Research Article

Negative Energy Solutions for a New Fractional $p(x)$ -Kirchhoff Problem without the (AR) Condition

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In this paper, we investigate the following Kirchhoff type problem involving the fractional $p(x)$ -Laplacian operator.
$$\begin{cases} (a - b \int_{\Omega \times \Omega} (|u(x) - u(y)|^{p(x,y)} / p(x,y) |x - y|^{N+sp(x,y)}) dx dy) \mathcal{L}u = \lambda |u|^{q(x)-2} u + f(x, u) & x \in \Omega \\ u = 0 & x \in \partial\Omega, \end{cases}$$
 where Ω is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary, $a \geq b > 0$ are constants, $p(x, y)$ is a function defined on $\bar{\Omega} \times \bar{\Omega}$, $s \in (0, 1)$, and $q(x) > 1$, $\mathcal{L}u$ is the fractional $p(x)$ -Laplacian operator, $N > sp(x, y)$, for any $(x, y) \in \bar{\Omega} \times \bar{\Omega}$, $p(x)^* = (p(x, x)N)/(N - sp(x, x))$, λ is a given positive parameter, and f is a continuous function. By using Ekeland's variational principle and dual fountain theorem, we obtain some new existence and multiplicity of negative energy solutions for the above problem without the Ambrosetti-Rabinowitz ((AR) for short) condition.

1. Introduction and the Main Results

In this article, we investigate the existence and multiplicity of solutions for the Kirchhoff type problem involving the fractional $p(x)$ -Laplacian operator. More precisely, we consider the following problem (P_λ) :

$$(P_\lambda): \begin{cases} \left(a - b \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{p(x,y) |x - y|^{N+sp(x,y)}} dx dy \right) \mathcal{L}u = \lambda |u|^{q(x)-2} u + f(x, u) & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases} \quad (1)$$

where Ω is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary and $a \geq b > 0$ are constants, $s \in (0, 1)$ and $N > sp(x, y)$ for all $(x, y) \in \bar{\Omega} \times \bar{\Omega}$. λ is a given positive parameter, and f is a continuous function. The operator $\mathcal{L}u = (-\Delta)_{p(\cdot)}^s u$ is the fractional $p(x)$ -Laplacian operator, which is defined by

$$\mathcal{L}u := P.V. \int_{\Omega} \frac{|u(x) - u(y)|^{p(x,y)-2} (u(x) - u(y))}{|x - y|^{N+sp(x,y)}} dy, \quad x \in \Omega, \quad (2)$$

where $P.V.$ stands for the Cauchy principal value. Especially, when $p(\cdot) \equiv \text{constant}$, the operator $\mathcal{L}u = (-\Delta)_{p(\cdot)}^s u$ reduced to the usual fractional p -Laplace operator. In the past few decades, many people studied the fractional p -Laplace operator problem, and we refer the readers to [1–9]. We point out that Behboudi et al. studied the existence of mountain pass solution for the nonlocal fractional (p, q) -Laplacian problem in [10] by using the variational method. The more general problem is the following $p(x)$ -Laplacian equation

$$\begin{cases} (-\Delta)_{p(x)}^s u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (3)$$

The study for this problem and the related knowledge on fractional Sobolev spaces with variable exponent please refer to [11, 12] and the references therein.

In the recent years, the fractional calculus and the related problems are studied widely, see for example, the literature [13, 14]. The elementary knowledge of fractional derivative and integral can be found in [15, 16]. Especially, the Caputo derivative is discussed in [17]. These references will be helpful for our research later.

The following Kirchhoff equation was introduced by Kirchhoff in [18]:

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{p_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial t} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0, \quad (4)$$

where ρ, p_0, h, E, L are constants that have physical meaning. We call the problem (P_λ) a new problem of Kirchhoff type because it contains the new Kirchhoff term:

$$a - b \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{p(x,y)|x - y|^{N+sp(x,y)}} dx dy, \quad (5)$$

which makes the problem P_λ more interesting, meaningful, and difficult. Up to now, the study on the Kirchhoff type problems involving $p(x)$ -Laplacian operator and the fractional $p(x)$ -Laplacian operator is very active, see [19–23]. Many results concerning the existence and multiplicity of solutions have been appeared. Especially, [24] proved the existence of two weak solutions by using the variational methods in Orlicz-Sobolev spaces, [25] studied a class of Kirchhoff nonlocal fractional equations, and obtained the existence of three solutions. We also mention that [26] discussed a class of p -Kirchhoff equations via the fountain theorem and dual fountain theorem, and [27] studied the existence of nonnegative solutions for a Kirchhoff type problem driven by a nonlocal integrodifferential operator.

It is well known that the (AR) condition plays an important role in verifying the Palais-Smale condition. However, there are a lot of functions not satisfying this condition. Hence, many people pay attention to find the new reasonable conditions instead of the (AR) condition, see [28–31] and the references therein. Motivated by these work, we use Ekeland's variational principle and dual fountain theorem to study the existence and multiplicity of negative energy solutions for a new fractional $p(x)$ -Kirchhoff problem without the (AR) condition. Our results generalize the related work in two ways. Firstly, we deal with the problem (P_λ) in the fractional framework and [32] consider only the integer framework. Secondly, compare to [33], we add a new Kirchhoff function and consider the case of variable exponents.

Throughout this paper, the nonlinearity $f(x, t): \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathe'odory function satisfying

(f1): There exists a positive constant C such that

$$|f(x, t)| \leq C \left(1 + |t|^{r(x)-1} \right), \text{ for all } (x, t) \in \Omega \times \mathbb{R}, \quad (6)$$

where $1 < r(x) < p(x)^* = (p(x, x)N)/(N - sp(x, x))$;

(f2): $\lim_{|t| \rightarrow 0} (f(x, t)t/|t|^{p^+}) = 0$, for $x \in \Omega$ uniformly;

(f3): $\lim_{|t| \rightarrow \infty} F(x, t)/|t|^{2p^+} = +\infty$, $x \in \Omega$, where $F(x, t) = \int_0^t f(x, s)ds \geq 0$;

(f4): There exists positive constants θ_0 such that $F(x, t) \leq (1/p^+)f(x, t)t + \theta_0|t|^{2p^-}$, for all $(x, t) \in \Omega \times \mathbb{R}$;

(f5): $f(x, -t) = -f(x, t)$, for all $(x, t) \in \Omega \times \mathbb{R}$.

Definition 1. We say $u \in X_0$ is a weak solution of problem (P_λ) , if

$$\begin{aligned} & \left(a - b \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{p(x,y)|x - y|^{N+sp(x,y)}} dx dy \right) \\ & \times \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)-2} (u(x) - u(y)) (\varphi(x) - \varphi(y))}{|x - y|^{N+sp(x,y)}} dx dy \\ & = \lambda \int_{\Omega} |u|^{q(x)-2} u \varphi dx + \int_{\Omega} f(x, u) \varphi dx, \end{aligned} \quad (7)$$

for any $\varphi \in X_0$, where X_0 will be introduced in Section 2.

It is well known that a weak solution for problem (P_λ) is a critical point of the following energy functional I defined on X_0 by

$$\begin{aligned} I(u) := & a \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{p(x,y)|x - y|^{N+sp(x,y)}} dx dy \\ & - \frac{b}{2} \left(\int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{p(x,y)|x - y|^{N+sp(x,y)}} dx dy \right)^2 \\ & - \lambda \int_{\Omega} \frac{1}{q(x)} |u|^{q(x)} dx - \int_{\Omega} F(x, u) dx, \end{aligned} \quad (8)$$

for all $u \in X_0$. Moreover, we have

$$\begin{aligned} \langle I'(u), \varphi \rangle := & \left(a - b \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{p(x,y)|x - y|^{N+sp(x,y)}} dx dy \right) \\ & \times \int_{\Omega} \frac{|u(x) - u(y)|^{p(x,y)-2} (u(x) - u(y)) (\varphi(x) - \varphi(y))}{|x - y|^{N+sp(x,y)}} dx \\ & - \lambda \int_{\Omega} |u|^{q(x)-2} u \varphi dx - \int_{\Omega} f(x, u) \varphi dx, \end{aligned} \quad (9)$$

for any $u, \varphi \in X_0$. Under our assumptions, I is well defined in X_0 and $I \in C^1(X_0, \mathbb{R})$. We say that a weak solution u for problem (P_λ) is a negative energy solution if the energy $I(u) < 0$.

In order to reduce our statements, we define the function space

$$C_+(\bar{\Omega}) := \{H(x): H(x) \in C(\bar{\Omega}, \mathbb{R}), 1 < H(x) < \infty \text{ for all } x \in \bar{\Omega}\}, \quad (10)$$

and q^-, q^+, p^-, p^+, r^- will be introduced in Section 2. The main results of this paper are as follows.

Theorem 2. Assume that the function f satisfies (f1)–(f4) and

$$1 < q^- < q^+ < p^- < p^+ < \sqrt{2}p^- < 2p^- < r^- < 2p^+ < p^*(x). \quad (11)$$

Then, there exists a $\lambda^* > 0$ such that for any $0 < \lambda < \lambda^*$, the problem (P_λ) has at least one solution u_0 with negative energy.

Theorem 3. Assume that the function f satisfies (f1)–(f5) and

$$1 < q^- < q^+ < p^- < p^+ < \sqrt{2}p^- < 2p^- < r^- < 2p^+ < p^*(x). \quad (12)$$

Then, for any $\lambda > 0$, problem (P_λ) has infinitely many solutions $\{u_n\}$ in X_0 with negative energy converging to 0.

The rest of this paper is organized as follows. In Section 2, some basic properties of the variable exponent fractional Sobolev spaces and Lebesgue spaces are given. In Section 3, it is proved that the functional satisfies the Cerami compactness condition in certain energy levels. In Section 4, by using Ekeland's variational principle, we give the proof of Theorem 2. Finally, in Section 5, we prove Theorem 3 by using the dual fountain theorem.

Throughout this paper, for simplicity, we use letters C_i ($i = 1, 2, \dots, N$) to denote positive constants in different cases, and we will specify them whenever it is necessary.

2. Preliminary Results

In this section, we recall some preliminary results of generalized Lebesgue spaces $L^{q(x)}(\Omega)$ with variable exponent and generalized fractional Sobolev spaces $W^{s,h(x),p(x,y)}(\Omega)$ which will be used later. The readers can consult [34–36] and the references therein for more details. Let $s \in (0, 1)$ and Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary, and $q(x)$, $h(x)$, $p(x, y)$ be continuous functions satisfying

$$1 < q^- := \min_{x \in \Omega} q(x) \leq q(x) \leq q^+ := \max_{x \in \Omega} q(x) < \infty, \quad (13)$$

$$1 < h^- := \min_{x \in \Omega} h(x) \leq h(x) \leq h^+ := \max_{x \in \Omega} h(x) < \infty. \quad (14)$$

We assume that $p(x, y)$ is symmetric; that is, $p(x, y) = p(y, x)$ for all $(x, y) \in \Omega \times \Omega$, such that

$$1 < p^- := \min_{(x,y) \in \Omega \times \Omega} p(x, y) \leq p(x, y) \leq p^+ := \max_{(x,y) \in \Omega \times \Omega} p(x, y) < \infty. \quad (15)$$

For any $q(x) \in C_+(\bar{\Omega})$, we introduce the variable exponent Lebesgue space as

$$L^{q(x)}(\Omega) := \left\{ u : u \text{ is measurable, } \int_{\Omega} |u|^{q(x)} dx < \infty \right\} \quad (16)$$

endowed with the so-called Luxemburg norm

$$|u|_{q(x)} := \inf \left\{ \mu > 0 : \int_{\Omega} \left| \frac{u}{\mu} \right|^{q(x)} dx \leq 1 \right\}. \quad (17)$$

Define a mapping $\rho : L^{q(x)}(\Omega) \longrightarrow \mathbb{R}$ by

$$\rho_{q(x)}(u) := \int_{\Omega} |u|^{q(x)} dx. \quad (18)$$

Lemma 4 (See [37]). The space $(L^{q(x)}(\Omega), |u|_{q(x)})$ is separable, uniformly convex, and reflexive and its conjugate space is $(L^{q'(x)}(\Omega), |u|_{q'(x)})$, where $q'(x)$ is the conjugate function of $q(x)$, i.e.,

$$\frac{1}{q(x)} + \frac{1}{q'(x)} = 1, \text{ for all } x \in \Omega, \quad (19)$$

for all $u \in L^{q(x)}(\Omega)$, $v \in L^{q'(x)}(\Omega)$, and the Hölder type inequality

$$\left| \int_{\Omega} uv dx \right| \leq \left(\frac{1}{q^-} + \frac{1}{(q')^-} \right) |u|_{q(x)} |v|_{q'(x)} \leq 2 |u|_{q(x)} |v|_{q'(x)} \quad (20)$$

holds.

Note that for any function $q_1(x), q_2(x) \in C_+(\bar{\Omega})$ and $q_1(x) < q_2(x)$, there exists a continuous embedding $L^{q_2(x)}(\Omega) \hookrightarrow L^{q_1(x)}(\Omega)$ for any $x \in \bar{\Omega}$.

Lemma 5 (see [38]). Suppose that $u_n, u \in L^{q(x)}(\Omega)$. Then, the following properties hold

$$|u|_{q(x)} > 1 \Rightarrow |u|_{q(x)}^{q^-} \leq \rho_{q(x)}(u) \leq |u|_{q(x)}^{q^+},$$

$$|u|_{q(x)} < 1 \Rightarrow |u|_{q(x)}^{q^+} \leq \rho_{q(x)}(u) \leq |u|_{q(x)}^{q^-},$$

$$|u|_{q(x)} < 1 (\text{respectively, } = 1, > 1) \Leftrightarrow \rho_{q(x)}(u) < 1 (\text{respectively, } = 1, > 1),$$

$$|u_n|_{q(x)} \longrightarrow 0 (\text{respectively, } \longrightarrow \infty) \Leftrightarrow \rho_{q(x)}(u_n) \longrightarrow 0 (\text{respectively, } \longrightarrow \infty),$$

$$\lim_{n \rightarrow \infty} |u_n - u|_{q(x)} = 0 \Leftrightarrow \lim_{n \rightarrow \infty} \rho_{q(x)}(u_n - u) = 0. \quad (21)$$

The fractional Sobolev spaces with variable exponent $X = W^{s,h(x),p(x,y)}(\Omega)$ are defined by

$$W^{s,h(x),p(x,y)}(\Omega) := \left\{ u : u \in L^{h(x)}(\Omega), \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{\mu^{p(x,y)} |x - y|^{N+sp(x,y)}} dx dy < \infty, \lambda > 0 \right\}. \quad (22)$$

For $u \in X$, let

$$[u]_{s,p(x,y)} := \inf \left\{ \mu > 0, \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{\mu^{p(x,y)} |x - y|^{N+sp(x,y)}} dx dy \leq 1 \right\}. \quad (23)$$

Define the corresponding variable exponent norm

$$\|u\|_X := |u|_{h(x)} + [u]_{s,p(x,y)} \quad (24)$$

and set

$$\rho_{p(x,y)}(u) := \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}} dx dy. \quad (25)$$

Then, the $(X, \|\cdot\|_X)$ becomes an uniformly convex and reflexive Banach spaces (see [39]). Let $X_0 = W_0^{s,h(x),p(x,y)}(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ in X endowed with the norm

$$\|u\|_{X_0} := [u]_{s,p(x,y)}, \text{ for all } u \in X_0, \quad (26)$$

and then $(X_0, \|\cdot\|_{X_0})$ is also an uniformly convex and reflexive Banach spaces. X_0^* is the dual spaces of X_0 .

Theorem 6 (see [39]). Let $\Omega \in \mathbb{R}^N$ be a smooth bounded domain and $s \in (0, 1)$, $h(x), p(x, y)$ be continuous variable exponents with $N > sp(x, y)$ for $(x, y) \in \Omega \times \Omega$ and $h(x) > p(x, x)$ for $x \in \bar{\Omega}$, and (13) and (15) be satisfied. Assume that $r' : \bar{\Omega} \rightarrow (1, \infty)$ is a continuous function and $r'^- = \min_{x \in \bar{\Omega}} r'(x)$ such that

$$p(x)^* = \frac{p(x, x)N}{N - sp(x, x)} > r'(x) \geq r'^- > 1, \quad (27)$$

for $x \in \bar{\Omega}$. Then, there exists a constant $C_0 = C_0(N, s, p, h, r', \Omega)$ such that for any $u \in X$,

$$|u|_r'^{(x)} \leq C_0 \|u\|_X. \quad (28)$$

Thus, the spaces X is continuously embedded in $L^{r'(x)}(\Omega)$ for any $r'(x) \in (1, p^*)$. Furthermore, this embedding is compact. If $u \in X_0$, then there exists a constant $C_0 = C_0(N, s, p, h, r', \Omega)$ such that

$$|u|_r'^{(x)} \leq C_0 [u]_{s,p(x,y)}. \quad (29)$$

Lemma 7 (See [40]). For all $u, \varphi \in X_0$, define the functional $\mathcal{L}u : X_0 \rightarrow X_0^*$ by

$$\langle \mathcal{L}u, \varphi \rangle := \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)-2} (u(x) - u(y)) (\varphi(x) - \varphi(y))}{|x - y|^{N+sp(x,y)}} dx dy. \quad (30)$$

Then,

(1) $\mathcal{L}u$ is a bounded and strictly monotone operator

(2) $\mathcal{L}u$ is a mapping of type (S_+) , i.e., if $u_n \rightarrow u$ in X_0 and $\limsup_{n \rightarrow \infty} \langle \mathcal{L}u_n - \mathcal{L}u, u_n - u \rangle \leq 0$, then $u_n \rightarrow u$ in X_0

(3) $\mathcal{L}u : X_0 \rightarrow X_0^*$ is a homeomorphism

Lemma 8 (see [41]).

(1) If $1 \leq \|u\|_{X_0} < \infty$, then

$$\|u\|_{X_0}^{p_-} \leq \rho_{p(x,y)}(u) \leq \|u\|_{X_0}^{p_+}. \quad (31)$$

(2) If $\|u\|_{X_0} < 1$, then

$$\|u\|_{X_0}^{p_+} \leq \rho_{p(x,y)}(u) \leq \|u\|_{X_0}^{p_-}. \quad (32)$$

We state Ekeland's variational principle and dual fountain theorem which will be used in the proofs of Theorems 2 and 3.

Theorem 9 (Ekeland's variational principle, see [42]). Let X_0 be a complete metric spaces with metric d , and let $I : X_0 \rightarrow (-\infty, +\infty]$ be a lower semicontinuous function, bounded from below and not identical to $+\infty$. Let ε_0 be given and $U \in X_0$ be such that

$$I(U) \leq \inf_{X_0} I + \varepsilon_0. \quad (33)$$

Then, there exists $V \in X_0$ such that

$$I(U) \leq I(V), d(U, V) \leq 1, \quad (34)$$

and for each $W \in X_0$, one has

$$I(U) \leq I(W) + \varepsilon_0 d(V, W). \quad (35)$$

The space X_0 is a separable and reflexive real Banach space, and there exists $\{e_j\} \in X_0$ and $\{e_j^*\} \in X_0^*$ such that $X_0 = \text{span}\{e_j : j = 1, 2, \dots\}$, $X_0^* = \text{span}\{e_j^* : j = 1, 2, \dots\}$ and

$$\langle e_j^*, e_i \rangle = \begin{cases} 1, & i \neq j; \\ 0, & i = j. \end{cases}$$

Set $X_i = \text{span}\{e_i\}$, we define $A_k = \oplus_{i=1}^k X_i$, $B_k = \oplus_{i=k}^\infty X_i$.

Theorem 10 (Dual Fountain Theorem, see [43]). Assume that $I \in C^1(X_0, \mathbb{R})$ satisfies $I(-u) = I(u)$, and for every $k \geq k_0$, there exist $\rho_k > \gamma_k > 0$ such that

(B1) $a_k = \inf\{I(u) : u \in B_k; \|u\| = \rho_k\} \geq 0$;

(B2) $b_k = \max\{I(u) : u \in A_k; \|u\| = \gamma_k\} < 0$;

(B3) $d_k = \inf\{I(u) : u \in B_k; \|u\| \leq \gamma_k\} \rightarrow 0$, as $k \rightarrow \infty$;

(B4) I satisfies the $(C)_c^*$ condition for every $c \in [d_{k_0}, 0]$.

Then, I has a sequence of negative critical values converging to 0.

3. Cerami Compactness Condition

We discuss the compactness properties of our energy functional I related to the $(Ce)_c$ condition and $(Ce)_c^*$ condition.

Definition 11. Let $I \in C^1(X, \mathbb{R})$, and we say that I satisfies the Cerami condition at the level $c \in \mathbb{R}$ ($(Ce)_c$ for short), if any sequence $\{u_n\} \subset X_0$ with

$$I(u_n) \rightarrow c, \left(1 + \|u_n\|_{X_0}\right) I'(u_n) \rightarrow 0, \text{ as } n \rightarrow \infty \quad (36)$$

possesses a convergent subsequence in X_0 .

Definition 12. Let $I \in C^1(X, \mathbb{R})$, and we say that I satisfies $(Ce)_c^*$ condition at the level $c \in \mathbb{R}$ ($(Ce)_c^*$ for short), if any sequence $\{u_{n_j}\}_{n_j} \subset X_0$, namely, $u_{n_j}^- \in A_{n_j}$, with

$$I(u_{n_j}^-) \rightarrow c, \left(1 + \|u_{n_j}^-\|_{X_0}\right) I'(u_{n_j}^-) \rightarrow 0, \text{ as } n_j \rightarrow \infty \quad (37)$$

possesses a convergent subsequence in X_0 .

Assume that $\{u_n\}$ is a bounded sequence in X_0 . By Theorem 6, there exists $u \in X_0$ such that

$$u_n \rightharpoonup u \text{ in } X_0, u_n \rightarrow u \text{ a.e. in } \Omega, u_n \rightarrow u \text{ in } L^{q(x)}(\Omega). \quad (38)$$

Lemma 13. Let $u_n, u, v \in X_0$ such that (38) holds. Then, passing to a subsequence, the following properties hold

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{\Omega} |u_n|^{q(x)-2} u_n (u_n - u) dx &= 0, \\ \lim_{n \rightarrow \infty} \int_{\Omega} f(x, u_n) (u_n - u) dx &= 0. \end{aligned} \quad (39)$$

Proof.

(i) By Hölder's inequality (Lemma 4), Theorem 6, and (38), we have

$$\begin{aligned} \left| \int_{\Omega} |u_n|^{q(x)-2} u_n (u_n - u) dx \right| &\leq \int_{\Omega} |u_n|^{q(x)-1} |u_n - u| dx \\ &\leq \|u_n\|_{q(x)}^{q(x)-1} \|u_n - u\|_{q(x)} \rightarrow 0, \text{ as } n \rightarrow \infty, \end{aligned} \quad (40)$$

and thus

$$\lim_{n \rightarrow \infty} \int_{\Omega} |u_n|^{q(x)-2} u_n (u_n - u) dx = 0. \quad (41)$$

(ii) By virtue of conditions (f1) and (f2), for any $\varepsilon \in (0, 1)$, there exists $C_\varepsilon > 0$ such that

$$|f(x, u_n)| \leq \varepsilon |u_n|^{p^+-1} + C_\varepsilon |u_n|^{r(x)-1}, \quad (42)$$

combining with Hölder's inequality, Theorem 6 and (38), and it follows that

$$\begin{aligned} &\left| \int_{\Omega} f(x, u_n) (u_n - u) dx \right| \\ &\leq \int_{\Omega} \left(\varepsilon |u_n|^{p^+-1} |u_n - u| + C_\varepsilon |u_n|^{r(x)-1} |u_n - u| \right) dx \\ &\leq \varepsilon \|u_n\|_{p^+}^{p^+-1} \|u_n - u\|_{p^+} + C_\varepsilon \|u_n\|_{r(x)}^{r(x)-1} \|u_n - u\|_{r(x)} \rightarrow 0, \text{ as } n \rightarrow \infty, \end{aligned} \quad (43)$$

which implies that

$$\lim_{n \rightarrow \infty} \int_{\Omega} f(x, u_n) (u_n - u) dx = 0. \quad (44)$$

The following lemma about the $(Ce)_c$ condition will play an important role in the proof of our main results.

Lemma 14. Let the function f satisfy (f1), (f3), and (f4), then the energy functional I satisfies the $(Ce)_c$ condition, where precisely $c < a^2/2b$.

Proof. Step 1. We prove that $\{u_n\}$ is bounded in X_0 . Let $\{u_n\}$ be a $(Ce)_c$ sequence, $I(u_n) \rightarrow c$, $(1 + \|u_n\|_{X_0}) I'(u_n) \rightarrow 0$, which implies that

$$c = I(u_n) + o(1), \langle I'(u_n), u_n \rangle = o(1), \quad (45)$$

where $o(1) \rightarrow 0$ as $n \rightarrow \infty$.

We claim that $\{u_n\}$ is a bounded sequence. Suppose to the contrary that

$$\|u_n\|_{X_0} \rightarrow \infty, \text{ as } n \rightarrow \infty. \quad (46)$$

Denote $\omega_n := u_n / \|u_n\|_{X_0}$, then $\omega_n \in X_0$ with $\|\omega_n\|_{X_0} = 1$. Up to subsequences, for some $\omega \in X_0$, we get $\omega_n \rightharpoonup \omega$ in X_0 , $\omega_n \rightarrow \omega$ a.e. in Ω , $\omega_n \rightarrow \omega$ in $L^{q(x)}(\Omega)$. There are only two cases need to be discussed $\omega = 0$ and $\omega \neq 0$.

First, let us consider the case $\omega = 0$. In view of (f4), (8), (9), (45), (46), and Theorem 6, we obtain

$$\begin{aligned}
0 &= \frac{1}{\|u_n\|_{X_0}^{2p^-}} \left[I(u_n) - \frac{1}{p^+} \langle I'(u_n), u_n \rangle \right] \\
&= \frac{1}{\|u_n\|_{X_0}^{2p^-}} \left[a \int_{\Omega \times \Omega} \frac{|u_n(x) - u_n(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}} dx dy \right. \\
&\quad - \frac{b}{2} \left(\int_{\Omega \times \Omega} \frac{|u_n(x) - u_n(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}} dx dy \right)^2 \\
&\quad - \lambda \int_{\Omega} \frac{1}{q(x)} |u_n|^{q(x)} dx - \int_{\Omega} F(x, u_n) dx \\
&\quad - \frac{a}{p^+} \int_{\Omega \times \Omega} \frac{|u_n(x) - u_n(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}} dx dy \\
&\quad + \frac{b}{p^+} \int_{\Omega \times \Omega} \frac{|u_n(x) - u_n(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}} dx dy \\
&\quad \times \int_{\Omega \times \Omega} \frac{|u_n(x) - u_n(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}} dx dy \\
&\quad \left. + \frac{\lambda}{p^+} \int_{\Omega} |u_n|^{q(x)} dx + \frac{1}{p^+} \int_{\Omega} f(x, u_n) u_n dx \right] \\
&\geq \frac{1}{\|u_n\|_{X_0}^{2p^-}} \left[\frac{a}{p^+} \int_{\Omega \times \Omega} \frac{|u_n(x) - u_n(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}} dx dy \right. \\
&\quad - \frac{b}{2p^{-2}} \left(\int_{\Omega \times \Omega} \frac{|u_n(x) - u_n(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}} dx dy \right)^2 \\
&\quad - \frac{\lambda}{q^-} \int_{\Omega} |u_n|^{q(x)} dx - \int_{\Omega} F(x, u_n) dx \\
&\quad - \frac{a}{p^+} \int_{\Omega \times \Omega} \frac{|u_n(x) - u_n(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}} dx dy \\
&\quad + \frac{b}{p^{+2}} \int_{\Omega \times \Omega} \frac{|u_n(x) - u_n(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}} dx dy \\
&\quad \times \int_{\Omega \times \Omega} \frac{|u_n(x) - u_n(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}} dx dy \\
&\quad \left. + \frac{\lambda}{p^+} \int_{\Omega} |u_n|^{q(x)} dx + \frac{1}{p^+} \int_{\Omega} f(x, u_n) u_n dx \right] \\
&\geq \frac{1}{\|u_n\|_{X_0}^{2p^-}} \left[\left(\frac{b}{p^{+2}} - \frac{b}{2p^{-2}} \right) \|u_n\|_{X_0}^{2p^-} - \left(\frac{\lambda}{q^-} - \frac{\lambda}{p^+} \right) C_{q^-} \|u_n\|_{X_0}^{q^-} \right. \\
&\quad \left. + \int_{\Omega} \left(\frac{1}{p^+} f(x, u_n) u_n - F(x, u_n) \right) dx \right] \\
&\geq \frac{1}{\|u_n\|_{X_0}^{2p^-}} \left[\left(\frac{b}{p^{+2}} - \frac{b}{2p^{-2}} \right) \|u_n\|_{X_0}^{2p^-} - \left(\frac{\lambda}{q^-} - \frac{\lambda}{p^+} \right) C_{q^-} \|u_n\|_{X_0}^{q^-} \right. \\
&\quad - \theta_0 \|u_n\|_{X_0}^{2p^-} \int_{\Omega} |\omega_n|^{2p^-} dx \Big] \geq \left(\frac{b}{p^{+2}} - \frac{b}{2p^{-2}} \right) \\
&\quad - \left(\frac{\lambda}{q^-} - \frac{\lambda}{p^+} \right) C_{q^-} \|u_n\|_{X_0}^{q^- - 2p^-} - \theta_0 \int_{\Omega} |\omega_n|^{2p^-} dx,
\end{aligned} \tag{47}$$

which implies $0 > b/p^{+2} - b/2p^{-2}$, and this is a contradiction according to the conditions $p^+ < \sqrt{2}p^-$ and $q^- < 2p^-$ (see (11)).

In the case of $\omega \neq 0$, setting $\Omega_1 := \{x \in \Omega : \omega \neq 0\}$, then $\lim_{n \rightarrow \infty} \omega_n = \lim_{n \rightarrow \infty} u_n / \|u_n\|_{X_0} = \omega \neq 0$, from (46) implies that $|u_n| = |\omega_n| \|u_n\|_{X_0} \rightarrow +\infty$, for $x \in \Omega_1$, we have $u_n \rightarrow \infty$ as $n \rightarrow \infty$. In view of (f3), we see that

$$\begin{aligned}
\frac{F(x, u_n)}{\|u_n\|_{X_0}^{2p^+}} &= \frac{F(x, u_n)}{|u_n|^{2p^+}} \frac{|u_n|^{2p^+}}{\|u_n\|_{X_0}^{2p^+}} \\
&= \frac{F(x, u_n)}{|u_n|^{2p^+}} |\omega_n|^{2p^+} \rightarrow +\infty, \text{ a.e. in } \Omega_1;
\end{aligned} \tag{48}$$

so, using Fatou's Lemma, we have

$$\lim_{n \rightarrow \infty} \int_{\Omega_1} \frac{F(x, u_n)}{\|u_n\|_{X_0}^{2p^+}} dx \rightarrow +\infty, \text{ a.e. in } \Omega_1. \tag{49}$$

From (f1) and (f3), there exist $M', M'', M > 0$, such that $|F(x, u_n)| \leq M' |u_n|^{r(x)}$ and $F(x, u_n) \geq -M'' |u_n|^{2p^+}$ as $|u_n| \rightarrow \infty$, and then we have $F(x, u_n) \geq -M |u_n|^{r^-}$, hence

$$\int_{\Omega \setminus \Omega_1} \frac{F(x, u_n)}{\|u_n\|_{X_0}^{2p^+}} dx \geq - \frac{\int_{\Omega \setminus \Omega_1} M |u_n|^{r^-} dx}{\|u_n\|_{X_0}^{2p^+}} \geq - \frac{M |u_n|^{r^-}}{\|u_n\|_{X_0}^{2p^+}}, \tag{50}$$

by (46), which implies that

$$\liminf_{n \rightarrow \infty} \int_{\Omega \setminus \Omega_1} \frac{F(x, u_n)}{\|u_n\|_{X_0}^{2p^+}} dx \geq 0. \tag{51}$$

Using (8), (45), (46), (49), (51), and Theorem 6, we obtain

$$\begin{aligned}
0 &= \lim_{n \rightarrow \infty} \frac{-c + o(1)}{\|u_n\|_{X_0}^{2p^+}} = \lim_{n \rightarrow \infty} \frac{-I(u_n)}{\|u_n\|_{X_0}^{2p^+}} \geq \lim_{n \rightarrow \infty} \frac{1}{\|u_n\|_{X_0}^{2p^+}} \\
&\quad \cdot \left(-\frac{a}{p^-} \|u_n\|_{X_0}^{p^+} + \frac{b}{2(p^+)^2} \|u_n\|_{X_0}^{2p^-} + \frac{\lambda C_q}{q^+} \|u_n\|_{X_0}^{q^-} + \int_{\Omega} F(x, u_n) dx \right) \\
&\geq -\frac{a}{p^-} \lim_{n \rightarrow \infty} \|u_n\|_{X_0}^{p^+ - 2p^+} + \frac{b}{2(p^+)^2} \lim_{n \rightarrow \infty} \|u_n\|_{X_0}^{2p^- - 2p^+} \\
&\quad + \frac{\lambda C_q}{q^+} \lim_{n \rightarrow \infty} \|u_n\|_{X_0}^{q^- - 2p^+} + \lim_{n \rightarrow \infty} \int_{\Omega_1} \frac{F(x, u_n)}{\|u_n\|_{X_0}^{2p^+}} dx \\
&\quad + \liminf_{n \rightarrow \infty} \int_{\Omega \setminus \Omega_1} \frac{F(x, u_n)}{\|u_n\|_{X_0}^{2p^+}} dx \rightarrow +\infty, \text{ as } n \rightarrow \infty,
\end{aligned} \tag{52}$$

and this is a contradiction according to the condition $q^- < p^- < p^+$ (see (11)). Then, the sequence $\{u_n\}$ is bounded in X_0 .

Step 2. We prove that $\{u_n\}$ has a convergent subsequence in X_0 . Since $(1 + \|u_n\|) I'(u_n) \rightarrow 0$, as $n \rightarrow \infty$, $\{u_n\}$ is bounded in X_0 , we have

$$\begin{aligned} \left| \langle I'(u_n), u_n - u \rangle \right| &\leq \left| \langle I'(u_n, u_n) \rangle \right| + \left| \langle I'(u_n, u) \rangle \right| \\ &\leq \|I'(u_n)\|_{X_0} \|u_n\|_{X_0} + \|I'(u_n)\|_{X_0} \|u\|_{X_0}, \end{aligned} \quad (53)$$

therefore, we obtain

$$\lim_{n \rightarrow \infty} \langle I'(u_n), u_n - u \rangle = 0, \quad (54)$$

or

$$\begin{aligned} &\langle I'(u_n), u_n - u \rangle \\ &= \left(a - b \int_{\Omega \times \Omega} \frac{|u_n(x) - u_n(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}} dx dy \right) \\ &\quad \times \int_{\Omega \times \Omega} \frac{|u_n(x) - u_n(y)|^{p(x,y)-2} (u_n(x) - u_n(y) - u(x) + u(y))}{|x-y|^{N+sp(x,y)}} dx dy \\ &\quad - \lambda \int_{\Omega} |u_n|^{q(x)-2} u_n (u_n - u) dx - \int_{\Omega} f(x, u_n) (u_n - u) dx, \end{aligned} \quad (55)$$

and we deduce from Lemma 13 that

$$\begin{aligned} &\left(a - b \int_{\Omega \times \Omega} \frac{|u_n(x) - u_n(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}} dx dy \right) \\ &\quad \times \int_{\Omega \times \Omega} \frac{|u_n(x) - u_n(y)|^{p(x,y)-2} (u_n(x) - u_n(y) - u(x) + u(y))}{|x-y|^{N+sp(x,y)}} dx dy \longrightarrow 0. \end{aligned} \quad (56)$$

Since $\{u_n\}$ is bounded in X_0 , passing to a subsequence, we may assume

$$\int_{\Omega \times \Omega} \frac{|u_n(x) - u_n(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}} dx dy \longrightarrow t_0 \geq 0 \text{ as } n \longrightarrow \infty. \quad (57)$$

(i) If $t_0 = 0$, then u_n strongly converges to $u = 0$ in X_0

(ii) If $t_0 > 0$, $t_0 \neq a/b$, then

$$a - b \int_{\Omega \times \Omega} \frac{|u_n(x) - u_n(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}} dx dy \longrightarrow 0 \quad (58)$$

is not true. Because $\{u_n\}$ is bounded in X_0 , so,

$$\left\{ a - b \int_{\Omega \times \Omega} \frac{|u_n(x) - u_n(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}} dx dy \right\} \quad (59)$$

is bounded.

(iii) If $t_0 > 0$, $t_0 = a/b$, then $a - b \int_{\Omega \times \Omega} (|u_n(x) - u_n(y)|^{p(x,y)} / p(x,y) |x-y|^{N+sp(x,y)}) dx dy \longrightarrow 0$. We define a function by

$$\psi(u) = \lambda \int_{\Omega} \frac{1}{q(x)} |u|^{q(x)} dx + \int_{\Omega} F(x, u) dx, \text{ for all } u \in X_0, \quad (60)$$

and then

$$\langle \psi'(u), \varphi \rangle = \lambda \int_{\Omega} |u|^{q(x)-2} u \varphi dx + \int_{\Omega} f(x, u) \varphi dx, \text{ for all } u, \varphi \in X_0, \quad (61)$$

and it follows that

$$\begin{aligned} \langle \psi'(u_n) - \psi'(u), \varphi \rangle &= \lambda \int_{\Omega} (|u_n|^{q(x)-2} u_n - |u|^{q(x)-2} u) \varphi dx \\ &\quad + \int_{\Omega} (f(x, u_n) - f(x, u)) \varphi dx. \end{aligned} \quad (62)$$

Since $u_n \longrightarrow u$ in $L^{q(x)}(\Omega)$, which implies that

$$|u_n|^{q(x)-2} u_n \longrightarrow |u|^{q(x)-2} u \text{ in } L^{q(x)/q(x)-1}(\Omega), \quad (63)$$

according to the Hölder's inequality, we get

$$\begin{aligned} &\left| \int_{\Omega} (|u_n|^{q(x)-2} u_n - |u|^{q(x)-2} u) \varphi dx \right| \\ &\leq \int_{\Omega} \left| |u_n|^{q(x)-2} u_n - |u|^{q(x)-2} u \right| |\varphi| dx \\ &\leq \left| |u_n|^{q(x)-2} u_n - |u|^{q(x)-2} u \right|_{q(x)/q(x)-1} \|\varphi\|_{q(x)} \longrightarrow 0. \end{aligned} \quad (64)$$

Hence, we obtain

$$\int_{\Omega} (|u_n|^{q(x)-2} u_n - |u|^{q(x)-2} u) \varphi dx = 0. \quad (65)$$

With a slight modification of the above proof, we can also prove the following conclusion; so, we have omitted the details. By Hölder's inequality, (38) and (42), we deduce that

$$\lim_{n \rightarrow \infty} \int_{\Omega} |f(x, u_n) - f(x, u)| |\varphi| dx = 0. \quad (66)$$

Combining with (65) and (66), we get

$$\|\psi'(u_n) - \psi'(u)\|_{X_0^*} \longrightarrow 0 \text{ and } \psi'(u_n) \longrightarrow \psi'(u). \quad (67)$$

While

$$\begin{aligned} \langle I'(u_n), \varphi \rangle &= \left(a - b \int_{\Omega \times \Omega} \frac{|u_n(x) - u_n(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}} dx dy \right) \\ &\quad \times \int_{\Omega \times \Omega} \frac{|u_n(x) - u_n(y)|^{p(x,y)-2} (u_n(x) - u_n(y)) (\varphi(x) - \varphi(y))}{|x-y|^{N+sp(x,y)}} dx dy \\ &\quad - \langle \psi'(u_n), \varphi \rangle, \end{aligned} \quad (68)$$

since $\langle I'(u_n), \varphi \rangle \rightarrow 0$, and $a - b \int_{\Omega \times \Omega} (|u_n(x) - u_n(y)|^{p(x,y)} / p(x,y) |x-y|^{N+sp(x,y)}) dx dy \rightarrow 0$, it follows that $\psi'(u_n) \rightarrow 0$ as $n \rightarrow \infty$, i.e., by virtue of (61), we obtain

$$-\lambda |u|^{q(x)-2} u - f(x, u) = 0, \text{ a.e. } x \in \Omega \quad (69)$$

by the fundamental lemma of the variational method (see [43]). It follows that $u = 0$. So,

$$\begin{aligned} \psi(u_n) &= \lambda \int_{\Omega} \frac{1}{q(x)} |u_n|^{q(x)} dx + \int_{\Omega} F(x, u_n) dx \rightarrow \lambda \int_{\Omega} \frac{1}{q(x)} |u|^{q(x)} dx \\ &\quad + \int_{\Omega} F(x, u) dx = 0. \end{aligned} \quad (70)$$

Hence, for $t_0 = a/b$, we have

$$\begin{aligned} I(u_n) &= a \int_{\Omega \times \Omega} \frac{|u_n(x) - u_n(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}} dx dy \\ &\quad - \frac{b}{2} \left(\int_{\Omega \times \Omega} \frac{|u_n(x) - u_n(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}} dx dy \right)^2 \\ &\quad - \lambda \int_{\Omega} \frac{1}{q(x)} |u_n|^{q(x)} dx - \int_{\Omega} F(x, u_n) dx \rightarrow \frac{a^2}{2b}. \end{aligned} \quad (71)$$

This is a contradiction with $I(u_n) \rightarrow c < a^2/2b$; then, $a - b \int_{\Omega \times \Omega} (|u_n(x) - u_n(y)|^{p(x,y)} / p(x,y) |x-y|^{N+sp(x,y)}) dx dy \rightarrow 0$ is not true and similarly to (ii), and we obtain

$$\left\{ a - b \int_{\Omega \times \Omega} \frac{|u_n(x) - u_n(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}} dx dy \right\} \quad (72)$$

that is bounded. So, it can be concluded from the above discussion (ii) and (iii) that

$$\int_{\Omega \times \Omega} \frac{|u_n(x) - u_n(y)|^{p(x,y)-2} (u_n(x) - u_n(y) - u(x) + u(y))}{|x-y|^{N+sp(x,y)}} dx dy \rightarrow 0. \quad (73)$$

Thus, invoking the S_+ condition (see (2) of Lemma 7), we deduce that the $\{u_n\}$ converges to u in X_0 .

Therefore, the energy functional I satisfies the $(Ce)_c$ condition.

Lemma 15. Let the function f satisfy (f1), (f3), and (f4), and then the energy functional I satisfies the $(Ce)_c^*$ condition, where precisely $c < a^2/2b$.

Proof. Consider a sequence $\{u_{n_j}^-\}_{n_j} \subset X_0$ such that $u_{n_j}^- \in A_{n_j}$, $I(u_{n_j}^-) \rightarrow c$, $I'(u_{n_j}^-)|'_{A_{n_j}} \rightarrow 0$, in X_0^* , as $n_j \rightarrow \infty$. By the same method used in the proof of Lemma 14, we can prove that $\{u_{n_j}^-\}_{n_j}$ has a strongly convergent subsequence in X_0 . The details are omitted.

Remark 16. Since the $(Ce)_c$ and $(Ce)_c^*$ conditions hold for the energy $c < a^2/2b$, we discuss the negative energy solutions for the problem (P_λ) .

4. Proof of Theorem 2

We firstly prove two lemmas.

Lemma 17. Let assumptions (f1), (f2), and (f3) hold, and then there are constants ρ, λ^* and $\alpha > 0$ such that $I(u) \geq \alpha$, where $\lambda \in (0, \lambda^*)$ and $\|u\|_{X_0} = \rho$.

Proof. For any $\varepsilon \in (0, 1)$ and $\lambda > 0$, by (f1) and (f2), there exists $C_\varepsilon > 0$ such that

$$|F(x, u)| \leq \frac{\varepsilon}{p^+} |u|^{p^+} + \frac{C_\varepsilon}{r(x)} |u|^{r(x)}. \quad (74)$$

Let $u \in X_0$ be such that $\|u\| = \rho \in (0, 1)$ and combining with (74), we deduce that

$$\begin{aligned} I(u) &= a \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}} dx dy \\ &\quad - \frac{b}{2} \left(\int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}} dx dy \right)^2 \\ &\quad - \lambda \int_{\Omega} \frac{1}{q(x)} |u|^{q(x)} dx - \int_{\Omega} F(x, u) dx \\ &\geq a \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}} dx dy \\ &\quad - \frac{b}{2} \left(\int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}} dx dy \right)^2 \\ &\quad - \lambda \int_{\Omega} \frac{1}{q(x)} |u|^{q(x)} dx - \int_{\Omega} \frac{\varepsilon}{p^+} |u|^{p^+} dx \\ &\quad - \int_{\Omega} \frac{C_\varepsilon}{r(x)} |u|^{r(x)} dx \geq \frac{a}{p^+} \rho_{p(x,y)}(u) - \frac{b}{2p^{+2}} \rho_{p(x,y)}^2(u) \\ &\quad - \frac{\lambda}{q^-} \rho_{q(x)}(u) - \frac{\varepsilon}{p^+} \rho_{p^+}(u) - \frac{C_\varepsilon}{r^-} \rho_{r(x)}(u). \end{aligned} \quad (75)$$

By Theorem 6, Lemma 5, and Lemma 8, there exist positive constants C_{q^-}, C_{p^+}, C_r such that

$$|u|_{q(x)}^{q^-} \leq C_{q^-} \|u\|_{X_0}^{q^-}, |u|_{p^+}^{p^+} \leq C_p \|u\|_{X_0}^{p^+}, |u|_{r(x)}^{r^-} \leq C_r \|u\|_{X_0}^{r^-}. \quad (76)$$

Hence, we obtain

$$\begin{aligned} I(u) &\geq \frac{a}{p^+} \|u\|_{X_0}^{p^+} - \frac{b}{2p^{-2}} \|u\|_{X_0}^{2p^-} - \frac{\lambda C_{q^-}}{q^-} \|u\|_{X_0}^{q^-} - \frac{\varepsilon C_p}{p^+} \|u\|_{X_0}^{p^+} \\ &\quad - \frac{C_\varepsilon C_r}{r^-} \|u\|_{X_0}^{r^-} = \|u\|_{X_0}^{p^+} \left(\frac{a}{p^+} - \frac{b}{2p^{-2}} \|u\|_{X_0}^{2p^- - p^+} \right. \\ &\quad \left. - \frac{\lambda C_{q^-}}{q^-} \|u\|_{X_0}^{q^- - p^+} - \frac{\varepsilon C_p}{p^+} - \frac{C_\varepsilon C_r}{r^-} \|u\|_{X_0}^{r^- - p^+} \right). \end{aligned} \quad (77)$$

Choose $\varepsilon = a/2C_p > 0$ and take

$$I(t) = \frac{a}{2p^+} - \frac{b}{2p^{-2}} t^{2p^- - p^+} - \frac{C_\varepsilon C_r}{r^-} t^{r^- - p^+}, \text{ for all } t \geq 0. \quad (78)$$

Note that $p^+ < 2p^- < r^-$ (see (11)), and we can conclude that there exists a constant $\rho > 0$ such that

$$I(\rho) = \max_{t \geq 0} I(t) > 0. \quad (79)$$

Therefore, taking $\lambda^* := (q^-/2C_{q^-} \rho^{q^- - p^+})I(\rho) > 0$, we have

$$I(u) \geq \frac{1}{2} \rho^{p^+} I(\rho) =: \alpha > 0, \quad (80)$$

whenever $\|u\|_{X_0} = \rho$ and $\lambda \in (0, \lambda^*)$. This completes the proof.

Lemma 18. *Let assumptions (f1), (f2,) and (f3) hold, and then*

$$-\infty < \inf \{I(u) : u \in \bar{B}_\rho\} < 0, \quad (81)$$

where ρ is given by Lemma 17 and $\bar{B}_\rho := \{u \in X_0 : \|u\|_{X_0} \leq \rho\}$.

Proof. By (f2) and (f3), for any $M_1 > 0$, there exists $C_{M_1} > 0$ such that

$$F(x, u) \geq M_1 |u|^{2p^+} - C_{M_1} |u|^{p^+}, \text{ for all } (x, u) \in \Omega \times \mathbb{R}. \quad (82)$$

Let $\phi \in X_0$, $\phi > 0$ and $t > 0$, combining with (82), we have

$$\begin{aligned} I(t\phi) &= a \int_{\Omega \times \Omega} \frac{|t\phi(x) - t\phi(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}} dx dy \\ &\quad - \frac{b}{2} \left(\int_{\Omega \times \Omega} \frac{|t\phi(x) - t\phi(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}} dx dy \right)^2 \\ &\quad - \lambda \int_{\Omega} \frac{1}{q(x)} |t\phi|^{q(x)} dx - \int_{\Omega} F(x, t\phi) dx \\ &\leq a \int_{\Omega \times \Omega} \frac{|t\phi(x) - t\phi(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}} dx dy \\ &\quad - \frac{b}{2} \left(\int_{\Omega \times \Omega} \frac{|t\phi(x) - t\phi(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}} dx dy \right)^2 \\ &\quad - \lambda \int_{\Omega} \frac{1}{q(x)} |t\phi|^{q(x)} dx - \int_{\Omega} M_1 |t\phi|^{2p^+} dx \\ &\quad + \int_{\Omega} C_{M_1} |t\phi|^{p^-} dx \leq \frac{at^{p^-}}{p^-} \int_{\Omega \times \Omega} \frac{|\phi(x) - \phi(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} dx dy \\ &\quad - \frac{bt^{2p^+}}{2p^{+2}} \left(\int_{\Omega \times \Omega} \frac{|\phi(x) - \phi(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} dx dy \right)^2 \\ &\quad - \frac{\lambda t^{q^+}}{q^+} \int_{\Omega} |\phi|^{q(x)} dx - M_1 t^{2p^+} \int_{\Omega} |\phi|^{2p^+} dx + C_{M_1} t^{p^+} \int_{\Omega} |\phi|^{p^+} dx. \end{aligned} \quad (83)$$

Since $q^+ < p^- < p^+$ (see (11)), for $t > 0$, it is small enough. Hence, we obtain that

$$-\infty < \inf \{I(u) : u \in \bar{B}_\rho\} < 0, \quad (84)$$

where ρ is given by Lemma 17 and $\bar{B}_\rho := \{u \in X_0 : \|u\|_{X_0} \leq \rho\}$.

Now, we give the proof of Theorem 2.

Proof. By Lemma 17 and Lemma 18, we know that

$$-\infty < c = \inf_{\bar{B}_\rho} I(u) < 0 < \alpha < \inf_{\partial \bar{B}_\rho} I(u), \quad (85)$$

whenever $\lambda \in (0, \lambda^*)$. Set

$$\frac{1}{n} \in \left(0, \inf_{\partial \bar{B}_\rho} I(u) - \inf_{\bar{B}_\rho} I(u) \right), n \in \mathbb{N}. \quad (86)$$

So, by Theorem 9 (Ekeland's variational principle), there exists $u_n \in \bar{B}_\rho$ such that

$$c \leq I(u_n) \leq c + \frac{1}{n}, \quad (87)$$

$$I(u_n) \leq I(u) + \frac{1}{n} \|u_n - u\|_{X_0}, \quad (88)$$

for all $u \in \bar{B}_\rho$, $u_n \neq u$. In that way, by (86)–(88), we have

$$I(u_n) \leq c + \frac{1}{n} \leq \inf_{u \in \bar{B}_\rho} I(u) + \frac{1}{n} < \inf_{u \in \partial B_\rho} I(u), \quad (89)$$

so that $u_n \in B_\rho$.

Define the functional $M_n : \bar{B}_\rho \longrightarrow \mathbb{R}$ by

$$M_n(u) = I(u) + \frac{1}{n} \|u_n - u\|_{X_0}, \text{ for all } u \in \bar{B}_\rho. \quad (90)$$

Like that, (88) implies that $M_n(u_n) < M_n(u)$ for all $u \in \bar{B}_\rho$, with $u_n \neq u$, and thus u_n is a strict local minimum of M_n . Furthermore, for all $v \in B_1$, where $B_1 = \{v \in X_0 : \|v\|_{X_0} = 1\}$, taking $t > 0$ small enough, we have

$$\frac{M_n(u_n + tv) - M_n(u_n)}{t} \geq 0, \quad (91)$$

which implies that

$$\frac{I(u_n + tv) - I(u_n)}{t} + \frac{1}{n} \|v\|_{X_0} \geq 0. \quad (92)$$

Thus,

$$\langle I'(u_n), v \rangle + \frac{1}{n} \|v\|_{X_0} \geq 0 \text{ for all } v \in B_1. \quad (93)$$

In the above inequality, replacing v with $-v$, we have

$$-\langle I'(u_n), v \rangle + \frac{1}{n} \|v\|_{X_0} \geq 0 \text{ for all } v \in B_1. \quad (94)$$

Hence,

$$\|I'(u_n)\|_{X_0^*} \leq \frac{1}{n}. \quad (95)$$

Passing to the limit in (87) and (95), we conclude that there is a sequence $\{u_n\} \subset \bar{B}_\rho$ such that

$$I(u_n) \longrightarrow c < 0 \text{ and } \|I'(u_n)\|_{X_0^*} \longrightarrow 0, \text{ as } n \longrightarrow \infty. \quad (96)$$

To sum up, by Lemma 14, there exists a strongly convergent sequence $\{u_n\}$, still denoted by $\{u_n\}$, such that $u_n \longrightarrow u_0$ in X_0 . Consequently, u_0 is a solution of (P_λ) , with $I(u_0) < 0$. The proof of Theorem 2 is completed.

5. Proof of Theorem 3

In order to prove Theorem 3, we need the following result.

Lemma 19 (see [44], Lemma 18). *Assume that $\zeta(x) \in C_+(\bar{\Omega})$, $\zeta(x) < p^*(x)$, for any $x \in \bar{\Omega}$ and denote*

$$\beta_k = \sup_{u \in B_k, \|u\|_{X_0} = 1} |u|_{\zeta(x)}, \quad (97)$$

then $\lim_{k \rightarrow \infty} \beta_k = 0$.

We give the proof of Theorem 3.

Proof. The function f satisfies hypotheses (f1)–(f5), and it is obvious that $I(-u) = I(u)$ for all $u \in X_0$. Together with Lemma 15, we only need to verify that the energy functional $I : X_0 \longrightarrow \mathbb{R}$ satisfies all conditions (B1)–(B3) of Theorem 10.

Step 1. Verification of condition (B1). According to (f1) and (f2), there exist $C_2 > 0$, $C_3 > 0$, such that

$$|F(x, u)| \leq \frac{C_2}{p^+} |u|^{p^+} + \frac{C_3}{r(x)} |u|^{r(x)}. \quad (98)$$

For any $u \in B_k$ with $\|u\| \leq \min(1, (1/C_0))$, combining with (98) Theorem 6 and Lemma 8, there exist positive constants C_q , C_p , C_r such that we obtain

$$\begin{aligned} I(u) &= a \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}} dx dy \\ &\quad - \frac{b}{2} \left(\int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}} dx dy \right)^2 \\ &\quad - \lambda \int_{\Omega} \frac{1}{q(x)} |u|^{q(x)} dx - \int_{\Omega} F(x, u) dx \\ &\geq a \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}} dx dy \\ &\quad - \frac{b}{2} \left(\int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}} dx dy \right)^2 \\ &\quad - \lambda \int_{\Omega} \frac{1}{q(x)} |u|^{q(x)} dx - \int_{\Omega} \frac{C_2}{p^+} |u|^{p^+} dx \\ &\quad - \int_{\Omega} \frac{C_3}{r(x)} |u|^{r(x)} dx \geq \frac{a}{p^+} \rho_{p(x,y)}(u) - \frac{b}{2p^{+2}} \rho_{p(x,y)}^2(u) \\ &\quad - \frac{\lambda}{q^-} \rho_{q(x)}(u) - \frac{C_2}{p^+} \rho_{p^+}(u) - \frac{C_3}{r^-} \rho_{r(x)}(u) \geq \frac{a}{p^+} \|u\|_{X_0}^{p^+} \\ &\quad - \frac{b}{2p^{+2}} \|u\|_{X_0}^{2p^+} - \frac{\lambda C_q}{q^-} \|u\|_{X_0}^{q^-} - \frac{C_2 C_p}{p^+} \|u\|_{X_0}^{p^+} \\ &\quad - \frac{C_3 C_r}{r^-} \|u\|_{X_0}^{r^-}. \end{aligned} \quad (99)$$

According to Lemma 19 and the conditions $q^- < p^+ < \sqrt{2}p^-$ (see (12)), it follows that

$$\begin{aligned}
I(u) &\geq \frac{a}{p^+} \|u\|_{X_0}^{p^+} - \frac{b}{2p^-} \|u\|_{X_0}^{2p^-} - \frac{\lambda C_{q^-} \beta_k^{q^-}}{q^-} \|u\|_{X_0}^{q^-} \\
&\quad - \frac{C_2 C_p \beta_k^{p^+}}{p^+} \|u\|_{X_0}^{p^+} - \frac{C_3 C_r \beta_k^{r^-}}{r^-} \|u\|_{X_0}^{r^-} \\
&\geq \left(\frac{a}{p^+} - \frac{b}{2p^-} \right) \|u\|_{X_0}^{2p^-} - \frac{(\lambda C_{q^-} + C_2 C_p) \beta_k^{q^-}}{q^-} \|u\|_{X_0}^{q^-} \\
&\quad - \frac{C_3 C_r \beta_k^{r^-}}{r^-} \|u\|_{X_0}^{r^-},
\end{aligned} \tag{100}$$

since $a \geq b$ and $p^+ < \sqrt{2}p^- < 2p^- < r^-$ (see (12)) imply that $a/p^+ - b/2p^- = 2p^-a - bp^+/2p^-p^+ > 0$.

Hence, we deduce

$$\begin{aligned}
I(u) &\geq \left(\frac{2p^-a - bp^+}{2p^-p^+} - \frac{C_3 C_r \beta_k^{r^-}}{r^-} \right) \|u\|_{X_0}^{r^-} \\
&\quad - \frac{(\lambda C_{q^-} + C_2 C_p) \beta_k^{q^-}}{q^-} \|u\|_{X_0}^{q^-}.
\end{aligned} \tag{101}$$

Choosing $2p^-a - bp^+/4p^-p^+ > C_3 C_r \beta_k^{r^-}/r^-$, we have

$$I(u) \geq \frac{2p^-a - bp^+}{4p^-p^+} \|u\|_{X_0}^{r^-} - \frac{(\lambda C_{q^-} + C_2 C_p) \beta_k^{q^-}}{q^-} \|u\|_{X_0}^{q^-}. \tag{102}$$

Obviously, there exists a large enough k such that

$$I(u) \geq \|u\|_{X_0}^{q^-} \left(\frac{2p^-a - bp^+}{4p^-p^+} \|u\|_{X_0}^{r^- - q^-} - \frac{(\lambda C_{q^-} + C_2 C_p) \beta_k^{q^-}}{q^-} \right). \tag{103}$$

We choose

$$\rho_k := \left(\frac{4(\lambda C_{q^-} + C_2 C_p) p^+ p^{-2}}{q^- 2ap^{-2} - bq^- p^+} \beta_k^{q^-} \right)^{1/r^- - q^-}, \tag{104}$$

by Lemma 19, and we know that $\rho_k \rightarrow 0$ as $k \rightarrow \infty$. So, there exists $k_0 > 0$, such that $k > k_0$ and $u \in B_k$ with $\|u\|_{X_0} = \rho_k$, and we have $I(u) \geq 0$. Therefore, the condition (B1) of Theorem 10 holds.

Step 2. Verification of condition (B2). By (f2) and (f3), there exist $C_4, C_5 > 0$ such that

$$F(x, u) \geq C_4 |u|^{2p^+} - C_5 |u|^{p^+}, \text{ for all } (x, u) \in \Omega \times \mathbb{R}. \tag{105}$$

For $u \in A_k$, when $\|u\| = \gamma_k$ with $\rho_k > \gamma_k > 0$, we have

$$\begin{aligned}
I(u) &= a \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{p(x,y) |x - y|^{N+sp(x,y)}} dx dy \\
&\quad - \frac{b}{2} \left(\int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{p(x,y) |x - y|^{N+sp(x,y)}} dx dy \right)^2 \\
&\quad - \lambda \int_{\Omega} \frac{1}{q(x)} |u|^{q(x)} dx - \int_{\Omega} F(x, u) dx \\
&\leq a \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{p(x,y) |x - y|^{N+sp(x,y)}} dx dy \\
&\quad - \frac{b}{2} \left(\int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{p(x,y) |x - y|^{N+sp(x,y)}} dx dy \right)^2 \\
&\quad - \lambda \int_{\Omega} \frac{1}{q(x)} |u|^{q(x)} dx - \int_{\Omega} C_4 |u|^{2p^+} dx + \int_{\Omega} C_5 |u|^{p^+} dx \\
&\leq \frac{a}{p^-} \rho_{p(x,y)}(u) - \frac{b}{2p^{+2}} \rho_{p(x,y)}^2(u) - \frac{\lambda}{q^+} \rho_{q(x)}(u) \\
&\quad - C_4 \rho_{2p^+}(u) + C_5 \rho_{p^+}(u).
\end{aligned} \tag{106}$$

Since all norms on the finite-dimensional space A_k are equivalent, we obtain

$$\begin{aligned}
I(u) &\leq \frac{a}{p^-} \|u\|_{X_0}^{p^-} - \left(\frac{b}{2p^{+2}} + C_4 C_{2p^+} \right) \|u\|_{X_0}^{2p^+} - \frac{\lambda C_{q^+}}{q^+} \|u\|_{X_0}^{q^+} \\
&\quad + C_5 C_{p^+} \|u\|_{X_0}^{p^+}.
\end{aligned} \tag{107}$$

According to the condition $q^+ < p^- < p^+$ (see (12)), for some $\gamma_k = \|u\|_{X_0}$ that is small enough, we deduce that

$$b_k = \max_{u \in A_k, \|u\|_{X_0} = \gamma_k} I(u) \leq 0. \tag{108}$$

Therefore, the condition (B2) of Theorem 10 holds.

Step 3. Verification of condition (B3). By verification of condition (B1), one has that for $k \geq k_0$ and $u \in B_k$ with $\|u(x)\|_{X_0} \leq \gamma_k$,

$$I(u) \geq - \frac{(\lambda + C_2 C_p) \beta_k^{q^-}}{q^-} \|u\|_{X_0}^{q^-} \geq - \frac{(\lambda + C_2 C_p) \beta_k^{q^-}}{q^-} \gamma_k^{q^-}, \tag{109}$$

since $\gamma_k \rightarrow 0$ as $k \rightarrow \infty$. Therefore, the condition (B3) of Theorem 10 also holds.

By above three steps and Lemma 15, I satisfies all conditions of Theorem 10. The proof of Theorem 3 is completed.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

Each of the authors contributed to each part of this study equally, and all authors read and approved the final manuscript.

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