Research Article

End Point Estimate of Littlewood-Paley Operator Associated to the Generalized Schrödinger Operator

Yanping Chen and Wenyu Tao

School of Mathematics and Physics, University of Science and Technology Beijing, Beijing 100083, China

Correspondence should be addressed to Wenyu Tao; wytao@xs.ustb.edu.cn

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Let $\mathcal{L} = -\Delta + \mu$ be the generalized Schrödinger operator on $\mathbb{R}^d$, $d \geq 3$, where $\mu \neq 0$ is a nonnegative Radon measure satisfying certain scale-invariant Kato conditions and doubling conditions. In this work, we give a new BMO space associated to the generalized Schrödinger operator $\mathcal{L}$, $\text{BMO}_{\theta, \mathcal{L}}$, which is bigger than the BMO spaces related to the classical Schrödinger operators $\mathcal{A} = -\Delta + V$, with V a potential satisfying a reverse Hölder inequality introduced by Dziubański et al. in 2005. Besides, the boundedness of the Littlewood-Paley operators associated to $\mathcal{L}$ in $\text{BMO}_{\theta, \mathcal{L}}$ also be proved.

1. Introduction

Consider the generalized Schrödinger operator

$$\mathcal{L} = -\Delta + \mu, \quad (1)$$

where μ is a nonnegative Radon measure on $\mathbb{R}^d$, $d \geq 3$. According to [1], there exist positive constants C_0 , C_1 , and δ such that μ satisfies the following conditions:

$$\mu(B(x, r)) \leq C_0 \left(\frac{r}{R}\right)^{d-2+\delta} \mu(B(x, R)), \quad (2)$$

$$\mu(B(x, 2r)) \leq C_1 \left\{ \mu(B(x, r)) + r^{d-2} \right\}, \quad (3)$$

for all $x \in \mathbb{R}^d$ and $0 < r < R$, where $B(x, r)$ denotes the open ball centered at x with radius r . Condition (2) is regarded as scale-invariant Kato-condition, and from (3), we can see that the measure μ is doubling on balls satisfying $\mu(B(x, r)) \geq cr^{d-2}$. We will also assume that $\mu \neq 0$. If $d\mu = V(x)dx$ and

$V \geq 0$ are in the reverse Hölder class, that is, there exists $C = C(d, V) > 0$ such that

$$\left\{ \frac{1}{|B(x, r)|} \int_{B(x, r)} V(y)^{d/2} dy \right\}^{2/d} \leq \frac{C}{|B(x, r)|} \int_{B(x, r)} V(y) dy, \quad (4)$$

then μ satisfies the conditions (2) and (3) for some $\delta > 0$. However, in general, measures which satisfy (2) and (3) need not be absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$, the counterexample is visible in ([1], Remark 0.10). Let $\mathcal{A} = -\Delta + V$, it is easy to know that $\mathcal{L}$ is more general than $\mathcal{A}$ from the above.

The boundedness of the operators associated to the classical operator $\mathcal{A}$ such as the Gauss-semigroup and Poisson-semigroup maximal functions, the Littlewood-Paley-square function and the fractional integral operator has attracted much interest [2–4]. It is worth noting that these operators fail to be bounded in BMO, even in the classic case (i.e. $V = 0$). In 2005, Dziubański et al. [2] identified the BMO-type space related to $\mathcal{A}$, $\text{BMO}_{\mathcal{A}}$, namely,

$$\text{BMO}_{\mathcal{A}} = \left\{ f \in \text{BMO} : \frac{1}{|B_r|} \int_{B_r} |f(y)| dy \leq C, \text{ for all } B_r = B(x, r), r \geq \rho(x, V) \right\}, \quad (5)$$

where $\rho(x, V)$ is the auxiliary function, defined by

$$\rho(x, V) = \sup \left\{ r > 0 : \frac{1}{r^{d-2}} \int_{B(x, r)} V(y) dy \leq 1 \right\}, \quad (6)$$

(see [2, 4–7]). They proved the above operators were bounded in this space. By the fact that $\text{BMO}_{\mathcal{A}}$ is a subspace of BMO , we know that it is very meaningful to consider the boundedness of these operators if we can expand the $\text{BMO}_{\mathcal{A}}$ space a little bit.

In this paper, we shall be interested in a new BMO space associated to the generalized Schrödinger operator $\mathcal{L}$. To give the definition of the new BMO space, we first recall the auxiliary function $\rho(x, \mu)$ (see [8]),

$$\rho(x, \mu) = \sup \left\{ r > 0 : \frac{\mu(B(x, r))}{r^{d-2}} \leq C_1 \right\}, \quad (7)$$

where C_1 is the constant in (3).

Here, we define the new BMO space, $\text{BMO}_{\theta, \mathcal{L}}$, namely,

$$\text{BMO}_{\theta, \mathcal{L}} = \left\{ f \in \text{BMO}_{\theta} : \frac{1}{|B_r|} \int_{B_r} |f(y)| dy \leq C \left(1 + \frac{r}{\rho(x, \mu)} \right)^{\theta}, \text{ for all } B_r = B(x, r), r \geq \rho(x, \mu) \right\}, \quad (8)$$

where

$$\text{BMO}_{\theta} = \left\{ f \in L^1_{\text{loc}} : \frac{1}{|B_r|} \int_{B_r} |f(y) - f_{B_r}| dy \leq C \left(1 + \frac{r}{\rho(x, \mu)} \right)^{\theta}, \text{ for all } B_r = B(x, r) \right\}, \quad (9)$$

for $\theta > 0$. The precise definition of the norm in the spaces $\text{BMO}_{\theta, \mathcal{L}}$ is given in Definition 1.

Definition 1. Let $\mu \neq 0$ be a nonnegative Radon measure in $\mathbb{R}^d, d \geq 3$. Assume that μ satisfies the conditions (2) and (3). For $\theta > 0$, we shall say that a locally integrable function f belongs to $\text{BMO}_{\theta, \mathcal{L}}$ whenever there is a constant $C \geq 0$ so that

$$\frac{1}{|B_R|} \int_{B_R} |f| \leq C \left(1 + \frac{R}{\rho(x, \mu)} \right)^{\theta} \text{ and } \frac{1}{|B_r|} \int_{B_r} |f(y) - f_{B_r}| dy \leq C, \quad (10)$$

for all balls $B_R = B(x, R), B_r = B(x, r)$ such that $R \geq \rho(x, \mu) \geq r$. The norm of $f \in \text{BMO}_{\theta, \mathcal{L}}$, denoted by $\|f\|_{\text{BMO}_{\theta, \mathcal{L}}}$, is the smallest C in (10) above. Here and subsequently, $f_B = 1/|B| \int_B f(x) dx$.

Remark 2. From the definition of $\text{BMO}_{\theta, \mathcal{L}}$, we have $\text{BMO}_{\theta, \mathcal{L}} \subseteq \text{BMO}_{\theta}$. Also, we emphasize that $\text{BMO}_{\theta, \mathcal{L}}$ is actually bigger than $\text{BMO}_{\mathcal{A}}$, which is defined by Dziubański et al. in [2], that is, $\text{BMO}_{\mathcal{A}} \subseteq \text{BMO}_{\theta, \mathcal{L}}$. From the definition of $\text{BMO}_{\theta, \mathcal{L}}$ and the fact that $\mathcal{L}$ is more general than $\mathcal{A}$, it is obvious that $\text{BMO}_{\mathcal{A}}$ is the subset of $\text{BMO}_{\theta, \mathcal{L}}$.

Since μ is nonnegative on $\mathbb{R}^d$, the Feynman-Kac formula implies that the kernel $k_t(x, y)$ of the semigroup $e^{-t\mathcal{L}}$ of linear operators generated by $-\mathcal{L}$ satisfies

$$0 \leq k_t(x, y) \leq h_t(x - y) = (4\pi t)^{-d/2} e^{-\frac{|x-y|^2}{4t}}, \quad (11)$$

also see in [9].

After Wang [10] considered the g -function defined on BMO functions, more and more scholars pay attention to the end point estimate of the Littlewood-Paley operator [3, 8, 11–13]. In this paper, we will also consider the follows Littlewood-Paley operators associated to the generality Schrödinger operator are bounded in $\text{BMO}_{\theta, \mathcal{L}}$.

$$\begin{aligned} s(f)(x) &:= \left(\int_0^{\infty} |t \mathcal{L} e^{-t\mathcal{L}} f(x)|^2 \frac{dt}{t} \right)^{1/2} \\ &= \left(\int_0^{\infty} \left| t \frac{dT_t}{dt} f(x) \right|^2 \frac{dt}{t} \right)^{1/2}, \end{aligned} \quad (12)$$

where $T_t f(x) = e^{-t\mathcal{L}} f(x) = \int_{\mathbb{R}^d} k_t(x, y) f(y) dy$.

The main theorem is as follows.

Theorem 3. Let $\theta > 0$ and s defined as in (12), then s is bounded in $\text{BMO}_{\theta, \mathcal{L}}$. That is, if $f \in \text{BMO}_{\theta, \mathcal{L}}$, there exist a constant $C > 0$ such that

$$\|s(f)\|_{\text{BMO}_{\theta, \mathcal{L}}} \leq C \|f\|_{\text{BMO}_{\theta, \mathcal{L}}}. \quad (13)$$

The paper is organized as follows. In Section 2, we give some necessary lemmas. In Section 3, we consider $s(f)$ and give the proof of Theorem 3.

Throughout this paper, give a ball B , we denote by B^* the ball with the same center and twice radius. c and C will denote positive constants that may not be the same in each occurrence.

2. Preliminaries

We first recall some basic properties of the auxiliary function $\rho(x, \mu)$ in (7) (see [1], Proposition 1.8). In the sequel, C_0, C_1 , and δ are positive constants in (2) and (3).

Lemma 4. *Suppose μ satisfies (2) and (3). Then*

- (i) *If $r = \rho(x, \mu)$, then $r^{d-2} \leq \mu(B(x, r)) \leq C_1 r^{d-2}$*
- (ii) *If $|x - y| \leq C\rho(x, \mu)$, then $\rho(x, \mu) \sim \rho(y, \mu)$*
- (iii) *There exist constants $C > 0$ such that for $x, y \in \mathbb{R}^d$*

$$\begin{aligned} C^{-1} \rho(x, \mu) \left(1 + \frac{|x - y|}{\rho(x, \mu)} \right)^{-k_0} \\ \leq \rho(y, \mu) \leq C \rho(x, \mu) \left(1 + \frac{|x - y|}{\rho(x, \mu)} \right)^{k_0/(k_0+1)}, \end{aligned} \quad (14)$$

with $k_0 = C_2/\delta > 0$ and $C_2 = \log_2(C_1 + 2^{d-2})$.

Next, we recall some results about covering $\mathbb{R}^d$ by critical balls, which can be found in (see [14], Lemma 2.3).

Lemma 5. *There exists a sequence of points $\{x_k\}_{k=1}^\infty$ in $\mathbb{R}^d$, so that the family of critical balls $Q_k = B(x_k, \rho(x_k, \mu))$, $k \geq 1$, satisfies*

- (i) $\bigcup_k Q_k = \mathbb{R}^d$
- (ii) *There exists N such that for every $k \in \mathbb{N}$, $\text{card} \{j : 4Q_j \cap 4Q_k \neq \emptyset\} \leq N$*

The kernel $k_t(x, y)$ of the semigroup $e^{-t\mathcal{L}}$ satisfies following upper bound in ([15], Theorem 1.1).

Lemma 6. *Let $\mu \neq 0$ be a nonnegative Radon measure in $\mathbb{R}^d$, $d \geq 3$. Assume that μ satisfies the conditions (2) and (3). There exist constants μ, C , the constant k_0 in (14), the heat kernel $k_t(x, y)$ of $e^{-t\mathcal{L}}$ satisfies*

$$k_t(x, y) \leq Ct^{-d/2} e^{-c(|x-y|^2/t)} e^{-c(1+(\sqrt{t}/\rho(x, \mu)))^{1/(k_0+1)}}. \quad (15)$$

Using the inequality (15) and repeating the proof process in ([2], Proposition 4), we can obtain the estimates for the integral kernel $Q_t(x, y) = t(\partial k_t(x, y)/\partial t)$.

Lemma 7. *There exist constants $c, \sigma > 0$ such that for k_0 in (14) and any N , there is a constant C so that*

- (i) $|Q_t(x, y)| \leq Ct^{-d/2} e^{-c(|x-y|^2/t)} e^{-c(1+(\sqrt{t}/\rho(x, \mu)))^{1/(k_0+1)}}$
- (ii) $|Q_t(x+h, y) - Q_t(x, y)| \leq C(|h|/\sqrt{t})^\sigma t^{-d/2} e^{-c(|x-y|^2/t)} e^{-c(1+(\sqrt{t}/\rho(x, \mu)))^{1/(k_0+1)}}$, for all $|h| \leq \sqrt{t}$

$$(iii) \left| \int_{\mathbb{R}^d} Q_t(x, y) dy \right| \leq C((\sqrt{t}/\rho(x, \mu))^\sigma / (1 + (\sqrt{t}/\rho(x, \mu)))^N)$$

Finally, following ([16], Proposition 3), we recall some basic properties about the norm of BMO_θ .

Lemma 8. *For $\theta > 0$, let $f \in BMO_\theta$, $B = B(x, r)$, and $m \geq 1$, then*

$$(i) \left(\frac{1}{|B|} \int_B |f(y) - f_B|^m dy \right)^{1/m} \leq C \|f\|_{BMO_\theta} \left(1 + \frac{r}{\rho(x, \mu)} \right)^{\theta'}, \quad (16)$$

and for all $k \in \mathbb{N}$

$$(ii) \left(\frac{1}{|2^k B|} \int_{2^k B} |f(y) - f_B|^m dy \right)^{1/m} \leq C \|f\|_{BMO_\theta} k \left(1 + \frac{2^k r}{\rho(x, \mu)} \right)^{\theta'}, \quad (17)$$

where $\theta' = (k_0 + 1)\theta$ and k_0 the constant appearing in (14).

3. Proof of Theorem 3

Before we prove Theorem 3, we first recall some basic facts about the nonnegative Radon measure μ .

Lemma 9.

- (i) *For all balls $B(x, r)$ in $\mathbb{R}^d$, let C_1 as in (3), then*

$$\int_{B(x, 2r)} d\mu \leq C_1 \left\{ \int_{B(x, r)} d\mu + r^{d-2} \right\}. \quad (18)$$

- (ii) *For $0 < r < R < \infty$, let δ and C_0 as in (3), then*

$$\frac{1}{r^{d-2}} \int_{B(x, r)} d\mu \leq C_0 \left(\frac{r}{R} \right)^\delta \frac{1}{R^{d-2}} \int_{B(x, R)} d\mu. \quad (19)$$

- (iii) *If $r \sim \rho(x, \mu)$, then*

$$\int_{B(x, r)} d\mu \sim r^{d-2}. \quad (20)$$

Proof. The proof of (i) and (ii) can be obtained directly from (2) and (3), respectively. From Lemma 4(i), the proof of (iii) can be obtained.

Next, we give the following result, which is similar to the proof of ([17], Corollary 1).

Lemma 10. *A function f belong to $BMO_{\theta, \mathcal{L}}$ with $\theta > 0$ if and only if*

$$\frac{1}{|B|} \int_B |f(y) - f_B| dy \leq C, \quad (21)$$

for every ball $B = B(x, r)$ with $x \in \mathbb{R}^d$ and $r < \rho(x, \mu)$ and

$$\frac{1}{|B(x, \rho(x, \mu))|} \int_{B(x, \rho(x, \mu))} |f(y)| dy \leq C, \quad (22)$$

for all $x \in \mathbb{R}^d$.

Proof. From the Definition 1, it is easy to see that if $f \in BM O_{\theta, \mathcal{L}}$, then f satisfies (21) and (22), furthermore, to prove $f \in BMO_{\theta, \mathcal{L}}$, we need (22) and

$$\frac{1}{|B|} \int_B |f(y)| dy \leq C \left(1 + \frac{r}{\rho(x, \mu)} \right)^\theta, \quad (23)$$

for every ball $B = B(x, r)$ with $x \in \mathbb{R}^d$ and $r \geq \rho(x, \mu)$.

Obviously, we just have to prove that if f satisfies (22) then f satisfies (23). Let $\{x_k\}_{k=1}^\infty$ be a sequence as in Lemma 5, it follows from (22) that

$$\frac{1}{|B_k|} \int_{B_k} |f(y)| dy \leq C, \quad (24)$$

where $B_k = B(x_k, \rho(x_k, \mu))$ for all $k \geq 1$. According to Lemma 5, we have the set $N = \{k : B \cap B_k \neq \emptyset\}$ is finite.

For every $k \in N$, and $z \in B \cap B_k$, by Lemma 4(iii) we get

$$\begin{aligned} \rho(x_k) &\leq C\rho(z) \left(1 + \frac{|x_k - z|}{\rho(x_k, \mu)} \right)^{k_0} \leq C2^{k_0} \rho(z) \\ &\leq C2^{k_0} \rho(x, \mu) \left(1 + \frac{|x - z|}{\rho(x, \mu)} \right)^{k_0/(k_0+1)} \\ &\leq C2^{k_0} \rho(x, \mu) \left(1 + \frac{r}{\rho(x, \mu)} \right) \leq C2^{k_0+1} r, \end{aligned} \quad (25)$$

then $B_k \subseteq CB$ for every $k \in N$. Thus

$$\int_B |f(y)| dy \leq \sum_{k \in N} \int_{B \cap B_k} |f(y)| dy \leq C \sum_{k \in N} |B_k| \leq C|B|. \quad (26)$$

That is for $r \geq \rho(x, \mu)$

$$\frac{1}{|B|} \int_B |f(y)| dy \leq C \leq C \left(1 + \frac{r}{\rho(x, \mu)} \right)^\theta. \quad (27)$$

Now, we turn to estimate the boundedness of the operator s from $BMO_{\theta, \mathcal{L}}$ to $BMO_{\theta, \mathcal{L}}$. According to Lemma 10, we only need to check that

$$(1) \frac{1}{|B(x_0, \rho(x_0, \mu))|} \int_{B(x_0, \rho(x_0, \mu))} |s(f)(x)| dx \leq C \|f\|_{BMO_{\theta, \mathcal{L}}}$$

$$(2) \frac{1}{|B|} \int_B |s(f) - (s(f))_B| \leq C \|f\|_{BMO_{\theta, \mathcal{L}}}, \text{ for } B = B(x_0, r) \text{ and } r < \rho(x_0, \mu)$$

Proof of Theorem 3. Without loss of generality, fix $f \in BM O_{\theta, \mathcal{L}}$ with $\|f\|_{BMO_{\theta, \mathcal{L}}} = 1$, we first to show that

$$\frac{1}{|B(x_0, \rho(x_0, \mu))|} \int_{B(x_0, \rho(x_0, \mu))} |s(f)(x)| dx \leq C. \quad (28)$$

We split

$$\begin{aligned} |s(f)(x)|^2 &\leq \int_0^{\rho(x_0, \mu)^2} |t \mathcal{L} e^{-t\mathcal{L}} f(x)|^2 \frac{dt}{t} \\ &\quad + \int_{\rho(x_0, \mu)^2}^\infty |t \mathcal{L} e^{-t\mathcal{L}} f(x)|^2 \frac{dt}{t} \\ &=: |s_1(f)(x)|^2 + |s_2(f)(x)|^2. \end{aligned} \quad (29)$$

For $|s_1(f)|^2$, we set

$$\begin{aligned} f &= (f - f_{B(x_0, 2\rho(x_0, \mu))}) \mathcal{X}_{B(x_0, 2\rho(x_0, \mu))} \\ &\quad + (f - f_{B(x_0, 2\rho(x_0, \mu))}) \mathcal{X}_{(B(x_0, 2\rho(x_0, \mu)))^c} \\ &\quad + f_{B(x_0, 2\rho(x_0, \mu))} =: \sum_{i=1}^3 f_i. \end{aligned} \quad (30)$$

Using the self-adjointness of $t \mathcal{L} e^{-t\mathcal{L}}$, we get

$$\|s(f)\|_{\mathcal{L}^2} \sim \|f\|_{\mathcal{L}^2}. \quad (31)$$

Then, combining Hölder's inequality, (31), Lemma 8(i), and Lemma 4(ii) with $\rho(x, \mu) \sim \rho(x_0, \mu)$ for $x \in B(x_0, \rho(x_0, \mu))$, we have

$$\begin{aligned} &\left[\frac{1}{|B(x_0, \rho(x_0, \mu))|} \int_{B(x_0, \rho(x_0, \mu))} |s_1(f_1)(x)| dx \right]^2 \\ &\leq C \frac{1}{|B(x_0, \rho(x_0, \mu))|} \int_{B(x_0, \rho(x_0, \mu))} |s_1(f_1)(x)|^2 dx \\ &\leq C \frac{1}{|B(x_0, \rho(x_0, \mu))|} \|f_1\|_{L^2}^2 \\ &= C \frac{1}{|B(x_0, \rho(x_0, \mu))|} \int_{B(x_0, 2\rho(x_0, \mu))} |f(x) - f_{B(x_0, 2\rho(x_0, \mu))}|^2 dx \\ &\leq C \|f\|_{BMO_\theta}^2 \left(1 + \frac{2\rho(x_0, \mu)}{\rho(x_0, \mu)} \right)^{2\theta'} \leq C, \end{aligned} \quad (32)$$

where the last inequality follows from Remark 2.

Next, for $t < \rho(x_0, \mu)^2$ and $x \in B(x_0, \rho(x_0, \mu))$, it follows from Lemma 7(i) and Lemma 8(ii) that

$$\begin{aligned}
 |t\mathcal{L}e^{-t\mathcal{L}}f_2(x)| &\leq C \int_{\mathbb{R}^d} |f_2(y)| |Q_t(x, y)| dy \leq C \int_{(B(x_0, 2\rho(x_0, \mu)))^c} \\
 &\quad \cdot |f(y) - f_{B(x_0, 2\rho(x_0, \mu))}| t^{-d/2} e^{-c'|x-y|^2/t} e^{-c''|x-y|^2/t} dy \\
 &\leq C \int_{(B(x_0, 2\rho(x_0, \mu)))^c} |f(y) - f_{B(x_0, 2\rho(x_0, \mu))}| t^{-d/2} \\
 &\quad \cdot \left(\frac{|x-y|}{\sqrt{t}}\right)^{-(d+1)} e^{-c''|x-y|^2/t} dy \\
 &\leq C \int_{(B(x_0, 2\rho(x_0, \mu)))^c} |f(y) - f_{B(x_0, 2\rho(x_0, \mu))}| \\
 &\quad \sqrt{t}|x-y|^{-(d+1)} e^{-c''|x-y|^2/t} dy \\
 &\leq C \sum_{j=1} \int_{B(x_0, 2^{j+1}\rho(x_0, \mu)) \setminus B(x_0, 2^j\rho(x_0, \mu))} \frac{\sqrt{t}}{(2^j\rho(x_0, \mu))^{d+1}} \\
 &\quad \cdot |f(y) - f_{B(x_0, 2\rho(x_0, \mu))}| e^{-c''|x_0-y|^2/t} dy \\
 &\leq C \frac{\sqrt{t}}{\rho(x_0, \mu)} \sum_{j=1} 2^{-d-2j} e^{-c''2^{2j}} \frac{1}{(2^{j+1}\rho(x_0, \mu))^d} \\
 &\quad \cdot \int_{B(x_0, 2^{j+1}\rho(x_0, \mu))} |f(y) - f_{B(x_0, 2\rho(x_0, \mu))}| dy \\
 &\leq C \frac{\sqrt{t}}{\rho(x_0, \mu)} \sum_{j=1} 2^{-d-2j} e^{-c''2^{2j}} j 2^{j\theta'} \|f\|_{\text{BMO}_{\theta'}} \leq C \frac{\sqrt{t}}{\rho(x_0, \mu)}. \tag{33}
 \end{aligned}$$

Then, we obtain

$$\begin{aligned}
 &\left[\frac{1}{|B(x_0, \rho(x_0, \mu))|} \int_{B(x_0, \rho(x_0, \mu))} |s_1(f_2)(x)| dx \right]^2 \\
 &\leq C \int_0^{\rho(x_0, \mu)^2} \left(\frac{\sqrt{t}}{\rho(x_0, \mu)} \right)^2 \frac{dt}{t} \leq C. \tag{34}
 \end{aligned}$$

Now, we turn to estimate the term for $s_1(f_3)$. Using $\rho(x, \mu) \sim \rho(x_0, \mu)$ for $x \in B(x_0, \rho(x_0, \mu))$ and Lemma 7(iii), we have

$$\begin{aligned}
 &\left[\frac{1}{|B(x_0, \rho(x_0, \mu))|} \int_{B(x_0, \rho(x_0, \mu))} |s_1(f_3)(x)| dx \right]^2 \\
 &\leq \frac{1}{|B(x_0, \rho(x_0, \mu))|} \int_0^{\rho(x_0, \mu)^2} \int_{B(x_0, \rho(x_0, \mu))} |t\mathcal{L}e^{-t\mathcal{L}}f_{B(x_0, 2\rho(x_0, \mu))}|^2 \frac{dx dt}{t} \\
 &\leq C \frac{|f_{B(x_0, 2\rho(x_0, \mu))}|^2}{|B(x_0, \rho(x_0, \mu))|} \int_0^{\rho(x_0, \mu)^2} \int_{B(x_0, \rho(x_0, \mu))} \left(\frac{\sqrt{t}}{\rho(x_0, \mu)} \right)^{2\sigma} \frac{dx dt}{t} \\
 &\leq C |f_{B(x_0, 2\rho(x_0, \mu))}|^2 \leq C \left(1 + \frac{2\rho(x_0, \mu)}{\rho(x_0)} \right)^{2\theta} \leq C. \tag{35}
 \end{aligned}$$

For $s_2(f)$, one can easily check using Lemma 7(i), Minkowski's inequality and $\rho(x, \mu) \sim \rho(x_0, \mu)$ for $x \in B(x_0, \rho(x_0, \mu))$ that

$$\begin{aligned}
 |s_2(f)(x)| &\leq C \left\{ \int_{\rho(x_0, \mu)^2}^{\infty} \left[\int_{\mathbb{R}^d} |Q_t(x, y)| |f(y)| dy \right]^2 \frac{dt}{t} \right\}^{1/2} \\
 &\leq C \left\{ \int_{\rho(x_0, \mu)^2}^{\infty} \left[\int_{\mathbb{R}^d} e^{-c\left(1+\frac{\sqrt{t}}{\rho(x_0, \mu)}\right)^{1/(k_0+1)}} t^{-\frac{d}{2}} e^{-\frac{c|x-y|^2}{t}} |f(y)| dy \right]^2 \frac{dt}{t} \right\}^{1/2} \\
 &\leq C \left\{ \int_{\rho(x_0, \mu)^2}^{\infty} \left[e^{-c\left(1+\frac{\sqrt{t}}{\rho(x_0, \mu)}\right)^{1/(k_0+1)}} \frac{1}{t^{d/2}} \int_{|x-y| \leq \sqrt{t}} |f(y)| dy + e^{-c\left(1+\frac{\sqrt{t}}{\rho(x_0, \mu)}\right)^{1/(k_0+1)}} \right. \right. \\
 &\quad \cdot \left. \sum_{j=1} e^{-2^{2j}} \frac{1}{t^{d/2}} \int_{|x-y| \sim 2^j\sqrt{t}} |f(y)| dy \right]^2 \frac{dt}{t} \Bigg\}^{1/2} \\
 &\leq C \left[\left(\int_{\rho(x_0, \mu)^2}^{\infty} e^{-c\left(1+\frac{\sqrt{t}}{\rho(x_0, \mu)}\right)^{2/(k_0+1)}} \left(1 + \frac{\sqrt{t}}{\rho(x_0, \mu)} \right)^{2\theta} \frac{dt}{t} \right)^{1/2} \right. \\
 &\quad \left. + \sum_{j=1} e^{-2^{2j}} 2^{jd} 2^{(j+1)\theta} \left(\int_{\rho(x_0, \mu)^2}^{\infty} e^{-c\left(1+\frac{\sqrt{t}}{\rho(x_0, \mu)}\right)^{2/(k_0+1)}} \left(\frac{\sqrt{t}}{\rho(x_0, \mu)} \right)^{2\theta} \frac{dt}{t} \right)^{1/2} \right] \leq C. \tag{36}
 \end{aligned}$$

Next, we deal with the following case

$$\frac{1}{|B|} \int_B |s(f) - (s(f))_B| \leq C, \quad \text{for } B = B(x_0, r) \quad \text{and} \quad r < \rho(x_0, \mu). \tag{37}$$

In this section, inspired by the methods in [2], we decompose f as follows

$$f = (f - f_{B^*})\mathcal{X}_{B^*} + (f - f_{B^*})\mathcal{X}_{(B^*)^c} + f_{B^*} =: \sum_{i=1}^3 \tilde{f}_i(x). \tag{38}$$

Set

$$s_B = \left\{ \int_0^{\rho(x_0, \mu)^2} |t\Delta e^{-t\Delta} \tilde{f}_2(x_0)|^2 \frac{dt}{t} \right\}^{1/2}. \tag{39}$$

It is easy to see that s_B is a finite constant and

$$\begin{aligned}
 |s(f)(x) - s_B| &\leq \left(\int_{\rho(x_0, \mu)^2}^{\infty} |t\mathcal{L}e^{-t\mathcal{L}}f(x)|^2 \frac{dt}{t} \right)^{1/2} \\
 &\quad + \left(\int_0^{\rho(x_0, \mu)^2} |t\mathcal{L}e^{-t\mathcal{L}}f(x) - t\Delta e^{-t\Delta}f(x)|^2 \frac{dt}{t} \right)^{1/2} \\
 &\quad + \left(\int_0^{\rho(x_0, \mu)^2} |t\Delta e^{-t\Delta}f(x) - t\Delta e^{-t\Delta}\tilde{f}_2(x_0)|^2 \frac{dt}{t} \right)^{1/2} \\
 &=: S_1(f)(x) + S_2(f)(x) + S_3(f)(x). \tag{40}
 \end{aligned}$$

To complete the proof, it suffices to show that

$$\frac{1}{|B|} \int_B S_i(f)(x) dx \leq C \quad \text{for } i = 1, 2, 3. \tag{41}$$

Note that $\rho(x, \mu) \sim \rho(x_0, \mu)$ for any $x \in B(x_0, r)$, by (36), we get for $x \in B(x_0, r)$

$$|S_1(f)(x)| \leq \left\{ \int_{\rho(x_0, \mu)^2}^{\infty} \left(\int_{\mathbb{R}^d} |Q_t(x, y)| |f(y)| dy \right)^2 \frac{dt}{t} \right\}^{1/2} \leq C. \quad (42)$$

It remains to estimate $S_2(f)$. Let $G_t(x, y) = t\partial_t k_t(x, y) - t\partial_t h_t(x, y)$. We claim

$$|G_t(x, y)| \leq Cs^{-d/2} e^{-c|x-y|^2/t} \left(\frac{\sqrt{t}}{\rho(x, \mu)} \right)^\sigma, \quad (43)$$

for $\sqrt{t} < \rho(x, \mu)$. It follows from the perturbation formula (see ([18], Chapter 9, (2.3))) that

$$\begin{aligned} G_t(x, y) &= \int_{\mathbb{R}^d} t h_{t/2}(w, x) k_{t/2}(w, y) d\mu(w) \\ &\quad + \int_{t/2}^t \int_{\mathbb{R}^d} t h_t(w, x) \partial_t k_{t-s}(w, y) d\mu(w) ds \\ &\quad + \int_0^{t/2} \int_{\mathbb{R}^d} t \partial_t h_{t-s}(w, x) k_t(w, y) d\mu(w) ds \\ &=: I_1 + I_2 + I_3. \end{aligned} \quad (44)$$

By Lemma 9, we have for $\sqrt{t} < \rho(x, \mu)$,

$$\begin{aligned} I_1 &= \int_{|x-w| < |x-y|/2} t h_{t/2}(w, x) k_{t/2}(w, y) d\mu(w) \\ &\quad + \int_{|x-w| \geq |x-y|/2} t h_{t/2}(w, x) k_{t/2}(w, y) d\mu(w) \\ &\leq Ct^{-\frac{d}{2}+1} e^{-c\frac{|x-y|^2}{t}} \int_{|x-w| < |x-y|/2} t^{-\frac{d}{2}} e^{-c\frac{|x-w|^2}{t}} d\mu(w) \\ &\quad + Ct^{-\frac{d}{2}+1} \int_{|x-w| \geq |x-y|/2} t^{-\frac{d}{2}} e^{-c\frac{|x-w|^2 + |x-y|^2}{t}} d\mu(w) \\ &\leq Ct^{-\frac{d}{2}} e^{-c\frac{|x-y|^2}{t}} \int_{\mathbb{R}^d} t^{-\frac{d-2}{2}} e^{-c\frac{|x-w|^2}{t}} d\mu(w) \\ &\leq Ct^{-\frac{d}{2}} e^{-c\frac{|x-y|^2}{t}} \left(\frac{1}{(\sqrt{t})^{d-2}} \int_{|x-w| < \sqrt{t}} d\mu(w) \right. \\ &\quad \left. + \sum_{k=0}^{\infty} 2^{-2k} \frac{1}{(\sqrt{t})^{d-2}} \int_{2^k \sqrt{t} < |x-w| < 2^{k+1} \sqrt{t}} d\mu(w) \right) \\ &\leq Ct^{-\frac{d}{2}} e^{-c\frac{|x-y|^2}{t}} \left(\frac{\sqrt{t}}{\rho(x, \mu)} \right)^\sigma. \end{aligned} \quad (45)$$

By the same argument as estimating I_1 , combining Lemma 7, we also obtain

$$\begin{aligned} I_2 &= \int_{t/2}^t \int_{\mathbb{R}^d} \frac{t}{t-s} h_t(w, x) Q_{t-s}(w, y) d\mu(w) ds \\ &\leq \int_{t/2}^t \int_{|w-y| < |x-y|/2} \frac{t}{t-s} h_t(w, x) Q_{t-s}(w, y) d\mu(w) ds \\ &\quad + \int_{t/2}^t \int_{|w-y| \geq |x-y|/2} \frac{t}{t-s} h_t(w, x) Q_{t-s}(w, y) d\mu(w) ds \\ &\leq Ct^{-\frac{d}{2}} e^{-c\frac{|x-y|^2}{t}} \int_{t/2}^t \int_{|w-y| < |x-y|/2} s^{-\frac{d}{2}} e^{-c\frac{|y-w|^2}{s}} d\mu(w) ds \\ &\quad + Ct^{-\frac{d}{2}} \int_{t/2}^t \int_{|w-y| \geq |x-y|/2} s^{-\frac{d}{2}} e^{-c\frac{|x-y|^2 + |y-w|^2}{s}} d\mu(w) ds \\ &\leq Ct^{-\frac{d}{2}} e^{-c\frac{|x-y|^2}{t}} \int_{t/2}^t \frac{1}{s} \left(\frac{\sqrt{s}}{\rho(x, \mu)} \right)^\sigma ds \\ &= Ct^{-\frac{d}{2}} e^{-c\frac{|x-y|^2}{t}} \left(\frac{\sqrt{t}}{\rho(x, \mu)} \right)^\sigma. \end{aligned} \quad (46)$$

The same estimate as I_2 , we have

$$I_3 \leq Ct^{-d/2} e^{-c|x-y|^2/t} \left(\frac{\sqrt{t}}{\rho(x, \mu)} \right)^\sigma. \quad (47)$$

The proof of (43) is finished. For $x \in B(x_0, r)$, $\rho(x_0, \mu) \sim \rho(x, \mu)$, it follows from Definition 1 and Lemma 8(ii) that

$$\begin{aligned} |S_2(f)(x)| &\leq \left\{ \int_0^{\rho(x_0, \mu)^2} \left(\int_{\mathbb{R}^d} |G_t(x, y)| |f(y)| dy \right)^2 \frac{dt}{t} \right\}^{1/2} \\ &\leq C \left\{ \int_0^{\rho(x_0, \mu)^2} \left(\int_{\mathbb{R}^d} t^{-d/2} e^{-c|x-y|^2/t} \left(\frac{\sqrt{t}}{\rho(x, \mu)} \right)^\sigma |f(y)| dy \right)^2 \frac{dt}{t} \right\}^{1/2} \\ &\leq C \left\{ \int_0^{\rho(x_0, \mu)^2} \left(\int_{B(x, \rho(x, \mu))} \rho(x, \mu)^{-\sigma} |x-y|^{-(d-\frac{\sigma}{2})} |f(y)| dy \right)^2 \frac{dt}{t^{1-\sigma/2}} \right. \\ &\quad \left. + \int_0^{\rho(x_0, \mu)^2} \left(\int_{B(x, \rho(x, \mu))^c} e^{-c\eta|x-y|^2/t} \cdot |x-y|^{-(d+1)} |f(y)| dy \right)^2 dt \right\}^{1/2} \\ &\leq C \sum_{j=0}^{\infty} \rho(x, \mu)^{-d} 2^{(j+1)(d-\sigma/2)} \int_{B(x, 2^{-j}\rho(x, \mu)) \setminus B(x, 2^{-(j+1)}\rho(x, \mu))} |f(y)| dy \\ &\quad + C \sum_{j=0}^{\infty} \rho(x, \mu)^{-d} 2^{-j(d+1)} e^{-c\eta 2^{2j}} \int_{B(x, 2^{j+1}\rho(x, \mu)) \setminus B(x, 2^j\rho(x, \mu))} |f(y)| dy \\ &\leq C \sum_{j=0}^{\infty} 2^d 2^{-(j+1)\delta/2} \frac{1}{(2^{-j}\rho(x, \mu))^d} \int_{B(x, 2^{-j}\rho(x, \mu))} |f(y)| dy \\ &\quad + C \sum_{j=0}^{\infty} (2^{j+1})^d 2^{-j(d+1)} e^{-c\eta 2^{2j}} \frac{1}{(2^{j+1}\rho(x, \mu))^d} \int_{B(x, 2^{j+1}\rho(x, \mu))} |f(y)| dy \\ &\leq C \sum_{j=0}^{\infty} 2^d 2^{-(j+1)} \left(\frac{1}{B(x, 2^{-j}\rho(x, \mu))} \int_{B(x, 2^{-j}\rho(x, \mu))} |f(y) - f_{B(x, 2^{-j}\rho(x, \mu))}| dy \right. \\ &\quad \left. + \sum_{i=0}^{j-1} |f_{B(x, 2^{-j+i}\rho(x, \mu))} - f_{B(x, 2^{-j+i+1}\rho(x, \mu))}| + |f_{B(x, \rho(x, \mu))}| \right) \\ &\quad + C \sum_{j=0}^{\infty} (2^{j+1})^d 2^{-j(d+1)} e^{-c\eta 2^{2j}} \left(1 + \frac{2^{j+1}\rho(x, \mu)}{\rho(x, \mu)} \right)^\theta \\ &\leq C \sum_{j=0}^{\infty} 2^d 2^{-(j+1)\delta/2} (j2^{\theta'} + 2^\theta) + C \sum_{j=0}^{\infty} 2^d 2^{-j} 2^{(j+1)\theta} e^{-c\eta 2^{2j}} \leq C. \end{aligned} \quad (48)$$

For $S_3(f)$, since the formula $t\Delta e^{-t\Delta}\mathbf{1} = 0$.

$$\begin{aligned} |S_3(f)(x)| &\leq \left(\int_0^{\rho(x_0, \mu)^2} \left| t\Delta e^{-t\Delta} \tilde{f}_1(x) \right|^2 \frac{dt}{t} \right)^{1/2} \\ &\quad + \left(\int_0^{\rho(x_0, \mu)^2} \left| t\Delta e^{-t\Delta} \tilde{f}_2(x) - t\Delta e^{-t\Delta} \tilde{f}_2(x_0) \right|^2 \frac{ds}{s} \right)^{1/2} \\ &= S_{3,1}(x) + S_{3,2}(x). \end{aligned} \quad (49)$$

It is known that $(\int_0^\infty |t\Delta e^{-t\Delta} f|^2 (dt/t))^{1/2} =: s_\Delta f$ is bounded on $L^2(\mathbb{R}^d)$ ([19]). Hence, combining with Lemma 8(i), we obtain

$$\begin{aligned} \frac{1}{|B|} \int_B S_{3,1}(x) dx &\leq \frac{1}{|B|} \int_B |s_\Delta \tilde{f}_1(x)| dx \\ &\leq \left(\frac{1}{|B|} \int_B |s_\Delta \tilde{f}_1(x)|^2 dx \right)^{1/2} \\ &\leq C \left(\frac{1}{|B|} \|\tilde{f}_1\|_{L^2}^2 \right)^{1/2} \leq C \left(\frac{1}{|B|} \int_{B^*} |f - f_{B^*}|^2 \right)^{1/2} \\ &\leq C \|f\|_{\text{BMO}_\theta} \leq C. \end{aligned} \quad (50)$$

We set the kernel of $t\partial_t h_t(x, y) =: \tilde{Q}_t(x, y)$. We recall this kernel satisfies the estimates

$$|\tilde{Q}_t(x, y)| \leq Ct^{-d/2} e^{-ct^{-1}|x-y|^2} \quad (51)$$

and

$$|\tilde{Q}_t(x+h, y) - \tilde{Q}_t(x, y)| \leq C|h|t^{-(d+1)/2} e^{-ct^{-1}|x-y|^2} \text{ if } |h| \leq \frac{|x-y|}{2}. \quad (52)$$

For $x \in B(x_0, r)$ and $r \leq \rho(x_0, \mu)$, choosing k_0 such that $2^{k_0}r < \rho(x_0, \mu) \leq 2^{k_0+1}r$, it follows from Minkowski's inequality and Lemma 8(ii) that

$$\begin{aligned} S_{3,2}(x) &\leq \left\{ \int_0^{\rho(x_0, \mu)^2} \left(\int_{(B^*)^c} |\tilde{Q}_t(x, y) - \tilde{Q}_t(x_0, y)| |\tilde{f}_2(y)| dy \right)^2 \frac{dt}{t} \right\}^{1/2} \\ &\leq C \left\{ \int_0^{\rho(x_0, \mu)^2} \left(\int_{(B^*)^c} |x - x_0| t^{-(d+1)/2} e^{-ct^{-1}|x_0-y|^2} |\tilde{f}_2(y)| dy \right)^2 \frac{dt}{t} \right\}^{1/2} \\ &\leq C \left\{ \int_0^{\rho(x_0, \mu)^2} \left(\sum_{j=1}^\infty \int_{B(x_0, 2^{j+1}r) \setminus B(x_0, 2^j r)} r t^{-(d+1)/2} e^{-c't^{-1}|x_0-y|^2} e^{-c''2^{2j}} \right. \right. \\ &\quad \cdot |\tilde{f}_2(y)| dy \left. \right)^2 \frac{dt}{t} \left. \right\}^{1/2} \\ &\quad + C \left\{ \int_{r^2}^{\rho(x_0, \mu)^2} \left[\left(\sum_{j=1}^{k_0} + \sum_{j=k_0+1}^\infty \right) \int_{B(x_0, 2^{j+1}r) \setminus B(x_0, 2^j r)} r t^{-(d/2+1/4)} e^{-ct^{-1}|x_0-y|^2} \right. \right. \\ &\quad \cdot |\tilde{f}_2(y)| dy \left. \right]^2 \frac{dt}{t^{3/2}} \left. \right\}^{1/2} \end{aligned}$$

$$\begin{aligned} &\leq C \sum_{j=1}^\infty e^{-c''2^{2j}} \left\{ \int_0^{r^2} \left[\int_{B(x_0, 2^{j+1}r) \setminus B(x_0, 2^j r)} r t^{-(d/2+1)} e^{-c't^{-1}|x_0-y|^2} \right. \right. \\ &\quad \cdot |\tilde{f}_2(y)| dy \left. \right]^2 dt \left. \right\}^{1/2} + C \sum_{j=1}^{k_0} \left\{ \int_{r^2}^{\rho(x_0, \mu)^2} \right. \\ &\quad \cdot \left[\int_{B(x_0, 2^{j+1}r) \setminus B(x_0, 2^j r)} r t^{-(d/2+1/4)} e^{-c't^{-1}|x_0-y|^2} |\tilde{f}_2(y)| dy \left. \right]^2 \frac{dt}{t^{3/2}} \left. \right\}^{1/2} \\ &\quad + C \sum_{j=k_0+1}^\infty e^{-c''2^{2j-(k_0+1)}} \left\{ \int_{r^2}^{\rho(x_0, \mu)^2} \right. \\ &\quad \cdot \left[\int_{B(x_0, 2^{j+1}r) \setminus B(x_0, 2^j r)} r t^{-(d/2+1/4)} e^{-c't^{-1}|x_0-y|^2} |\tilde{f}_2(y)| dy \left. \right]^2 \frac{dt}{t^{3/2}} \left. \right\}^{1/2} \\ &\leq C \sum_{j=1}^\infty e^{-c''2^{2j}} \left\{ r^4 \left[\int_{B(x_0, 2^{j+1}r) \setminus B(x_0, 2^j r)} |x_0 - y|^{-(d+2)} |\tilde{f}_2(y)| dy \right]^2 \right. \\ &\quad + C \sum_{j=1}^{k_0} \left\{ r \left[\int_{B(x_0, 2^{j+1}r) \setminus B(x_0, 2^j r)} |x_0 - y|^{-(d+1/2)} |\tilde{f}_2(y)| dy \right]^2 \right. \\ &\quad + C \sum_{j=k_0+1}^\infty e^{-c''2^{2j-(k_0+1)}} \left\{ r \left[\int_{B(x_0, 2^{j+1}r) \setminus B(x_0, 2^j r)} |x_0 - y|^{-(d+1/2)} |\tilde{f}_2(y)| dy \right]^2 \right. \\ &\quad \leq C \left(\sum_{j=1}^\infty e^{-c''2^{2j}} 2^{-2j+d} + \sum_{j=1}^{k_0} 2^{-1/2j+d} + \sum_{j=k_0+1}^\infty e^{-c''2^{2j-(k_0+1)}} 2^{-1/2j+d} \right) \\ &\quad \cdot \frac{1}{|B(x_0, 2^{j+1}r)|} \int_{B(x_0, 2^{j+1}r)} |\tilde{f}_2(y)| dy \\ &\leq C \sum_{j=1}^\infty e^{-c''2^{2j}} 2^{-2j+d} j 2^{j\theta'} + C \sum_{j=1}^{k_0} 2^{-1/2j+d} j 2^{j\theta'} + C \\ &\quad \cdot \sum_{j=k_0+1}^\infty e^{-c''2^{2j-(k_0+1)}} 2^{-1/2j+d} j 2^{j\theta'} \leq C. \end{aligned} \quad (53)$$

Therefore,

$$\frac{1}{|B|} \int_B S_3(f)(x) dx \leq C. \quad (54)$$

By (42), (48), and (54), we finish the proof of (37). Then, combining (28) and (37), we finish the proof of Theorem 3.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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