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Research Article

Some Results on Iterative Proximal Convergence and Chebyshev Center

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In this paper, we prove a sufficient condition that every nonempty closed convex bounded pair (M, N) in a reflexive Banach space B satisfying Opial's condition has proximal normal structure. We analyze the relatively nonexpansive self-mapping T on $M \cup N$ satisfying $T(M) \subseteq M$ and $T(N) \subseteq N$, to show that Ishikawa's and Halpern's iteration converges to the best proximity point. Also, we prove that under relatively isometry self-mapping T on $M \cup N$ satisfying $T(N) \subseteq N$ and $T(M) \subseteq M$, Ishikawa's iteration converges to the best proximity point in the collection of all Chebyshev centers of N relative to M. Some illustrative examples are provided to support our results.

1. Introduction and Preliminaries

Let M and N be nonempty subsets of a Banach space B. A mapping $T : M \cup N \rightarrow M \cup N$ satisfying $\|Ts - Tt\| \leq \|s - t\|$ (respectively, $\|Ts - Tt\| = \|s - t\|$) for all $s \in M$, $t \in N$ is called relatively nonexpansive mapping (respectively, relatively isometry mapping). For more results on relatively nonexpansive (respectively, relatively isometry) mappings, readers can see the research papers in [1, 2] and references therein.

For any two nonempty bounded subsets M and N of a Banach space B, we denote some notations as follows:

$$\begin{aligned} R(s, N) &:= \sup \{\|s - t\| : t \in N\}, \\ M_0 &:= \{s \in M : \|s - t\| = \operatorname{dist}(M, N) \text{ for some } t \in N\}, \\ N_0 &:= \{t \in N : \|s - t\| = \operatorname{dist}(M, N) \text{ for some } s \in M\}, \end{aligned} \tag{1}$$

where $\operatorname{dist}(M, N) := \inf \{\|s - t\| : s \in M \text{ and } t \in N\}$. Here, it is to note that if $M \cap N \neq \emptyset$, then $M_0 = N_0 = M \cap N$.

Let M be a nonempty convex subset of a normed linear space X, and let $T : M \rightarrow M$ be a mapping with $\operatorname{Fix}(T) \neq \emptyset$, where $\operatorname{Fix}(T) = \{s \in M : Ts = s\}$. A set M is said to have approximate fixed point property (AFPP) if the nonexpansive mapping T has an approximate fixed point sequence, that is, a sequence $\{p_n\}$ in M satisfies $\lim_{n \rightarrow +\infty} \|Tp_n - p_n\| = 0$.

Definition 1 [3]. A normed space X is said to be uniformly convex (or uniformly rotund) if and only if for every $\varepsilon \in (0, 2]$ there exists $\delta > 0$ such that $(\|s + t\|/2) \leq 1 - \delta$ whenever s, $t \in X$ implies $\|s\| = 1$, $\|t\| = 1$, and $\|s - t\| \geq \varepsilon$.

Definition 2 [4]. A nonempty convex subset M of a Banach space B is said to have normal structure if for any nonempty convex closed bounded subset S of M with $\operatorname{diam}(S) > 0$ there exists $s \in S$ such that $R(s, S) < \operatorname{diam}(S)$, where $\operatorname{diam}(S) = \operatorname{diam}(S, S) = \sup \{R(s, S): s \in S\}$.

Eldred et al. [1] introduced the notions of proximal pair and proximal normal structure.

Definition 3 [1]. *A nonempty pair (M, N) of a normed linear space X is known as a proximal pair if, for every $(s, t) \in M \times N$*

,*there exists* $(s', t') \in M \times N$ *such that*

$$\|s - t'\| = \text{dist}(M, N) = \|s' - t\|. \quad (2)$$

A nonempty convex pair (M, N) in a Banach space B is said to have proximal normal structure if $(M_1, N_1) \subseteq (M, N)$ is a closed bounded convex pair for which $\text{dist}(M_1, N_1) = \text{dist}(M, N)$ and $\text{diam}(M_1, N_1) > \text{dist}(M_1, N_1)$, there exists $(s_1, t_1) \in M_1 \times N_1$ such that

$$\begin{aligned} R(s_1, N_1) &< \text{diam}(M_1, N_1) \\ &= \sup \{R(s_1, N_1) \colon s_1 \in M_1\} \text{ and } R(t_1, M_1) \\ &< \text{diam}(M_1, N_1). \end{aligned} \quad (3)$$

Here, it is to note that every nonempty convex weakly compact pair in a uniformly convex Banach space has proximal normal structure. If $M = N$, then proximal normal structure becomes normal structure of Definition 2.

Definition 4 [5]. *A proximal pair* (M, N) *in a Banach space* B *is known as a proximal parallel pair if*

(1) for every element (s, t) in $M \times N$, there exists a unique element (s_1, t_1) in $M \times N$ such that $\|s - t_1\| = \|t - s_1\| = \text{dist}(M, N)$ and

(2) $N = M + h$, where h is a unique element in B

Further, Espinola [5] proved the following lemma.

Lemma 5 [5]. *If* (M, N) *is a nonempty proximal pair in a strictly convex Banach space* B *, then proximal pair* (M, N) *is a proximal parallel pair.*

Definition 6 [6]. *The nonempty proximal parallel pair* (M, N) *in a Banach space* B *is said to have rectangle property if for any* $s, t \in M$,

$$\|s + h - t\| = \|t + h - s\|, \quad (4)$$

where $h \in B$ and $N = M + h$.

Eldred et al. [1] proved the following result.

Theorem 7 [1]. *Let* (M, N) *be a nonempty closed bounded convex pair in a uniformly convex Banach space* B *. Let* T *be a relatively nonexpansive self-mapping on* $M \cup N$ *satisfying* $T(M) \subseteq M$ *and* $T(N) \subseteq N$ *. Let* $s_0 \in M$ *be an initial point, and define a sequence (Krasnoselskii's iteration formula) by* $s_{n+1} = (s_n + Ts_n/2)$, $n \geq 0$ *. Then,* $\lim_{n\to+\infty} \|Ts_n - s_n\| = 0$ *. If* $T(M)$ *is a subset of some compact set in* B *, then the limit point of* $\{s_n\}$ *under the norm topology is the best proximity point of* T.

It is ascertained that the geometric property, that is, proximal normal structure, was used in the following result of Eldred et al. [1].

Theorem 8 [1]. *Let* B *be a strictly convex Banach space, and let* (M, N) *be a nonempty weakly compact convex pair having proximal normal structure. Let* T *be a self-mapping on* $M \cup N$ *satisfying*

$$T(M) \subseteq M, T(N) \subseteq N \text{ and} \|Ts - Tt\| \leq \|s - t\| \text{for all } s \in M, t \in N, \quad (5)$$

then T *has fixed points* $s \in M$, $t \in N$, *and* $\|s - t\| = dist(M, N)$.

Definition 9 [7]. *Let* M *be a nonempty convex subset of a real Hilbert space* H *, and let* T *be a self-mapping on* M *. Let* $s_0 \in M$ *be an initial point, and* $\{s_n\}$ *is a sequence defined by*

$$s_{n+1} = (1 - \xi_n)s_n + \xi_n Tt_n, t_n = (1 - \eta_n)s_n + \eta_n Ts_n, \quad (6)$$

where $0 \leq \xi_n \leq 1$, $0 \leq \eta_n \leq 1$, $n \geq 0$.

The iterative sequence defined in (6) is called Ishikawa's iteration. If $\eta_n = 0$, then Ishikawa's iteration sequence reduces to Mann's iteration sequence. Eldred and Praveen [8] generalized and extended Theorem 7 of Eldred et al. [1] by using Mann's iteration method.

Definition 10 [9]. *Let* M *be a nonempty convex subset of a real Hilbert space* H *, and let* T *be a self-mapping on* M *. Fix* $u \in M$ *. Let* $p_0 \in M$ *be an initial point, and a sequence* $\{p_n\}$ *is defined by*

$$p_{n+1} = \xi_n u + (1 - \xi_n)Tp_n, \quad 0 \leq \xi_n \leq 1, n \geq 0. \quad (7)$$

The iterative sequence defined in (7) is called Halpern's iteration.

The following interesting result will be used extensively in the sequel.

Proposition 11 [10]. *Let* X *be a uniformly convex normed linear space,* $0 < \alpha < 1$ *, and* $\varepsilon > 0$ *. For any* $r > 0$ *, if* $s, t \in X$ *are such that* $\|s\| \leq r$, $\|t\| \leq r$, $\|s - t\| \geq \varepsilon$ *, then there exists* $\delta = \delta(\varepsilon/r) > 0$ *such that*

$$\|\alpha s + (1 - \alpha)t\| \leq \left(1 - 2\delta\left(\frac{\varepsilon}{r}\right) \min\{\alpha, 1 - \alpha\}\right)r. \quad (8)$$

Almezel et al. [11] modified the result of Xu [12] in the following way.

Lemma 12 [11, 12]. *Let* $\{x_n\}$ *be a sequence of nonnegative real numbers satisfying*

$$x_{n+1} \leq (1 - \eta_n)x_n + \eta_n \nu_n, n \geq 0, \quad (9)$$

where $\{\eta_n\}$ *and* $\{\nu_n\}$ *satisfy the following conditions.*

(1) $\{\eta_n\}$ *is a sequence in* $]0, 1[$, *where* $\sum_{n=1}^{+\infty} \eta_n = +\infty$

(2) $\{\nu_n\}$ *is a sequence in* $\mathbb{R}$; *either* $\limsup_{n\to+\infty} \eta_n \leq 0$ *or* $\sum_{n=0}^{+\infty} |\eta_n \nu_n| < +\infty$

Then, $x_n \to 0$ as $n \to +\infty$.

Let M and N be nonempty bounded subsets of a Banach space B. The number $R(M,N) := \inf\{R(s,N): s \in M\}$ is the Chebyshev radius of N relative to M and $C_M(N) := \{s \in M : R(s,N) = R(M,N)\}$ is the set of all Chebyshev centers of N relative to M. Since the function R is convex and continuous on X, R is lower semicontinuous with respect to the weak topology. Consequently, if M is a nonempty weakly compact convex set, then $C_M(N)$ is a nonempty convex weakly compact subset of M. Rajesh and Veeramani [2] proved the following proposition.

Proposition 13 [2]. *Let (M,N) be a nonempty convex weakly compact proximal parallel pair in a Banach space B. Let the nonempty pair (M,N) have the rectangle property. Then, $R(s,N) = R(s+h, M)$ for $s \in M$, and $R(t,M) = R(t-h, N)$ for $t \in N$. Moreover, $C_N(M) = C_M(N) + h$.*

Definition 14 [13]. *Let B be a Banach space. We say that B satisfies Opial's condition if for any sequence $\{p_n\}$ in B converges weakly to some s, then $\limsup_{n\to+\infty}\|p_n - p\| > \limsup_{n\to+\infty}\|p_n - s\|$ for all $p \neq s \in B$. If a reflexive Banach space B satisfies Opial's condition, then B has a normal structure.*

Proposition 15 (demiclosed principle [13]). *Let B be a Banach space, and let M be a nonempty weakly compact subset of B. Also, let T be a nonexpansive self-mapping on M with Fix $(T) \neq \emptyset$. If a sequence $\{p_n\}$ in M converges weakly to s and a sequence $\{(I-T)p_n\}$ converges strongly to p, then $(I-T)s = p$. Moreover, if $p = 0$, then $I - T$ is demiclosed at zero.*

We need the following result of Dutta and Veeramani [14] to prove Proposition 17.

Theorem 16 [14]. *If a nonempty convex pair (M,N) in a Banach space B does not have a proximal normal structure, then there exist sequences $\{s_n\} \subset M$, $\{t_n\} \subset N$ such that $\|s_n - t_n\| = dist(M,N)$ for all n, $\|s_m - t_n\| > dist(M,N)$ for some m, n and*

$$\lim_{n\to+\infty} dist(s_{n+1}, conv\{t_1, t_2, \cdots, t_n\}) = diam(\{s_n\}, \{t_n\}), \quad (10)$$

or

$$\lim_{n\to+\infty} dist(t_{n+1}, conv\{s_1, s_2, \cdots, s_n\}) = diam(\{s_n\}, \{t_n\}), \quad (11)$$

where $diam(\{s_n\}, \{t_n\}) = diam(\{s_1, \cdots, s_n, \cdots\}, \{t_1, \cdots, t_n, \cdots\})$.

2. Opial's Condition and Ishikawa's Iteration for Relatively Nonexpansive Mappings

The geometrical property, that is, the proximal normal structure, is the sufficient condition for the existence of the best proximity [1]. For details about the best proximity point, one can see research papers in [1, 2, 5, 15–19]. We now prove the following result, which shows that the above condition can be dropped if a reflexive Banach space satisfies Opial's condition.

Proposition 17. *Every closed bounded convex pair (M,N) in reflexive Banach space B satisfying Opial's condition has proximal normal structure.*

Proof. Suppose the pair (M,N) does not have a proximal normal structure. Then, by Theorem 16, there exist sequences $\{s_n\} \subset M$, $\{t_n\} \subset N$ such that $\|s_n - t_n\| = \text{dist}(M,N)$ for all n, $\|s_m - t_n\| > \text{dist}(M,N)$ for some m, n, and $\lim_{n\to+\infty} \text{dist}(t_{n+1}, \text{conv}\{s_1, s_2, \cdots, s_n\}) = R(\{s_n\}, \{t_n\})$. Let the sequence $\{s_n\}$ converges weakly to 0. Therefore, $0 \in \overline{\text{conv}}\{s_1, s_2, \cdots, s_n\}$.

Suppose $s \in \text{conv}\{s_1, s_2, \cdots, s_n\}$, then $\lim_{n\to+\infty}\|s - t_n\| = R(\{s_n\}, \{t_n\})$, and the same holds as $s \in \overline{\text{conv}}\{s_1, s_2, \cdots, s_n\}$. Therefore, when taking $s = 0$, we get $\lim_{n\to+\infty}\|t_n\| = R(\{s_n\}, \{t_n\})$, and $\lim_{n\to+\infty}\|s_1 - t_n\| = R(\{s_n\}, \{t_n\})$, which is a contradiction, hence the result.

After analyzing the theorems, definitions, lemma, and propositions mentioned above, we have some impressive new results herewith.

Theorem 18. *Let (M,N) be a nonempty convex closed bounded proximal pair of B, a uniformly convex Banach space. Let T be a relatively nonexpansive self-mapping on $M \cup N$ satisfying $T(M) \subseteq M$ and $T(N) \subseteq N$. Let $s_0 \in M$ be an initial point, and a sequence $\{s_n\}$ is defined as*

$$s_{n+1} = (1-\xi_n)s_n + \xi_n T t_n, t_n = (1-\eta_n)s_n + \eta_n T s_n, \theta < \eta_n \le \xi_n < 1-\theta, 0 < \theta \le \frac{1}{2}, \lim_{n\to+\infty}\xi_n\eta_n = 0. \quad (12)$$

Then, $\lim_{n\to+\infty}\|Ts_n - s_n\| = 0$. If $T(M)$ is a subset of a compact set, then the limit point of $\{s_n\}$ under the norm topology is the best proximity point of T.

Proof. If $\text{dist}(M,N) = 0$, then it is not necessary to discuss. Suppose $\text{dist}(M,N) > 0$, then by applying the result of Theorem 8, there exists $t \in N$ such that $Tt = t$. Since

$$\begin{aligned}\|s_{n+1} - t\| &\le (1-\xi_n)\|s_n - t\| + \xi_n\|Tt_n - t\| \\ &\le (1-\xi_n)\|s_n - t\| + \xi_n(1-\eta_n)\|s_n - t\| \\ &+ \xi_n\eta_n\|Ts_n - t\| \le \|s_n - t\|,\end{aligned} \quad (13)$$

$\{\|s_n - t\|\}$ is a nonincreasing sequence, there exists $k > 0$ such that $\lim_{n\to+\infty} \|s_n - t\| = k$.

Suppose $\lim_{n\to+\infty} \|Ts_n - s_n\| \neq 0$, then there exists a subsequence $\{s_{n_i}\}$ of $\{s_n\}$ such that

$$\|s_{n_i} - Ts_{n_i}\| \geq \varepsilon > 0. \tag{14}$$

Let $\alpha \in]0, 1[$ and ε_1 such that $\varepsilon/\alpha > k$ and $0 < \varepsilon_1 < \min\{(\varepsilon/\alpha) - k, (k\delta(\alpha)/(1 - \delta(\alpha)))\}$. Since B is a uniformly convex Banach space, the modulus of convexity function $\delta(.)$ is strictly increasing and continuous. Hence, $0 < \delta(\alpha) < \delta(\varepsilon/(k + \varepsilon_1))$. So, we can choose a small positive number $\varepsilon_1 > 0$ such that $(1 - a\delta(\varepsilon/(k + \varepsilon_1)))(k + \varepsilon_1) < k$, where $a > 0$.

Let $\|s_{n_i} - t\| \leq k + \varepsilon_1$ and $\|t_{n_i} - t\| \leq k + \varepsilon_1$ for some i. Now,

$$\begin{aligned} \|t - Tt_{n_i}\| &\leq \|t - t_{n_i}\| = \left\|t - \left\{\left(1 - \eta_{n_i}\right)s_{n_i} + \eta_{n_i} Ts_{n_i}\right\}\right\| \\ &= \left\|\left(1 - \eta_{n_i}\right)\left(t - s_{n_i}\right) + \eta_{n_i}\left(t - Ts_{n_i}\right)\right\| \\ &\leq \left(1 - 2\delta\left(\frac{\varepsilon}{k + \varepsilon_1}\right)\min\left\{\eta_{n_i}, 1 - \eta_{n_i}\right\}\right)(k + \varepsilon_1) \\ &\leq \left(1 - a_1\delta\left(\frac{\varepsilon}{k + \varepsilon_1}\right)\right)(k + \varepsilon_1), \end{aligned} \tag{15}$$

where $0 < a_1 \leq 2\min\{\eta_{n_i}, 1 - \eta_{n_i}\}$. Further

$$\begin{aligned} \|t - s_{n_{i+1}}\| &= \left\|t - \left\{\left(1 - \xi_{n_i}\right)s_{n_i} + \xi_{n_i} Tt_{n_i}\right\}\right\| \\ &= \left\|\left(1 - \xi_{n_i}\right)\left(t - s_{n_i}\right) + \xi_{n_i}\left(t - Tt_{n_i}\right)\right\| \\ &\leq \left(1 - 2\delta\left(\frac{\varepsilon}{k + \varepsilon_1}\right)\min\left\{\xi_{n_i}, 1 - \xi_{n_i}\right\}\right)(k + \varepsilon_1) \\ &\leq \left(1 - a_2\delta\left(\frac{\varepsilon}{k + \varepsilon_1}\right)\right)(k + \varepsilon_1), \end{aligned} \tag{16}$$

where $0 < a_2 \leq 2\min\{\xi_{n_i}, 1 - \xi_{n_i}\}$.

By choosing $\varepsilon_1 > 0$ as small as we wish, we get

$$\max\left\{\left(1 - a_1\delta\left(\frac{\varepsilon}{k + \varepsilon_1}\right)\right)(k + \varepsilon_1), \left(1 - a_2\delta\left(\frac{\varepsilon}{k + \varepsilon_1}\right)\right)(k + \varepsilon_1)\right\} < k, \tag{17}$$

which is a contradiction. Hence, $\lim_{n\to+\infty} \|Ts_n - s_n\| = 0$ and $\lim_{n\to+\infty} \|s_{n+1} - s_n\| = 0$.

If $T(M)$ is compact, then the sequence $\{s_n\}$ has a subsequence $\{s_{n_i}\}$ such that $\lim_{i\to+\infty} s_{n_i} = s \in M$. Since (M, N) is a proximal pair, there exists $v \in N$ such that $\|s - v\| = \operatorname{dist}(M, N)$.

Now, we have $\lim_{i\to+\infty} \|s_{n_i} - v\| = \operatorname{dist}(M, N)$, and $\{\|s_n - v\|\}$ is a nonincreasing sequence; it implies that $\lim_{n\to+\infty} \|s_n - v\| = \operatorname{dist}(M, N)$. This shows that $\lim_{n\to+\infty} s_n = s \in M$. By strict convexity of the norm, $\lim_{n\to+\infty} \|Ts_n - v\| = \operatorname{dist}(M, N)$ and $\|Ts_n - Tv\| \leq \|s_n - v\| \to \operatorname{dist}(M, N)$ as $n \to +\infty$ give $Tv = v$. Since $\lim_{n\to+\infty} \|Ts_n - s_n\| = 0$, it follows that $Ts = s$.

We obtain the following result from Theorem 18 by taking $\eta_n = 0$ for $n \in \mathbb{N}$.

Corollary 19 [8]. *Let (M, N) be a nonempty convex closed bounded proximal pair of B , a uniformly convex Banach space, and let T be a relatively nonexpansive self-mapping on $M \cup N$ satisfying $T(N) \subseteq N$ and $T(M) \subseteq M$. Let $s_0 \in M$ be an initial point, and a sequence $\{s_n\}$ is defined as*

$$s_{n+1} = (1 - \xi_n)s_n + \xi_n Ts_n, \varepsilon < \xi_n < 1 - \varepsilon, 0 < \varepsilon \leq \frac{1}{2}(\text{Mann's iteration}). \tag{18}$$

Then, $\lim_{n\to+\infty} \|Ts_n - s_n\| = 0$. Moreover, if $T(M)$ is a subset of a compact set, then the limit point of $\{s_n\}$ under norm topology is the best proximity point of T.

3. Halpern's Iteration and Relatively Nonexpansive Mapping

Let M be a nonempty subset of a real Hilbert space H, and let $P_M : H \to 2^M$ be the nearest point projection mapping from H onto M that is, $P_M(s) \coloneqq \{s' \in M : \|s' - s\| = \operatorname{dist}(s, M)\}$. If M is nonempty convex closed, then P_M is nonexpansive giving unique image for all s in H, and hence by Kolmogorov's criterion $< P_M t - s, P_M t - t > \leq 0$ for all $t \in X, s \in M$. Here, we use the following notation $M_{\mathrm{Fix}T} = \{s \in M : Ts = s\}$.

Theorem 20. *Let (M, N) be a nonempty closed bounded convex proximal pair of a real Hilbert space H, and let T be a relatively nonexpansive self-mapping on $M \cup N$ satisfying $T(M) \subseteq M$ and $T(N) \subseteq N$. Let $0 < \xi_n < 1$, and $s_0 \in M$ be an initial point. A sequence $\{s_n\}$ is defined as*

$$s_{n+1} = \xi_n u + (1 - \xi_n)Ts_n, \tag{19}$$

where $u \in M$ such that $\|s_n - t\| \geq \|u - t\|$ for all $t \in N$.

If $\lim_{n\to+\infty} \xi_n = 0$, $\sum_{n=1}^{+\infty} \xi_n = +\infty$, and either $\sum_{n=1}^{+\infty} |\xi_{n+1} - \xi_n| < +\infty$ or $\lim_{n\to+\infty} (\xi_n/\xi_{n+1}) = 1$, then the sequence $\{s_n\}$ under the norm topology converges to $s \in M_{\mathrm{Fix}T}$, closest to point u such that $\|s - t\| = \operatorname{dist}(M, N)$ for some $t \in N_{\mathrm{Fix}T}$.

Proof. By applying Theorem 8, it is found that there exists $t \in N$ such that $Tt = t$. Now, we have

$$\begin{aligned} \|s_{n+1} - t\| &= \|\xi_n u + (1 - \xi_n)Ts_n - t\| \leq \xi_n\|u - t\| \\ &\quad + (1 - \xi_n)\|s_n - t\| \leq \xi_n\|u - t\| \\ &\quad + (1 - \xi_n)\|s_n - t\| \leq \xi_n\|s_n - t\| \\ &\quad + (1 - \xi_n)\|s_n - t\| (\text{since} \|s_n - t\| \geq \|u - t\|). \end{aligned} \tag{20}$$

Hence $\{\|s_n - t\|\}$ is nonincreasing and $\lim_{n\to+\infty} \|s_n - t\| = k > 0$.

Suppose $\lim_{n\to+\infty} \|Ts_n - s_n\| \neq 0$, then there exists a subsequence $\{s_{n_i}\}$ of $\{s_n\}$ such that $\|s_{n_i} - Ts_{n_i}\| \geq \varepsilon > 0$. Since H is a Hilbert space (and hence uniformly convex space), it is possible to choose a small positive number $\varepsilon_1 > 0$, such that $(1 - a\delta(\varepsilon/(k+\varepsilon_1)))(k+\varepsilon_1) < k$, where $a > 0$.

Let $\|s_{n_i} - t\| \leq k + \varepsilon_1$ for some i. Now,

$$\begin{aligned} \|t - s_{n_{i+1}}\| &= \|t - \{\xi_{n_i} u + (1-\xi_{n_i}) Ts_{n_i}\}\| \\ &= \|(1-\xi_{n_i})(t - Ts_{n_i}) + \xi_{n_i}(t-u)\| \\ &\leq \left(1 - 2\delta\left(\frac{\varepsilon}{k+\varepsilon_1}\right) \min\{\xi_{n_i}, 1-\xi_{n_i}\}\right)(k+\varepsilon_1) \\ &\leq \left(1 - a_1\delta\left(\frac{\varepsilon}{k+\varepsilon_1}\right)\right)(k+\varepsilon_1), \end{aligned} \tag{21}$$

where $0 < a_1 \leq 2 \min\{\xi_{n_i}, 1-\xi_{n_i}\}$.

By choosing $\varepsilon_1 > 0$ as small as we wish, we have

$$\left(1 - a_1\delta\left(\frac{\varepsilon}{k+\varepsilon_1}\right)\right)(k+\varepsilon_1) < k, \tag{22}$$

which is a contradiction. Hence, $\lim_{n\to+\infty} \|Ts_n - s_n\| = 0$ and $\lim_{n\to+\infty} \|s_{n+1} - s_n\| = 0$.

Let $\{s_{n_i}\}$ be a subsequence of $\{s_n\}$ such that

$$\limsup_{n\to+\infty} < s_n - s, s - u > = \limsup_{i\to+\infty} < s_{n_i} - s, s - u > . \tag{23}$$

Without loss of generality, we assume that subsequence $\{s_{n_i}\}$ converges weakly to $p \in M$ such that $\|p - t\| = \mathrm{dist}(M, N)$ for some $t \in N_{\mathrm{Fix}T}$. Since $\lim_{n\to+\infty} \|Ts_n - s_n\| = 0$, by applying the demiclosed principle, we have $p \in M_{\mathrm{Fix}T}$. Hence, by applying Kolmogorov's criterion, we have

$$\limsup_{i\to+\infty} < s_{n_i} - s, s - u > = < p - s, s - u > \geq 0. \tag{24}$$

Now, we have

$$\begin{aligned} \|s_{n+1} - s\|^2 &= <\xi_n u + (1-\xi_n) Ts_n - s, s_{n+1} - s> \\ &\leq \xi_n s < u - s, s_{n+1} - s > + (1-\xi_n)\|s_n - s\| . \|s_{n+1} - s\| \\ &\leq \frac{(1-\xi_n)}{2}\left(\|s_n - s\|^2 + \|s_{n+1} - s\|^2\right) \\ &\quad + \xi_n < u - s, s_{n+1} - s > . \end{aligned} \tag{25}$$

Hence,

$$\begin{aligned} \Rightarrow \|s_{n+1} - s\|^2 &\leq \frac{2\xi_n}{1+\xi_n} < u - s, s_{n+1} - s > + \left(1 - \frac{2\xi_n}{1+\xi_n}\right)\|s_n - s\|^2 \\ &= (1-\eta_n)\|s_n - s\|^2 + \xi_n \nu_n, \end{aligned} \tag{26}$$

where $\eta_n = 2\xi_n/(1+\xi_n)$ and $\nu_n = (2/(1+\xi_n)) < u - s, s_{n+1} - s >$.

Since $\sum_{n=1}^{+\infty} \eta_n = +\infty$ and $\limsup_{n\to+\infty} \nu_n \leq 0$, by Lemma 12, we have $\lim_{n\to+\infty} s_n = s \in M_{\mathrm{Fix}T}$, closest to point u so that $\|s - t\| = \mathrm{dist}(M, N)$ for some $t \in N_{\mathrm{Fix}T}$.

We obtain the following corollary from Theorem 20 when $M = N$.

Corollary 21 [9]. *Let M be nonempty closed bounded convex subsets of a real Hilbert space H and T be a nonexpansive self-mapping on M. Let $s_0 \in M$ be an initial point, and $\{s_n\}$ is a sequence defined as*

$$s_{n+1} = \xi_n u + (1-\xi_n) Ts_n, \tag{27}$$

where $u \in M$ and $0 < \xi_n < 1$ (Halpern's iteration).

If $\lim_{n\to+\infty} \xi_n = 0$, $\sum_{n=1}^{+\infty} \xi_n = +\infty$, and either $\sum_{n=1}^{+\infty} |\xi_{n+1} - \xi_n| < +\infty$ or $\lim_{n\to+\infty} \xi_n/\xi_{n+1} = 1$, then the sequence $\{s_n\}$ under the norm topology converges to $s \in M_{\mathrm{Fix}T}$, closest to point u.

4. Ishikawa's Iteration and Chebyshev Centre

Lim et al. [20] proved the following interesting theorem in the year 2003, by using the geometrical property, viz., normal structure.

Theorem 22 [20]. *Let B be a Banach space, and let T be an isometry self-mapping on M, a nonempty weakly compact convex subset of B. It is assumed that M has a normal structure. Then, there exists $s \in C(M) = C_M(M)$, the set of all Chebyshev centers of M such that $Ts = s$.*

Let (M, N) be a nonempty convex weakly compact proximal parallel pair in a Banach space B. Suppose the pair (M, N) has the rectangle property. Let $T : M \cup N \to M \cup N$ be a relatively isometry mapping satisfying $T(M) \subseteq M$ and $T(N) \subseteq N$. It is ascertained that $T(C_M(N)) \subseteq C_M(N)$ if and only if $R(s, N) = R(Ts, N) = R(M, N)$ for all $s \in C_M(N)$. Similarly, $T(C_N(M)) \subseteq C_N(M)$ if and only if $R(t, M) = R(Tt, M) = R(M, N)$ for all $t \in C_N(M)$. It is affirmed that $C_N(M) = C_M(N) + h$ for some $h \in B$ (for details, see [2, 21, 22]). We establish the following result.

Lemma 23. *Let (M, N) be a nonempty weakly compact convex proximal pair in a strictly convex Banach space B. Suppose T is a relatively isometry self-mapping on $M \cup N$ satisfying $T(M) \subseteq M$ and $T(N) \subseteq N$. If $s \in M$ and $\{T^n s\}$ has a Cauchy subsequence in M, then $R(s, N) = R(Ts, N)$. Similarly, if $t \in$*

N and $\{T^n t\}$ has a Cauchy subsequence in N, then $R(t, M) = R(Tt, M)$.

Proof. Let $s \in M$. Then,

$$R(s, N) = R(Ts, TN) \le R(Ts, N). \quad (28)$$

Let $(s, t) \in M \times N$ such that $\|s - t\| = \text{dist}(M, N)$. Suppose $\|T^{j_n}s - T^{i_n}t\| < \text{dist}(M, N) + 1/n$, where $i_n, j_n \in \mathbb{Z}^+$, with $i_n < j_n$, for every $n \in \mathbb{Z}^+$. Since T is a relatively isometry mapping, we get $\lim_{n\to+\infty} T^{j_n - i_n}s = s$.

Let $\{a_n\}$ be a nondecreasing subsequence of $\{j_n - i_n\}$. Since R is a nonnegative continuous real valued function, then the sequence $\{R(T^{a_n}s, N)\}$ is nondecreasing, and $\lim_{n\to+\infty} T^{a_n}s = s$. Therefore, $\lim_{n\to+\infty} R(T^{a_n}s, N) = R(s, N)$. Thus,

$$R(Ts, N) \le R(T^{a_1}s, N) \le \lim_{n\to+\infty} R(T^{a_n}s, N) = R(s, N). \quad (29)$$

From, (28) and (29), we have $R(s, N) = R(Ts, N)$. Similarly, we can show that $R(t, M) = R(Tt, M)$.

Lemma 24. *Let (M, N) be a nonempty weakly compact convex proximal parallel pair in a strictly convex Banach space B. It is assumed that the pair (M, N) has the rectangle property. Suppose T is a relatively isometry self-mapping on $M \cup N$ satisfying $T(M) \subseteq M$ and $T(N) \subseteq N$. If $(C_M(N), C_N(M))$ is nonempty and contained in a totally bounded proximal parallel pair (M_1, N_1) of (M, N) such that $T(M_1) \subseteq M_1$ and $T(N_1) \subseteq N_1$, then $T(C_M(N)) \subseteq C_M(N)$ and $T(C_N(M)) \subseteq C_N(M)$.*

Proof. Let $s \in C_M(N)$, where $C_M(N) \subseteq M_1$, $T(M_1) \subseteq M_1$, and $C_N(M) = C_M(N) + h$, for some $h \in B$. Then, $\{T^n s : n \in \mathbb{Z}^+\} \subseteq M_1$, and $\{T^n(s + h) : n \in \mathbb{Z}^+\} \subseteq N_1$.

As (M_1, N_1) is a totally bounded proximal pair, the sequences $\{T^n(s)\}$ and $\{T^n(s + h)\}$, respectively, have Cauchy subsequences in M_1 and N_1. So, by Lemma 23, we have $R(s, N) = R(M, N) = R(Ts, N)$.

Hence, $T(C_M(N)) \subseteq C_M(N)$. Similarly, $T(C_N(M)) \subseteq C_N(M)$.

Example 25. Let $X = (\mathbb{R}^2, \|.\|_1)$. Let $M = \{(s, 10 - 10s) : s \in [0, 1]\}$, and $N = M + h$, where $h = (0, 1) \in X$. Let $(s, 10 - 10s) \in M$ and $(1 + t, 10 - 10t) \in N$, where $s, t \in [0, 1]$. Now, we have

$$\begin{aligned} \|(1 + t, 10 - 10t) - (s, 10 - 10s)\|_1 &= \|(1 + t - s, 10s - 10t)\|_1 \\ &= 1 + t - s + |10s - 10t| \\ &= \begin{cases} 1 + 9s - 9t, & \text{if } t \le s, \\ 1 + 11t - 11s, & \text{if } s \le t. \end{cases} \end{aligned} \quad (30)$$

In particular, take $(1, 0), (0, 10) \in M$, and $(2, 0), (1, 10) \in N$, we have $\|\{(1, 0) + (1, 0)\} - (0, 10)\|_1 = 12$ and $\|\{(0, 10) + (1, 0)\} - (1, 0)\|_1 = 10$.

It shows that there exists a proximal parallel pair (M, N) with $\text{dist}(M, N) = \|h\|_1 = 1$ which does not satisfy the rectangle property.

Theorem 26. *Let (M, N) be a nonempty totally bounded convex closed proximal pair in a uniformly convex (and hence reflexive) Banach space B. It is also assumed that the pair (M, N) has the rectangle property. Suppose T is a relatively isometry self-mapping on $M \cup N$ satisfying $T(N) \subseteq N$ and $T(M) \subseteq M$. Let $s_0 \in C_M(N)$ be an initial point, and a sequence $\{s_n\}$ is defined as*

$$\begin{aligned} s_{n+1} &= (1 - \xi_n)s_n + \xi_n Tt_n, t_n = (1 - \eta_n)s_n + \eta_n Ts_n, 0 \le \eta_n \\ &\le \xi_n < 1, \lim_{n\to+\infty} \xi_n\eta_n = 0, -\theta, 0 < \theta \le \frac{1}{2} \end{aligned} \quad (31)$$

Then, $\lim_{n\to+\infty} \|Ts_n - s_n\| = 0$. If $T(C_M(N))$ is a subset of a compact set, then the limit point $s \in C_M(N)$ of the sequence $\{s_n\}$ under norm topology is the best proximity point of T.

Proof. It is easy to see that $(C_M(N), C_N(M))$ is a nonempty convex weakly compact proximal parallel pair having the rectangle property in a uniformly convex Banach space B.

Since (M, N) is totally bounded and T is a relatively isometry self-mapping on $M \cup N$ satisfying $T(N) \subseteq N$ and $T(M) \subseteq M$, by applying Lemma 24, we have $T(C_M(N)) \subseteq C_M(N)$ and $T(C_N(M)) \subseteq C_N(M)$.

Now, by Theorem 8, there exist $s \in C_M(N)$ and $t \in C_N(M)$ such that $Ts = s$, $Tt = t$, and $\|s - t\| = \text{dist}(M, N)$.

By applying Theorem 18, it is found that the sequence $\{s_n\}$ under norm topology converges to $Ts = s \in C_M(N)$, such that $\|s - t\| = \text{dist}(M, N)$ for some $t \in N_{\text{Fix}T}$.

We obtain the following result from Theorem 26 if $\eta_n = 0$ for $n \in \mathbb{N}$.

Theorem 27. *Let (M, N) be a nonempty totally bounded convex closed proximal pair in a uniformly convex (and hence reflexive) Banach space B. It is also assumed that (M, N) has the rectangle property. Suppose T is a relatively isometry self-mapping on $M \cup N$ satisfying $T(N) \subseteq N$ and $T(M) \subseteq M$. Let $s_0 \in C_M(N)$ be an initial point, and a sequence $\{s_n\}$ is defined as*

$$s_{n+1} = (1 - \xi_n)s_n + \xi_n Ts_n, \varepsilon < \xi_n < 1 - \varepsilon, 0 < \varepsilon \le \frac{1}{2}. \quad (32)$$

Then, $\lim_{n\to+\infty} \|Ts_n - s_n\| = 0$. If $T(C_M(N))$ is a subset of a compact set, then the limit point $s \in C_M(N)$ of $\{s_n\}$ under the norm topology is the best proximity point of T.

Proof. The result is similar to that of Theorem 26.

Example 28. Let $X = (\mathbb{R}^2, \|.\|)$, a Euclidean space. Let

$$\begin{aligned} M &= \{(s, t) : s = -2, -1 \le t \le 1\}, \\ N &= \{(s, t) : s = 2, -1 \le t \le 1\}. \end{aligned} \quad (33)$$

Here, (M, N) is a proximal parallel pair having the rectangle property, $R(M, N) = \sqrt{17}$, $C_M(N) = \{(-2, 0)\}$, $C_N(M) = \{(2, 0)\}$, and $C_N(M) = C_M(N) + h$, where $h = (4, 0)$.

Define

$$T : M \to M \text{ by } T(s, t) = (T_1 s, T_2 t) = (-2, -t), \quad (34)$$

where $T_1 : \{-2\} \to \{-2\}$ and $T_2 : [-1, 1] \to [-1, 1]$.

Let $(s, t) \in M$, and $(s', t') \in N$. Then

$$\begin{aligned} \|T(s, t) - T(s', t')\| &= \|(-2, -t) - (2, -t')\| \\ &= \|(-4, t' - t)\| \\ &= \sqrt{(-4)^2 + (t' - t)^2} \\ &= \|(s, t) - (s', t')\|. \end{aligned} \quad (35)$$

Hence, T is a relatively isometry (and hence relatively nonexpansive) mapping on $M \cup N$ satisfying $T(N) \subseteq N$ and $T(M) \subseteq M$.

From Theorem 18, we take the initial point $(s, t) \in M$ and set $s_1 = (1 - \xi_0)s + \xi_0 T_1 s'_0$ and $s'_0 = (1 - \eta_0)s + \eta_0 T_1 s$. We have $T_1 s = -2$. Since $s = -2$, we obtain $s'_0 = -2$ which implies $s_1 = -2$.

Similarly, set $s_2 = (1 - \xi_1)s_1 + \xi_1 T_1 s'_1$ and $s'_1 = (1 - \eta_1) s_1 + \eta_1 T_1 s_1$. Since $s_1 = -2$, we obtain $s'_1 = -2$ which implies $s_2 = -2$. In general, we obtain $s_{n+1} = -2$. Therefore, $s_n \to -2$ as $n \to +\infty$.

Again, set $t_1 = (1 - \xi_0)t + \xi_0 T_2 t'_0$ and $t'_0 = (1 - \eta_0)t + \eta_0 T_2 t$. Since $T_2 t = -t$, we obtain $t'_0 = (1 - 2\eta_0)t$ which implies $t_1 = (1 - 2\xi_0 + 2\xi_0\eta_0)t$. Similarly, set $t_2 = (1 - \xi_1)t_1 + \xi_1 T_2 t'_1$ and $t'_1 = (1 - \eta_1)t_1 + \eta_1 T_2 t_1$. Since $T_2 t_1 = -t_1$, we obtain

$$\begin{aligned} t'_1 &= (1 - \eta_1)t_1 + \eta_1 T_2 t_1 \\ &= (1 - \eta_1)(1 - 2\xi_0 + 2\xi_0\eta_0)t + \eta_1 T_2[(1 - 2\xi_0 + 2\xi_0\eta_0)t] \\ &= (1 - \eta_1)(1 - 2\xi_0 + 2\xi_0\eta_0)t - \eta_1(1 - 2\xi_0 + 2\xi_0\eta_0)t \\ &= (1 - 2\eta_1)(1 - 2\xi_0 + 2\xi_0\eta_0)t, \end{aligned} \quad (36)$$

which implies

$$\begin{aligned} t_2 &= (1 - \xi_1)t_1 + \xi_1 T_2 t'_1 \\ &= (1 - \xi_1)(1 - 2\xi_0 + 2\xi_0\eta_0)t - \xi_1(1 - 2\eta_1)(1 - 2\xi_0 + 2\xi_0\eta_0)t \\ &= (1 - 2\xi_0 + 2\xi_0\eta_0)(1 - 2\xi_1 + 2\xi_1\eta_1)t. \end{aligned} \quad (37)$$

In general, $t_{n+1} = (1 - 2\xi_0 + 2\xi_0\eta_0)(1 - 2\xi_1 + 2\xi_1\eta_1) \cdots (1 - 2\xi_n + 2\xi_n\eta_n)t$. Therefore, $t_n \to 0$ as $n \to +\infty$. Hence, $\lim_{n\to+\infty} (s_n, t_n) = (-2, 0)$, a fixed point of T. In a similar way, if $(s', t') \in N$, then $\lim_{n\to+\infty} (s'_n, t'_n) = (2, 0)$, a fixed point of T and $\|(-2, 0) - (2, 0)\| = \text{dist}(M, N)$.

From Theorem 26, if we take the initial point $(x, y) \in C_M(N)$, then it is trivial that $\lim_{n\to+\infty} (s_n, t_n) = (-2, 0)$, a fixed point of T. In a similar way, if $(s', t') \in C_N(M)$, then $\lim_{n\to+\infty} (s'_n, t'_n) = (2, 0)$, a fixed point of T and $\|(-2, 0) - (2, 0)\| = \text{dist}(M, N)$.

5. Open Problem

Let (M, N) be a nonempty weak compact convex pair in a Banach space (or Hilbert space) B. Can Ishikawa's iteration and Halpern's iteration converge to the best proximity point of relatively nonexpansive (or relatively isometry) mapping $T : M \cup N \to M \cup N$ satisfying $T(M) \subseteq N$ and $T(N) \subseteq M$?

6. Conclusion

If a reflexive Banach space satisfies Opial's condition, then every bounded convex pair (M, N) has a proximal normal structure. Also, we show that Ishikawa's and Halpern's iterative sequences converge to the best proximity point. Finally, we show that Ishikawa's iterative sequence converges to the best proximity point, which is a Chebyshev center.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors have no conflicts of interest regarding the publication of this article.

Authors' Contributions

All authors contributed equally in writing this article.

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