Research Article

On Generalized (p, q) -Euler Matrix and Associated Sequence Spaces

Taja Yaying ¹, Bipan Hazarika ² and Mohammad Mursaleen ^{3,4}

¹Department of Mathematics, Dera Natung Government College, Itanagar 791113, India

²Department of Mathematics, Gauhati University, Gauhati 781014, India

³Department of Medical Research, China Medical University Hospital, China Medical University (Taiwan), Taichung, Taiwan

⁴Al-Qaryah, Street No. 1 (West), Doharra, Aligarh 202002, India

Correspondence should be addressed to Mohammad Mursaleen; mursaleenm@gmail.com

Received 19 September 2020; Accepted 26 May 2021; Published 21 June 2021

Academic Editor: Mitsuru Sugimoto

Copyright © 2021 Taja Yaying et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In this study, we introduce new BK-spaces $b_s^{r,t}(p, q)$ and $b_\infty^{r,t}(p, q)$ derived by the domain of (p, q) -analogue $B^{r,t}(p, q)$ of the binomial matrix in the spaces ℓ_s and ℓ_∞ , respectively. We study certain topological properties and inclusion relations of these spaces. We obtain a basis for the space $b_s^{r,t}(p, q)$ and obtain Köthe-Toeplitz duals of the spaces $b_s^{r,t}(p, q)$ and $b_\infty^{r,t}(p, q)$. We characterize certain classes of matrix mappings from the spaces $b_s^{r,t}(p, q)$ and $b_\infty^{r,t}(p, q)$ to space $\mu \in \{\ell_\infty, c, c_0, \ell_1, bs, cs, cs_0\}$. Finally, we investigate certain geometric properties of the space $b_s^{r,t}(p, q)$.

1. Introduction and Preliminaries

The (p, q) -calculus has been a wide and interesting area of research in recent times. Several researchers have worked in the field of (p, q) -calculus due to its vast applications in mathematics, physics, and engineering sciences. In the field of mathematics, it is widely used by researchers in operator theory, approximation theory, hypergeometric functions, special functions, quantum algebras, combinatorics, etc. By (p, q) -analogue of a known mathematical expression, we mean the generalization of that expression using two independent variables p and q rather than a single variable q as in q -calculus. If we put $p = 1$ in the (p, q) -analogue of a known mathematical expression, we get q -analogue of that expression. Furthermore, when $q \rightarrow 1$, we receive the original expression. Chakrabarti and Jagannathan [1] introduced (p, q) -number to generalize several forms of q -oscillator algebras. Since then, several researchers used (p, q) -theory in different fields of mathematics to extend the theory of single parameter q -calculus. We strictly refer to [1–8] for studies in (p, q) -calculus and [9] in q -calculus.

1.1. Notations and Definitions on (p, q) -Calculus

Definition 1 (see [5]). Let $0 < q < p \leq 1$. Then, twin basic number or (p, q) -number is defined by

$$[i]_{pq} = \begin{cases} \frac{p^i - q^i}{p - q} & (i > 0), \\ 0 & (i = 0). \end{cases} \quad (1)$$

Clearly, when $p = 1$, $[i]_{pq}$ reduces to its q version $[i]_q$.

Definition 2 (see [4]). The (p, q) -analogue of binomial coefficient or (p, q) -binomial coefficient is defined by

$$\begin{bmatrix} i \\ j \end{bmatrix}_{pq} = \begin{cases} \frac{[i]_{pq}!}{[i-j]_{pq}![j]!} & (i \geq j), \\ 0 & (j > i), \end{cases} \quad (2)$$

where (p, q) -factorial $[i]_{pq}$ of i is given by

$$[i]_{pq}! = [i]_{pq}[i-1]_{pq} \cdots [2]_{pq}[1]_{pq}. \quad (3)$$

Lemma 3. The (p, q) -binomial formula is defined by

$$(x \oplus y)^i = \begin{cases} (x+y)(qx+py)(q^2x+p^2y) \cdots (q^{i-1}x+p^{i-1}y) & (i \geq 1), \\ 1 & (i = 0) \end{cases}$$

$$= \sum_{j=0}^i \begin{bmatrix} i \\ j \end{bmatrix}_{pq} p^{\binom{i-j}{2}} q^{\binom{j}{2}} x^j y^{i-j}. \quad (4)$$

1.2. Sequence Spaces. Let w denote the set of all real-valued sequences. Any linear subspace of w is called sequence space. The following are some sequence spaces which we shall be frequently used throughout this paper:

$$\begin{aligned} \ell_\infty &= \left\{ x = (x_i) \in w : \sup_{i \in \mathbb{N}_0} |x_i| < \infty \right\}, \ell_s = \left\{ x = (x_i) \in w : \sum_{i=0}^{\infty} |x_i|^s < \infty \right\}, (1 \leq s < \infty), \\ c &= \left\{ x = (x_i) \in w : \lim_{i \rightarrow \infty} x_i \text{ exists} \right\}, c_0 = \left\{ x = (x_i) \in w : \lim_{i \rightarrow \infty} x_i = 0 \right\}, \\ cs &= \left\{ x = (x_i) \in w : \lim_{i \rightarrow \infty} \sum_{j=0}^i x_j \text{ exists} \right\}, cs_0 = \left\{ x = (x_i) \in w : \lim_{i \rightarrow \infty} \sum_{j=0}^i x_j = 0 \right\}, \end{aligned} \quad (5)$$

and bs denotes the space of all bounded series.

Here, $\mathbb{N}_0$ denotes the set of all natural numbers including zero. The sequence spaces ℓ_s and ℓ_∞ are Banach spaces equipped with the norms

$$\|x\|_{\ell_s} = \left(\sum_{i=0}^{\infty} |x_i|^s \right)^{1/s} \quad \text{and} \quad \|x\|_{\ell_\infty} = \sup_{i \in \mathbb{N}_0} |x_i|, \quad (6)$$

respectively.

Let λ and μ be two sequence spaces and $\Phi = (\phi_{ij})$ be an infinite matrix of real entries. By Φ_i , we denote the i^{th} row of the matrix Φ . We say that Φ defines a matrix mapping from λ to μ if $\Phi x \in \mu$ for every $x = (x_j) \in \lambda$, where $\Phi x = \{(\Phi x)_i\} = \{\sum_{j=0}^{\infty} \phi_{ij} x_j\}$ is Φ -transform of the sequence x . The notation $(\lambda : \mu)$ will denote the family of all matrices that map from λ to μ .

The matrix domain λ_Φ of the matrix Φ in the space λ is defined by

$$\lambda_\Phi = \{x \in w : \Phi x \in \lambda\}, \quad (7)$$

which itself is a sequence space. Using this notation, several authors in the past have constructed sequence spaces using some special matrices. For relevant literature, we refer to the papers [10–15] and textbooks [16–18]. For some recent publications dealing with the domain of triangles in classical spaces, we refer [19–28].

1.3. Literature Review. We give a short survey of literature concerning Euler sequence spaces. Altay and Başar [10] introduced Euler sequence space $e_0^r = (c_0)_{E^r}$ and $e_\infty^r = (\ell_\infty)_{E^r}$, obtained their α -, β -, γ -, and continuous duals, and characterized certain class of matrix mappings on the space

$(c)_{E^r}$, where $E^r = (e_{ij}^r)$ denotes the Euler matrix of order r and is defined by

$$e_{ij}^r = \begin{cases} \binom{i}{j} (1-r)^{i-j} r^j & (0 \leq j \leq i), \\ 0 & (j > i), \end{cases} \quad (8)$$

for all $i, j \in \mathbb{N}_0$ and $0 < r < 1$. The Euler matrix E^r is regular for $0 < r < 1$ and is invertible with $(E^r)^{-1} = E^{1/r}$.

Altay et al. [11] introduced the Euler space $e_s^r = (\ell_s)_{E^r}$, $1 \leq s \leq \infty$, and obtained certain inclusion relations, Schauder basis and Köthe-Toeplitz duals of the space e_s^r . As a natural continuation of [11], Mursaleen et al. [14] characterized various classes of matrix mappings from the space e_s^r to other spaces and examined certain geometric properties of the space e_s^r . Further, Altay and Polat [29] introduced Euler difference spaces $e_0^r(\Delta^B) = (c_0)_{E^r \Delta^B}$ and $e_c^r(\Delta^B) = (c)_{E^r \Delta^B}$, where Δ^B is backward difference operator defined by $(\Delta^B v)_j = v_j - v_{j-1}$ for all $j \in \mathbb{N}_0$. Extending these spaces, Polat and Başar [30] studied Euler difference spaces $e_0^r(\Delta^{Bm}), e_c^r(\Delta^{Bm})$, and $e_\infty^r(\Delta^{Bm})$ of m^{th} ($m \in \mathbb{N}$) order defined as the set of all sequences whose m^{th} order backward differences are in the spaces e_0^r, e_c^r , and e_∞^r , respectively. Kadak and Baliarsingh [31] further generalized these spaces by introducing Euler difference spaces $e_s^r(\Delta^{Bq}), e_0^r(\Delta^{Bq}), e_c^r(\Delta^{Bq})$, and $e_\infty^r(\Delta^{Bq})$ of fractional order q , where Δ^{Bq} is the backward fractional difference operator defined by $(\Delta^{Bq} x)_j = \sum_{i=0}^{\infty} (-1)^i \Gamma(q+1)/i! \Gamma(q-i+1) x_{j-i}$. Kara et al. [32] introduced paranormed Euler space $e^r(p) = (\ell(p))_{E^r}$ and studied its topological and geometric properties. Aftermore, Karakaya and Polat [33] studied paranormed Euler difference sequence spaces $e_0^r(\Delta^B, p)$,

$e_c^r(\Delta^B, p)$, and $e_\infty^r(\Delta^B, p)$. Extending these spaces, Karakaya et al. [34] studied paranormed Euler backward difference spaces $e_0^r(\Delta^{Bm}, p)$, $e_c^r(\Delta^{Bm}, p)$, and $e_\infty^r(\Delta^{Bm}, p)$ of m^{th} order. Besides, Demiriz and Çakan [35] introduced paranormed Euler difference spaces $e_0^r(u, p)$ and $e_c^r(u, p)$. Furthermore, Kirisci [36] introduced Euler almost null $f_0(E^r)$ and Euler almost convergent $f(E^r)$ sequence spaces. Later on, Kara and Başar [37] introduced generalized difference Euler spaces $e_0^r(B^{(m)}) = (c_0)_{B^{(m)}}$, $e_c^r(B^{(m)}) = (c)_{B^{(m)}}$ and $e_\infty^r(B^{(m)}) = (\ell_\infty)_{B^{(m)}}$, where $B^{(m)} = (b_{ij}^{(m)})$ is a generalized difference matrix defined by

$$b_{ij}^{(m)} = \begin{cases} \binom{m}{i-j} r^{m-i+j} s^{i-j} & (\max\{0, i-m\} \leq j \leq i), \\ 0 & (0 \leq j \leq \max\{0, i-m\}) \text{ or } (j > i), \end{cases} \quad (9)$$

and characterized certain classes of compact operators on the spaces $e_0^r(B^{(m)})$ and $e_\infty^r(B^{(m)})$. Meng and Mei [38] gave a further generalization of [37] by introducing Euler difference spaces $e_0^r(B_v^{(m)})$ and $e_c^r(B_v^{(m)})$, where the difference operator $B_v^{(m)} = (b_{ij}^{(m),v})$ is defined by

$$b_{ij}^{(m),v} = \begin{cases} \binom{m}{i-j} r^{m-i+j} s^{i-j} v_j & (\max\{0, i-m\} \leq j \leq i), \\ 0 & (0 \leq j \leq \max\{0, i-m\}) \text{ or } (j > i), \end{cases} \quad (10)$$

where $v = (v_j)$ is a fixed sequence of nonzero real numbers. Recently, Bisgin [39, 40] introduced more generalized Euler space by defining binomial spaces $b_s^{r,t} = (\ell_s)_{B^{r,t}}$, $b_0^{r,t} = (c_0)_{B^{r,t}}$, $b_c^{r,t} = (c)_{B^{r,t}}$, and $b_\infty^{r,t} = (\ell_\infty)_{B^{r,t}}$, and $B^{r,t} = (b_{ij}^{r,t})$ is the binomial matrix defined by

$$b_{ij}^{r,t} = \begin{cases} \frac{1}{(r+t)^i} \binom{i}{j} r^i s^{i-j} & (0 \leq j \leq i), \\ 0 & (j > i). \end{cases} \quad (11)$$

Meng and Song [41] further generalized these spaces by introducing binomial $B^{(m)}$ -difference sequence spaces $b_0^{r,t}(B^{(m)}) = (c_0)_{B^{r,t}B^{(m)}}$, $b_c^{r,t}(B^{(m)}) = (c)_{B^{r,t}B^{(m)}}$, and $e_\infty^r(B^{(m)}) = (\ell_\infty)_{B^{r,t}B^{(m)}}$. Meng and Mei [42] studied binomial backward difference sequence spaces $b_0^{r,t}(\Delta^{Bq}) = (c_0)_{B^{r,t}\Delta^{Bq}}$, $b_c^{r,t}(\Delta^{Bq}) = (c)_{B^{r,t}\Delta^{Bq}}$, and $b_\infty^{r,t}(\Delta^{Bq}) = (\ell_\infty)_{B^{r,t}\Delta^{Bq}}$ of fractional order q . Besides, Yaying and Hazarika [27] also studied binomial backward difference spaces $b_s^{r,t}(\Delta^{Bq}) = (\ell_s)_{B^{r,t}\Delta^{Bq}}$ of fractional order q .

For $0 < q < 1$, the q -Cesàro matrix $C(q) = (c_{ij}^q)$ [43, 44] is defined by

$$c_{ij}^q = \begin{cases} \frac{q^j}{[i+1]_q} & (0 \leq j \leq i), \\ 0 & (j > i). \end{cases} \quad (12)$$

Demiriz and Sahin [45] studied the domain of q -Cesàro mean in the spaces c and c_0 . Very recently, Yaying et al. [28] studied Banach sequence spaces $\mathcal{X}_s^q$ and $\mathcal{X}_\infty^q$ defined as the domain of q -Cesàro mean in the spaces ℓ_s and ℓ_∞ , respectively, and studied associated operator ideals.

Motivated by the above studies, we generalize Euler mean E^r and Binomial mean $B^{r,t}$ in the sense of (p, q) -theory to $B^{r,t}(p, q)$ and study its domain $b_s^{r,t}(p, q)$ and $b_\infty^{r,t}(p, q)$ in the spaces ℓ_s and ℓ_∞ , respectively. We investigate some topological properties and inclusion relations of the spaces $b_s^{r,t}(p, q)$ and $b_\infty^{r,t}(p, q)$ and obtain a basis for the space $b_s^{r,t}(p, q)$. In Section 3, we obtain the Köthe-Toeplitz duals (α -, β -, and γ -duals) of the spaces $b_s^{r,t}(p, q)$ and $b_\infty^{r,t}(p, q)$. In Section 4, we characterize some matrix mappings from $b_s^{r,t}(p, q)$ and $b_\infty^{r,t}(p, q)$ spaces to space $\mu \in \{\ell_\infty, c, c_0, \ell_1, cs, cs_0, bs\}$. Section 5 is devoted to investigation of certain geometric properties like Banach-Saks of type s and modulus of convexity of the space $b_s^{r,t}(p, q)$.

In the rest of the paper, $1 \leq s < \infty$, unless stated otherwise.

2. Generalized Euler Sequence Spaces $b_s^{r,t}(p, q)$ and $b_\infty^{r,t}(p, q)$

In this section, we introduce sequence spaces $b_s^{r,t}(p, q)$ and $b_\infty^{r,t}(p, q)$, study their topological properties and some inclusion relations, and obtain a basis for the space $b_s^{r,t}(p, q)$.

Let r, t be nonnegative real numbers and $0 < q < p \leq 1$ holds, then the generalized (p, q) -Euler matrix $B^{r,t}(p, q) = (b_{ij}^{r,t})$ of order (r, t) is defined by

$$b_{ij}^{r,t} = \begin{cases} \frac{1}{(r \oplus t)^i} \begin{bmatrix} i \\ j \end{bmatrix}_{pq} p^{\binom{i-j}{2}} q^{\binom{j}{2}} r^j t^{i-j} & (0 \leq j \leq i), \\ 0 & (j > i). \end{cases} \quad (13)$$

One can clearly observe that the matrix $B^{r,t}(p, q)$ reduces to the binomial matrix $B^{r,t}$ when $p = q = 1$. Thus, $B^{r,t}(p, q)$ generalizes binomial matrix $B^{r,t}$. We may call the matrix $B^{r,t}(p, q)$ as the (p, q) -analogue of the binomial matrix $B^{r,t}$. We also realise that when $p = 1$, the matrix $B^{r,t}(p, q)$ reduces to its q -version $B^{r,t}(q)$ with entries

$$\begin{bmatrix} i \\ j \end{bmatrix}_q \binom{j}{2} r^j t^{i-j} / (r \oplus t)_q^i \text{ if } 0 \leq j \leq i \text{ and } 0, \text{ otherwise. We}$$

call $B^{r,t}(q)$ as the q -analogue of the binomial matrix $B^{r,t}$. Moreover, when $t = 1 - r$, then the matrix $B^{r,t}(p, q)$

reduces to $E^r(p, q)$ with entries $1/(r \oplus (1 - r))^i \begin{bmatrix} i \\ j \end{bmatrix}_{pq}$

$p \begin{pmatrix} i-j \\ 2 \end{pmatrix} q \begin{pmatrix} j \\ 2 \end{pmatrix} r^j (1-r)^{i-j}$ if $0 \leq j \leq i$ and 0, otherwise. The generalized (p, q) -Euler sequence spaces $b_s^{r,t}(p, q)$ and $b_\infty^{r,t}(p, q)$ are defined by

$$b_s^{r,t}(p, q) = \left\{ x = (x_j) \in w : \sum_{i=0}^{\infty} \left| \frac{1}{(r \oplus t)^i} \sum_{j=0}^i \begin{bmatrix} i \\ j \end{bmatrix}_{pq} p \begin{pmatrix} i-j \\ 2 \end{pmatrix} q \begin{pmatrix} j \\ 2 \end{pmatrix} r^j t^{i-j} \right|^s < \infty \right\},$$

$$b_\infty^{r,t}(p, q) = \left\{ x = (x_j) \in w : \sup_{i \in \mathbb{N}_0} \left| \frac{1}{(r \oplus t)^i} \sum_{j=0}^i \begin{bmatrix} i \\ j \end{bmatrix}_{pq} p \begin{pmatrix} i-j \\ 2 \end{pmatrix} q \begin{pmatrix} j \\ 2 \end{pmatrix} r^j t^{i-j} \right| < \infty \right\}. \quad (14)$$

The above sequence spaces can be redefined in the notation of (7) by

$$b_s^{r,t}(p, q) = (\ell_s)_{B^{r,t}(p,q)} \text{ and } b_\infty^{r,t}(p, q) = (\ell_\infty)_{B^{r,t}(p,q)}. \quad (15)$$

The spaces $b_s^{r,t}(p, q)$ and $b_\infty^{r,t}(p, q)$ reduce to the following classes of spaces in the special cases of (p, q) and (r, t) :

- (1) When $p = 1$, the spaces $b_s^{r,t}(p, q)$ and $b_\infty^{r,t}(p, q)$ reduce to q -binomial sequence spaces $b_s^{r,t}(q) = (\ell_s)_{B^{r,t}(q)}$ and $b_\infty^{r,t}(q) = (\ell_\infty)_{B^{r,t}(q)}$, respectively, which further reduce to binomial sequence spaces $b_s^{r,t}$ and $b_\infty^{r,t}$, respectively, when $q \rightarrow 1$, as studied by Bisgin [40]
- (2) When $p = 1$ and $r + t = 1$, the spaces $b_s^{r,t}(p, q)$ and $b_\infty^{r,t}(p, q)$ reduce to q -Euler space $e_s^r(q) = (\ell_s)_{E^r(q)}$ and $e_\infty^r(q) = (\ell_\infty)_{E^r(q)}$, respectively, which further reduce to well known Euler sequence spaces e_s^r and e_∞^r , respectively, when $q \rightarrow 1$, as studied by Altay et al. [11]
- (3) When $r + t = 1$, the spaces $b_s^{r,t}(p, q)$ and $b_\infty^{r,t}(p, q)$ reduce to (p, q) -Euler sequence spaces $e_s^r(p, q) = (\ell_s)_{E^r(p,q)}$ and $e_\infty^r(p, q) = (\ell_\infty)_{E^r(p,q)}$

Let us define a sequence $y = (y_i)$ in terms of sequence $x = (x_j)$ by

$$y_i = (B^{r,t}(p, q)x)_i = \frac{1}{(r \oplus t)^i} \sum_{j=0}^i \begin{bmatrix} i \\ j \end{bmatrix}_{pq} p \begin{pmatrix} i-j \\ 2 \end{pmatrix} q \begin{pmatrix} j \\ 2 \end{pmatrix} r^j t^{i-j} x_j, \quad (16)$$

for each $i \in \mathbb{N}_0$. The sequence y is called $B^{r,t}(p, q)$ -transform of the sequence x . Further, on using (16), we write

$$x_i = \sum_{j=0}^i (-1)^{i-j} \frac{\begin{bmatrix} i \\ j \end{bmatrix}_{pq} q \begin{pmatrix} i-j \\ 2 \end{pmatrix} t^{i-j} (r \oplus t)^j}{r^i q \begin{pmatrix} i \\ 2 \end{pmatrix}} y_j, \quad (17)$$

for each $i \in \mathbb{N}_0$.

It is known that if λ is a BK-space and Φ is a triangle then the domain λ_Φ of the matrix Φ in the space λ is also a BK-space equipped with the norm $\|x\|_{\lambda_\Phi} = \|\Phi x\|_\lambda$. In the light of this, we have the following result.

Theorem 4. The sequence spaces $b_s^{r,t}(p, q)$ and $b_\infty^{r,t}(p, q)$ are BK-spaces equipped with the norms defined by

$$\|x\|_{b_s^{r,t}(p,q)} = \|B^{r,t}(p, q)x\|_{\ell_s} = \left(\sum_{i=0}^{\infty} \left| \frac{1}{(r \oplus t)^i} \sum_{j=0}^i \begin{bmatrix} i \\ j \end{bmatrix}_{pq} p \begin{pmatrix} i-j \\ 2 \end{pmatrix} q \begin{pmatrix} j \\ 2 \end{pmatrix} r^j t^{i-j} x_j \right|^s \right)^{1/s},$$

$$\|x\|_{b_\infty^{r,t}(p,q)} = \|B^{r,t}(p, q)x\|_{\ell_\infty} = \sup_{i \in \mathbb{N}_0} \left| \frac{1}{(r \oplus t)^i} \sum_{j=0}^i \begin{bmatrix} i \\ j \end{bmatrix}_{pq} p \begin{pmatrix} i-j \\ 2 \end{pmatrix} q \begin{pmatrix} j \\ 2 \end{pmatrix} r^j t^{i-j} x_j \right|, \quad (18)$$

respectively.

Proof. The proof is a routine exercise and hence omitted. $\square$

Theorem 5. The sequence spaces $b_s^{r,t}(p, q)$ and $b_\infty^{r,t}(p, q)$ are linearly isomorphic to ℓ_s and ℓ_∞ , respectively.

Proof. We provide the proof for the space $b_s^{r,t}(p, q)$. Define the mapping $T : b_s^{r,t}(p, q) \longrightarrow \ell_s$ by $Tx = B^{r,t}(p, q)x$ for all $x \in b_s^{r,t}(p, q)$. It is easy to observe that T is linear and one to one. Let $y = (y_i) \in \ell_s$ and $x = (x_i)$ is as defined in (17). Then, we have

$$\begin{aligned} \|x\|_{b_s^{r,t}(p,q)} &= \left(\sum_{i=0}^{\infty} \left| \frac{1}{(r \oplus t)^i} \sum_{j=0}^i \begin{bmatrix} i \\ j \end{bmatrix}_{pq} p \binom{i-j}{2} q \binom{j}{2} r^j t^{i-j} x_j \right|^s \right)^{1/s} \\ &= \left(\sum_{i=0}^{\infty} \left| \frac{1}{(r \oplus t)^i} \sum_{j=0}^i \begin{bmatrix} i \\ j \end{bmatrix}_{pq} p \binom{i-j}{2} q \binom{j}{2} r^j t^{i-j} x_j \right|^s \right)^{1/s} \\ &\quad \cdot \left(\sum_{k=0}^j (-1)^{j-k} \frac{\begin{bmatrix} i \\ j \end{bmatrix}_{pq} q \binom{j-k}{2} t^{j-k} (r \oplus t)^k}{r^j q \binom{j}{2}} y_k \right)^s \right)^{1/s} \\ &= \left(\sum_{i=0}^{\infty} |y_i|^s \right)^{1/s} = \|y\|_{\ell_s} < \infty. \end{aligned} \quad (19)$$

Thus, $x \in b_s^{r,t}(p, q)$ and the mapping $T : b_s^{r,t}(p, q) \longrightarrow \ell_s$ is onto and norm preserving. Hence, the space $b_s^{r,t}(p, q)$ is linearly isomorphic to ℓ_s . This completes the proof. $\square$

Theorem 6. The space $b_s^{r,t}(p, q)$, $1 \leq s \leq \infty$, is not a Hilbert space, except for the case $s = 2$.

Proof. Define the sequences $x = (x_i)$ and $y = (y_i)$ by

$$x_i = \begin{cases} 1 & (i = 0), \\ (-1)^i \left(\left(\frac{t}{r} \right)^i - \frac{(r \oplus t) q \binom{i-1}{2} t^{i-1} \begin{bmatrix} i \\ 1 \end{bmatrix}_{pq}}{r^i q \binom{i}{2}} \right) & (i > 0), \end{cases}$$

$$y_i = \begin{cases} 1 & (i = 0), \\ (-1)^i \left(\left(\frac{t}{r} \right)^i + \frac{(r \oplus t) q \binom{i-1}{2} t^{i-1} \begin{bmatrix} i \\ 1 \end{bmatrix}_{pq}}{r^i q \binom{i}{2}} \right) & (i > 0). \end{cases} \quad (20)$$

We realise that $(B^{r,t}(p, q)x)_i = (1, 1, 0, 0, \dots)$ and $(B^{r,t}(p, q)y)_i = (1, -1, 0, 0, \dots)$. Then

$$\begin{aligned} \|x + y\|_{b_s^{r,t}(p,q)}^2 + \|x - y\|_{b_s^{r,t}(p,q)}^2 &= 8 \neq 2^{2+2/s} \\ &= 2 \left(\|x\|_{b_s^{r,t}(p,q)}^2 + \|y\|_{b_s^{r,t}(p,q)}^2 \right). \end{aligned} \quad (21)$$

Thus, $b_s^{r,t}(p, q)$ norm violates the parallelogram identity. Hence, $b_s^{r,t}(p, q)$ is not a Hilbert space, except for the case $s = 2$. $\square$

Now we give certain inclusion relations related to the spaces $b_s^{r,t}(p, q)$ and $b_\infty^{r,t}(p, q)$.

Theorem 7. The inclusion $\ell_s \subset b_s^{r,t}(p, q)$, $1 \leq s \leq \infty$, strictly holds.

Proof. We provide proof of the inclusion $\ell_s \subset b_s^{r,t}(p, q)$, $1 \leq s < \infty$. Let $x = (x_i) \in \ell_s$ for $1 < s < \infty$. Applying Hölder's inequality, we have

$$\begin{aligned} \sum_{i=0}^{\infty} |(B^{r,t}(p, q)x)_i|^s &\leq \sum_{i=0}^{\infty} \left(\sum_{j=0}^i \frac{\begin{bmatrix} i \\ j \end{bmatrix}_{pq} p \binom{i-j}{2} q \binom{j}{2} r^j t^{i-j}}{(r \oplus t)^i} |x_j| \right)^s \\ &\leq \sum_{i=0}^{\infty} \left(\sum_{j=0}^i \frac{\begin{bmatrix} i \\ j \end{bmatrix}_{pq} p \binom{i-j}{2} q \binom{j}{2} r^j t^{i-j}}{(r \oplus t)^i} |x_j|^s \right)^s \end{aligned}$$

$$\begin{aligned}
& \left(\sum_{j=0}^i \frac{\begin{bmatrix} i \\ j \end{bmatrix}_{pq} p \binom{i-j}{2} q \binom{j}{2} r^j t^{i-j}}{(r \oplus t)^i} \right)^{s-1} \\
&= \sum_{i=0}^{\infty} \left(\sum_{j=0}^i \frac{\begin{bmatrix} i \\ j \end{bmatrix}_{pq} p \binom{i-j}{2} q \binom{j}{2} r^j t^{i-j}}{(r \oplus t)^i} |x_j|^s \right) \\
&= \sum_{j=0}^{\infty} |x_j|^s \left(\sum_{i=j}^{\infty} \frac{\begin{bmatrix} i \\ j \end{bmatrix}_{pq} p \binom{i-j}{2} q \binom{j}{2} r^j t^{i-j}}{(r \oplus t)^i} \right). \tag{22}
\end{aligned}$$

Thus, $\|x\|_{b_s^{r,t}(p,q)}^s \leq K \|x\|_{\ell_s}^s < \infty$, where $K = \sup_{j \in \mathbb{N}_0} \left(\sum_{i=j}^{\infty} \begin{bmatrix} i \\ j \end{bmatrix}_{pq} \right)$

$p \binom{i-j}{2} q \binom{j}{2} r^j t^{i-j} / (r \oplus t)^i$, provided K exists. This yields the fact that $x \in b_s^{r,t}(p, q)$. Thus, $\ell_s \subset b_s^{r,t}(p, q)$. Similarly, we can show that $\ell_1 \subset b_1^{r,t}(p, q)$.

Now consider the sequence $x_i = (-1)^i$, then it is easy to see that $x_i \in b_s^{r,t}(p, q) \setminus \ell_s$. Hence, the inclusion $\ell_s \subset b_s^{r,t}(p, q)$ is strict. This completes the proof. $\square$

Theorem 8. The inclusion $b_s^{r,t}(p, q) \subset b_k^{r,t}(p, q)$, $1 \leq s < k < \infty$, strictly holds.

Proof. It is known that inclusion $\ell_s \subset \ell_k$ holds for $1 \leq s < k < \infty$ and the mapping $B^{r,t}(p, q): b_s^{r,t}(p, q) \longrightarrow \ell_s$ is isomorphic, therefore, the inclusion $b_s^{r,t}(p, q) \subset b_k^{r,t}(p, q)$ holds. To prove the strictness part, we recall that the inclusion $\ell_s \subset \ell_k$ strictly holds for $1 \leq s < k < \infty$. We choose $y \in \ell_k \setminus \ell_s$ and x as defined in (17). Then, $B^{r,t}(p, q)x = y \in \ell_k \setminus \ell_s$. This implies that $x \in b_k^{r,t}(p, q) \setminus b_s^{r,t}(p, q)$. Hence, the inclusion $b_s^{r,t}(p, q) \subset b_k^{r,t}(p, q)$ is strict. $\square$

Theorem 9. The inclusion $b_s^{r,t}(p, q) \subset b_{\infty}^{r,t}(p, q)$ strictly holds.

Proof. The proof is similar to the proof of Theorem 8. To show the strictness part, we consider the sequence $x_k = (1, 1,$

$1, \dots)$. Then, it is clear that $x \in b_{\infty}^{r,t}(p, q) \setminus b_s^{r,t}(p, q)$. Hence, the inclusion $b_s^{r,t}(p, q) \subset b_{\infty}^{r,t}(p, q)$ strictly holds. $\square$

We recall that domain λ_{Φ} of a triangle Φ in space λ has a basis if and only if λ has a basis. This statement together with Theorem 5 gives us the following result.

Theorem 10. Let $\xi_j = (B^{r,t}(p, q)x)_j$ for each $j \in \mathbb{N}_0$. Define the sequence $b^{(j)}(p, q) = (b_i^{(j)}(p, q))$ of elements of the space $b_s^{r,t}(p, q)$ for every fixed $j \in \mathbb{N}_0$ by

$$b_i^{(j)}(p, q) = \begin{cases} (-1)^{i-j} \frac{\begin{bmatrix} i \\ j \end{bmatrix}_{pq} q \binom{i-j}{2} t^{i-j} (r \oplus t)^j}{r^i q \binom{i}{2}} & (j \leq i), \\ 0 & (j > i). \end{cases} \tag{23}$$

Then, the sequence $\{b^{(j)}(p, q)\}$ forms a basis for the space $b_s^{r,t}(p, q)$ and every $x \in b_s^{r,t}(p, q)$ can be uniquely expressed in the form $x = \sum_{j=0}^{\infty} \xi_j b^{(j)}(p, q)$ for each $j \in \mathbb{N}_0$.

3. Köthe-Toeplitz Duals

In this section, we obtain Köthe-Toeplitz duals (α -, β -, and γ -duals) of the spaces $b_s^{r,t}(p, q)$ and $b_{\infty}^{r,t}(p, q)$. We omit the proofs for cases $s = 1$ and $s = \infty$ as these can be obtained by analogy and provide proofs for only the case $1 < s < \infty$ in the current section. First, we recall the definitions of Köthe-Toeplitz duals.

Definition 11. The Köthe-Toeplitz duals or α -, β -, and γ -duals of subset $\lambda \subset w$ are defined by

$$\begin{aligned}
\lambda^{\alpha} &= \{a = (a_i) \in w : ax = (a_i x_i) \in \ell_1 \text{ for all } x \in \lambda\}, \\
\lambda^{\beta} &= \{a = (a_i) \in w : ax = (a_i x_i) \in cs \text{ for all } x \in \lambda\} \text{ and,} \\
\lambda^{\gamma} &= \{a = (a_i) \in w : ax = (a_i x_i) \in bs \text{ for all } x \in \lambda\}, \tag{24}
\end{aligned}$$

respectively.

Quite recently, Talebi [25] obtained Köthe-Toeplitz duals of the domain of an arbitrary invertible summability matrix in ℓ_s space. We follow his approach to find the Köthe-Toeplitz duals of the spaces $b_s^{r,t}(p, q)$ and $b_{\infty}^{r,t}(p, q)$. In the rest of the paper, $\mathcal{N}$ will denote the family of all finite subsets of $\mathbb{N}_0$ and $k = s/1 - s$ is the complement of s .

Theorem 12. Define the sets $d^{(k)}(p, q)$ and $d_{\infty}(p, q)$ by

$$d^{(k)}(p, q) = \left\{ a = (a_i) \in w : \sup_{N \in \mathcal{N}} \sum_{j=0}^{\infty} \left| \sum_{i \in N} (-1)^{i-j} \frac{\begin{bmatrix} i \\ j \end{bmatrix}_{pq} q^{\binom{i-j}{2}} t^{i-j}(r \oplus t)^j}{r^i q^{\binom{i}{2}}} a_i \right|^k < \infty \right\} \text{ and,} \quad (25)$$

$$d_{\infty}(p, q) = \left\{ a = (a_i) \in w : \sup_{j \in \mathbb{N}_0} \sum_{i=0}^{\infty} (-1)^{i-j} \frac{\begin{bmatrix} i \\ j \end{bmatrix}_{pq} q^{\binom{i-j}{2}} t^{i-j}(r \oplus t)^j}{r^i q^{\binom{i}{2}}} a_i < \infty \right\}.$$

Then, $[b_i^{r,t}(p, q)]^{\alpha} = d_{\infty}(p, q)$, $[b_s^{r,t}(p, q)]^{\alpha} = d^{(k)}(p, q)$ and $[b_{\infty}^{r,t}(p, q)]^{\alpha} = d^{(1)}(p, q)$.

for all $i \in \mathbb{N}_0$, where the matrix $G^{r,t}(p, q) = (g_{ij}^{r,t}(p, q))$ is defined by

Proof. Let $1 < s < \infty$. Let $(a_i) \in w$ and $y = (y_i)$ be the $B^{r,t}(p, q)$ -transform of sequence $x = (x_i)$. Then, from the equality (17), we have

$$a_i x_i = \sum_{j=0}^i (-1)^{i-j} \frac{\begin{bmatrix} i \\ j \end{bmatrix}_{pq} q^{\binom{i-j}{2}} t^{i-j}(r \oplus t)^j}{r^i q^{\binom{i}{2}}} a_j y_j = (G^{r,t}(p, q) y)_i, \quad (26)$$

$$g_{ij}^{r,t}(p, q) = \begin{cases} (-1)^{i-j} \frac{\begin{bmatrix} i \\ j \end{bmatrix}_{pq} q^{\binom{i-j}{2}} t^{i-j}(r \oplus t)^j}{r^i q^{\binom{i}{2}}} a_i & (0 \leq j \leq i), \\ 0 & (j > i). \end{cases} \quad (27)$$

Applying Theorem 2.1 of [25], we immediately obtained that

$$[b_s^{r,t}(p, q)]^{\alpha} = \left\{ a = (a_i) \in w : \sup_{N \in \mathcal{N}} \sum_{j=0}^{\infty} \left| \sum_{i \in N} (-1)^{i-j} \frac{\begin{bmatrix} i \\ j \end{bmatrix}_{pq} q^{\binom{i-j}{2}} t^{i-j}(r \oplus t)^j}{r^i q^{\binom{i}{2}}} a_i \right|^k < \infty \right\}. \quad (28)$$

This completes the proof. $\square$

Theorem 13. Define the sets $d_1(p, q), d_2(p, q), d_3(p, q)$, and $d^{[k]}(p, q)$ by

$$\begin{aligned}
 d_1(p, q) &= \left\{ a = (a_l) \in w : \sum_{l=j}^{\infty} (-1)^{l-j} \frac{\begin{bmatrix} l \\ j \end{bmatrix}_{pq} q^{\binom{l-j}{2}} t^{l-j}(r \oplus t)^j}{r^l q^{\binom{l}{2}}} a_l \text{ exists for each } j \in \mathbb{N}_0 \right\}, \\
 d_2(p, q) &= \left\{ a = (a_l) \in w : \sup_{i, j \in \mathbb{N}_0} \left| \sum_{l=j}^i (-1)^{l-j} \frac{\begin{bmatrix} l \\ j \end{bmatrix}_{pq} q^{\binom{l-j}{2}} t^{l-j}(r \oplus t)^j}{r^l q^{\binom{l}{2}}} a_l \right| < \infty \right\}, \\
 d_3(p, q) &= \left\{ a = (a_l) \in w : \lim_{i \rightarrow \infty} \sum_{j=0}^i \left| \sum_{l=j}^i (-1)^{l-j} \frac{\begin{bmatrix} l \\ j \end{bmatrix}_{pq} q^{\binom{l-j}{2}} t^{l-j}(r \oplus t)^j}{r^l q^{\binom{l}{2}}} a_l \right| = \sum_{j=0}^i \lim_{i \rightarrow \infty} \sum_{l=j}^i (-1)^{l-j} \frac{\begin{bmatrix} l \\ j \end{bmatrix}_{pq} q^{\binom{l-j}{2}} t^{l-j}(r \oplus t)^j}{r^l q^{\binom{l}{2}}} a_l \right\}, \\
 d^{[k]}(p, q) &= \left\{ a = (a_l) \in w : \sup_{i \in \mathbb{N}_0} \sum_{j=0}^i \left| \sum_{l=j}^i (-1)^{l-j} \frac{\begin{bmatrix} l \\ j \end{bmatrix}_{pq} q^{\binom{l-j}{2}} t^{l-j}(r \oplus t)^j}{r^l q^{\binom{l}{2}}} a_l \right|^k < \infty \right\}. \tag{29}
 \end{aligned}$$

Then, $[b_i^{r,t}(p, q)]^\beta = d_1(p, q) \cap d_2(p, q), [b_s^{r,t}]^\beta = d_1(p, q) \cap d^{[k]}(p, q)$ and $[b_\infty^{r,t}(p, q)]^\beta = d_1(p, q) \cap d_3(p, q)$.

Proof. Let $(a_i) \in w$ and $y = (y_i)$ be the $B^{r,t}(p, q)$ -transform of sequence $x = (x_i)$. Then, from the equality (17), we get

$$\sum_{j=0}^i a_j x_j = \sum_{j=0}^i \left\{ \sum_{l=0}^j (-1)^{j-l} \frac{\begin{bmatrix} j \\ l \end{bmatrix}_{pq} q^{\binom{j-l}{2}} t^{j-l}(r \oplus t)^l}{r^j q^{\binom{j}{2}}} y_l \right\} a_j$$

$$\begin{aligned}
 &= \sum_{j=0}^i \left\{ \sum_{l=j}^i (-1)^{l-j} \frac{\begin{bmatrix} l \\ j \end{bmatrix}_{pq} q^{\binom{l-j}{2}} t^{l-j}(r \oplus t)^j}{r^l q^{\binom{l}{2}}} a_l \right\} y_j, \\
 &= (H^{r,t}(p, q)y)_i, \tag{30}
 \end{aligned}$$

for each $i \in \mathbb{N}_0$, where the matrix $H^{r,t}(p, q) = (h_{ij}^{r,t}(p, q))$ is defined by

$$h_{ij}^{r,t}(p, q) = \begin{cases} \sum_{l=j}^i (-1)^{l-j} \frac{\begin{bmatrix} l \\ j \end{bmatrix}_{pq} q^{\binom{l-j}{2}} t^{l-j}(r \oplus t)^j}{r^l q^{\binom{l}{2}}} a_l & (0 \leq j \leq i), \\ 0 & (j > i), \end{cases} \quad (31)$$

for all $i, j \in \mathbb{N}_0$.

Thus, by applying Theorem 2.2 of [25], we straightly get

$$[b_s^{r,t}(p, q)]^\beta = d_1(p, q) \cap d^{[k]}(p, q). \quad (32)$$

This completes the proof. $\square$

Theorem 14. Define the set $d_4(p, q)$ by

$$d_4(p, q) = \left\{ a = (a_i) \in w : \sup_{i \in \mathbb{N}_0} \sum_{j=0}^i \left| \sum_{l=j}^i (-1)^{l-j} \frac{\begin{bmatrix} l \\ j \end{bmatrix}_{pq} q^{\binom{l-j}{2}} t^{l-j}(r \oplus t)^j}{r^l q^{\binom{l}{2}}} a_l \right| < \infty \right\}. \quad (33)$$

Then, $[b_i^{r,t}(p, q)]^\gamma = d_2(p, q), [b_s^{r,t}(p, q)]^\gamma = d^{[k]}(p, q)$ and $[b_\infty^{r,t}(p, q)]^\gamma = d_4(p, q)$.

Proof. The proof is similar to the previous theorem except that Theorem 2.3 of [25] is employed instead of Theorem 2.2 of [25]. $\square$

4. Matrix Mappings

In this section, we characterize a certain class of matrix mappings from the spaces $b_s^{r,t}(p, q)$ and $b_\infty^{r,t}(p, q)$ to space $\mu \in \{\ell_\infty, c, c_0, \ell_1, bs, cs, cs_0\}$. The following theorem is fundamental in our investigation.

Theorem 15. Let $1 \leq s \leq \infty$ and λ be an arbitrary subset of w . Then, $\Phi = (\phi_{ij}) \in (b_s^{r,t}(p, q) : \lambda)$ if and only if $\Psi^{(i)} = (\psi_{mj}^{(i)}) \in (\ell_s : c)$ for each $i \in \mathbb{N}_0$, and $\Psi = (\psi_{ij}) \in (\ell_s : \lambda)$, where

$$\psi_{mj}^{(i)} = \begin{cases} 0 & (j > m), \\ \sum_{l=j}^m (-1)^{l-j} \frac{\begin{bmatrix} l \\ j \end{bmatrix}_{pq} q^{\binom{l-j}{2}} t^{l-j}(r \oplus t)^j}{r^l q^{\binom{l}{2}}} \phi_{il} & (0 \leq j \leq m), \end{cases}$$

for all $i, j \in \mathbb{N}_0$.

Proof. The proof is similar to the proof of Theorem 4.1 of [13]. Hence, we omit details. $\square$

Now, using the results presented in Stielglitz and Tietz [46] together with Theorem 15, we obtain the following results:

Corollary 16. The following statements hold:

(1) $\Phi \in (b_1^{r,t}(p, q) : \ell_\infty)$ if and only if

$$\lim_{m \rightarrow \infty} \psi_{mj}^{(i)} \text{ exists for all } i, j \in \mathbb{N}_0, \quad (35)$$

$$\sup_{i, j \in \mathbb{N}_0} |\psi_{mj}^{(i)}| < \infty, \quad (36)$$

$$\sup_{i, j \in \mathbb{N}_0} |\psi_{ij}| < \infty, \quad (37)$$

- (2) $\Phi \in (b_1^{r,t}(p, q): c)$ if and only if (35) and (36) hold, and (37) and

$$\lim_{i \rightarrow \infty} \psi_{ij} \text{ exists for all } j \in \mathbb{N}_0, \quad (38)$$

also hold

- (3) $\Phi \in (b_1^{r,t}(p, q): c_0)$ if and only if (35) and (36) hold, and (37) and

$$\lim_{i \rightarrow \infty} \psi_{ij} = 0 \text{ for all } j \in \mathbb{N}_0, \quad (39)$$

also hold

- (4) $\Phi \in (b_1^{r,t}(p, q): \ell_1)$ if and only if (35) and (36) hold, and

$$\sup_{j \in \mathbb{N}_0} \sum_{i=0}^{\infty} |\psi_{ij}| < \infty, \quad (40)$$

also holds

- (5) $\Phi \in (b_1^{r,t}(p, q): bs)$ if and only if (35) and (36) hold, and (37) also holds with $\Psi(i, j)$ instead of ψ_{ij} , where $\Psi(i, j) = \sum_{l=0}^i \psi_{lj}$

- (6) $\Phi \in (b_1^{r,t}(p, q): cs)$ if and only if (35) and (36) hold, and (37) and (38) also hold with $\Psi(i, j)$ instead of ψ_{ij} , where $\Psi(i, j) = \sum_{l=0}^i \psi_{lj}$

- (7) $\Phi \in (b_1^{r,t}(p, q): cs_0)$ if and only if (35) and (36) hold, and (37) and (39) also hold with $\Psi(i, j)$ instead of ψ_{ij} , where $\Psi(i, j) = \sum_{l=0}^i \psi_{lj}$

Corollary 17. The following statements hold:

- (1) $\Phi \in (b_s^{r,t}(p, q): \ell_{\infty})$ if and only if (35) holds, and

$$\sup_{m \in \mathbb{N}_0} \sum_{j=0}^m |\psi_{mj}^{(i)}|^k < \infty, \quad (41)$$

$$\sup_{i \in \mathbb{N}_0} \sum_{j=0}^i |\psi_{ij}|^k < \infty, \quad (42)$$

also hold

- (2) $\Phi \in (b_s^{r,t}(p, q): c)$ if and only if (35) and (41) hold, and (38) and (42) also hold

- (3) $\Phi \in (b_s^{r,t}(p, q): c_0)$ if and only if (35) and (41) hold, (39) and (42) also hold

- (4) $\Phi \in (b_s^{r,t}(p, q): \ell_1)$ if and only if (35) and (41) hold, and

$$\sup_{N \in \mathcal{N}} \sum_{j=0}^{\infty} \left| \sum_{i \in N} \psi_{ij} \right|^k < \infty, \quad (43)$$

also holds

- (5) $\Phi \in (b_s^{r,t}(p, q): bs)$ if and only if (35) and (41) hold, and (42) also holds with $\Psi(i, j)$ instead of ψ_{ij} , where $\Psi(i, j) = \sum_{l=0}^i \psi_{lj}$

- (6) $\Phi \in (b_s^{r,t}(p, q): cs)$ if and only if (35) and (41) hold, and (38) and (42) also hold

- (7) $\Phi \in (b_s^{r,t}(p, q): cs_0)$ if and only if (35) and (41) hold, and (39) and (42) also hold with $\Psi(i, j)$ instead of ψ_{ij} , where $\Psi(i, j) = \sum_{l=0}^i \psi_{lj}$

Corollary 18. The following statements hold:

- (1) $\Phi \in (b_{\infty}^{r,t}(p, q): \ell_{\infty})$ if and only if (35) and

$$\lim_{m \rightarrow \infty} \sum_{j=0}^m |\psi_{mj}^{(i)}| = \sum_{j=0}^m \left| \lim_{m \rightarrow \infty} \psi_{mj}^{(i)} \right| \text{ for each } i \in \mathbb{N}_0, \quad (44)$$

hold, and (42) also holds with $k = 1$

- (2) $\Phi \in (b_{\infty}^{r,t}(p, q): c)$ if and only if (35) and (44) hold, and (38) and

$$\lim_{i \rightarrow \infty} \sum_{j=0}^i |\psi_{ij}| = \sum_{j=0}^i \left| \lim_{i \rightarrow \infty} \psi_{ij} \right|, \quad (45)$$

also hold

- (3) $\Phi \in (b_{\infty}^{r,t}(p, q): c_0)$ if and only if (35) and (44) hold, and

$$\lim_{i \rightarrow \infty} \sum_{j=0}^i \psi_{ij} = 0, \quad (46)$$

also holds

- (4) $\Phi \in (b_{\infty}^{r,t}(p, q): \ell_1)$ if and only if (35) and (44) hold, and (43) also holds with $k = 1$

- (5) $\Phi \in (b_{\infty}^{r,t}(p, q): bs)$ if and only if (35) and (44) hold, and (42) also hold with $k = 1$, and $\Psi(i, j)$ instead of ψ_{ij} , where $\Psi(i, j) = \sum_{l=0}^i \psi_{lj}$

- (6) $\Phi \in (b_{\infty}^{r,t}(p, q): cs)$ if and only if (35) and (44) hold, and (45) also holds with $\Psi(i, j)$ instead of ψ_{ij} , where $\Psi(i, j) = \sum_{l=0}^i \psi_{lj}$

- (7) $\Phi \in (b_{\infty}^{r,t}(p, q): cs_0)$ if and only if (35) and (44) hold, and (46) also holds with $\Psi(i, j)$ instead of ψ_{ij} , where $\Psi(i, j) = \sum_{l=0}^i \psi_{lj}$

We recall a basic lemma due to Basar and Altay [47] that will help in characterizing certain classes of matrix mappings from the spaces $b_s^{r,t}(p, q)$ and $b_{\infty}^{r,t}(p, q)$ to any arbitrary space μ .

Lemma 19 (see [47]). Let λ and μ be any two sequence spaces, Φ be an infinite matrix and Ω be a triangular matrix. Then, $\Phi \in (\lambda : \mu_\Omega)$ if and only if $\Omega\Phi \in (\lambda : \mu)$.

Now, by combining Lemma 19 with Corollaries 16, 17, and 18, we derive the following classes of matrix mappings:

Corollary 20. Let $\Phi = (\phi_{ij})$ be an infinite matrix and define the matrix $C^\alpha = (c_{ij}^\alpha)$ by

$$c_{ij}^\alpha = \sum_{l=0}^i \frac{\binom{\alpha+i-l-1}{i-l}}{\binom{\alpha+i}{i}} \phi_{lj}, \quad (\alpha > 1), \quad (47)$$

for all $i, j \in \mathbb{N}_0$. Then, the necessary and sufficient conditions that Φ belongs to any one of the classes $(b_1^{r,t}(p, q): C_s^\alpha)$, $(b_1^{r,t}(p, q): C_\infty^\alpha)$, $(b_s^{r,t}(p, q): C_\infty^\alpha)$, $(b_\infty^{r,t}(p, q): C_s^\alpha)$, and $(b_\infty^{r,t}(p, q): C_\infty^\alpha)$ can be obtained from the respective ones in Corollaries 16, 17, and 18, by replacing the entries of the matrix Φ by those of matrix C^α , where C_s^α and C_∞^α are generalized Cesàro sequence spaces of order α defined by Roopaei et al. [48].

Corollary 21. Let $\Phi = (\phi_{ij})$ be an infinite matrix and define the matrix $\tilde{C} = (C_{ij})$ by

$$\tilde{C}_{ij} = \sum_{l=0}^i \frac{C_l C_{i-l}}{C_{i+1}} \phi_{lj}, \quad (i, j \in \mathbb{N}_0), \quad (48)$$

where (C_i) is the sequence of Catalan numbers. Then, the necessary and sufficient conditions that Φ belongs to any one of the classes $(b_1^{r,t}(p, q): c_0(\tilde{C}))$, $(b_1^{r,t}(p, q): c(\tilde{C}))$, $(b_s^{r,t}(p, q): c_0(\tilde{C}))$, $(b_s^{r,t}(p, q): c(\tilde{C}))$, $(b_\infty^{r,t}(p, q): c_0(\tilde{C}))$, and $(b_\infty^{r,t}(p, q): c(\tilde{C}))$ can be obtained from the respective ones in Corollaries 16, 17, and 18, by replacing the entries of the matrix Φ by those of matrix $\tilde{C}$, where $c(\tilde{C})$ and $c_0(\tilde{C})$ are Catalan sequence spaces defined by İlkhani [49].

Corollary 22. Let $\Phi = (\phi_{ij})$ be an infinite matrix and define the matrix $C^{(q)} = (c_{ij}^{(q)})$ by

$$c_{ij}^{(q)} = \sum_{l=0}^i \frac{q^l}{[i+1]_q} \phi_{lj}, \quad (i, j \in \mathbb{N}_0), \quad (49)$$

where $[i]_q$ is the q -analogue of $i \in \mathbb{N}_0$. Then, the necessary and sufficient conditions that Φ belongs to any one of the classes $(b_1^{r,t}(p, q): \mathcal{X}_s^q)$, $(b_1^{r,t}(p, q): \mathcal{X}_\infty^q)$, $(b_s^{r,t}(p, q): \mathcal{X}_s^q)$, $(b_s^{r,t}(p, q): \mathcal{X}_\infty^q)$, $(b_\infty^{r,t}(p, q): \mathcal{X}_s^q)$, and $(b_\infty^{r,t}(p, q): \mathcal{X}_\infty^q)$ can be obtained from the respective ones in Corollaries 16, 17, and 18, by replacing the entries of the matrix Φ by those of matrix $C^{(q)}$, where $\mathcal{X}_s^q$ and $\mathcal{X}_\infty^q$ are q -Cesàro sequence spaces defined by Yaying et al. [28].

5. Geometric Properties

In this section, we examine some geometric properties of the space $b_s^{r,t}(p, q)$. Before proceeding, we recall some notions in Banach spaces which are necessary for this investigation. We use the notation $B(\lambda)$ for unit ball in λ .

Definition 23 (see [50]). A Banach space λ has the weak Banach-Saks property if every weakly null sequence (x_i) in λ has a subsequence (x_{i_j}) whose Cesàro means sequence is norm convergent to zero, that is,

$$\lim_{i \rightarrow \infty} \left\| \frac{1}{i+1} \sum_{j=0}^i x_{i_j} \right\| = 0. \quad (50)$$

Further, λ has the Banach-Saks property if every bounded sequence in λ has a subsequence whose Cesàro means sequence is norm convergent.

Definition 24 (see [51]). A Banach space λ has the Banach-Saks type s , if every weakly null sequence (x_i) has a subsequence (x_{i_j}) such that, for some $K > 0$,

$$\left\| \sum_{j=0}^i x_{i_j} \right\| \leq K(i+1)^{1/s}, \quad (51)$$

for all $i \in \mathbb{N}_0$.

Theorem 25. The sequence space $b_s^{r,t}(p, q)$ is of Banach-Saks type s .

Proof. Let (ς_i) be a sequence of positive numbers satisfying $\sum_{i=0}^\infty \varsigma_i \leq 1/2$. Let (x_i) be a weakly null sequence in $B(b_s^{r,t}(p, q))$. We set $z_0 = x_0 = 0$ and $z_1 = x_{i_1} = x_1$. Then, there exists $u_1 \in \mathbb{N}_0$ such that

$$\left\| \sum_{j=u_1+1}^\infty z_1(j) e^{(j)} \right\|_{b_s^{r,t}(p, q)} < \varsigma_1. \quad (52)$$

Since (x_i) is a weakly null sequence, we realise that $x_i \rightarrow 0$ coordinatewise. Thus, there exists an $i_2 \in \mathbb{N}_0$ such that

$$\left\| \sum_{j=0}^{u_1} x_i(j) e^{(j)} \right\|_{b_s^{r,t}(p, q)} < \varsigma_1, \quad (53)$$

when $i \geq i_2$. We again set $z_2 = x_{i_2}$. Then, there exists $u_2 > u_1$ such that

$$\left\| \sum_{j=u_2+1}^\infty z_2(j) e^{(j)} \right\|_{b_s^{r,t}(p, q)} < \varsigma_2. \quad (54)$$

We again use the fact that $x_i \rightarrow 0$ coordinatewise, which implies that there exists $i_3 > i_2$ such that

$$\left\| \sum_{j=0}^{u_2} x_i(j) e^{(j)} \right\|_{b_s^{r,t}(p,q)} < \varsigma_2, \quad (55)$$

when $i \geq i_3$.

Continuing this process will lead us to two increasing sequences (i_j) and (u_j) such that

$$\left\| \sum_{j=0}^{u_l} x_i(j) e^{(j)} \right\|_{b_s^{r,t}(p,q)} < \varsigma_l, \quad (56)$$

for all $i \geq i_{j+1}$ and

$$\left\| \sum_{j=u_l+1}^{\infty} z_l(j) e^{(j)} \right\|_{b_s^{r,t}(p,q)} < \varsigma_l. \quad (57)$$

where $z_l = x_{i_l}$. Thus

$$\begin{aligned} & \left\| \sum_{l=0}^i z_l \right\|_{b_s^{r,t}(p,q)} \\ &= \left\| \sum_{l=0}^i \left(\sum_{j=0}^{u_{l-1}} z_l(j) e^{(j)} + \sum_{j=u_{l-1}+1}^{u_l} z_l(j) e^{(j)} + \sum_{j=u_l}^{\infty} z_l(j) e^{(j)} \right) \right\|_{b_s^{r,t}(p,q)} \\ &\leq \left\| \sum_{l=0}^i \left(\sum_{j=u_{l-1}+1}^{u_l} z_l(j) e^{(j)} \right) \right\|_{b_s^{r,t}(p,q)} + 2 \sum_{l=0}^i \varsigma_l. \end{aligned} \quad (58)$$

Now, since $x_i \in B(b_s^{r,t}(p, q))$ and $\|x\|_{b_s^{r,t}(p,q)} = \sum_{i=0}^{\infty} |\sum_{j=0}^i b_{ij}^{r,t}(p, q) x_j|$, we realise that $\|x\|_{b_s^{r,t}(p,q)} \leq 1$. Therefore, we have

$$\begin{aligned} & \left\| \sum_{l=0}^i \left(\sum_{j=u_{l-1}+1}^{u_l} z_l(j) e^{(j)} \right) \right\|_{b_s^{r,t}(p,q)}^p \\ &= \sum_{l=0}^i \sum_{j=u_{l-1}+1}^{u_l} \left| \sum_{m=0}^j b_{jm}^{r,t}(p, q) z_l(m) \right|^p \\ &\leq \sum_{l=0}^i \sum_{j=0}^{\infty} \left| \sum_{m=0}^j b_{jm}^{r,t}(p, q) z_l(m) \right|^p \leq i+1. \end{aligned} \quad (59)$$

Now using the fact that $1 \leq (i+1)^{1/s}$ for all $i \in \mathbb{N}_0$ and $1 \leq s < \infty$, we obtain

$$\left\| \sum_{l=0}^i z_l \right\|_{b_s^{r,t}(p,q)} \leq (i+1)^{1/s} + 1 \leq 2(i+1)^{1/s}. \quad (60)$$

Thus, we conclude that $b_s^{r,t}(p, q)$ is of the Banach-Saks type s . $\square$

Definition 26. The Gurarii's modulus of convexity of a normed linear space λ is defined by

$$\beta_\lambda(\varsigma) = \inf \left\{ 1 - \inf_{0 \leq t \leq 1} \|tx + (1-t)y\| : x, y \in B(\lambda), \|x - y\| = \varsigma \right\}, \text{ where } 0 \leq \varsigma \leq 2. \quad (61)$$

Theorem 27. The Gurarii's modulus of convexity of the normed space $b_s^{r,t}(p, q)$ is

$$\beta_{b_s^{r,t}(p,q)} \leq 1 - \left(1 - \left(\frac{\varsigma}{2} \right)^s \right)^{1/s}, \text{ where } 0 \leq \varsigma \leq 2. \quad (62)$$

Proof. Let $x \in b_s^{r,t}(p, q)$. Then

$$\|x\|_{b_s^{r,t}(p,q)} = \|B^{r,t}(p, q)x\|_{\ell_s} = \left(\sum_{i=0}^{\infty} |(B^{r,t}(p, q)x)_i|^s \right)^{1/s}. \quad (63)$$

Let $0 \leq \varsigma \leq 2$ and consider the following two sequences:

$$\begin{aligned} x &= \left(\left(C^{r,t}(p, q) \left(1 - \left(\frac{\varsigma}{2} \right)^s \right) \right)^{1/s}, C^{r,t}(p, q) \left(\frac{\varsigma}{2} \right), 0, 0, \dots \right), \\ y &= \left(\left(C^{r,t}(p, q) \left(1 - \left(\frac{\varsigma}{2} \right)^s \right) \right)^{1/s}, C^{r,t}(p, q) \left(\frac{-\varsigma}{2} \right), 0, 0, \dots \right), \end{aligned} \quad (64)$$

where the matrix $C^{r,t}(p, q) = (c_{ij}^{r,t}(p, q))$ is the inverse of the matrix $B^{r,t}(p, q)$. Then, we observe that

$$\begin{aligned} \|x\|_{b_s^{r,t}(p,q)} &= \|B^{r,t}(p, q)x\|_{\ell_s} = \left| \left(1 - \left(\frac{\varsigma}{2} \right)^s \right)^{1/s} \right|^s + \left| \frac{\varsigma}{2} \right|^s \\ &= 1 - \left(\frac{\varsigma}{2} \right)^s + \left(\frac{\varsigma}{2} \right)^s = 1, \\ \|y\|_{b_s^{r,t}(p,q)} &= \|B^{r,t}(p, q)y\|_{\ell_s} = \left| \left(1 - \left(\frac{\varsigma}{2} \right)^s \right)^{1/s} \right|^s + \left| \frac{-\varsigma}{2} \right|^s \\ &= 1 - \left(\frac{\varsigma}{2} \right)^s + \left(\frac{\varsigma}{2} \right)^s = 1. \end{aligned}$$

$$\begin{aligned} \|x - y\|_{b_s^{r,t}(p,q)} &= \|B^{r,t}(p, q)x - B^{r,t}(p, q)y\|_{\ell_s} \\ &= \left(\left| \left(1 - \left(\frac{\varsigma}{2} \right)^s \right)^{1/s} \right|^s - \left| \left(1 - \left(\frac{\varsigma}{2} \right)^s \right)^{1/s} \right|^s \right. \\ &\quad \left. + \left| \frac{\varsigma}{2} - \left(\frac{-\varsigma}{2} \right) \right|^s \right)^{1/s} = \varsigma. \end{aligned} \quad (65)$$

Finally, for $0 \leq \iota \leq 1$, we have

$$\begin{aligned}
 & \inf_{0 \leq \iota \leq 1} \|\iota x + (1 - \iota)y\|_{b_s^{r,t}(p,q)} \\
 &= \inf_{0 \leq \iota \leq 1} \|\iota B^{r,t}(p,q)x + (1 - \iota)B^{r,t}(p,q)y\|_{\ell_s} \\
 &= \inf_{0 \leq \iota \leq 1} \left\{ \left| \iota \left(1 - \left(\frac{\varsigma}{2}\right)^s\right)^{1/s} + (1 - \iota) \left(1 - \left(\frac{\varsigma}{2}\right)^s\right)^{1/s} \right|^s \right. \\
 &\quad \left. + \left| \iota \left(\frac{\varsigma}{2}\right) + (1 - \iota) \left(\frac{-\varsigma}{2}\right) \right|^s \right\}^{1/s} \quad (66) \\
 &= \inf_{0 \leq \iota \leq 1} \left\{ 1 - \left(\frac{\varsigma}{2}\right)^s + |2\iota - 1|^s \left(\frac{\varsigma}{2}\right)^s \right\}^{1/s} \\
 &= \left(1 - \left(\frac{\varsigma}{2}\right)^s\right)^{1/s}.
 \end{aligned}$$

Consequently, $\beta_{b_s^{r,t}(p,q)}(\varsigma) \leq 1 - (1 - (\varsigma/2)^s)^{1/s}$. This completes the proof. $\square$

Corollary 28. *The following results hold:*

- (1) If $\varsigma = 2$, then $\beta_{b_s^{r,t}(p,q)}(\varsigma) \leq 1$. Hence, $b_s^{r,t}(p,q)$ is strictly convex
- (2) If $0 < \varsigma < 2$, then $0 < \beta_{b_s^{r,t}(p,q)}(\varsigma) < 1$. Hence, $b_s^{r,t}(p,q)$ is uniformly convex

6. Conclusion

The (p, q) -Euler matrix $B^{r,t}$ of order (r, t) generalizes some of the well-known matrices presented in the literature, for instance, Binomial matrix $B^{r,t}$ of order (r, t) [39, 40], Euler matrix of order r [10, 11], etc. Thus, the results presented in this paper strengthen the results of [11, 14, 40, 52–55]. As for future scope, we shall study the domain of the matrix $B^{r,t}(p, q)$ in the spaces c and c_0 of convergent and null sequences, respectively.

Data Availability

All the data are included within the article.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Acknowledgments

The authors are grateful to the anonymous reviewers for careful reading and making necessary corrections which have improved the readability of the paper. The first author (T. Yaying) is thankful to Dr. M.Q. Khan, Principal, Dera Natung Government College, Itanagar, for constant encour-

agement and administrative supports. The research of the first author (T. Yaying) is supported by SERB, DST (Department of Science and Technology), New Delhi, India, under the grant number EEQ/2019/000082.

References

- [1] R. Chakrabarti and R. Jagannathan, "A (p, q) -oscillator realization of two parameters quantum algebras," *Journal of Physics A: Mathematical and General*, vol. 24, no. 13, pp. L711–L718, 1991.
- [2] S. Araci, U. Duran, M. Acikgoz, and H. M. Srivastava, "A certain (p, q) -derivative operator and associated divided differences," *Journal of Inequalities and Applications*, vol. 2016, no. 1, 2016.
- [3] M. Burban and A. U. Kilmyk, "P, Q-differentiation, P, Q-integration and P, Q-hypergeometric functions related to quantum groups," *Integral Transforms and Special Functions*, vol. 2, no. 1, pp. 15–36, 1994.
- [4] R. B. Corcino, "On P, Q-binomial coefficients," *Integers: Electronic Journal of Combinatorial Number Theory*, vol. 8, p. A29, 2008.
- [5] R. Jagannathan and K. S. Rao, "Two-parameter quantum algebras, twin basic numbers, and associated generalized hypergeometric series," in *Proceedings of the international Conference on Number Theory and Mathematical Physics, Srinivasa Ramanujan Centre*, pp. 20–21, Kumbakonam, India, December 2005.
- [6] M. Mursaleen, K. J. Ansari, and A. Khan, "On (p, q) -analogue of Bernstein operators," *Applied Mathematics and Computation*, vol. 266, pp. 874–882, 2015.
- [7] P. N. Sadjang, "On the fundamental theorem of (p, q) -calculus and some (p, q) -Taylor formulas," *Results in Mathematics*, vol. 73, p. 39, 2018.
- [8] P. N. Sadjang, "On two (p, q) -analogues of Laplace transform," *Journal of Difference Equations and Applications*, vol. 23, no. 9, pp. 1562–1583, 2016.
- [9] V. Kac and P. Cheung, *Quantum Calculus*, Springer, New York, 2002.
- [10] B. Altay and F. Başar, "On some Euler sequence spaces of non-absolute type," *Ukrainian Mathematical Journal*, vol. 57, no. 1, pp. 1–17, 2005.
- [11] B. Altay, F. Başar, and M. Mursaleen, "On the Euler sequence spaces which include the spaces ℓ_p and ℓ_∞ I," *Information Sciences*, vol. 176, no. 10, pp. 1450–1462, 2006.
- [12] E. E. Kara and M. Başarır, "An application of Fibonacci numbers into infinite Toeplitz matrices," *Caspian Journal of Mathematical Sciences*, vol. 1, no. 1, pp. 43–47, 2012.
- [13] M. Kirişçi and F. Başar, "Some new sequence spaces derived by the domain of generalized difference matrix," *Computers & Mathematics with Applications*, vol. 60, no. 5, pp. 1299–1309, 2010.
- [14] M. Mursaleen, F. Başar, and B. Altay, "On the Euler sequence spaces which include the spaces ℓ_p and ℓ_∞ II," *Nonlinear Analysis*, vol. 65, no. 3, pp. 707–717, 2006.
- [15] P.-N. Ng and P.-Y. Lee, "Cesàro sequence spaces of non-absolute type," *Polskiego Towarzystwa Matematycznego*, vol. 20, no. 2, pp. 429–433, 1978.
- [16] F. Başar and R. Çolak, *Summability Theory and its Applications*, Bentham Science Publisher, İstanbul, Turkey, 2012.

- [17] M. Mursaleen and F. Başar, "Sequence spaces: topic in modern summability theory," in *Series: Mathematics and its Applications*, CRC Press, Taylor & Francis Group, Boca Raton, London, New York, 2020.
- [18] A. Wilansky, *Summability through Functional Analysis*, North-Holland Mathematics Studies, vol. 85, Elsevier, Amsterdam, 1984.
- [19] S. Demiriz, M. İlkan, and E. E. Kara, "Almost convergence and Euler totient matrix," *Annals of Functional Analysis*, vol. 11, no. 3, pp. 604–616, 2020.
- [20] M. İlkan and E. E. Kara, "A new Banach space defined by Euler totient matrix operator," *Operators and Matrices*, vol. 13, no. 2, pp. 527–544, 2019.
- [21] M. İlkan, N. Simsek, and E. E. Kara, "A new regular infinite matrix defined by Jordan totient function and its matrix domain in ℓ_p ," *Mathematical Methods in the Applied Sciences*, vol. 44, no. 9, pp. 7622–7633, 2021.
- [22] M. İlkan, "Matrix domain of a regular matrix derived by Euler totient function in the spaces c_0 and c ," *Mediterranean Journal of Mathematics*, vol. 17, no. 1, p. 27, 2020.
- [23] H. Roopaei, "Norm of Hilbert operator on sequence spaces," *Journal of Inequalities and Applications*, vol. 2020, no. 1, 2020.
- [24] H. Roopaei, "A study on Copson operator and its associated sequence space," *Journal of Inequalities and Applications*, vol. 2020, no. 1, 2020.
- [25] G. Talebi, "On multipliers of matrix domains," *Journal of Inequalities and Applications*, vol. 2018, no. 1, 2018.
- [26] T. Yaying and B. Hazarika, "On sequence spaces defined by the domain of a regular Tribonacci matrix," *Mathematica Slovaca*, vol. 70, no. 3, pp. 697–706, 2020.
- [27] T. Yaying and B. Hazarika, "On sequence spaces generated by binomial difference operator of fractional order," *Mathematica Slovaca*, vol. 69, no. 4, pp. 901–918, 2019.
- [28] T. Yaying, B. Hazarika, and M. Mursaleen, "On sequence space derived by the domain of q -Cesàro matrix in ℓ_p space and the associated operator ideal," *Journal of Mathematical Analysis and Applications*, vol. 493, no. 1, article 124453, 2021.
- [29] B. Altay and H. Polat, "On some new Euler difference sequence spaces," *Southeast Asian Bulletin of Mathematics*, vol. 30, pp. 209–220, 2006.
- [30] H. Polat and F. Başar, "Some Euler spaces of difference sequences of order m ," *Acta Mathematica Scientia B*, vol. 27, no. 2, pp. 254–266, 2007.
- [31] U. Kadak and P. Baliarsingh, "On certain Euler difference sequence spaces of fractional order and related dual properties," *Journal of Nonlinear Sciences and Applications*, vol. 8, no. 6, pp. 997–1004, 2015.
- [32] E. E. Kara, M. Öztürk, and M. Başarır, "Some topological and geometric properties of generalized Euler sequence space," *Mathematica Slovaca*, vol. 60, no. 3, pp. 385–398, 2010.
- [33] V. Karakaya and H. Polat, "Some new paranormed sequence spaces defined by Euler and difference operators," *Acta Scientiarum Mathematicarum (Szeged)*, vol. 76, pp. 87–100, 2010.
- [34] V. Karakaya, E. Savas, and H. Polat, "Some paranormed Euler sequence spaces of difference sequences of m^{th} order," *Mathematica Slovaca*, vol. 63, no. 4, pp. 849–862, 2013.
- [35] S. Demiriz and C. Çakan, "On some new paranormed Euler sequence spaces and Euler core," *Acta Mathematica Sinica, English Series*, vol. 26, no. 7, pp. 1207–1222, 2010.
- [36] M. Kirisci, "On the spaces of Euler almost null and Euler almost convergent sequences," *Communications Faculty Of Science University of Ankara Series A1 Mathematics and Statistics*, vol. 62, no. 2, pp. 85–100, 2013.
- [37] E. E. Kara and M. Başarır, "On compact operators and some Euler $B^{(m)}$ -difference sequence spaces," *Journal of Mathematical Analysis and Applications*, vol. 379, no. 2, pp. 499–511, 2011.
- [38] J. Meng and L. Mei, "The matrix domain and the spectra of a generalized difference operator," *Journal of Mathematical Analysis and Applications*, vol. 470, no. 2, pp. 1095–1107, 2019.
- [39] M. C. Bisgin, "The binomial sequence spaces of nonabsolute type," *Journal of Inequalities and Applications*, vol. 2016, no. 1, 2016.
- [40] M. C. Bisgin, "The binomial sequence spaces which include the spaces ℓ_p and ℓ_∞ and geometric properties," *Journal of Inequalities and Applications*, vol. 2016, no. 1, 2016.
- [41] J. Meng and M. Song, "On some binomial $B^{(m)}$ -difference sequence spaces," *Journal of Inequalities and Applications*, vol. 2017, no. 1, 2017.
- [42] J. Meng and L. Mei, "Binomial difference sequence spaces of fractional order," *Journal of Inequalities and Applications*, vol. 2018, no. 1, 2018.
- [43] H. Aktuğlu and S. Bekar, "On q -Cesàro matrix and q -statistical convergence," *Journal of Computational and Applied Mathematics*, vol. 235, no. 16, pp. 4717–4723, 2011.
- [44] S. Bekar, "q-matrix summability methods," *Applied Mathematics and Computer Science*, [Ph.D. Thesis], Eastern Mediterranean University, 2010.
- [45] S. Demiriz and A. Sahin, "q-Cesàro sequence spaces derived by q-analogues," *Advances in Mathematics*, vol. 5, no. 2, pp. 97–110, 2016.
- [46] M. Stieglitz and H. Tietz, "Matrixtransformationen von Folgenräumen eine Ergebnisübersicht," *Mathematische Zeitschrift*, vol. 154, no. 1, pp. 1–16, 1977.
- [47] F. Başar and B. Altay, "On the space of sequences of p -bounded variation and related matrix mappings," *Ukrainian Mathematical Journal*, vol. 55, no. 1, pp. 136–147, 2003.
- [48] H. Roopaei, D. Foroutannia, M. İlkan, and E. E. Kara, "Cesàro spaces and norm of operators on these matrix domains," *Mediterranean Journal of Mathematics*, vol. 17, no. 4, p. 121, 2020.
- [49] M. İlkan, "A new conservative matrix derived by Catalan numbers and its matrix domain in the spaces c and c_0 ," *Linear and Multilinear Algebra*, vol. 68, no. 2, pp. 417–434, 2020.
- [50] B. Beauzamy, "Banach-Saks properties and spreading models," *Mathematica Scandinavica*, vol. 44, pp. 357–384, 1997.
- [51] H. Knaust, "Orlicz sequence spaces of Banach-Saks type," *Archiv der Mathematik*, vol. 59, no. 6, pp. 562–565, 1992.
- [52] M. İlkan, "Certain geometric properties and matrix transformations on a newly introduced Banach space," *Fundamental Journal of Mathematics and Applications*, vol. 3, no. 1, pp. 45–51, 2020.
- [53] M. İlkan and E. E. Kara, "Matrix transformations and compact operators on Catalan sequence spaces," *Journal of Mathematical Analysis and Applications*, vol. 498, no. 1, p. 124925, 2021.
- [54] E. E. Kara and M. Başarır, "On the m^{th} order difference sequence space of generalized weighted mean and compact operators," *Acta Mathematica Scientia*, vol. 33, no. 3, pp. 797–813, 2013.
- [55] P. Z. Alp and M. İlkan, "On the difference sequence space $\ell^p(T \wedge^q)$," *Mathematical Sciences and Applications E-Notes*, vol. 7, no. 2, pp. 161–173, 2019.