

Research Article

Fixed Point Results for \mathcal{C} -Contractive Mappings in Generalized Metric Spaces with a Graph

Karim Chaira, Mustapha Kabil , and Abdessamad Kamouss 

Laboratory of Mathematics and Applications, Faculty of Sciences and Technologies Mohammedia, University Hassan II, Casablanca, Morocco

Correspondence should be addressed to Abdessamad Kamouss; akamouss@gmail.com

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In this paper, we establish fixed point theorems for Chatterjea contraction mappings on a generalized metric space endowed with a graph. Our results extend, generalize, and improve many of existing theorems in the literature. Moreover, some examples and an application to matrix equations are given to support our main result.

1. Introduction

Fixed point theorems for contraction mappings and their generalizations play a crucial role in the determination of the existence and uniqueness of solutions of certain problems in mathematics and applied sciences, such as variational and linear inequalities, mathematical models, optimization, and mathematical economics. In 1922, Banach [1] proved the contraction principle, today named after him, which states any contraction on a complete metric space has a unique fixed point. In 1972, Chatterjea [2] proved that a self-mapping on a complete metric space X has a unique fixed point whenever there exists $0 \leq k < 1/2$ such that

$$d(Tx, Ty) \leq k[d(x, Ty) + d(y, Tx)], \text{ for all } x, y \in X. \quad (1)$$

On the other hand, different generalizations of the usual notion of metric space were proposed by a number of mathematicians (see [3, 4]). Recently, Jleli and Samet [5] introduced a new concept of generalized metric space that, in fact, recovers various topological spaces. The class of such metric spaces is larger than the class of standard metric spaces, than the class of b -metric spaces, than that of dislocated metric spaces, than that of dislocated b -metric spaces, and than the class of modular spaces with the Fatou property. The interested reader is referred to [5] for further details.

This work is the continuation of [6]. Motivated by the ideas recently introduced in [7–12]), we extend the Chatterjea fixed point theorem to the setting of generalized metric spaces with a graph. As corollaries, we obtain Chatterjea fixed point theorems in the setting of partially ordered metric spaces. Furthermore, we generalize the common fixed point result given in [13]. We provide an example to illustrate our main result.

2. Preliminaries

We recall the definition of generalized metric space and some related topological concepts, as introduced firstly by Jleli and Samet in [5].

Definition 1 [5]. Let X be a nonempty set and $\mathcal{D} : X \times X \rightarrow [0, +\infty]$ be a given mapping.

For every $x \in X$, define the set

$$\mathcal{C}(\mathcal{D}, X, x) = \left\{ \{x_n\} \subset X : \lim_{n \rightarrow \infty} \mathcal{D}(x_n, x) = 0 \right\}. \quad (2)$$

We say that \mathcal{D} is a generalized metric on X if it satisfies the following conditions:

(\mathcal{D}_1) For every $(x, y) \in X \times X$, $\mathcal{D}(x, y) = 0$ implies $x = y$

(\mathcal{D}_2) For every $(x, y) \in X \times X$, $\mathcal{D}(x, y) = \mathcal{D}(y, x)$

(\mathcal{D}_3) There exists $C > 0$ such that for all $(x, y) \in X \times X$, if there exists $\{x_n\} \in \mathcal{C}(\mathcal{D}, X, x)$, then

$$\mathcal{D}(x, y) \leq \text{Clim sup}_{n \rightarrow \infty} \mathcal{D}(x_n, y). \quad (3)$$

The pair (X, \mathcal{D}) is called a generalized metric space.

Definition 2 [5].

- (i) A sequence $\{x_n\}$ in a generalized metric space (X, \mathcal{D}) is said to be \mathcal{D} -convergent to $x \in X$ if $\{x_n\} \in \mathcal{C}(\mathcal{D}, X, x)$
- (ii) A sequence $\{x_n\}$ in a generalized metric space (X, \mathcal{D}) is said to be a \mathcal{D} -Cauchy sequence if $\lim_{m, n \rightarrow \infty} \mathcal{D}(x_n, x_m) = 0$
- (iii) The space (X, \mathcal{D}) is said to be \mathcal{D} -complete if every \mathcal{D} -Cauchy sequence in X is \mathcal{D} -convergent to some element in X
- (iv) The space (X, \mathcal{D}) is said to be \mathcal{D} -compact if every sequence in X has a \mathcal{D} -convergent subsequence to some element in X

The basic concepts, notation, and terminology related to graph theory can be found, for example, in [14, 15]. A directed graph or digraph G consists of a nonempty set $V(G)$, whose elements are called the vertices of G , and a set $E(G) \subset V(G) \times V(G)$, called the set of directed edges of G . The diagonal of the cartesian product $V(G) \times V(G)$ will be denoted by Δ . A digraph is said to be reflexive if $E(G)$ contains all loops, i.e., if $\Delta \subset E(G)$. G is said to be transitive if, for any $x, y, z \in V(G)$

$$[(x, y) \in E(G) \text{ and } (y, z) \in E(G)] \implies (x, z) \in E(G). \quad (4)$$

Given a digraph $G = (V, E)$, a directed path in G is a sequence of vertices.

$a_0, a_1, \dots, a_n \dots$, with $(a_i, a_{i+1}) \in E(G)$ for each $i \in \mathbb{N}$. A finite path (a_0, a_1, \dots, a_n) is said to have length n . The transitive closure of G is the digraph G' such that $V(G') = V(G)$ and that (i, j) is an edge in G' if there is a directed path from i to j in G .

We say that a vertex x in $V(G)$ is isolated if for any vertex y in $V(G)$ such that $x \neq y$, neither $(x, y) \in E(G)$ nor $(y, x) \in E(G)$.

In the sequel, given a graph G , G^{-1} will stand for it, that is, for the graph obtained from G by reversing the direction of its edges. Thus,

$$E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}. \quad (5)$$

In addition, \tilde{G} will stand for the undirected graph obtained from G by ignoring the direction of its edges. In other words,

$$E(\tilde{G}) = E(G) \cup E(G^{-1}). \quad (6)$$

Throughout this paper, the triplet (X, \mathcal{D}, G) will stand for the generalized metric space (X, \mathcal{D}) endowed with a reflexive digraph G such that $V(G) = X$. In [16], Alfuraidan et al. introduced the idea of G -monotonicity of sequences and the G -completeness of the metric space. Specifically,

Definition 3 [16]. Let G be a digraph. A sequence $\{x_n\} \in V(G)$ is said to be

- (i) G -increasing, if $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$
- (ii) G -decreasing, if $(x_{n+1}, x_n) \in E(G)$ for all $n \in \mathbb{N}$
- (iii) G -monotone, if it is either G -increasing or G -decreasing

The preceding notion of G -completeness can naturally be extended to the setting of generalized metric spaces as follows:

Definition 4. A generalized metric space (X, \mathcal{D}) is said to be G -complete if any \mathcal{D} -Cauchy, G -monotone sequence $\{x_n\} \subset V(G)$ is \mathcal{D} -convergent to an element in $V(G)$.

Remark 5. It is shown in [16] (Example 3.3) that G -completeness is finer than usual completeness.

The following definitions of some useful types of continuity are borrowed from [11].

Definition 6. A self-mapping T on the generalized metric space X is called

- (i) Subsequentially continuous, if for every sequence $\{x_n\} \subset X$, \mathcal{D} -convergent to $x \in X$, there exists a subsequence $\{x_{n_q}\}$ of $\{x_n\}$ such that $\{Tx_{n_q}\}$ \mathcal{D} -converges to Tx (as $q \rightarrow \infty$)
- (ii) Orbitally G -continuous, if for all $x, y \in V(G)$ and any sequence $\{k_n\}$ of positive integers

$$\{T^{k_n}x\} \rightarrow y \text{ and } (T^{k_n}x, T^{k_{n+1}}x) \in E(\tilde{G}) \implies \{T(T^{k_n}x)\} \rightarrow Ty. \quad (7)$$

The following property, initially introduced in [17] for partially ordered sets and in [11] for metric spaces with a graph, is often assumed to relax continuity assumptions.

Property (JNRL). The digraph G is said to satisfy the property (JNRL), if for any G -monotone increasing (decreasing) sequence $\{x_n\}$, which \mathcal{D} -converges to some $x \in V(G)$, it holds that $(x_n, x) \in E(G)$ ($(x, x_n) \in E(G)$), for any $n \in \mathbb{N}$.

Let (X, \mathcal{D}, G) be a generalized metric space endowed with a reflexive graph. Motivated by [11, 18], we define G -Chatterjea mappings on a generalized metric space (X, \mathcal{D}) with a graph, as follows:

Definition 7. A mapping $T : X \longrightarrow X$ is said to be a G -Chatterjea mapping if the following conditions are satisfied:

- (i) T is G -monotone (edge-preserving), that is, if:

$$(Tx, Ty) \in E(G), \text{ for every } ((x, y) \in E(G)), \quad (8)$$

- (ii) There exists $k \in [0, 1/2)$ such that for every $(x, y) \in E(G)$,

$$\mathcal{D}(Tx, Ty) \leq k(\mathcal{D}(Tx, y) + \mathcal{D}(x, Ty)). \quad (9)$$

Remark 8. It follows immediately from the above definition that:

- (i) If T is a G -Chatterjea mapping, then T is both a G^{-1} -Chatterjea and a G -Chatterjea mapping
- (ii) Any Chatterjea mapping is a G_0 -Chatterjea mapping, where the complete graph G_0 is defined by $V(G_0) = X$ and $E(G_0) = X \times X$

The following example shows that a G -Chatterjea mapping is not necessarily a Chatterjea mapping.

Example 1. Let $X = \{0, 1, 2, 3\}$. Consider the function \mathcal{D} defined on X by $\mathcal{D}(x, y) = (x - y)^2$. It can be shown that \mathcal{D} is a generalized metric with constant $C \geq 2$.

Consider the mapping $f : X \longrightarrow X$ defined by

$$\begin{cases} f(0) = 1 \\ f(1) = f(2) = 0, \\ f(3) = 1. \end{cases} \quad (10)$$

Since $\mathcal{D}(f(0), f(2)) = 1$ and $\mathcal{D}(f(0), 2) + \mathcal{D}((0, f(2))) = 1$, f is not a Chatterjea mapping.

On the other hand, consider the digraph G with $V(G) = X$ and edges

$$E(G) = \{(0, 0), (1, 1), (2, 2), (3, 3), (0, 3), (1, 2), (1, 3), (2, 3)\}. \quad (11)$$

It can be easily seen that f is a G -Chatterjea mapping with constant $k \in [1/9, 1/2)$.

3. Main Results

In this section, we extend the fixed point theorems for G -Chatterjea mappings to the setting of a generalized metric space with a digraph.

Let $T : X \longrightarrow X$ be a mapping. Let $x_0 \in X$. Let $G[\mathbb{O}_T(x_0)]$ be the subgraph of G induced on the orbit $\mathbb{O}_T(x_0) := \{T^n x_0$

$: n \in \mathbb{N}\}$. The following technical lemmas are necessary for the proof of the main result in this work.

Lemma 9. Let $T : X \longrightarrow X$ be a G -monotone mapping and suppose that there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$ (respectively, $(Tx_0, x_0) \in E(G)$) and that the subgraph $G[\mathbb{O}_T(x_0)]$ is transitive. Then, $\{T^n x_0\}$ is a G -increasing (respectively, G -decreasing) sequence and $(T^m x_0, T^n x_0) \in E(G)$ (respectively, $(T^n x_0, T^m x_0) \in E(G)$) for any $m, n \in \mathbb{N}$ such that $m \leq n$.

Proof. Without loss of generality, assume that $(x_0, Tx_0) \in E(G)$. Since T is G -monotone, it follows that $(Tx_0, T^2 x_0) \in E(G)$. Induction on n yields $(T^n x_0, T^{n+1} x_0) \in E(G)$ for all $n \in \mathbb{N}$. Therefore, $\{T^n x_0\}$ is a G -monotone increasing sequence. Since $(T^m x_0, T^{m+1} x_0), (T^{m+1} x_0, T^{m+2} x_0), \dots, (T^{n-1} x_0, T^n x_0) \in E(G)$ and $G[\mathbb{O}_T(x_0)]$ is transitive, it follows that $(T^m x_0, T^n x_0) \in E(G)$.

The following notation will be used in the sequel:

$$\delta(\mathcal{D}, T, x_0) := \sup \{\mathcal{D}(T^i x_0, x_0) : i \in \mathbb{N}^*\}. \quad (12)$$

Lemma 10. Under the assumptions of Lemma 9, if T is a G -Chatterjea mapping with constant $k \in [0, 1/2)$, then

- (i) For every $(m, n) \in \mathbb{N}^* \times \mathbb{N}^*$, we have

$$\mathcal{D}(T^n x_0, T^m x_0) \leq \delta_0 \left(\sum_m^{n+m-1} k^j \binom{j-1}{m-1} + \sum_n^{n+m-1} k^j \binom{j-1}{n-1} \right), \quad (13)$$

- (ii) For every $(m, n) \in \mathbb{N}^{*2}$ such that $m \leq n$, we have

$$\mathcal{D}(T^n x_0, T^m x_0) \leq \frac{\delta_0}{1-2k} (2k)^m, \quad (14)$$

where $\delta_0 = \delta(\mathcal{D}, T, x_0)$.

Proof.

- (i) The proof of this statement follows from the application of two-dimensional induction on $p = n + m$, for every $p \geq 2$

Since

$$\mathcal{D}(T^1 x_0, T^1 x_0) \leq k\mathcal{D}(Tx_0, x_0) + k\mathcal{D}(x_0, Tx_0) \leq \delta_0(2k), \quad (15)$$

it is clear that the inequality (13) holds for $p = 2$ with $(m, n) = (1, 1)$.

Assume next that inequality (13) holds for any $(m', n') \in \mathbb{N}^{*2}$ be chosen in such a way that $n' + m' = p$; let $(m, n) \in \mathbb{N}^{*2}$ with $n + m = p + 1$.

Since T is a G -Chatterjea mapping and $(T^{n-1}x_0, T^{m-1}x_0) \in E(\tilde{G})$, it holds that

$$\mathcal{D}(T^n x_0, T^m x_0) \leq k(\mathcal{D}(T^n x_0, T^{m-1} x_0) + \mathcal{D}(T^{n-1} x_0, T^m x_0)). \quad (16)$$

Since $n + (m - 1) = p$ and $(n - 1) + m = p$, the inductive hypothesis yields

$$\begin{aligned} \mathcal{D}(T^n x_0, T^m x_0) &\leq k\delta_0 \left(\sum_{j=m}^{n+m-2} k^j \binom{j-1}{m-1} + \sum_{j=n-1}^{n+m-2} k^j \binom{j-1}{n-2} \right) \\ &\quad + \sum_{j=m-1}^{n+m-2} k^j \binom{j-1}{m-2} + \sum_{j=n}^{n+m-2} k^j \binom{j-1}{n-1} \\ &\leq k\delta_0 \left(\sum_{j=m}^{n+m-2} k^j \left(\binom{j-1}{m-1} + \binom{j-1}{m-2} \right) + k^{m-1} \right) \\ &\quad + \sum_{j=n}^{n+m-2} k^j \left(\binom{j-1}{n-1} + \binom{j-1}{n-2} \right) + k^{n-1} \\ &\leq k\delta_0 \left(\sum_{j=m}^{n+m-2} k^j \binom{j}{m-1} + \sum_{j=n}^{n+m-2} k^j \right) \\ &\quad \cdot \left(\binom{j}{n-1} + k^{m-1} + k^{n-1} \right) \\ &\leq k\delta_0 \left(\sum_{j=m-1}^{n+m-2} k^j \binom{j}{m-1} + \sum_{j=n-1}^{n+m-2} k^j \binom{j}{n-1} \right) \\ &\leq \delta_0 \left(\sum_{j=m}^{n+m-1} k^j \binom{j-1}{m-1} + \sum_{j=n}^{n+m-1} k^j \binom{j-1}{n-1} \right), \end{aligned} \quad (17)$$

that inequality (13) holds for $(n, m) \in \mathbb{N}^*$ such that $n + m = p + 1$.

(ii) Let $n, m \in \mathbb{N}^*$; assume that $m \leq n$. Since $\binom{j-1}{m-1} \leq 2^{j-1}$ for any $j \in m, n + m - 1$ and $\binom{j-1}{n-1} \leq 2^{j-1}$ for any $j \in n, n + m - 1$, it follows that

$$\sum_{j=m}^{n+m-1} k^j \binom{j-1}{m-1} \leq \frac{1}{2} \sum_{j=m}^{n+m-1} (2k)^j \leq \frac{1}{2(1-2k)} (2k)^m, \quad (18)$$

and that

$$\sum_{j=n}^{n+m-1} k^j \binom{j-1}{n-1} \leq \frac{1}{2(1-2k)} (2k)^n \leq \frac{1}{2(1-2k)} (2k)^m. \quad (19)$$

It follows from inequality (13) that

$$\mathcal{D}(T^n x_0, T^m x_0) \leq \frac{\delta_0}{1-2k} (2k)^m. \quad (20)$$

Theorem 11. Let (X, \mathcal{D}, G) be a generalized, G -complete metric space and $T : X \rightarrow X$ be a G -Chatterjea mapping with constant $k \in [0, 1/2)$. Suppose that there exists $x_0 \in X$ such that $\delta(\mathcal{D}, T, x_0) < \infty$, that $(x_0, Tx_0) \in E(\tilde{G})$, and that the subgraph $G[O_T(x_0)]$ is transitive. Under these assumptions, the sequence $\{T^n x_0\}$ converges to some $\omega \in X$. Moreover, if one of the following conditions (i) – (iii) holds, namely

(i) T is subsequentially continuous

(ii) T is orbitally G -continuous

(iii) G satisfies property (JNRL) and $\mathcal{D}(x_0, T\omega) < \infty$

then ω is a fixed point of T .

Proof. Without loss of generality, it may be assumed that $(x_0, Tx_0) \in E(G)$. Select $(m, n) \in \mathbb{N}^* \times \mathbb{N}^*$ such that $m \leq n$. From Lemma 9, it is clear that $(T^m x_0, T^n x_0) \in E(G)$. If T is a G -Chatterjea mapping, Lemma 10 yields

$$\mathcal{D}(T^m x_0, T^n x_0) \leq \delta_0 / (1 - 2k) (2k)^m. \quad (21)$$

Thus, $\{T^n x_0\}$ is a \mathcal{D} -Cauchy sequence. Since (X, \mathcal{D}, G) is G -complete, the sequence $\{T^n x_0\}$ \mathcal{D} -converges to some $\omega \in X$.

(i) It follows from the subsequential continuity assumption on T that there exists a subsequence $\{T^{n_q} x_0\}$ such that $\{T^{n_q+1} x_0\}$ \mathcal{D} -converges to $T\omega$ as $n_q \rightarrow \infty$. The uniqueness of the limit yields $T\omega = \omega$.

(ii) Assume that T is orbitally G -continuous. Since $\{T^n x_0\}$ \mathcal{D} -converges to ω and $(T^n x_0, T^{n+1} x_0) \in E(G)$, it follows that $T(T^n x_0) \rightarrow T\omega$. Likewise, $T(T^n x_0) = T^{n+1} x_0 \rightarrow \omega$. Hence, $\omega = T\omega$.

(iii) Assume that G satisfies Property (JNRL) and that $\mathcal{D}(x_0, T\omega) < \infty$. Since $\{T^n x_0\}$ is G -increasing and it \mathcal{D} -converges to $\omega \in X$, it follows that $(T^n x_0, \omega) \in E(G)$, for any $n \in \mathbb{N}$.

Select $n \in \mathbb{N}$, $n \geq 1$. If T is a G -Chatterjea mapping, then necessarily

$$\begin{aligned} \mathcal{D}(T^n x_0, T\omega) &\leq k\mathcal{D}(T^{n-1} x_0, T\omega) + k\mathcal{D}(T^n x_0, \omega) \\ &\leq k^2 \mathcal{D}(T^{n-2} x_0, T\omega) + k^2 \mathcal{D}(T^{n-1} x_0, \omega) \\ &\quad + k\mathcal{D}(T^n x_0, \omega). \end{aligned} \quad (22)$$

It follows by induction on n , that for any $n \geq 1$,

$$\mathcal{D}(T^n x_0, T\omega) \leq k^n \mathcal{D}(x_0, T\omega) + \sum_{j=1}^n k^j \mathcal{D}(T^{n+1-j} x_0, \omega). \quad (23)$$

Let $j \in 1, n$. Since $\{T^p x_0\}_{p \geq n}$ \mathcal{D} -converges to ω using (\mathcal{D}_3) , it follows that

$$\mathcal{D}(T^{n+1-j}x_0, \omega) \leq \text{Clim sup}_{p \rightarrow \infty} \mathcal{D}(T^{n+1-j}x_0, T^p x_0). \quad (24)$$

Applying Lemma 10, we obtain

$$\mathcal{D}(T^{n+1-j}x_0, \omega) \leq \frac{C\delta_0}{1-2k} \limsup_{p \rightarrow \infty} (2k)^{n+1-j} \leq \frac{C\delta_0}{1-2k} (2k)^{n+1-j}. \quad (25)$$

Then,

$$k^j \mathcal{D}(T^{n+1-j}x_0, \omega) \leq \frac{C\delta_0}{1-2k} (k)^j (2k)^{n+1-j} \leq \frac{C\delta_0}{1-2k} (2k)^{n+1} \frac{1}{2^j}. \quad (26)$$

Hence

$$\sum_{j=1}^n k^j \mathcal{D}(T^{n+1-j}x_0, \omega) \leq \frac{C\delta_0}{1-2k} (2k)^{n+1} \left(\sum_{j=1}^n \frac{1}{2^j} \right) \leq \frac{C\delta_0}{1-2k} (2k)^{n+1}. \quad (27)$$

Finally, inequality (23) becomes

$$\mathcal{D}(T^n x_0, T\omega) \leq k^n \mathcal{D}(x_0, T\omega) + \frac{C\delta_0}{1-2k} (2k)^{n+1}. \quad (28)$$

Since $\mathcal{D}(x_0, T\omega) < \infty$, it follows that $\lim_{n \rightarrow \infty} \mathcal{D}(T^n x_0, T\omega) = 0$. Therefore, $\{T^n x_0\}$ \mathcal{D} -converges to $T\omega$. Uniqueness of the limit yields $T\omega = \omega$.

Proposition 12. *Suppose that T is G -Chatterjea. If T has two fixed points ω and ω' in X , such that $\mathcal{D}(\omega, \omega') < \infty$ and $(\omega, \omega') \in E(G)$, then $\omega = \omega'$.*

Proof. Suppose that $\omega, \omega' \in X$ are two fixed points of T such that $\mathcal{D}(\omega, \omega') < \infty$. Since T is a G -Chatterjea mapping, we have

$$\mathcal{D}(\omega, \omega') = \mathcal{D}(T\omega, T\omega') \leq k \left(\mathcal{D}(T\omega, \omega') + \mathcal{D}(\omega, T\omega') \right), \quad (29)$$

which implies that

$$\mathcal{D}(\omega, \omega') \leq 2k \mathcal{D}(\omega, \omega'). \quad (30)$$

Hence

$$(1-2k) \mathcal{D}(\omega, \omega') \leq 0. \quad (31)$$

Therefore, $\mathcal{D}(\omega, \omega') = 0$, i.e., $\omega = \omega'$.

The following example illustrates Theorem 11.

Example 2. Let X be the open interval $(-1, 1)$. Consider the function \mathcal{D} defined on X as follows:

$$\mathcal{D}(x, y) = \begin{cases} 2(|x|+|y|) & \text{if either of } x = 0 \text{ or } y = 0, \\ \frac{|x|+|y|}{3} & \text{otherwise.} \end{cases} \quad (32)$$

It can be easily verified that (D_1) and (D_2) hold. For the validity of (D_3) , observe first that for all $x \neq 0$, we have $\mathcal{E}(\mathcal{D}, X, x) = \emptyset$. If $x = 0$, then there exists a sequence $\{x_n\}$ such that $\lim_{n \rightarrow \infty} \mathcal{D}(x_n, x) = 0$. Consider the sets $P := \{n \in \mathbb{N} : x_n \neq 0\}$ and $Q := \{n \in \mathbb{N} : x_n = 0\}$. We distinguish three cases:

If P is finite, then there exists $C \geq 1$ such that for any $y \in X$ it holds that

$$\mathcal{D}(0, y) = 2|y| \leq 2C|y| = \text{Clim sup}_{n \rightarrow \infty} \mathcal{D}(x_n, y). \quad (33)$$

If Q is finite, then there exists $C \geq 6$ such that for any $y \in X$

$$\mathcal{D}(0, y) = 2|y| \leq C \frac{|y|}{3} = \text{Clim sup}_{n \rightarrow \infty} \mathcal{D}(x_n, y). \quad (34)$$

If P and Q are infinite, there exist two increasing functions $\varphi, \psi : \mathbb{N} \rightarrow \mathbb{N}$ such that, for all $n \in \mathbb{N}$, $x_{\varphi(n)} \neq 0$, $x_{\psi(n)} = 0$, and $\{x_n\} = \{x_{\varphi(n)}\} \cup \{x_{\psi(n)}\}$. Then, for any $y \in X$

$$\limsup_{n \rightarrow \infty} \mathcal{D}(x_{\varphi(n)}, y) = \frac{|y|}{3} \text{ and } \limsup_{n \rightarrow \infty} \mathcal{D}(x_{\psi(n)}, y) = 2|y|. \quad (35)$$

Thus, \mathcal{D} is a generalized metric with $C \geq 6$. Note that X is not a \mathcal{D} -compact space. Indeed, let $\{x_n\}_{n \in \mathbb{N}^*}$ be a sequence of X such that $x_n = 1 - 1/n$ and suppose that there exists a subsequence $\{x_{\varphi(n)}\}$ of $\{x_n\}$ which \mathcal{D} -converges to an element x in X . Since $\lim_{n \rightarrow \infty} \mathcal{D}(x_{\varphi(n)}, x) = 0$, we have

$$\lim_{n \rightarrow \infty} \left| x_{\varphi(n)} \right| + |x| = 0. \quad (36)$$

Thus, $|x| = -1$. Contradiction.

Consider the graph G on X consisting of the transitive closure of the graph represented in Figure 1.

Note that

$$E(G) = \Delta \cup \left\{ \left(\frac{(-1)^n}{2^n}, 0 \right), \left(\frac{(-1)^m}{2^m}, \frac{(-1)^n}{2^n} \right) : n, m \in \mathbb{N}^* \text{ and } n > m \right\}. \quad (37)$$

Let us prove that the space (X, \mathcal{D}) is G -complete. Let $\{x_n\}$ be a G -monotone, \mathcal{D} -Cauchy sequence in X . We have two cases:

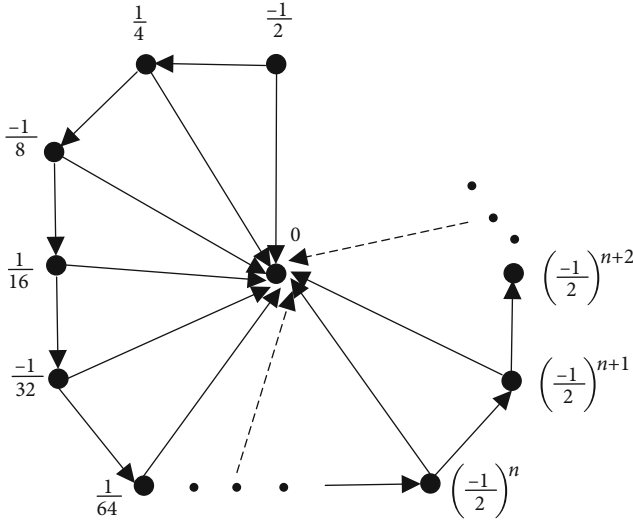


FIGURE 1: All loops and isolated vertices are not represented.

Case 1. If there exists $n_0 \in \mathbb{N}$ such that $x_n = 0$, for any $n \geq n_0$. We have $\lim_{n \rightarrow \infty} \mathcal{D}(x_n, 0) = 0$. Therefore, the sequence $\{x_n\}$ \mathcal{D} -converges to 0.

Case 2. If there exists $n_0 \in \mathbb{N}$ and a nondecreasing sequence $\{p_n\} \subset \mathbb{N}^*$ such that $x_n = (-1/2)^{p_n}$, for any $n \geq n_0$. We have $\lim_{n \rightarrow \infty} \mathcal{D}(x_n, 0) = \lim_{n \rightarrow \infty} (1/3)(1/2)^{p_n} = 0$. Therefore, the sequence $\{x_n\}$ is \mathcal{D} -convergent to 0.

(Note that in the case where there exists a nonzero element α in X such that $x_n = \alpha$ for any $n \in \mathbb{N}$, the sequence $\{x_n\}$ is not \mathcal{D} -Cauchy).

Now, consider the self-mapping T on X defined by

$$Tx = \begin{cases} -\frac{x}{2} & \text{if } x \in X \cap \mathbb{Q}, \\ \frac{x}{5} & \text{otherwise.} \end{cases} \quad (38)$$

One can easily see that:

$$\begin{cases} \left(T \frac{(-1)^n}{2^n}, T0 \right) = \left(\frac{(-1)^{n+1}}{2^{n+1}}, 0 \right) \in E(G), \text{ for any } n \in \mathbb{N}^*, \\ \left(T \frac{(-1)^m}{2^m}, T \frac{(-1)^n}{2^n} \right) = \left(\frac{(-1)^{m+1}}{2^{m+1}}, \frac{(-1)^{n+1}}{2^{n+1}} \right) \in E(G), \text{ for any } n, m \in \mathbb{N}^* \text{ with } n > m. \end{cases} \quad (39)$$

It is therefore apparent that T is G -monotone. For $x_0 = -1/2$, we have that $(x_0, Tx_0) \in E(G)$, $G[\mathbb{O}_T(x_0)]$ is transitive and

$$\delta(T, \mathcal{D}, x_0) = \sup \left\{ \mathcal{D} \left(\frac{(-1)^i}{2^i}, \frac{-1}{2} \right) : i \in \mathbb{N} \text{ and } i \geq 2 \right\} = \frac{1}{4}. \quad (40)$$

Pick $x, y \in X$ such that $(x, y) \in E(G)$. If $x = y \in X \cap \mathbb{Q}$, then

$$\mathcal{D}(Tx, Tx) = \frac{|x|}{3} \leq k|x| = k(\mathcal{D}(Tx, x) + \mathcal{D}(x, Tx)). \quad (41)$$

Observe that if $x = y \in X \setminus \mathbb{Q}$, then

$$\mathcal{D}(Tx, Tx) = \frac{2|x|}{15} \leq k \frac{12|x|}{15} = k(\mathcal{D}(Tx, x) + \mathcal{D}(x, Tx)). \quad (42)$$

On the other hand, if $(x, y) = ((-1)^n/2^n, 0)$, then

$$\mathcal{D} \left(T \frac{(-1)^n}{2^n}, T0 \right) = \frac{1}{2^n} \leq k \frac{3}{2^n} = k \left(\mathcal{D} \left(T \frac{(-1)^n}{2^n}, 0 \right) + \mathcal{D} \left(\frac{(-1)^n}{2^n}, T0 \right) \right). \quad (43)$$

Finally, in the case $(x, y) = ((-1)^m/2^m, (-1)^n/2^n)$, then

$$\begin{aligned} \mathcal{D} \left(T \frac{(-1)^m}{2^m}, T \frac{(-1)^n}{2^n} \right) &= \frac{2^n + 2^m}{3 \times 2^{n+m+1}} \leq k \frac{2^n + 2^m}{2^{n+m+1}} \\ &= k \left(\mathcal{D} \left(T \frac{(-1)^m}{2^m}, \frac{(-1)^n}{2^n} \right) + \mathcal{D} \left(\frac{(-1)^m}{2^m}, T \frac{(-1)^n}{2^n} \right) \right). \end{aligned} \quad (44)$$

In all cases, $\mathcal{D}(Tx, Ty) \leq k(\mathcal{D}(Tx, y) + \mathcal{D}(y, Tx))$. Thus, T is a G -Chatterjea mapping with constant $k \in [1/3, 1/2]$.

The sequence $\{T^n x_0\} = \{(-1)^{n+1}/2^{n+1}\}$ is G -increasing, \mathcal{D} -convergent to 0, and furthermore, $((-1)^{n+1}/2^{n+1}, 0) \in E(G)$. Hence, G satisfies Property (JNRL). On account of Theorem 11, T has a fixed point, namely 0.

Next, we present a version of Theorem 11 in the setting of a partially ordered generalized metric space. Let (X, \mathcal{D}, \leq) be a generalized metric space endowed with a partial order. We define the directed graph G_{\leq} on X as follows: $V(G_{\leq}) = X$ and $E(G_{\leq}) = \{(x, y) \in X \times X : x \leq y\}$. In this setting, we say that $T : X \rightarrow X$ is a monotone Chatterjea mapping if it is a G_{\leq} -Chatterjea mapping. We also say that T is orbitally monotone continuous if T is orbitally G_{\leq} -continuous. The generalized metric space (X, \mathcal{D}, \leq) satisfies the Property (JNRL) if whenever $\{x_n\}$ is a decreasing (respectively increasing) sequence such that $x_n \rightarrow x$ in X , then for all $n \in \mathbb{N}$, $x \leq x_n$ (respectively $x_n \leq x$).

Theorem 13. Let (X, \mathcal{D}, \leq) be a generalized D -complete metric space endowed with a partial order and $T : X \rightarrow X$ be a monotone Chatterjea mapping with constant $k \in [0, 1/2]$. Suppose that there exists $x_0 \in X$ such that $\delta(\mathcal{D}, T, x_0) < \infty$ and that either $x_0 \leq Tx_0$ or $Tx_0 \leq x_0$. Then, the sequence $\{T^n x_0\}$ converges to some $\omega \in X$. Moreover, if any one of the conditions (i) – (iii) in Theorem 11 holds, then ω is a fixed point of T .

Proof. Since the subgraph $G_{\leq}[\mathbb{O}_T(x_0)]$ is transitive, Theorem 13 is a direct consequence of Theorem 11.

We remark that Theorem 3.9 in [19] is a corollary of the preceding theorem, from which it can be derived simply by removing the ordering.

Corollary 14. *Let (X, \mathcal{D}) be a D -complete generalized metric space and $T : X \rightarrow X$ be a Chatterjea contraction with constant $k \in [0, 1/2)$. Suppose that there exists $x_0 \in X$ such that $\mathcal{D}(x_0, Tx_0) < \infty$. Then, $\{T^n x_0\}$ converges to some $\omega \in X$. If $\mathcal{D}(x_0, T\omega) < \infty$, then ω is a fixed point of T with $\mathcal{D}(\omega, \omega) = 0$. Moreover, if $\omega' \in X$ is another fixed point of T such that $\mathcal{D}(\omega, \omega') < \infty$, then $\omega = \omega'$.*

Proof. Taking $G = G_0$, where G_0 is the complete graph, i.e., $V(G_0) = X$ and $E(G_0) = X \times X$, the proof follows from Theorem 11 and Proposition 12.

We next set to show that the fixed point result given in Theorem 11 is, in fact, a generalization of the analogue common fixed point theorem established in [13]. To this effect, we state and prove the following lemma, introduced by Haghi et al. in [20].

Lemma 15. *Let X be a nonempty set and $f : X \rightarrow X$ a function. Then, there exists a subset $E \subset X$ such that $f(E) = f(X)$. Moreover, $f : E \rightarrow X$ is one-to-one.*

Let $T, S : X \rightarrow X$ be two self mappings. We recall the definition of G -Chatterjea S -contraction and the property (P) given in [13].

Definition 16. We say that T is G -Chatterjea S -contraction if there exists $k \in [0, 1/2)$ such that for every $x, y \in V(G)$, it holds that

$$(Sx, Sy) \in E(\tilde{G}) \mathcal{D}(Tx, Ty) \leq k(\mathcal{D}(Tx, Sy) + \mathcal{D}(Sx, Ty)). \quad (45)$$

We recall that x^* is said to be a point of coincidence of T and S , if there exists a in X such that $x^* = Ta = Sa$.

Property (P). The digraph G is said to satisfy the property (P) for T and S , if whenever x^*, y^* are points of coincidence of T and S in $V(G)$, then $(x^*, y^*) \in E(\tilde{G})$ and $\mathcal{D}(x^*, y^*) < \infty$.

Suppose that $T(X) \subseteq S(X)$. If $x_0 \in X$ is arbitrary, we can choose a point x_1 in X such that $Tx_0 = Sx_1$. Proceeding in this manner, assuming that x_n in X is given, we can define $x_{n+1} \in X$ by the recurrence relation

$$Tx_n = Sx_{n+1}, n = 0, 1, 2, \dots \quad (46)$$

By $C(T, S)$, we denote the set of all elements x_0 of X such that $(Sx_n, Sx_m) \in E(\tilde{G})$, for $n, m = 1, 2, \dots$. The following notation will be used in the sequel:

$$\delta(\mathcal{D}, S, T, x_0) = \sup \{ \mathcal{D}(Sx_p, Sx_1) : p \geq 2 \}. \quad (47)$$

Corollary 17. *Let (X, \mathcal{D}) be a generalized metric space endowed with a reflexive digraph G . Assume that $V(G) = X$, that G has no parallel edges, and that it satisfies the (JNRL) property. Let T and S be two self mappings on X such that T is a G -Chatterjea S -contraction, $S(X)$ is a \mathcal{D} -complete subspace of X and that $T(X) \subseteq S(X)$.*

- (1) *Suppose that there exists $x_0 \in C(T, S)$ such that $\delta(\mathcal{D}, S, T, x_0) < \infty$. Then, the sequence $\{Sx_n\}$ defined by (46) \mathcal{D} -converges to $x^* = Sa$, with $a \in X$. Moreover, if $\mathcal{D}(Tx_0, Ta) < \infty$, then x^* is a point of coincidence of T and S in X .*
- (2) *In addition, T and S have a unique point of coincidence in X if the digraph G has the property (P) for T and S . Finally, if T and S are weakly compatible, then T and S have a unique common fixed point in X .*

Proof. By Lemma 15, there exists $X_0 \subset X$ such that $S(X_0) = S(X) = Y$; moreover, $S : X_0 \rightarrow X$ is one-to-one. Define the mapping $F : Y \rightarrow Y$ as

$$F(Sx) = Tx. \quad (48)$$

Since S is one-to-one on X_0 , F is well defined.

Let $u, v \in Y$. There exist $x, y \in X$ such that $u = Sx$ and $v = Sy$. If $(u, v) \in E(\tilde{G})$, then $(Sx, Sy) \in E(\tilde{G})$. Since T is a G -Chatterjea S -contraction, there exists $k \in [0, 1/2)$ such that

$$\mathcal{D}(Tx, Ty) \leq k(\mathcal{D}(Tx, Sy) + \mathcal{D}(Sx, Ty)), \quad (49)$$

i.e., $\mathcal{D}(F(Sx), F(Sy)) \leq k(\mathcal{D}(F(Sx), Sy) + \mathcal{D}(Sx, F(Sy)))$. Then

$$\mathcal{D}(Fu, Fv) \leq k(\mathcal{D}(Fu, v) + \mathcal{D}(u, Fv)). \quad (50)$$

Consequently, F is a G -Chatterjea mapping on Y .

Suppose that there exists $x_0 \in C(T, S)$ such that $\delta(\mathcal{D}, S, T, x_0) < \infty$. Setting $y_0 = Sx_1$, it is clear that $Fy_0 = F(Sx_1) = Tx_1 = Sx_2$. It follows easily by induction that $F^p y_0 = Sx_{p+1}$, for any $p \in \mathbb{N}$. Moreover,

$$\begin{aligned} \delta(\mathcal{D}, F, y_0) &= \sup \{ \mathcal{D}(F^i y_0, y_0) : i \geq 1 \} \\ &= \sup \{ \mathcal{D}(F^{p-1} y_0, y_0) : p \geq 2 \} \\ &= \sup \{ \mathcal{D}(Sx_p, Sx_1) : p \geq 2 \} = \delta(\mathcal{D}, S, T, x_0) < \infty. \end{aligned} \quad (51)$$

From $(Sx_n, Sx_m) \in E(\tilde{G})$ for $n, m = 1, 2, \dots$, it follows that $(F^i y_0, F^j y_0) \in E(\tilde{G})$ for any i, j in \mathbb{N} . Hence, $G[\mathbb{O}_F(y_0)]$ is transitive. Furthermore, $(Sx_1, Sx_2) = (y_0, Fy_0) \in E(\tilde{G})$.

By virtue of Theorem 11, the sequence $\{Sx_n\} = \{F^{n-1} y_0\}$ \mathcal{D} -converges to $x^* = Sa$ with $a \in Y \subset X$.

Moreover, we have

$$\mathcal{D}(y_0, Fx^*) = \mathcal{D}(Sx_1, F(Sa)) = \mathcal{D}(Tx_0, Ta) < \infty, \quad (52)$$

and since G satisfies property (JNRL), on account of Theorem 11, x^* is a fixed point of F . Hence $Ta = F(Sa) = Fx^* = x^* = Sa$, and x^* is a point of coincidence of T and S in X , as claimed.

Assume next that there exists another point of coincidence $y^* \in S(X)$, that $b \in X$, and that $y^* = Sb = Tb = F(Sb) = Fy^*$. Since the digraph G has the property (P) for T and S , then $(x^*, y^*) \in E(\tilde{G})$ and $\mathcal{D}(x^*, y^*) < \infty$. By Proposition 12, necessarily $x^* = y^*$, which implies that T and S have a unique point of coincidence in X . It follows from [21] (Proposition 1.4) that if T and S are weakly compatible, then T and S have a unique common fixed point in X .

4. Application

In this section, we study the existence and uniqueness of solution for the following general nonlinear matrix equation in the set of all $n \times n$ Hermitian-positive definite matrices $\mathcal{P}(n)$:

$$X^q - A^* \mathcal{F}(X)^s A = B, \quad q > \sqrt{2}, s \in (0, 1], X \in \mathcal{P}(n), \quad (53)$$

where A is $n \times n$ nonsingular matrix, A^* is the Hermitian transpose of the matrix A , the matrix B is $n \times n$ positive definite matrix, and $\mathcal{F} : \mathcal{E}(n) \rightarrow \mathcal{E}(n)$ is a self-adjoint operator such that $\mathcal{E}(n)$ is a nonempty subset of $\mathcal{P}(n)$. This type of matrix equation arises in control theory, ladder networks, dynamic programming, stochastic filtering and statistics, etc.

For $M, N \in \mathcal{P}(n)$, we denote

$$M < N \Leftrightarrow N - M \text{ is positive definite.} \quad (54)$$

We denote by $\|\cdot\|$ the spectral norm $\|A\| = \sqrt{\rho(A^*A)} = \|A^*\|$, where $\rho(A^*A)$ is the largest eigenvalue of A^*A . We recall that the Thompson metric is defined on $\mathcal{P}(n)$ by:

$$d : \mathcal{P}(n) \times \mathcal{P}(n) \rightarrow \mathbb{R}_+, \quad (55)$$

such that

$$\begin{aligned} d(A, B) &= \max \left\{ \ln \left(\mathcal{W} \left(\frac{A}{B} \right) \right), \ln \left(\mathcal{W} \left(\frac{B}{A} \right) \right) \right\} \\ &= \left\| \ln \left(A^{-(1/2)} B A^{-(1/2)} \right) \right\|, \end{aligned} \quad (56)$$

where $\mathcal{W}(A/B) = \inf \{ \lambda > 0 : A \leq \lambda B \} = \lambda \max (A^{-(1/2)} B A^{-(1/2)})$. It is easy to verify that $(\mathcal{P}(n), d)$ is a complete metric space (see [22]). In the sequel, we consider the space $\mathcal{P}(n)$ endowed by the Thompson generalized metric \mathcal{D} defined by

$$\mathcal{D}(A, B) = \left\| \ln \left(A^{-(1/2)} B A^{-(1/2)} \right) \right\|^2, \quad (57)$$

for any $A, B \in \mathcal{P}(n)$. In the following lemmas, we extend some properties of the Thompson metric given in [23] to the Thompson generalized metric space.

Lemma 18. Let $\mathcal{D} : \mathcal{P}(n) \times \mathcal{P}(n) \rightarrow \mathbb{R}_+$ be a Thompson generalized metric on the open convex cone $\mathcal{P}(n)$; then, for any $A, B \in \mathcal{P}(n)$ and nonsingular matrix M , we have the following conditions:

- (i) $\mathcal{D}(A, B) = \mathcal{D}(A^{-1}, B^{-1}) = \mathcal{D}(M^* A M, M^* B M)$, where A^{-1}, B^{-1} are the inversion of matrices A and B , respectively
- (ii) $\mathcal{D}(A^r, B^r) \leq r^2 \mathcal{D}(A, B)$, $r \in [-1, 1]$
- (iii) $\mathcal{D}(M^* A^r M, M^* B^r M) \leq r^2 \mathcal{D}(A, B)$, $r \in [-1, 1]$

Proof. Let d be the Thompson metric defined by (56). Since $\mathcal{D}(A, B) = (d(A, B))^2$, we get (i), (ii), and (iii) by the invariant under the matrix inversion, congruence transformations for nonsingular matrix M , and the nonpositive curvature property of the Thompson metric d .

Lemma 19. For any $A, B, C, D \in \mathcal{P}(n)$,

$$\mathcal{D}(A + C, B + D) \leq \max \{ \mathcal{D}(A, B), \mathcal{D}(C, D) \}. \quad (58)$$

Especially, $\mathcal{D}(A + C, B + C) \leq \mathcal{D}(A, B)$.

Proof. Let d be the Thompson metric defined by (56). By using [23] (Lemma 2.1), we have $d(A + C, B + D) \leq \max \{ d(A, B), d(C, D) \}$. Since $\mathcal{D}(A, B) = (d(A, B))^2$, we deduce our result.

We endow $\mathcal{P}(n)$ by the graph G defined by:

$$V(G) = \mathcal{P}(n) \text{ and } E(G) = \Delta \cup \{ (M, N) \in \mathcal{P}(n) \times \mathcal{P}(n), M < N \}. \quad (59)$$

We give a graphical version of [24] (Lemma 4.3) in $\mathcal{P}(n)$ endowed with the graph G .

Lemma 20. For any $A, B \in \mathcal{P}(n)$, if $(A, B) \in E(G)$, then $(A^r, B^r) \in E(G)$ for all $r \in]0, 1]$, and $(B^r, A^r) \in E(G)$ for all $r \in [-1, 0[$.

Proof. Let $A, B \in \mathcal{P}(n)$ such that $(A, B) \in E(G)$. If $A = B$ then $(A^r, B^r) \in E(G)$ for all $r \in]0, 1]$. If $A < B$, then by using the Löwner-Heinz inequality [25, 26], we get $A^r < B^r$. Thus, $(A^r, B^r) \in E(G)$ for all $r \in]0, 1]$.

Theorem 21. Let $X_0 \in \mathcal{P}(n)$ and $\mathcal{E}(n) = \{ X \in \mathcal{P}(n) : (X_0, X) \in E(G) \}$. If the operator \mathcal{F} is nondecreasing and for all $X, Y \in \mathcal{E}(n)$ such that $(X, Y) \in E(G)$, we have:

$$\begin{aligned} & \left\| \ln \left(\mathcal{F}(X)^{-(1/2)} \mathcal{F}(Y) \mathcal{F}(X)^{-(1/2)} \right) \right\| \\ & \leq \left\| \ln \left(X^{-(1/2)} (B + A^* \mathcal{F}(Y)^s A)^{1/q} X^{-(1/2)} \right) \right\| \end{aligned} \quad (60)$$

and $(X_0^q, B) \in E(G)$; then, the matrix equation (53) has a unique solution.

Proof. Let $T : \mathcal{E}(n) \longrightarrow \mathcal{E}(n)$ be a mapping defined by

$$T(X) = (B + A^* \mathcal{F}(X)^s A)^{1/q}, X \in \mathcal{E}(n). \quad (61)$$

Let $X \in \mathcal{E}(n)$. Since $(X_0, X) \in E(G)$ and F is nondecreasing,

$$(B + A^* F(X_0)^s A, B + A^* \mathcal{F}(X)^s A) \in E(G). \quad (62)$$

As $(X_0^q, B) \in E(G)$ and $A^* F(X_0)^s A \in \mathcal{P}(n)$, then $(X_0^q, B + A^* F(X_0)^s A) \in E(G)$. By Lemma 20, we have

$$\left(X_0, (B + A^* F(X)^s A)^{1/q} \right) = (X_0, T(X)) \in E(G). \quad (63)$$

Thus, $T(X) \in \mathcal{E}(n)$ and so T is well defined.

Let $X, Y \in \mathcal{E}(n)$ such that $(X, Y) \in E(G)$, we have $(T(Y))^q - (T(X))^q = A^*(F(Y)^s - F(X)^s)A$, then $((T(X))^q, (T(Y))^q) \in E(G)$. Since $0 \leq 1/q < 1$, by Lemma 20 we have $(T(X), T(Y)) \in E(G)$.

Let X, Y be two elements in $\mathcal{E}(n)$ such that $(X, Y) \in E(G)$. By using Lemmas 18 and 19 we have

$$\begin{aligned} \mathcal{D}(F(X), F(Y)) &\geq \frac{1}{s^2} \mathcal{D}(F(X)^s, F(Y)^s) \geq \frac{1}{s^2} \mathcal{D}(A^* F(X)^s A, A^* F(Y)^s A) \\ &\geq \frac{1}{s^2} \mathcal{D}(B + A^* F(X)^s A, B + A^* F(Y)^s A) \\ &\geq \frac{1}{s^2} \mathcal{D}(T(X)^q, T(Y)^q) \geq \left(\frac{s}{q}\right)^2 \mathcal{D}(T(X), T(Y)). \end{aligned} \quad (64)$$

Thus,

$$\mathcal{D}(T(X), T(Y)) \leq \left(\frac{s}{q}\right)^2 \mathcal{D}(F(X), F(Y)). \quad (65)$$

If $\|\ln(\mathcal{F}(X)^{-(1/2)} F(Y) F(X)^{-(1/2)})\| \leq \|\ln(X^{-(1/2)}(B + A^* \mathcal{F}(Y)^s A)^{1/q} X^{-(1/2)})\|$, then

$$\mathcal{D}(F(X), F(Y)) \leq \mathcal{D}(X, T(Y)). \quad (66)$$

From (65) and (66), we get

$$\mathcal{D}(T(X), T(Y)) \leq \left(\frac{s}{q}\right)^2 (\mathcal{D}(X, T(Y)) + \mathcal{D}(Y, T(X))). \quad (67)$$

Thus, there exists $k = (s/q)^2 \in [0, 1/2)$ such that T is a G -Chatterjea mapping on $\mathcal{E}(n)$.

Since $(X_0^q, B) \in E(G)$, $(X_0^q, B + A^* F(X_0)^s A) \in E(G)$. Thus, $(X_0, T(X_0)) \in E(G)$.

Next, we show that $\mathcal{E}(n)$ satisfies the Property (JNRL) for the generalized metric \mathcal{D} . In fact, let $(X_k)_k$ be a nondecreasing sequence of $\mathcal{E}(n)$ which converges to $X \in \mathcal{E}(n)$. If the set $\{k \in \mathbb{N} : X_k = X_0\}$ is infinite, there exists a nondecreasing function $\phi : \mathbb{N} \longrightarrow \mathbb{N}$ such that $X_{\phi(k)} = X_0, \forall k \in \mathbb{N}$, then $X = X_0 \in \mathcal{E}(n)$. If not, $X_k \neq X_0$, for large integer k . Fix $m \in \mathbb{N}$ arbitrary. For all $k > m$,

$$(X_m, X_k) \in E(G) \implies X_k - X_m \in \mathcal{P}(n) \implies X_k \in \mathcal{P}(n) + X_m. \quad (68)$$

Since $\mathcal{P}(n) + X_m$ is closed, $X \in \mathcal{P}(n) + X_m$. Thus, $(X_m, X) \in E(G)$, for all $m \in \mathbb{N}$. Thus, according to Theorem 11, we can show that there exists $X^* \in \mathcal{E}(n)$ such that $T(X^*) = X^*$ which is a solution of the matrix equation (53).

If equation (53) has another solution Y^* such that $(X^*, Y^*) \in E(\tilde{G})$, then using Proposition 12, we have $Y^* = X^*$.

5. Conclusion

Summarizing the present work enhances the area in many directions

- (1) Using Theorem 13, we can improve the following results
 - (i) Theorem 2 in [27]
 - (ii) Theorem 2.12 in [18]
 - (iii) Theorem 3.9 in [19]
 - (iv) Theorem 8 in [13]
- (2) Establish or improve Chatterjea fixed point theorems in the setting of standard metric spaces, dislocated metric spaces, b -metric spaces, and modular spaces with the Fatou property, also in these spaces endowed with a partial order and more generally with a graph

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

- [1] S. Banach, "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales," *Fundamenta Mathematicae*, vol. 3, pp. 133–181, 1922.
- [2] S. K. Chatterjea, "Fixed-point theorems," *Dokladi na Bolgarskata Akademiya na Naukite*, vol. 25, pp. 727–730, 1972.
- [3] E. Jawhari, M. Pouzet, and D. Misane, "Retracts: graphs and ordered sets from the metric point of view," *Contemporary Mathematics*, vol. 57, pp. 175–226, 1986.
- [4] M. Kabil, M. Pouzet, and I. G. Rosenberg, "Free monoids and generalized metric spaces," *European Journal of Combinatorics*, vol. 80, pp. 339–360, 2019.
- [5] M. Jleli and B. Samet, "A generalized metric space and related fixed point theorems," *Fixed Point Theory and Applications*, vol. 2015, no. 1, Article ID 61, 2015.
- [6] K. Chaira, A. Eladraoui, M. Kabil, and A. Kamouss, "Kannan fixed point theorem on generalized metric space with a graph," *Applied Mathematical Sciences*, vol. 13, no. 6, pp. 263–274, 2019.

- [7] N. Hussain, S. Al-Mezel, and P. Salimi, “Fixed points for ψ -graphic contractions with application to integral equations,” *Abstract and Applied Analysis*, vol. 2013, Article ID 575869, 11 pages, 2013.
- [8] Z. D. Mitrovic and N. Hussain, “On results of Hardy-Rogers and Reich in cone b -metric space over Banach algebra and applications,” *UPB Scientific Bulletin, Series A*, vol. 81, pp. 147–154, 2019.
- [9] W. Shatanawi, Z. D. Mitrović, N. Hussain, and S. Radenović, “On generalized Hardy–Rogers type α -admissible mappings in cone b -metric spaces over Banach algebras,” *Symmetry*, vol. 12, no. 1, p. 81, 2020.
- [10] C. Vetro and F. Vetro, “Metric or partial metric spaces endowed with a finite number of graphs: a tool to obtain fixed point results,” *Topology and its Applications*, vol. 164, pp. 125–137, 2014.
- [11] J. Jachymski, “The contraction principle for mappings on a metric space with a graph,” *Proceedings of the American Mathematical Society*, vol. 136, no. 4, pp. 1359–1373, 2008.
- [12] T. Kamran, M. Postolache, Fahimuddin, and M. U. Ali, “Fixed point theorems on generalized metric space endowed with graph,” *Journal of Nonlinear Sciences and Applications*, vol. 9, no. 6, pp. 4277–4285, 2016.
- [13] K. Chaira, M. Kabil, and A. Kamouss, “Common fixed points in generalized metric spaces with a graph,” *Journal of Mathematics*, vol. 2019, 8 pages, 2019.
- [14] J. A. Bondy and U. S. R. Murty, “Graph theory,” in *Graduate Texts in Mathematics*, Springer-Verlag, London, UK, 1st edition, 2008.
- [15] W. D. Wallis, *A beginner’s Guide to Graph Theory*, Birkhäuser Basel, Basel, Switzerland, 2nd edition, 2007.
- [16] M. R. Alfuraidan, M. Bachar, and M. A. Khamsi, “Almost monotone contractions on weighted graphs,” *Journal of Nonlinear Sciences and Applications*, vol. 9, no. 8, pp. 5189–5195, 2016.
- [17] J. J. Nieto and R. Rodríguez-López, “Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations,” *Order*, vol. 22, no. 3, pp. 223–239, 2005.
- [18] K. Fallahi and A. Aghanians, “Fixed points for Chatterjea contractions on a metric space with a graph,” *International Journal of Nonlinear Analysis and Applications*, vol. 7, pp. 49–58, 2016.
- [19] Y. ElKouch and E. L. Marhrani, “On some fixed point theorems in generalized metric spaces,” *Fixed Point Theory and Applications*, vol. 2017, no. 1, Article ID 23, 2017.
- [20] R. H. Haghi, S. Rezapour, and N. Shahzad, “Some fixed point generalizations are not real generalizations,” *Nonlinear Analysis: Theory, Methods and Applications*, vol. 74, no. 5, pp. 1799–1803, 2011.
- [21] M. Abbas and G. Jungck, “Common fixed point results for noncommuting mappings without continuity in cone metric spaces,” *Journal of Mathematical Analysis and Applications*, vol. 341, no. 1, pp. 416–420, 2008.
- [22] R. D. Nussbaum and D. Hilbert, “Hilbert’s projective metric and iterated nonlinear maps,” *American Mathematical*, vol. 1, 1988.
- [23] Y. Lim, “Solving the nonlinear matrix equation $X = Q + \sum_{i=1}^m M_i X^\delta M_i$ via a contraction principle,” *Linear Algebra and Its Applications*, vol. 430, no. 4, pp. 1380–1383, 2009.
- [24] O. Rida, C. Karim, and M. el Miloudi, “Related Suzuki-type fixed point theorems in ordered metric space,” *Fixed Point Theory and Applications*, vol. 2020, no. 1, Article ID 7, 2020.
- [25] E. Heinz, “Beiträge zur störungstheorie der spektralzerleung,” *Mathematische Annalen*, vol. 123, no. 1, pp. 415–438, 1951.
- [26] K. Löwner, “Über monotone matrixfunktionen,” *Mathematische Zeitschrift*, vol. 38, no. 1, pp. 177–216, 1934.
- [27] M. Kir and H. Kiziltunc, “On some well known fixed point theorems in b -Metric spaces,” *Turkish Journal of Analysis and Number Theory*, vol. 1, no. 1, pp. 13–16, 2013.