa \in [0, \infty)$, we have

- (i) $\mathcal{F}(v, \kappa) \leq v$
- (ii) $\mathcal{F}(v, \kappa) = v$ yields that either $v = 0$, or $\kappa = 0$

Denote by $\mathcal{C}$ the set of C -class functions. The following functions are elements in $\mathcal{C}$ (for all $v, \kappa \in [0, \infty)$):

- (1) $\mathcal{F}(v, \kappa) = v - \kappa$
- (2) $\mathcal{F}(v, \kappa) = \mu v$ (with $0 < \mu < 1$)
- (3) $\mathcal{F}(v, \kappa) = \Theta(v)$ (with $\Theta : [0, \infty) \longrightarrow [0, 1]$ is continuous)

Denote by Ψ the set of functions [29] $\psi : [0, \infty) \longrightarrow [0, \infty)$ verifying that:

- (i) ψ is monotone and nondecreasing
- (ii) $\psi(v) = 0 \iff v = 0$

Let Φ be the set of functions $\varphi : [0, \infty) \longrightarrow [0, \infty)$ so that

- (i) φ is continuous
- (ii) $\varphi(v) = 0 \iff v = 0$

In this paper, we initiate the concept of generalized α - η - ψ - φ - F -contraction self-mappings via C -class functions and the connotation of triangular weak α -admissibility

with regard to η . The goal is to prove some related fixed point results in the context of partial b -metric spaces.

2. Main Results

We start with the following.

$$\psi\left(s\rho_{pb}(Su, Sk)\right) \leq \begin{cases} F(\psi(\mathcal{P}(u, \kappa)), \varphi(\mathcal{P}(u, \kappa))) & \text{if } \rho_{pb}(u, S\kappa) + \rho_{pb}(Su, \kappa) \neq 0, \\ 0 & \text{otherwise } \rho_{pb}(u, S\kappa) + \rho_{pb}(Su, \kappa) = 0, \end{cases} \quad (5)$$

where either

$$\mathcal{P}(u, \kappa) = \max \left\{ \rho_{pb}(u, \kappa), \rho_{pb}(u, Su), \rho_{pb}(\kappa, S\kappa), \frac{\rho_{pb}(u, S\kappa) + \rho_{pb}(\kappa, Su)}{2s} \right\}, \quad (6)$$

or

$$\mathcal{P}(u, \kappa) = \left[\frac{\rho_{pb}(u, Su) + \rho_{pb}(\kappa, S\kappa)}{2} \right], \quad (7)$$

then S is called a generalized $\alpha - \eta - \psi - \varphi - F$ -contraction.

Theorem 8. Let (W, ρ_{pb}) be a complete b -partial metric space (with $s \geq 1$) and $S : W \longrightarrow W$. Assume that

- (a) S is a generalized $\alpha - \eta - \psi - \varphi - F$ -contraction
- (b) S is triangular weak α -admissible
- (c) There is $v_0 \in W$ such that $\alpha(v_0, Sv_0) \geq \eta(v_0, Sv_0)$
- (d) S is $\alpha - \eta$ -continuous

Then, S possesses a unique fixed point.

Proof. Let $v_0 \in W$ be in order that $\alpha(v_0, Sv_0) \geq \eta(v_0, Sv_0)$. Realize a sequence $\{v_n\}$ in W in order that $v_{n+1} = Sv_n$ for all $n \in \mathbb{N}$. If for some n , $v_n = v_{n+1}$, the v_n is a fixed point. Now, suppose that $v_n \neq v_{n+1}$ for all $n \in \mathbb{N}$. By using assumption (b), the triangular weak α -admissibility and Lemma 4, we get that for all integers $m > n$,

$$\alpha(v_n, v_m) \geq \eta(v_n, v_m). \quad (8)$$

In particular, $\alpha(v_n, v_{n+1}) \geq \eta(v_n, v_{n+1})$ for all $n \in \mathbb{N}$.

Case 1. Here, we consider $\mathcal{P}(u, \kappa) = \max \{ \rho_{pb}(u, \kappa), \rho_{pb}(u, Sv), \rho_{pb}(\kappa, S\kappa), ((\rho_{pb}(u, S\kappa) + \rho_{pb}(\kappa, Su))/2s) \}$. We need three steps.

Definition 7. Let (W, ρ_{pb}) be a complete b -partial metric space with coefficient $s \geq 1$ and $S : W \longrightarrow W$. If there are $\alpha, \eta : W \times W \longrightarrow [0, \infty)$, $F \in \mathcal{C}$, $\psi \in \Psi$, and $\varphi \in \Phi$, so that for all $u, \kappa \in W$ with $\alpha(u, \kappa) \geq \eta(u, \kappa)$,

Step 1. In view of assumption (a), one writes

$$\begin{aligned} \psi\left(\rho_{pb}(v_{n+1}, v_n)\right) &= \psi\left(\rho_{pb}(Sv_n, Sv_{n-1})\right) \leq \psi\left(s\rho_{pb}(Sv_n, Sv_{n-1})\right) \\ &\leq F(\psi(\mathcal{P}(v_n, v_{n-1})), \varphi(\mathcal{P}(v_n, v_{n-1}))) \\ &\leq \psi(\mathcal{P}(v_n, v_{n-1})), \end{aligned} \quad (9)$$

for all $n \in \mathbb{N}$, where

$$\begin{aligned} \mathcal{P}(v_n, v_{n-1}) &= \max \left\{ \rho_{pb}(v_n, v_{n-1}), \rho_{pb}(v_n, Sv_n), \rho_{pb}(v_{n-1}, Sv_{n-1}), \right. \\ &\quad \left. \frac{\rho_{pb}(v_n, Sv_{n-1}) + \rho_{pb}(v_{n-1}, Sv_n)}{2s} \right\} \\ &= \max \left\{ \rho_{pb}(v_n, v_{n-1}), \rho_{pb}(v_n, v_{n+1}), \rho_{pb}(v_{n-1}, v_n), \right. \\ &\quad \left. \frac{\rho_{pb}(v_n, v_n) + \rho_{pb}(v_{n-1}, v_{n+1})}{2s} \right\} \\ &= \max \left\{ \rho_{pb}(v_n, v_{n-1}), \rho_{pb}(v_n, v_{n+1}) \right\}. \end{aligned} \quad (10)$$

If $\mathcal{P}(v_n, v_{n-1}) = \rho_{pb}(v_n, v_{n+1})$ for some n , then we get

$$\begin{aligned} \psi\left(\rho_{pb}(v_{n+1}, v_n)\right) &\leq F(\psi(\mathcal{P}(v_n, v_{n-1})), \varphi(\mathcal{P}(v_n, v_{n-1}))) \\ &= F\left(\psi\left(\rho_{pb}(v_{n+1}, v_n)\right), \varphi\left(\rho_{pb}(v_{n+1}, v_n)\right)\right) \\ &\leq \psi\left(\rho_{pb}(v_{n+1}, v_n)\right). \end{aligned} \quad (11)$$

This implicates that either $\psi(\rho_{pb}(v_{n+1}, v_n)) = 0$ or $\varphi(\rho_{pb}(v_{n+1}, v_n)) = 0$, so $\rho_{pb}(v_{n+1}, v_n) = 0$; hence, $v_{n+1} = v_n$. It is a contradiction. Thus, $\mathcal{P}(v_n, v_{n-1}) = \rho_{pb}(v_n, v_{n-1})$ for all $n \geq 1$. By (5), we have

$$\psi\left(\rho_{pb}(v_{n+1}, v_n)\right) \leq \psi\left(\rho_{pb}(v_n, v_{n-1})\right). \quad (12)$$

Since ψ is nondecreasing, we get $\rho_{pb}(v_{n+1}, v_n) \leq \rho_{pb}(v_n, v_{n-1})$ for all $n \geq 1$. Thus, $\{\rho_{pb}(v_n, v_{n+1})\}$ is nonincreasing. Thus, there is $\omega \geq 0$ in order that $\lim_{n \rightarrow \infty} \rho_{pb}(v_n, v_{n+1}) = \omega$.

We claim that $\omega = 0$. We have

$$\begin{aligned} \psi(\rho_{pb}(v_{n+1}, v_{n+2})) &\leq \psi(s\rho_{pb}(Sv_n, Sv_{n+1})) \\ &\leq F(\psi(\rho_{pb}(v_n, v_{n+1})), \varphi(\rho_{pb}(v_n, v_{n+1}))) \\ &\leq \psi(\rho_{pb}(v_n, v_{n+1})). \end{aligned} \quad (13)$$

At the limit, we acquire that

$$\psi(\omega) \leq F(\psi(\omega), \varphi(\omega)) \leq \psi(\omega). \quad (14)$$

This infers that either $\psi(\omega) = 0$, or $\varphi(\omega) = 0$. That is, $\omega = 0$, i.e.,

$$\lim_{n \rightarrow \infty} \rho_{pb}(v_n, v_{n+1}) = 0. \quad (15)$$

To prove that $\{v_n\}$ is a Cauchy sequence in the b -partial metric space (W, ρ_{pb}) , we argue by contradiction. For this, suppose that there exist $\varepsilon > 0$ and subsequences $\{m(l)\}_{l=1}^\infty$ and $\{n(l)\}_{l=1}^\infty$ of positive integers with $n(l) > m(l) > l$ such that

$$\rho_{pb}(v_{n(l)}, v_{m(l)}) \geq \varepsilon \text{ and } \rho_{pb}(v_{m(l)}, v_{n(l)-1}) < \varepsilon. \quad (16)$$

Using (ρ_{pb4}) and applying (15) and (16), one easily gets

$$\frac{\varepsilon}{s^2} \leq \limsup_{l \rightarrow \infty} \rho_{pb}(v_{n(l)-1}, v_{m(l)-1}) \leq \varepsilon s, \quad (17)$$

$$\frac{\varepsilon}{s} \leq \limsup_{l \rightarrow \infty} \rho_{pb}(v_{n(l)-1}, v_{m(l)}) \leq \varepsilon, \quad (18)$$

$$\frac{\varepsilon}{s} \leq \limsup_{l \rightarrow \infty} \rho_{pb}(v_{n(l)}, v_{m(l)-1}) \leq \varepsilon s^2. \quad (19)$$

In view of assumption (a), we have

$$\begin{aligned} \psi(s\rho_{pb}(v_{n(l)}, v_{m(l)})) &= \psi(s\rho_{pb}(Sv_{n(l)-1}, Sv_{m(l)-1})) \\ &\leq F(\psi(\rho_{pb}(v_{n(l)-1}, v_{m(l)-1})), \varphi(\rho_{pb}(v_{n(l)-1}, v_{m(l)-1}))) \\ &\leq \psi(\rho_{pb}(v_{n(l)-1}, v_{m(l)-1})), \end{aligned} \quad (20)$$

where

$$\begin{aligned} \rho_{pb}(v_{n(l)-1}, v_{m(l)-1}) &= \max \left\{ \rho_{pb}(v_{n(l)-1}, v_{m(l)-1}), \rho_{pb}(v_{n(l)-1}, Sv_{n(l)-1}), \right. \\ &\quad \left. \rho_{pb}(v_{m(l)-1}, Sv_{m(l)-1}), \frac{\rho_{pb}(v_{n(l)-1}, Sv_{m(l)-1}) + \rho_{pb}(v_{m(l)-1}, Sv_{n(l)-1})}{2s} \right\} \\ &= \max \left\{ \rho_{pb}(v_{n(l)-1}, v_{m(l)-1}), \rho_{pb}(v_{n(l)-1}, v_{n(l)}), \rho_{pb}(v_{m(l)-1}, v_{m(l)}), \right. \\ &\quad \left. \frac{\rho_{pb}(v_{n(l)-1}, v_{m(l)}) + \rho_{pb}(v_{m(l)-1}, v_{n(l)})}{2s} \right\}. \end{aligned} \quad (21)$$

By (15)–(19), one gets

$$\limsup_{l \rightarrow \infty} \rho_{pb}(v_{n(l)-1}, v_{m(l)-1}) \leq \varepsilon s. \quad (22)$$

Letting $l \rightarrow \infty$ and having in mind properties of ψ , φ and F , we deduce that

$$\psi(s\varepsilon) \leq F(\psi(s\varepsilon), \varphi(s\varepsilon)) \leq \psi(s\varepsilon). \quad (23)$$

That is,

$$F(\psi(s\varepsilon), \varphi(s\varepsilon)) = \psi(s\varepsilon), \quad (24)$$

so either $\psi(s\varepsilon) = 0$ or $\varphi(s\varepsilon) = 0$. Thus, $\varepsilon = 0$, which is a contradiction. Hence, $\{v_n\}$ is a Cauchy sequence.

Step 2. We claim that S has a fixed point. The sequence $\{v_n\}$ is Cauchy in the complete b -partial metric space, so there is $v^* \in W$ in order that

$$\lim_{n, m \rightarrow \infty} \rho_{pb}(v_n, v^*) = \lim_{n, m \rightarrow \infty} \rho_{pb}(v_n, v_m) = \rho_{pb}(v^*, v^*) = 0. \quad (25)$$

For each $n \in \mathbb{N}$, we have

$$\begin{aligned} \rho_{pb}(v^*, Sv^*) &\leq s[\rho_{pb}(v^*, v_{n+1}) + \rho_{pb}(v_{n+1}, Sv^*)] - \rho_{pb}(v_{n+1}, v_{n+1}) \\ &\leq s[\rho_{pb}(v^*, v_{n+1}) + \rho_{pb}(Sv_n, Sv^*)] \leq s\rho_{pb}(v^*, v_{n+1}) \\ &\quad + s\rho_{pb}(Sv_n, Sv^*). \end{aligned} \quad (26)$$

By assumption (d) and Definition 5, $\rho_{pb}(Sv_n, Sv^*) \rightarrow \rho_{pb}(Sv^*, Sv^*)$ as $n \rightarrow \infty$. Using (25), we deduce that $\rho_{pb}(v^*, Sv^*) \leq s\rho_{pb}(Sv^*, Sv^*)$. We will show that $v^* = Sv^*$. Suppose that $v^* \neq Sv^*$, then we obtain that $\rho_{pb}(v^*, Sv^*) \neq 0$. This implies that

$$\begin{aligned} \psi(\rho_{pb}(Sv^*, Sv^*)) &\leq \psi(s\rho_{pb}(Sv^*, Sv^*)) \\ &\leq F(\psi(\rho_{pb}(Sv^*, Sv^*)), \varphi(\rho_{pb}(Sv^*, Sv^*))) \\ &\leq \psi(\rho_{pb}(Sv^*, Sv^*)), \end{aligned} \quad (27)$$

where

$$\begin{aligned} \rho_{pb}(v^*, v^*) &= \max \left\{ \rho_{pb}(v^*, v^*), \rho_{pb}(v^*, Sv^*), \rho_{pb}(v^*, Sv^*), \right. \\ &\quad \left. \frac{\rho_{pb}(v^*, Sv^*) + \rho_{pb}(v^*, Sv^*)}{2s} \right\} = \rho_{pb}(v^*, Sv^*). \end{aligned} \quad (28)$$

So,

$$\psi(\rho_{pb}(Su^*, Su^*)) \leq \psi(\rho_{pb}(v^*, Su^*)) \leq \psi(\rho_{pb}(Su^*, Su^*)). \quad (29)$$

Hence, $\psi(\rho_{pb}(Su^*, Su^*)) = \psi(\rho_{pb}(Su^*, v^*))$. We can write

$$\begin{aligned} \psi(\rho_{pb}(Su^*, v^*)) &\leq F(\psi(\rho_{pb}(Su^*, v^*)), \varphi(\rho_{pb}(Su^*, v^*))) \\ &\leq \psi(\rho_{pb}(Su^*, v^*)). \end{aligned} \quad (30)$$

Then,

$$F(\psi(\rho_{pb}(Su^*, v^*)), \varphi(\rho_{pb}(Su^*, v^*))) = \psi(\rho_{pb}(Su^*, v^*)). \quad (31)$$

Thus, $\psi(\rho_{pb}(Su^*, v^*)) = 0$ or $\varphi(\rho_{pb}(Su^*, v^*)) = 0$. Hence, $v^* = Su^*$. It is a contradiction. Consequently, S has a fixed point.

Step 3. We demonstrate that the fixed point of S is unique.

Given $u \neq \hat{u} \in W$ so that $Su = u$ and $S\hat{u} = \hat{u}$. From (5),

$$\begin{aligned} \psi(\rho_{pb}(u, \hat{u})) &\leq \psi(s\rho_{pb}(u, \hat{u})) = \psi(s\rho_{pb}(Su, S\hat{u})) \\ &\leq F\left(\psi\left(\max\left\{\rho_{pb}(u, \hat{u}), \rho_{pb}(u, Su), \rho_{pb}(\hat{u}, S\hat{u}), \frac{\rho_{pb}(u, S\hat{u}) + \rho_{pb}(\hat{u}, Su)}{2s}\right\}\right), \right. \\ &\quad \left. \varphi\left(\max\left\{\rho_{pb}(u, \hat{u}), \rho_{pb}(u, Su), \rho_{pb}(\hat{u}, S\hat{u}), \frac{\rho_{pb}(u, S\hat{u}) + \rho_{pb}(\hat{u}, Su)}{2s}\right\}\right)\right) \\ &= F\left(\psi\left(\max\left\{\rho_{pb}(u, \hat{u}), \rho_{pb}(u, u), \rho_{pb}(\hat{u}, \hat{u}), \frac{\rho_{pb}(u, \hat{u}) + \rho_{pb}(\hat{u}, u)}{2s}\right\}\right), \right. \\ &\quad \left. \varphi\left(\max\left\{\rho_{pb}(u, \hat{u}), \rho_{pb}(u, u), \rho_{pb}(\hat{u}, \hat{u}), \frac{\rho_{pb}(u, \hat{u}) + \rho_{pb}(\hat{u}, u)}{2s}\right\}\right)\right) \\ &= F\left(\psi\left(\max\left\{\rho_{pb}(u, \hat{u}), \rho_{pb}(u, u), \rho_{pb}(\hat{u}, \hat{u}), \frac{\rho_{pb}(u, \hat{u}) + \rho_{pb}(\hat{u}, u)}{2s}\right\}\right), \right. \\ &\quad \left. \varphi\left(\max\left\{\rho_{pb}(u, \hat{u}), \rho_{pb}(u, u), \rho_{pb}(\hat{u}, \hat{u}), \frac{\rho_{pb}(u, \hat{u}) + \rho_{pb}(\hat{u}, u)}{2s}\right\}\right)\right) \\ &= F(\psi(\rho_{pb}(u, \hat{u})), \varphi(\rho_{pb}(u, \hat{u}))) \leq \psi(\rho_{pb}(u, \hat{u})). \end{aligned} \quad (32)$$

This infers that either $\psi(\rho_{pb}(u, \hat{u})) = 0$ or $\varphi(\rho_{pb}(u, \hat{u})) = 0$. This implies that $\rho_{pb}(u, \hat{u}) = 0$, so $u = \hat{u}$.

Case 1. We consider $\mathcal{P}(v, \kappa) = [\rho_{pb}(v, Sv) + \rho_{pb}(\kappa, S\kappa)]/2$.

Again, in view of assumption (a), one writes

$$\begin{aligned} \psi(\rho_{pb}(v_{n+1}, v_n)) &\leq \psi(s\rho_{pb}(v_{n+1}, v_n)) = \psi(s\rho_{pb}(Sv_n, Sv_{n-1})) \\ &\leq F(\psi(\mathcal{P}(v_n, v_{n-1})), \varphi(\mathcal{P}(v_n, v_{n-1}))) \leq \psi(\mathcal{P}(v_n, v_{n-1})), \end{aligned} \quad (33)$$

for all $n \geq 1$, where

$$\begin{aligned} \mathcal{P}(v_n, v_{n-1}) &= \left[\frac{\rho_{pb}(v_n, Sv_n) + \rho_{pb}(v_{n-1}, Sv_{n-1})}{2} \right] \\ &= \left[\frac{\rho_{pb}(v_n, v_{n+1}) + \rho_{pb}(v_{n-1}, v_n)}{2} \right]. \end{aligned} \quad (34)$$

Then,

$$\begin{aligned} \psi(\rho_{pb}(v_{n+1}, v_n)) &\leq F(\psi(\mathcal{P}(v_n, v_{n-1})), \varphi(\mathcal{P}(v_n, v_{n-1}))) \\ &= F\left(\psi\left(\frac{\rho_{pb}(v_n, v_{n+1}) + \rho_{pb}(v_{n-1}, v_n)}{2}\right), \right. \\ &\quad \left. \varphi\left(\frac{\rho_{pb}(v_n, v_{n+1}) + \rho_{pb}(v_{n-1}, v_n)}{2}\right)\right) \\ &\leq \psi\left(\frac{\rho_{pb}(v_n, v_{n+1}) + \rho_{pb}(v_{n-1}, v_n)}{2}\right). \end{aligned} \quad (35)$$

We deduce that

$$\rho_{pb}(v_{n+1}, v_n) \leq \frac{\rho_{pb}(v_n, v_{n+1}) + \rho_{pb}(v_{n-1}, v_n)}{2}. \quad (36)$$

That is,

$$\rho_{pb}(v_{n+1}, v_n) \leq \rho_{pb}(v_{n-1}, v_n), \quad n \geq 1. \quad (37)$$

Let $\{\rho_{pb}(v_{n+1}, v_n)\} = \theta_n$. Then, $\theta_n < \theta_{n-1}$.

This emphasizes that $\{\theta_n\}$ is a bounded decreasing sequence and bounded below, so it must be a convergent sequence.

For each $n \in \mathbb{N}$, we must have,

$$\theta_n < \theta_{n-1} < \dots < \theta_1 = M \text{ (bounded value)}. \quad (38)$$

Again, for all $n, m \in \mathbb{N}$, we get

$$\rho_{pb}(v_n, v_m) \leq \frac{1}{2}(\theta_{n-1} + \theta_{m-1}) < M. \quad (39)$$

Therefore, $\{v_n\}$ is a bounded sequence in W . By Bolzano Weierstrass theorem that every bounded sequence of real numbers has a convergent subsequence, $\{v_n\}$ must have a convergent subsequence, like $\{v_{n_k}\}$ which converges to some $z \in W$. So,

$$\lim_{k \rightarrow \infty} \rho_{pb}(v_{n_k}, v_{n_{k+1}}) = \rho_{pb}\left(\lim_{k \rightarrow \infty} v_{n_k}, \lim_{k \rightarrow \infty} v_{n_{k+1}}\right) = \rho_{pb}(z, z) = 0. \quad (40)$$

This confirms that the subsequence $\{\theta_{n_k}\}$ itself converges to 0. So, the sequence $\{\theta_n\}$ must converge to 0. Hence, for all $n, m \in \mathbb{N}$,

$$\rho_{pb}(v_n, v_m) \leq \frac{1}{2} \left\{ \rho_{pb}(v_{n-1}, v_n) + \rho_{pb}(v_{m-1}, v_m) \right\} \longrightarrow 0, \quad (41)$$

as $n, m \longrightarrow \infty$.

This concludes that $\{v_n\}$ is a Cauchy sequence. As the subsequence $\{v_{n_k}\}$ of $\{v_n\}$ converges to z , the limit of $\{v_n\}$ must be z . Also, we have

$$\begin{aligned} \rho_{pb}(z, Sz) &\leq \rho_{pb}(z, Sv_{n+1}) + \rho_{pb}(Sv_{n+1}, Sz) < \rho_{pb}(z, Sv_{n+1}) \\ &+ \frac{1}{2} \left\{ \rho_{pb}(v_n, Sv_{n+1}) + \rho_{pb}(z, Sz) \right\} \Rightarrow \frac{1}{2} \rho_{pb}(z, Sz) \\ &\leq \rho_{pb}(z, Sv_{n+1}) + \frac{1}{2} \rho_{pb}(v_n, Sv_{n+1}) \longrightarrow 0 \text{ as } n \longrightarrow \infty. \end{aligned} \quad (42)$$

This infers that $z = Sz$. Hence, z is a fixed point of S .

Next, we assay the uniqueness of z . Debating by contradiction, let $\dot{z} \neq z$ be another fixed point of S , then $\rho_{pb}(z, \dot{z}) > 0$. We have

$$\begin{aligned} \psi(\rho_{pb}(z, \dot{z})) &\leq \psi(s\rho_{pb}(Sz, S\dot{z})) \leq F(\psi(\mathcal{P}(z, \dot{z}), \varphi(\mathcal{P}(z, \dot{z})))) \\ &= F\left(\psi\left(\frac{\rho_{pb}(z, Sz) + \rho_{pb}(\dot{z}, S\dot{z})}{2}\right), \varphi\left(\frac{\rho_{pb}(z, Sz) + \rho_{pb}(\dot{z}, S\dot{z})}{2}\right)\right) \\ &\leq \psi\left(\frac{\rho_{pb}(z, z) + \rho_{pb}(\dot{z}, \dot{z})}{2}\right) \leq \psi(\rho_{pb}(z, \dot{z})). \end{aligned} \quad (43)$$

Thus, $\rho_{pb}(z, \dot{z}) = 0$, which is a contradiction. Then, $z = \dot{z}$. Now, the continuity of S in Theorem 8 has been dropped.

Theorem 9. Let (W, ρ_{pb}) be a complete b -partial metric space (with $s \geq 1$) and $S : W \longrightarrow W$. Assume that:

- (1) S is a generalized $\alpha - \eta - \psi - \phi - F$ -contraction
- (2) S is triangular weak α -admissible
- (3) There is $v_0 \in W$ in order that $\alpha(v_0, Sv_0) \geq \eta(v_0, Sv_0)$
- (4) If $\{v_n\}$ is a sequence in W in order that $\alpha(v_n, v_{n+1}) \geq \eta(v_n, v_{n+1})$ and $v_n \longrightarrow v^* \in W$ as $n \longrightarrow \infty$, then there is $\{v_{n(k)}\}$ in order that $\alpha(v_{n(k)}, v^*) \geq \eta(v_{n(k)}, v^*)$

Then, S possesses a unique fixed point.

Proof. As in the proof of Theorem 8, we build the sequence $\{v_n\}$ by $v_{n+1} = Sv_n$ for all $n \in \mathbb{N}$, which converges to $v^* \in W$ in order that $\alpha(v_n, v_{n+1}) \geq \eta(v_n, v_{n+1})$ for all $n \in \mathbb{N}$. By condition (4), there is a subsequence $\{v_{n(k)}\}$ of $\{v_n\}$ in order that $\alpha(v_{n(k)}, v^*) \geq \eta(v_{n(k)}, v^*)$ for all $k \in \mathbb{N}$. Consequently,

$$\begin{aligned} \psi(\rho_{pb}(v_{n(k)+1}, Sv^*)) &\leq \psi(s\rho_{pb}(Sv_{n(k)}, Sv^*)) \\ &= \psi(s\rho_{pb}(Sv_{n(k)}, Sv^*)), \leq F(\psi(\mathcal{P}(v_{n(k)}, v^*), \varphi(\mathcal{P}(v_{n(k)}, v^*)))), \\ &\leq F(\psi(\mathcal{P}(v_{n(k)}, v^*))), \end{aligned} \quad (44)$$

for all $n \in \mathbb{N}$, where
Case 1.

$$\begin{aligned} \mathcal{P}(v_{n(k)}, v^*) &= \max \left\{ \rho_{pb}(v_n, v^*), \rho_{pb}(v_{n(k)}, Sv_{n(k)}), \rho_{pb}(v^*, Sv^*), \right. \\ &\quad \left. \frac{\rho_{pb}(v_{n(k)}, Sv^*) + \rho_{pb}(v^*, Sv_{n(k)})}{2s} \right\}, \\ &= \max \left\{ \rho_{pb}(v_{n(k)}, v^*), \rho_{pb}(v_{n(k)}, v_{n(k)+1}), \rho_{pb}(v^*, v^*), \right. \\ &\quad \left. \frac{\rho_{pb}(v_{n(k)}, v^*) + \rho_{pb}(v^*, Sv_{n(k)})}{2s} \right\}, \\ &\leq \max \left\{ \rho_{pb}(v_{n(k)}, v^*), \rho_{pb}(v_{n(k)}, v_{n(k)+1}), \rho_{pb}(v^*, Sv_{n(k)}) \right\}. \end{aligned} \quad (45)$$

By letting $n \longrightarrow \infty$, then

$$\mathcal{P}(v_{n(k)}, v^*) \leq \rho_{pb}(v^*, Sv^*). \quad (46)$$

Case 2.

$$\begin{aligned} \mathcal{P}(v_{n(k)}, v^*) &= \frac{\rho_{pb}(v_{n(k)}, Sv_{n(k)}) + \rho_{pb}(v^*, Sv^*)}{2}, \\ &= \frac{\rho_{pb}(v_{n(k)}, v_{n(k)+1}) + \rho_{pb}(v^*, Sv^*)}{2}. \end{aligned} \quad (47)$$

By letting $n \longrightarrow \infty$, then

$$\mathcal{P}(v_{n(k)}, v^*) \leq \frac{\rho_{pb}(v^*, Sv^*)}{2} \leq \rho_{pb}(v^*, Sv^*). \quad (48)$$

By using (44), we get

$$\psi(\rho_{pb}(v^*, Sv^*)) \leq F(\psi(\rho_{pb}(v^*, Sv^*)), \phi(\rho_{pb}(v^*, Sv^*))), \quad (49)$$

which infers that $\rho_{pb}(v^*, Sv^*) = 0$. Hence, $Sv^* = v^*$.

Following Theorem 8 and Theorem 9, we have

Corollary 10. Let (W, ρ_{pb}) be a complete b -partial metric space (with $s \geq 1$). Let S be a self-mapping on W . Given $\alpha : W \times W \longrightarrow [0, \infty)$. Assume that

- (1) for all $v, \kappa \in W$ with $\alpha(v, \kappa) \geq 1$ and $\rho_{pb}(v, S\kappa) + \rho_{pb}(Sv, \kappa) \neq 0$, we have $\psi(s\rho_{pb}(Sv, S\kappa)) \leq F(\psi(\rho_{pb}(v, \kappa)), \varphi(\rho_{pb}(v, \kappa)))$

- (2) S is triangular weak α -admissible
- (3) There is $v_0 \in W$ in order that $\alpha(v_0, Sv_0) \geq 1$
- (4) S is $\alpha - \eta$ - continuous

Then, S admits a unique fixed point.

Proof. We consider $\eta : W \times W \longrightarrow \mathbb{R}$ as $\eta(v, \kappa) = 1$.

Corollary 11. Let (W, ρ_{pb}) be a complete b -partial metric space (with $s \geq 1$). Let S be a self-mapping on W . Given $\alpha : W \times W \longrightarrow [0, \infty)$. Assume that

- (1) for all $v, \kappa \in W$ with $\alpha(v, \kappa) \geq 1$ and $\rho_{pb}(v, S\kappa) + \rho_{pb}(Sv, \kappa) \neq 0$, we have $\psi(s\rho_{pb}(Sv, S\kappa)) \leq \lambda F(\psi(\rho_{pb}(v, \kappa)), \varphi(\rho_{pb}(v, \kappa)))$
- (2) S is triangular α -admissible
- (3) There is $v_0 \in W$ in order that $\alpha(v_0, Sv_0) \geq 1$
- (4) If $\{v_n\}$ is a sequence in W in order that $\alpha(v_n, v_{n+1}) \geq 1$ and $v_n \longrightarrow v^* \in W$ as $n \longrightarrow \infty$, then there is $\{v_{n(k)}\}$ in order that $\alpha(v_{n(k)}, v^*) \geq 1$

Then, S possesses a unique fixed point.

Proof. Take $\eta(v, \kappa) = 1$ in Theorem 9.

Now, by taking $\mathcal{F}(v, \kappa) = v\beta(v)$ where $\Theta : [0, \infty) \longrightarrow [0, 1)$ is continuous, we have

Corollary 12. Let (W, ρ_{pb}) be a complete b -partial metric space (with $s \geq 1$). Assume S is a self-mapping on W . Given $\alpha, \eta : W \times W \longrightarrow [0, \infty)$. Assume that for all $v, \kappa \in W$ with $\alpha(v, \kappa) \geq \eta(v, \kappa)$ and $\rho_{pb}(v, S\kappa) + \rho_{pb}(Sv, \kappa) \neq 0$

$$\psi(s\rho_{pb}(Sv, S\kappa)) \leq \Theta(\psi(\rho_{pb}(v, \kappa)))\psi(\rho_{pb}(v, \kappa)), \quad (50)$$

where $\psi \in \Psi$ and $\Theta : [0, \infty) \longrightarrow [0, 1)$ is continuous. If

- (1) S is triangular weak α -admissible
- (2) There is $v_0 \in W$ in order that $\alpha(v_0, Sv_0) \geq \eta(v_0, Sv_0)$
- (3) S is $\alpha - \eta$ - continuous

then S possesses a unique fixed point.

Corollary 13. Let (W, ρ_{pb}) be a complete b -partial metric space (with $s \geq 1$) and S be a self-mapping on W . Given $\alpha, \eta : W \times W \longrightarrow [0, \infty)$. Assume that

$$\psi(s\rho_{pb}(Sv, S\kappa)) \leq \Theta(\psi(\rho_{pb}(v, \kappa))), \psi(\rho_{pb}(v, \kappa)), \quad (51)$$

where $\psi \in \Psi$ and $\Theta : [0, \infty) \longrightarrow [0, 1)$ is continuous, for all $v, \kappa \in W$ with $\alpha(v, \kappa) \geq \eta(v, \kappa)$ and $\rho_{pb}(v, S\kappa) + \rho_{pb}(Sv, \kappa) \neq 0$. If

- (1) S is triangular weak α -admissible
- (2) There is $v_0 \in W$ in order that $\alpha(v_0, Sv_0) \geq \eta(v_0, Sv_0)$
- (3) $\{v_n\}$ is a sequence in W so that $\alpha(v_n, v_{n+1}) \geq \eta(v_n, v_{n+1})$ and $v_n \longrightarrow v^* \in W$ as $n \longrightarrow \infty$, then there is $\{v_{n(k)}\}$ of $\{v_n\}$ in order that $\alpha(v_{n(k)}, v^*) \geq \eta(v_{n(k)}, v^*)$

then S possesses a unique fixed point.

3. Consequences

Let Δ be the family of nonnegative functions δ defined on $[0, \infty)$ so that:

- (1) δ is Lebesgue-integrable on every compact of $[0, \infty)$
- (2) for every $\omega > 0$

$$\int_0^\omega \delta(t) dt > 0. \quad (52)$$

Theorem 14. Let (W, ρ_{pb}) be a complete b -partial metric space (with $s \geq 1$) and S be a self-mapping on W . Suppose there are $F \in \mathcal{C}$, $\delta_1, \delta_2 \in \Delta$, and $\alpha, \eta : W \times W \longrightarrow [0, \infty)$ such for all $v, \kappa \in W$ with $\alpha(v, \kappa) \geq \eta(v, \kappa)$ and $\rho_{pb}(v, S\kappa) + \rho_{pb}(Sv, \kappa) \neq 0$, we get

$$\begin{aligned} & \int_0^{s\rho_{pb}(Sv, T\kappa)} \delta_1(z) dz \\ & \leq F \left(\int_0^{\max \{ \rho_{pb}(v, \kappa), \rho_{pb}(v, Sv), \rho_{pb}(Sv, Sv), (\rho_{pb}(v, S\kappa) + \rho_{pb}(\kappa, Sv))/2s \}} \delta_1(z) dz, \right. \\ & \quad \left. \int_0^{\max \{ \rho_{pb}(v, \kappa), \rho_{pb}(v, Sv), \rho_{pb}(Sv, Sv), (\rho_{pb}(v, S\kappa) + \rho_{pb}(\kappa, Sv))/2s \}} \delta_2(z) dz \right). \end{aligned} \quad (53)$$

Also, suppose that

- (1) S is triangular weak α -admissible
- (2) There is $v_0 \in W$ in order that $\alpha(v_0, Sv_0) \geq \eta(v_0, Sv_0)$
- (3) S is $\alpha - \eta$ - continuous

Then, S possesses a unique fixed point.

Proof. Take $\psi(t) = \int_0^t \delta_1(t) dt$ and $\varphi(t) = \int_0^t \delta_2(t) dt$. By using Theorem 8, S possesses a fixed point.

Theorem 15. Let (W, ρ_{pb}) be a complete b -partial metric space (with $s \geq 1$) and $S : W \longrightarrow W$. Suppose there are $F \in \mathcal{C}$, $\delta_1, \delta_2 \in \Delta$, and $\alpha, \eta : W \times W \longrightarrow [0, \infty)$ such for all $v, \kappa \in W$ with $\alpha(v, \kappa) \geq \eta(v, \kappa)$ and $\rho_{pb}(v, S\kappa) + \rho_{pb}(Sv, \kappa) \neq 0$, we have

$$\begin{aligned}
& \int_0^{s\rho_{pb}(Sv, S\kappa)} \delta_1(z) dz \\
& \leq F \left(\int_0^{\max \{ \rho_{pb}(v, \kappa), \rho_{pb}(v, Sv), \rho_{pb}(Sv, S\kappa), (\rho_{pb}(v, S\kappa) + \rho_{pb}(\kappa, Sv))/2s \}} \delta_1(z) dz, \right. \\
& \quad \left. \int_0^{\max \{ \rho_{pb}(v, \kappa), \rho_{pb}(v, Sv), \rho_{pb}(Sv, S\kappa), (\rho_{pb}(v, S\kappa) + \rho_{pb}(\kappa, Sv))/2s \}} \delta_2(z) dz \right). \quad (54)
\end{aligned}$$

Assume that:

- (1) S is triangular weak α -admissible
- (2) There is $v_0 \in W$ in order that $\alpha(v_0, Sv_0) \geq \eta(v_0, Sv_0)$
- (3) If $\{v_n\}$ is a sequence in W in order that $\alpha(v_n, v_{n+1}) \geq \eta(v_n, v_{n+1})$ and $v_n \longrightarrow v^* \in W$ as $n \longrightarrow \infty$, then there is $\{v_{n(k)}\}$ in order that $\alpha(v_{n(k)}, v^*) \geq \eta(v_{n(k)}, v^*)$

Then, S possesses a unique fixed point.

Proof. Consider $\psi(t) = \int_0^t \delta_1(u) du$ and $\varphi(t) = \int_0^t \delta_2(u) du$ in Theorem 9.

Example 1. Given $W = [0, \infty)$ and $\rho_{pb} : W \times W \longrightarrow \mathbb{R}$ as

$$\rho_{pb}(v, \kappa) = \begin{cases} [\max \{v, \kappa\}]^2 & \text{if } v \neq \kappa, \\ 0 & \text{if } v = \kappa, \end{cases} \quad (55)$$

for all $v, \kappa \in W$. Choose $\psi(t) = t$ and $\phi(t) = (13/16)t$. Consider $Sv = e^v/4$. Given $\alpha, \eta : W \times W \longrightarrow 0, \infty)$ as

$$\alpha(v, \kappa) = \begin{cases} 2 & \text{if } v, \kappa \in [0, 1], \\ 0 & \text{if not,} \end{cases} \quad \eta(v, \kappa) = \begin{cases} 1 & \text{if } v, \kappa \in [0, 1], \\ 0 & \text{if not.} \end{cases} \quad (56)$$

Put $F(\tau, \omega) = \tau - \omega$ for all $\tau, \omega \in W$. We are going to demonstrate the following:

- (1) (W, ρ_{pb}) is a complete partial b -metric space
- (2) S is triangular weak α -admissible
- (3) There is $v_0 \in W$ in order that $\alpha(v_0, Sv_0) \geq \eta(v_0, Sv_0)$
- (4) If $\{v_n\}$ is a sequence in W in order that $\alpha(v_n, v_{n+1}) \geq \eta(v_n, v_{n+1})$ and $v_n \longrightarrow v^* \in W$ as $n \longrightarrow \infty$

Proof. Note that (W, ρ_{pb}) is a complete partial b -metric space (Here, $s = 3$).

If $\alpha(v, Sv) \geq \eta(v, Sv)$, then $\alpha(Sv, S^2v) \geq \eta(Sv, S^2v)$. So $\alpha(v, (e^v/4)) = 2 > 1 = \eta(v, (e^v/4))$, then $\alpha((e^v/4), ((e^{(e^v/4)})/4)) = 2 \geq 1 = \eta((e^v/4), ((e^{(e^v/4)})/4))$. So $v \geq 0$ and so $Sv \leq 0$. Hence, $\alpha(v, Sv) \geq \eta(v, Sv)$.

Recall that $\alpha(v, Sv) \geq \eta(v, Sv)$ for all $v, \kappa \in [0, 1]$. In this case,

$$\begin{aligned}
\psi(s\rho_{pb}(Sv, S\kappa)) &= 3 \left[\max \frac{e^v}{4}, \frac{e^\kappa}{4} \right]^2 = \frac{3}{16} [\max e^v, e^\kappa]^2 \\
&= \frac{3}{16} \rho_{pb}(v, \kappa) \leq \frac{3}{16} \mathcal{P}(v, \kappa) = \mathcal{P}(v, \kappa) \\
&\quad - \frac{13}{16} \mathcal{P}(v, \kappa) = \psi(\mathcal{P}(v, \kappa)) - \phi(\mathcal{P}(v, \kappa)) \\
&= F(\psi(\mathcal{P}(v, \kappa)), \phi(\mathcal{P}(v, \kappa))). \quad (57)
\end{aligned}$$

All hypotheses of Corollary 11 hold, and so, S admits a unique fixed point.

4. Conclusion

We ensured the existence of a unique fixed point for generalized contraction type mappings involving triangular weak α -admissibility with regard to a function η and C -class functions in the class of partial b -metric spaces. Some illustrated examples have been also provided.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no competing interests regarding the publication of this paper.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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