Hindawi
Journal of Function Spaces
Volume 2021, Article ID 9081623, 9 pages
https://doi.org/10.1155/2021/9081623

Research Article

A Cantilever Beam Problem with Small Deflections and Perturbed Boundary Data

Ammar Khanfer [1] and Lazhar Bougoffa [2]

[1] *Department of Mathematics and General Sciences, Prince Sultan University, Riyadh, Saudi Arabia*
[2] *Imam Mohammad Ibn Saud Islamic University (IMSIU), Faculty of Science, Department of Mathematics and Statistics, Riyadh 11623, Saudi Arabia*

Correspondence should be addressed to Ammar Khanfer; akhanfer@psu.edu.sa

Received 2 September 2021; Accepted 21 October 2021; Published 8 November 2021

Academic Editor: Dr. Azhar Hussain

The boundary value problem of a fourth-order beam equation $u^{(4)} = \lambda f(x, u, u', u'', u''')$, $0 \le x \le 1$ is investigated. We formulate a nonclassical cantilever beam problem with perturbed ends. By determining appropriate values of λ and estimates for perturbation measurements on the boundary data, we establish an existence theorem for the problem under integral boundary conditions $u(0) = u'(0) = \int_0^1 p(x)u(x)dx$, $u''(1) = u'''(1) = \int_0^1 q(x)u''(x)dx$, where $p, q \in L^1[0, 1]$, and f is continuous on $[0, 1] \times [0, \infty) \times [0, \infty) \times (-\infty, 0] \times (-\infty, 0]$.

1. Introduction and Preliminaries

Beams are one of the main structural elements in construction engineering. One of the objectives of the beam theory is to study the behavior of beams to analyze deformations under loads. The deformation of beams occurs when the beam is under a load, which causes the beam to develop bending moment and shear force. The deformation of the beam is modeled by the fourth-order Euler–Bernoulli equation

$$u^{(4)} = f(x), \tag{1}$$

where u represents the deflection of the beam, u' represents the slope, u'' is the bending moment (torque), u''' is the shear force, and $u^{(4)}$ is the load density stiffness. The function f is a load on the beam; it is uniformly distributed if the loading is only the weight of the beam, without any further concentrated mass at the free end. The boundary conditions are governed by the particular type of beams under study and the way in which the beam is supported. The most important type of beams that have many useful applications in industry is the cantilever beam, where the beam is fixed (clamped, anchored to a support, or built into a wall) at one end (say at $x = 0$) and free at the other end (say $x = 1$). The fixed end must have zero transitional and zero rotational motions, whereas the free end must have zero bending and zero shearing force. The bending moment is equal to the applied force multiplied by the distance to the point of application; so, it becomes zero at the free end because there is no stress at this end. So, we must have the following boundary conditions

$$u(0) = u'(0) = 0, u''(1) = u'''(1) = 0. \tag{2}$$

The cantilever beams are widely used in construction engineering and can be found in many structures such as buildings and bridges. Some of the common examples of cantilever beams are aircraft wings, cranes, suspended bridges, balconies, diving board, electronic spring connectors, shelves, basketball backboard, road signs, and many other examples. Because of its importance, the cantilever beam problem received a wide and considerable attention from researchers, and a large number of research about cantilever can be found from a quick literature search [see and the references therein]. In particular, in [1–3], the authors investigated the existence of deflections (solutions) for the case

$$u^{(4)} = f(x, u(x)), 0 \le x \le 1, \tag{3}$$

under conditions (2). In [4–6], the case

$$u^{(4)} = f\left(x, u, u'\right), 0 \le x \le 1 \tag{4}$$

was investigated under conditions (2). In [7–11], the fully fourth-order nonlinear boundary value problem

$$u^{(4)} = f\left(x, u, u', u'', u'''\right), 0 \le x \le 1 \tag{5}$$

was investigated under conditions (2). The authors in [8, 9] assumed a Nagumo-type condition on f, and [10] assumed a L^1 Caratheodory condition on f.

In all the aforementioned research, the cantilever problem was investigated under the classical condition (2). Some special types of relaxed, restrained, or propped cantilever beams violate these conditions. If the fixed end is loose, the support (or the anchor) is relaxed then $u(0) \neq 0$ or/and $u'(0) \neq 0$. If the free end is rolled or pinned support, then $u''(1) \neq 0, u'''(1) \neq 0$. If a concentrated force, or translational elastic spring is attached at the free end, then $u''(1) = 0$ and $u'''(1) \neq 0$, while if a concentrated moment, or rotational elastic spring is attached, then $u''(1) \neq 0$. If a concentrated mass is placed at the free end, it will develop a shear force of the form $u'''(1) = -mg$, and when $m = 0$, this reduces the condition to the classical one.

The present paper deals with the fully fourth-order nonlinear boundary value problem

$$u^{(4)} = \lambda f\left(x, u, u', u'', u'''\right), 0 \le x \le 1, \tag{6}$$

where f is continuous on $[0, 1] \times [0, \infty) \times [0, \infty) \times (-\infty, 0] \times (-\infty, 0]$, under a perturbed homogeneous conditions, where the two ends are perturbed (the fixed end is slightly relaxed, and the free end is slightly supported). This can be formulated using integral conditions of the form

$$\begin{aligned} u(0) = u'(0) = \int_0^1 p(x)u(x)dx, u''(1) = u'''(1) \\ = \int_0^1 q(x)u''(x)dx, \end{aligned} \tag{7}$$

where $p, q \in L^1[0, 1]$. Here, f satisfies a growth condition with the variable parameters:

$$\begin{aligned} |f(x, u, v, w, z)| \le a(x)|u| + b(x)|v| + c(x)|w| \\ + d(x)|z| + e(x), \end{aligned} \tag{8}$$

where $a, b, c, d,$ and e are positive continuous functions on $[0, 1]$. Moreover, letting

$$\alpha = \int_0^1 p^2(x)dx, \beta = \int_0^1 q^2(x)dx, \tag{9}$$

then we have the following assumptions

$$0 \le \alpha \le \delta_1 < \frac{1}{4}, 0 \le \beta \le \delta_2 < \frac{1}{4}. \tag{10}$$

Letting

$$\lambda_0 = \frac{1/2 - \delta_2/1 - 2\delta_2}{M((2C_1/1 - 2\delta_1) + 2C_1 + (2/1 - 2\delta_2))}, \tag{11}$$

where $M = \max\{a, b, c, d, e\}$, and

$$C_1 = \frac{2/1 - 2\beta}{(1/2 - \alpha/1 - 2\alpha)^2}. \tag{12}$$

Then, we impose the following condition

$$\lambda \le \lambda_0. \tag{13}$$

Condition (13) provides small deflections. The parameter λ represents the reciprocal of the flexural rigidity which measures the resistance to bend; so, smaller values of λ indicate large flexural rigidity for the material of the beam, and this causes small deflections when load is applied on the beam. Hook's law is valid as long as the deflection is small. Problems with large deflections cannot be solved in terms of the linear beam theory—in which Euler-Bernoulli Equation applies, since Hook's law is no longer valid. When the beam is assumed to be homogeneous, made of high rigid material, and behaves in a linear elastic manner, its deflection under bending is usually small. When thin flexible beams are used, large deflections are expected to occur. Many researchers investigated cantilever beams with large deflections [12–26], and others investigated small deflections [27, 28]. Most of the beams used in industry and constructions (buildings, bridges, aircrafts, etc.) undergo small deflections [29]. This is of utmost importance since large deflections can cause cracks in the beams, and this may eventually lead to disastarous damages. So, best efforts are made to limit deflections in the ceilings and walls and in the design of aircraft, see [29] for more details on beam theory. Small deflections usually occur when either the loaded force f is small (hence, M is small) or the material of the beam has high flexural rigidity, which implies that λ is small.

Condition (10) provides small values for p, q to produce the required slight changes to the default boundary settings of a cantilever beam. The functions p, q represent perturbation measurements related to bearings, rollers, springs, or any other mechanical settings that will perturb the boundary data.

Condition (8) is a growth condition imposed on the load function f. This condition generalizes the boundedness condition and the growth condition with constant parameters

$$|f(x,u,v,w,z)| \le a|u| + b|v| + c|w| + d|z| + e \tag{14}$$

for some positive constants a, b, c, d, and e.

The integral boundary conditions (7) results from perturbing conditions (2) by imposing small measurements p, q related. Note that if $p = q = 0$, the conditions in (7) reduce to (2). The integral boundary conditions have been studied and applied extensively in beam theory by many authors, for example, see [30–41].

The purpose of this paper is to prove the existence of solutions (small deflections) of the cantilever beam boundary value problem (6) under the conditions (7)-(13). This interesting problem has not been studied in research. The result gives an affirmative answer to the question of existence of solutions of (6)-(13), i.e., the existence of small deflections of general types of perturbed cantilever beams under conditions (7)-(13), including the simple cantilever beam problem when $p = q = 0$.

2. Existence and Uniqueness Theorems

The problem (1.6)-(7) can be converted into the following system:

$$\begin{cases} u'' = v, u(0) = u'(0) = \int_0^1 p(x)u(x)dx, \\ v'' = \lambda f\left(x, u, u', v, v'\right), v(1) = v'(1) = \int_0^1 q(x)v(x)dx. \end{cases} \tag{15}$$

We need the following lemma.

Lemma 1. *If $h \in \mathscr{C}^1[0,1]$ with $h(0) = \int_0^1 r(x)h(x)dx$ or $h(1) = \int_0^1 r(x)h(x)dx, x \in [0,1]$, where $r \in \mathscr{C}[0,1]$, then*

$$\int_0^1 h^2(x)dx \le \frac{2}{1 - 2\int_0^1 r^2(x)dx} \int_0^1 \left(h'\right)^2(x)dx \tag{16}$$

provided $1 - 2\int_0^1 r^2(x)dx > 0$.

Proof. For $h(0) = \int_0^1 r(x)h(x)dx$, we note that

$$h(x) = \int_0^x h'(\xi)d\xi + h(0) = \int_0^x h'(\xi)d\xi + \int_0^1 r(x)h(x)dx. \tag{17}$$

Since $x \le 1$, we have

$$|h(x)| \le \int_0^1 |h'(x)|dx + \int_0^1 |r(x)h(x)|dx. \tag{18}$$

Hence,

$$h^2(x) \le 2\left[\left(\int_0^1 |h'(x)|dx\right)^2 + \left(\int_0^1 |r(x)h(x)|dx\right)^2\right]. \tag{19}$$

Using the Cauchy-Schwarz inequality, we obtain

$$h^2(x) \le 2\left[\int_0^1 \left(h'\right)^2(x)dx + \left(\int_0^1 r^2(x)dx\right)\left(\int_0^1 h^2(x)dx\right)\right]. \tag{20}$$

Consequently,

$$\int_0^1 h^2(x)dx \le 2\left[\int_0^1 \left(h'\right)^2(x)dx + \left(\int_0^1 r^2(x)dx\right)\left(\int_0^1 h^2(x)dx\right)\right]. \tag{21}$$

The proof is complete. Similarly, for $h(1) = \int_0^1 r(x)h(x)dx$, we also note that

$$-h(x) = \int_x^1 h'(\xi)d\xi - h(1) = \int_x^1 h'(\xi)d\xi - \int_0^1 r(x)h(x)dx. \tag{22}$$

Then, the argument is similar to the proof of the above.□ □

Proposition 2. *If (8)-(13) hold, then there exists a constant $L > 0$ such that for any $x \in [0,1]$ and any solution u to Eq.(6), we have*

$$\max_{0\le x\le 1}|u(x)| + \max_{0\le x\le 1}\left|u''(x)\right| \le L. \tag{23}$$

Proof. Multiplying both sides of the first equation of (15) by $\varphi(x)u'$, where $\varphi(x) = 1 - x$ and integrating the resulting equation from 0 to 1, then employing integration by parts with $\varphi(0) = 1, \varphi(1) = 0$ and $\varphi'(x) = -1$, we obtain

$$\frac{1}{2}\int_0^1 \left(u'(x)\right)^2 dx - \frac{1}{2}\left(u'(0)\right)^2 = \int_0^1 (1-x)v(x)u'(x)dx. \tag{24}$$

Taking into account $u'(0) = \int_0^1 p(x)u(x)dx$, we have

$$\frac{1}{2}\int_0^1 \left(u'(x)\right)^2 dx = \frac{1}{2}\left[\int_0^1 p(x)u(x)dx\right]^2 + \int_0^1 (1-x)v(x)u'(x)dx. \tag{25}$$

The integrals $\int_0^1 p(x)u(x)dx$ and $\int_0^1 (1-x)v(x)u'(x)dx$ can be estimated by means of the Cauchy-Schwarz inequality

$$\left[\int_0^1 p(x)u(x)dx\right]^2 \le \left(\int_0^1 p^2(x)dx\right)\left(\int_0^1 u^2(x)dx\right),$$
$$\left|\int_0^1 (1-x)v(x)u'(x)dx\right| \le \left(\int_0^1 v^2(x)dx\right)^{1/2}\left(\int_0^1 \left(u'(x)\right)^2 dx\right)^{1/2}. \tag{26}$$

Thus,

$$\frac{1}{2}\int_0^1 \left(u'(x)\right)^2 dx \le \frac{1}{2}\left(\int_0^1 p^2(x)dx\right)\left(\int_0^1 u^2(x)dx\right) + \left(\int_0^1 v^2(x)dx\right)^{1/2}\left(\int_0^1 \left(u'(x)\right)^2 dx\right)^{1/2}. \tag{27}$$

Applying now Lemma 1 to the functions u and v appearing in the right-hand side of this inequality with $u'(0)=\int_0^1 p(x)u(x)dx$ and $v(1)=\int_0^1 q(x)v(x)dx$, respectively, we obtain

$$\frac{1}{2}\int_0^1 \left(u'(x)\right)^2 dx \le \frac{\int_0^1 p^2(x)dx}{1-2\int_0^1 p^2(x)dx}\int_0^1 \left(u'(x)\right)^2 dx + \left(\frac{2}{1-2\int_0^1 q^2(x)dx}\right)^{1/2} \cdot \left(\int_0^1 \left(v'(x)\right)^2 dx\right)^{1/2}\left(\int_0^1 \left(u'(x)\right)^2 dx\right)^{1/2}. \tag{28}$$

It follows that

$$\left(\frac{1}{2} - \frac{\int_0^1 p^2(x)dx}{1-2\int_0^1 p^2(x)dx}\right)\left(\int_0^1 \left(u'(x)\right)^2 dx\right)^{1/2} \le \left(\frac{2}{1-2\int_0^1 q^2(x)dx}\right)^{1/2}\left(\int_0^1 \left(v'(x)\right)^2 dx\right)^{1/2}. \tag{29}$$

Consequently,

$$\left(\frac{1}{2} - \frac{\int_0^1 p^2(x)dx}{1-2\int_0^1 p^2(x)dx}\right)^2 \int_0^1 \left(u'(x)\right)^2 dx \le \frac{2}{1-2\int_0^1 q^2(x)dx}\int_0^1 \left(v'(x)\right)^2 dx \tag{30}$$

provided $1/2 - \int_0^1 p^2(x)dx/1 - 2\int_0^1 p^2(x)dx > 0$; that is, $1 - 4\int_0^1 p^2(x)dx > 0$.

It follows that

$$\int_0^1 \left(u'(x)\right)^2 dx \le C_1 \int_0^1 \left(v'(x)\right)^2 dx, \tag{31}$$

where

$$C_1 = \frac{2/1 - 2\int_0^1 q^2(x)dx}{\left(1/2 - \int_0^1 p^2(x)dx/1 - 2\int_0^1 p^2(x)dx\right)^2} = \frac{2/1-2\beta}{\left(1/2-\alpha/1-2\alpha\right)^2}. \tag{32}$$

Proceeding as before, multiplying both sides of the second equation of (15) by $\psi(x)v'$, where $\psi(x)=-x$ and integrating the resulting equation from 0 to 1, then employing integration by parts, taking into account $\psi(0)=0$, $\psi(1)=-1, \psi'(x)=-1$ and the nonlocal boundary condition $v(1)=\int_0^1 q(x)v(x)dx$, we obtain

$$\frac{1}{2}\int_0^1 \left(v'(x)\right)^2 dx = \frac{1}{2}\left[\int_0^1 q(x)v(x)dx\right]^2 - \lambda\int_0^1 xf\left(x,u,u',v,v'\right)v'(x)dx. \tag{33}$$

Applying the growth condition (8) to $f(x,u,u',v,v')$ by assuming that $a(x)\le a, b(x)\le b, c(x)\le c, d(x)\le d, e(x)\le e$, $\forall x\in[0,1]$ with $a,b,c,d,e>0$, we obtain

$$\frac{1}{2}\int_0^1 \left(v'(x)\right)^2 dx \le \frac{1}{2}\left(\int_0^1 q^2(x)dx\right)\left(\int_0^1 v^2(x)dx\right) + a\lambda\int_0^1 \left|u(x)v'(x)\right|dx + b\lambda\int_0^1 \left|u'(x)v'(x)\right|dx + d\lambda\int_0^1 \left(v'(x)\right)^2 dx + \lambda\int_0^1 \left|e(x)v'(x)\right|dx. \tag{34}$$

The integrals appearing in the right-hand side of this inequality can be estimated by means of the $\varepsilon-$inequality:

$$\int_0^1 |A(x)B(x)|dx \le \frac{1}{\varepsilon}\int_0^1 A^2(x)dx + \varepsilon\int_0^1 B^2(x)dx, \varepsilon>0. \tag{35}$$

Thus,

$$\begin{aligned}\frac{1}{2}\int_0^1 \left(v'(x)\right)^2 dx &\le \frac{1}{2}\left(\int_0^1 q^2(x)dx\right)\left(\int_0^1 v^2(x)dx\right) + \frac{a\lambda}{\varepsilon_1}\int_0^1 u^2(x)dx \\ &+ a\lambda\varepsilon_1\int_0^1 \left(v'(x)\right)^2 dx + \frac{b\lambda}{\varepsilon_2}\int_0^1 \left(u'(x)\right)^2 dx \\ &+ b\lambda\varepsilon_2\int_0^1 \left(v'(x)\right)^2 dx + \frac{c\lambda}{\varepsilon_3}\int_0^1 v^2(x)dx \\ &+ c\lambda\varepsilon_3\int_0^1 \left(v'(x)\right)^2 dx + d\lambda\int_0^1 \left(v'(x)\right)^2 dx \\ &+ \lambda\varepsilon_4\int_0^1 \left(v'(x)\right)^2 dx + \frac{e^2\lambda}{\varepsilon_4}, \varepsilon_i>0, i=1,\cdots,4.\end{aligned} \tag{36}$$

Applying Lemma 1 to the functions u and v, we obtain

$$\begin{aligned}
\frac{1}{2}\int_0^1\left(v'(x)\right)^2dx
&\le \frac{\int_0^1 q^2(x)dx}{1-2\int_0^1 q^2(x)dx}\int_0^1\left(v'(x)\right)^2dx\,\frac{2a\lambda}{\varepsilon_1\left(1-2\int_0^1 p^2(x)dx\right)}\\
&\cdot\int_0^1\left(u'(x)\right)^2dx + a\lambda\varepsilon_1\int_0^1\left(v'(x)\right)^2dx\\
&+\frac{b\lambda}{\varepsilon_2}\int_0^1\left(u'(x)\right)^2(x)dx + b\lambda\varepsilon_2\int_0^1\left(v'(x)\right)^2dx\\
&+\frac{2c\lambda}{\varepsilon_3\left(1-2\int_0^1 q^2(x)dx\right)}\int_0^1\left(v'(x)\right)^2dx\\
&+c\lambda\varepsilon_3\int_0^1\left(v'(x)\right)^2dx + d\lambda\int_0^1\left(v'(x)\right)^2dx\\
&+\lambda\varepsilon_4\int_0^1\left(v'(x)\right)^2dx + \frac{e^2\lambda}{\varepsilon_4},\ \varepsilon_i>0,\ i=1,\cdots,4..
\end{aligned} \tag{37}$$

From (31), the inequality (37) becomes

$$\begin{aligned}
\frac{1}{2}\int_0^1\left(v'(x)\right)^2dx
&\le \frac{\int_0^1 q^2(x)dx}{1-2\int_0^1 q^2(x)dx}\int_0^1\left(v'(x)\right)^2dx\\
&+\frac{2a\lambda C_1}{\varepsilon_1\left(1-2\int_0^1 p^2(x)dx\right)}\int_0^1\left(v'(x)\right)^2dx\\
&+a\lambda\varepsilon_1\int_0^1\left(v'(x)\right)^2dx + \frac{b\lambda C_1}{\varepsilon_2}\int_0^1\left(v'(x)\right)^2dx\\
&+b\lambda\varepsilon_2\int_0^1\left(v'(x)\right)^2dx + \frac{2c\lambda}{\varepsilon_3\left(1-2\int_0^1 q^2(x)dx\right)}\\
&\cdot\int_0^1\left(v'(x)\right)^2dx + c\lambda\varepsilon_3\int_0^1\left(v'(x)\right)^2dx\\
&+d\lambda\int_0^1\left(v'(x)\right)^2dx + \lambda\varepsilon_4\int_0^1\left(v'(x)\right)^2dx + \frac{e^2\lambda}{\varepsilon_4}.
\end{aligned} \tag{38}$$

Choosing $\varepsilon_i = 1, i = 1, \cdots, 3, \varepsilon_4 = M$, and using the hypothesis (10) with $M = \max\{a, b, c, d, e\}$, we obtain

$$\begin{aligned}
\frac{1}{2}\Delta &\le \frac{\delta_2}{1-2\delta_2}\Delta + \lambda\left(\frac{2MC_1}{1-2\delta_1} + MC_1 + 5M + \frac{2M}{1-2\delta_2}\right)\Delta\\
&+M\lambda,
\end{aligned} \tag{39}$$

where $\Delta = \int_0^1 (v'(x))^2 dx$. Note that $C_1 \ge 8$. This gives

$$\begin{aligned}
\lambda_0 &= \frac{1/2-\delta_2/1-2\delta_2}{M((2C_1/1-2\delta_1)+2C_1+(2/1-2\delta_2))}\\
&\le \frac{1/2-\delta_2/1-2\delta_2}{M((2C_1/1-2\delta_1)+C_1+5+(2/1-2\delta_2))}.
\end{aligned} \tag{40}$$

Let

$$\gamma = \frac{1}{2} - \frac{\delta_2}{1-2\delta_2} - \lambda\left(\frac{2MC_1}{1-2\delta_1} + MC_1 + 5M + \frac{2M}{1-2\delta_2}\right). \tag{41}$$

We see that $\gamma > 0$. Thus,

$$\int_0^1\left(v'(x)\right)^2dx \le \frac{M\lambda}{\gamma} = C_2 \tag{42}$$

and consequently from (31) and (42), we obtain

$$\int_0^1\left(u'(x)\right)^2dx \le C_1C_2. \tag{43}$$

On the other hand, we have

$$u(x) = \int_0^x u'(\xi)d\xi + u(0) = \int_0^x u'(\xi)d\xi + \int_0^1 p(x)u(x)dx. \tag{44}$$

Thus,

$$\begin{aligned}
|u(x)| &\le \left(\int_0^1\left(u'(x)\right)^2dx\right)^{1/2}\\
&+\left(\int_0^1 p^2(x)dx\right)^{1/2}\left(\int_0^1 u^2(x)dx\right)^{1/2}.
\end{aligned} \tag{45}$$

Hence,

$$\begin{aligned}
|u(x)| &\le (C_1C_2)^{1/2} + \left(2\int_0^1 p^2(x)dx\right)^{1/2}\\
&\cdot\left(\frac{\int_0^1 p^2(x)dx}{1-2\int_0^1 p^2(x)dx}\right)^{1/2}(C_1C_2)^{1/2} = L_1.
\end{aligned} \tag{46}$$

Similarly,

$$|v(x)| \le L_2 = (C_2)^{1/2} + \left(\frac{2\int_0^1 q^2(x)dx}{1-2\int_0^1 q^2(x)dx}\right)^{1/2}(C_2)^{1/2}. \tag{47}$$

These two inequalities imply the required result and complete the proof of the proposition.□ □

The fundamental theorem used in proving the existence of the solution is Schauder's fixed theorem. In order to make use of this theorem, it is sufficient to present the following lemmas.

Lemma 3. *Let $g : [0, 1] \longrightarrow \mathbb{R}$ be a continuous function. The unique solution z of the following initial value problem*

$$z'' = h(x) \tag{48}$$

subject to the nonlocal boundary conditions $z(0) = z'(0) = \int_0^1 r(x)z(x)dx$, is given by

$$z(x) = \int_0^1 G_1(x, y)h(y)dy + \int_0^1 G_2(x, y)h(y)dy, \tag{49}$$

where $G_1(x\,;y)$ is the Green function given by

$$\begin{aligned} G_1(x, y) &= \begin{cases} x - y, 0 \le y \le x \le 1, \\ 0, 0 \le x \le y \le 1 \end{cases} \\ G_2(x, y) &= \frac{(x+1)}{1 - \int_0^1 (x+1)r(x)dx} \int_0^1 r(x)G_1(x, y)dx. \end{aligned} \tag{50}$$

Proof. Integrating this equation twice, we obtain

$$z(x) = \int_0^x \left[\int_1^y h(s)ds\right] dy + k_1 x + k_2, \tag{51}$$

where $k_i, i = 1, 2$ are constants of integration. Integrations by parts of the integral with respect to y in this equation give

$$z(x) = -x\int_x^1 h(y)dy - \int_0^x yh(y)dy + k_1 x + k_2. \tag{52}$$

We determine k_1 and k_2 from $z(0) = z'(0) = \int_0^1 r(x)z(x)dx$. It follows that

$$z(x) = \int_0^x (x - y)h(y)dy + (x + 1)\int_0^1 r(x)z(x)dx. \tag{53}$$

Multiplying both sides of this equation by $r(x)z(x)$ and integrating the resulting from 0 to 1, we obtain

$$\begin{aligned} \int_0^1 r(x)z(x)dx = \frac{1}{1 - \int_0^1 (x+1)r(x)dx} \int_0^1 r(x) \\ \cdot \left(\int_0^x (x - y)h(y)dy\right) dx. \end{aligned} \tag{54}$$

Thus,

$$\begin{aligned} z(x) = \int_0^x (x - y)h(y)dy + \frac{(x+1)}{1 - \int_0^1 (x+1)r(x)dx} \\ \cdot \int_0^1 r(x)\int_0^1 r(x)\left(\int_0^x (x - y)h(y)dy\right) dx, \end{aligned} \tag{55}$$

$$\begin{aligned} z(x) = \int_0^1 G_1(x, y)h(y)dy + \frac{x+1}{1 - \int_0^1 (x+1)r(x)dx} \\ \cdot \int_0^1 r(x)\int_0^1 r(x)\left(\int_0^1 G_1(x, y)h(y)dy\right) dx. \end{aligned} \tag{56}$$

The proof is complete.□ □

We also have the following lemma.

Lemma 4. *Let $g : [0, 1] \longrightarrow \mathbb{R}$ be a continuous function. The unique solution z of the following initial value problem*

$$z'' = h(x) \tag{57}$$

subjects to the nonlocal boundary conditions $z(1) = z'(1) = \int_0^1 r(x)z(x)dx$, is given by

$$z(x) = \int_0^1 G_3(x, y)h(y)dy + \int_0^1 G_4(x, y)h(y)dy, \tag{58}$$

where $G_3(x\,;y)$ is the Green function given by

$$\begin{aligned} G_3(x, y) &= \begin{cases} 0, 0 \le y \le x \le 1, \\ y, 0 \le x \le y \le 1, \end{cases} \\ G_4(x, y) &= \frac{x}{1 - \int_0^1 xr(x)dx} \int_0^1 r(x)G_3(x, y)dx. \end{aligned} \tag{59}$$

Thus, problem (15) is equivalent to the following system of integral equations

$$\begin{cases} u = \int_0^1 G_1(x, y)v(y)dy + \int_0^1 G_2(x, y)v(y)dy, \\ v = \lambda\int_0^1 G_3(x, y)f\left(x, u(y), u'(y), v(y), v'(y)\right)dy + \lambda\int_0^1 G_4(x, y)f\left(x, u(y), u'(y), v(y), v'(y)\right)dy. \end{cases} \tag{60}$$

Define the Banach space $\mathbb{X} = \mathscr{C}[0,1] \times \mathscr{C}[0,1]$ *with norm* $\|u, v\|_\infty = \|u\|_\infty + \|v\|_\infty$, *where* $\|u\|_\infty = \max_{0 \le x \le 1} |u(x)|$. *Also, define the operator* $T : \mathbb{X} \longrightarrow \mathbb{X}$ *by* $T(u, v) = (T_1(u, v), T_2(u, v))$, *where*

$$T_1(u, v) = \int_0^1 G_1(x, y) v(y) dy + \int_0^1 G_2(x, y) v(y) dy, \tag{61}$$

$$T_2(u, v) = \lambda \int_0^1 G_3(x, y) f\left(x, u(y), u'(y), v(y), v'(y)\right) dy + \lambda \int_0^1 G_4(x, y) f\left(x, u(y), u'(y), v(y), v'(y)\right) dy. \tag{62}$$

Lemma 5. *Under the hypothesis of Proposition 2, there exists* $K_i > 0, i = 1, 2$ *such that*

$$\max_{0 \le x \le 1} |T_1(u, v)| \le K_1 \text{ and } \max_{0 \le x \le 1} |T_2(u, v)| \le K_2. \tag{63}$$

Proof. Since $|G_i(x, y)| \le 2, i = 1, 3, |G_2(x, y)| \le \int_0^1 p(x) dx / 1 - \int_0^1 (x+1) p(x) dx = \alpha^*$ and $|G_4(x, y)| \le 2 \int_0^1 q(x) dx / 1 - \int_0^1 xq(x) dx = \beta^*, \forall x, y \in [0, 1]$, thus,

$$|T_1(u, v)| \le \int_0^1 |G_1(x, y)||v(y)| dy + \int_0^1 |G_2(x, y)||v(y)| dy. \tag{64}$$

Using $|v(x)| \le L_2$, we get

$$|T_1(u, v)| \le (2 + \alpha^*) L_2 = K_1. \tag{65}$$

For $T_2(u, v)$, we have

$$|T_2(u, v)| \le \lambda \int_0^1 |G_3(x, y)| \left| f\left(x, u(y), u'(y), v(y), v'(y)\right) \right| dy + \lambda \int_0^1 |G_4(x, y)| \left| f\left(x, u(y), u'(y), v(y), v'(y)\right) \right| dy, \tag{66}$$

$$|T_2(u, v)| \le \lambda (2 + \beta *) \int_0^1 \left| f\left(x, u(y), u'(y), v(y), v'(y)\right) \right| dy. \tag{67}$$

Applying the growth condition,

$$\int_0^1 \left| f\left(x, u(y), u'(y), v(y), v'(y)\right) \right| dy \le a \int_0^1 |u(y)| dy + b \int_0^1 \left|u'(y)\right| dy + c \int_0^1 |v(y)| dy + d \int_0^1 \left|v'(y)\right| + e, \tag{68}$$

$$\int_0^1 \left| f\left(x, u(y), u'(y), v(y), v'(y)\right) \right| dy \le a \int_0^1 |u(y)| dy + b \left(\int_0^1 \left(u'(y)\right)^2 \right)^{1/2} + c \int_0^1 |v(y)| dy + d \left(\int_0^1 \left(v'(y)\right)^2 \right)^{1/2} + e, \tag{69}$$

and employing the following inequalities that used in the proof of Proposition 2:

$\int_0^1 (v'(x))^2 dx \le C_2, \int_0^1 (u'(x))^2 dx \le C_1 C_2, |u(x)| \le L_1$, and $|v(x)| \le L_2$, we obtain

$$\int_0^1 \left| f\left(x, u(y), u'(y), v(y), v'(y)\right) \right| dy \le M \left(L_1 + (C_1 C_2)^{\frac{1}{2}} + L_2 + (C_2)^{\frac{1}{2}} + 1 \right). \tag{70}$$

Hence,

$$|T_2(u, v)| \le \lambda (2 + \beta^*) M \left(L_1 + (C_1 C_2)^{1/2} + L_2 + (C_2)^{1/2} + 1 \right) = K_2. \tag{71}$$

Consider now the closed and convex set

$$\mathbb{S} = \left\{ (u, v) \in \mathbb{X} : \|u, v\|_\infty \le K_1 + K_2 = K \right\}. \tag{72}$$

□ □

Based on the above results, we have

Lemma 6. *For any* $(u, v) \in \mathbb{S}, T(u, v)$ *is contained in* $\mathbb{S}$.

Proof. From Lemma 5, we have $|T_1(u, v)| \le K_1$ and $|T_2(u, v)| \le K_2$. Since $T(u, v) = (T_1(u, v), T_2(u, v))$, we obtain $|T(u, v)| \le K_1 + K_2 = K$. This shows that $T(u, v)$ is contained in $\mathbb{S}$.

To prove that $T(u, v)$ is compact, we use the Arzela-Ascoli lemma; that is, $T(\mathbb{S})$ must be closed, bounded, and equicontinuous. Consequently, $T(u, v)$ has a fixed point by the Schauder's fixed point theorem.□ □

Thus, we have the following theorem.

Theorem 7. *There exists a continuous solution* (u, v) *which satisfies problem (15) with (8)-(13).*

3. Conclusion

The cantilever beam problem modeled by the nonlinear fourth-order equation (6) is investigated under the integral conditions (7) and assuming the conditions (8)-(13). As illustrated in Section 2, the proposed conditions stand for perturbed conditions at the boundary, which occurs if the cantilever beam is not perfectly cantilevered in the sense that the free end is rolled and/or the fixed end is loose. The integral condition generalizes the standard boundary conditions that are usually proposed in literature for a classical cantilever beam problem. Moreover, these boundary conditions are more practical, in the sense that they represent the actual conditions that may arise in a real-world cantilever beam, which is very useful to researchers in construction engineering. The objective of this research is to determine whether small deflections occur in the cantilever beam under perturbed boundary data. The result shows that the solution to the problem exists, which implies that small deflections continue to exist on the beam whether it is perfectly cantilevered or not.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The authors would like to acknowledge the support of Prince Sultan University, Saudi Arabia, for paying the article processing charges (APC) of this publication. The authors would like to thank Prince Sultan University for their support.

References

[1] R. Ma and H. Wang, "On the existence of positive solutions of fourth-order ordinary differential equations," *Applicable Analysis*, vol. 59, no. 1-4, pp. 225–231, 1995.

[2] D. R. Anderson and J. Hoffacker, "Existence of solutions for a cantilever beam problem," *Journal of Mathematical Analysis and Applications*, vol. 323, no. 2, pp. 958–973, 2006.

[3] G. Infante and P. Pietramala, "A cantilever equation with nonlinear boundary conditions," *Electronic Journal of Qualitative Theory of Differential Equations*, vol. 15, no. 15, pp. 1–14, 2009.

[4] Q. Yao, "Monotonically iterative method of nonlinear cantilever beam equations," *Applied Mathematics and Computation*, vol. 205, no. 1, pp. 432–437, 2008.

[5] R. Ma, "Multiple positive solutions for a semipositone fourth-order boundary value problem," *Hiroshima Mathematical Journal*, vol. 33, no. 2, pp. 217–227, 2003.

[6] Q. Yao, "Local existence of multiple positive solutions to a singular cantilever beam equation," *Journal of Mathematical Analysis and Applications*, vol. 363, no. 1, pp. 138–154, 2010.

[7] Y. Li, "Existence of positive solutions for the cantilever beam equations with fully nonlinear terms," *Nonlinear Analysis: Real World Applications*, vol. 27, pp. 221–237, 2016.

[8] Y. Li and X. Chen, "Solvability for fully cantilever beam equations with superlinear nonlinearities," *Li and Chen Boundary Value Problems*, vol. 2019, no. 1, p. 83, 2019.

[9] Q. A. Dang and T. K. Q. Ngo, "Existence results and iterative method for solving the cantilever beam equation with fully nonlinear term," *Nonlinear Analysis: Real World Applications*, vol. 36, pp. 56–68, 2017.

[10] A. Cabada, J. Fialho, and F. Minhós, "Extremal solutions to fourth order discontinuous functional boundary value problems," *Mathematische Nachrichten*, vol. 286, no. 17-18, pp. 1744–1751, 2013.

[11] Y. Li and Y. Gao, "The method of lower and upper solutions for the cantilever beam equations with fully nonlinear terms," *Journal of Inequalities and Applications*, vol. 2019, no. 1, 2019.

[12] G. Lewis and F. Monasa, "Large deflections of cantilever beams of nonlinear materials," *Journal of Computers and Structures*, vol. 14, no. 5-6, pp. 357–360, 1981.

[13] Y.-A. Kang and X.-F. Li, "Bending of functionally graded cantilever beam with power-law non-linearity subjected to an end force," *International Journal of Non-Linear Mechanics*, vol. 44, no. 6, pp. 696–703, 2009.

[14] F. Monasa and G. Lewis, "Large deflections of point loaded cantilevers with nonlinear behaviour," *Zeitschrift für angewandte Mathematik und Physik*, vol. 34, no. 1, pp. 124–130, 1983.

[15] K. E. Bisshopp and D. C. Drucker, "Large deflection of cantilever beams," *Quarterly of Applied Mathematics*, vol. 3, no. 3, pp. 272–275, 1945.

[16] H. J. Barten, "On the deflection of a cantilever beam," *Quarterly of Applied Mathematics*, vol. 2, no. 2, pp. 168–171, 1944.

[17] H. Ozturk, "In-plane free vibration of a pre-stressed curved beam obtained from a large deflected cantilever beam," *Finite Elements in Analysis and Design*, vol. 47, no. 3, pp. 229–236, 2011.

[18] B. N. Rao and G. V. Rao, "large deflections of a cantilever beam subjected to a rotational distributed loading," *Forschung im Ingenieurwesen A*, vol. 55, no. 4, pp. 116–120, 1989.

[19] K. Lee, "Large deflections of cantilever beams of non-linear elastic material under a combined loading," *International Journal of Non-Linear Mechanics*, vol. 37, no. 3, pp. 439–443, 2002.

[20] M. Dado and S. Al-Sadder, "A new technique for large deflection analysis of non-prismatic cantilever beams," *Mechanics Research Communications*, vol. 32, no. 6, pp. 692–703, 2005.

[21] M. A. Rahman, M. A. Kowser, and S. M. M. Hossain, "Large deflection anaIysis of cantilever beams with an opening," *International Journal of Applied Mechanics and Engineering*, vol. 12, no. 1, pp. 169–181, 2007.

[22] D. B. Holland, L. N. Virgin, and R. H. Plaut, "Large deflections and vibration of a tapered cantilever pulled at its tip by a cable," *Journal of Sound and Vibration*, vol. 310, no. 1-2, pp. 433–441, 2008.

[23] A. K. Nallathambi, C. Lakshmana Rao, and S. M. Srinivasan, "Large deflection of constant curvature cantilever beam under follower load," *International Journal of Mechanical Sciences*, vol. 52, no. 3, pp. 440–445, 2010.

[24] M. Mutyalarao, D. Bharathi, and B. Nageswara Rao, "Large deflections of a cantilever beam under an inclined end load," *Applied Mathematics and Computation*, vol. 217, no. 7, pp. 3607–3613, 2010.

[25] B. S. Shavrtsman, "Large deflections of a cantilever beam subjected to a follower force," *Journal of Sound and Vibration*, vol. 304, no. 3-5, pp. 969–973, 2007.

[26] S. Bouadjadja, A. Tati, and A. Sadgui, "Nonlinear bending analysis of composite cantilever beams," *Australian Journal of Basic and Applied Sciences*, vol. 13, no. 7, pp. 28–34, 2019.

[27] D. Singhal and V. Narayanamurthy, "Large and small deflection analysis of a cantilever beam," *Journal of The Institution of Engineers (India): Series A*, vol. 100, no. 1, pp. 83–96, 2019.

[28] T. Belendez, C. Neipp, and A. Belendez, "Large and small deflections of a cantilever beam," *European Journal of Physics*, vol. 23, no. 3, pp. 371–379, 2002.

[29] J. M. Gere and B. L. Goodno, "Mechanics of materials," *8th edition SI. Chapter 9*, Cengage Learning, 2012.

[30] A. Khanfer and L. Bougoffa, "On the fourth-order nonlinear beam equation of a small deflection with nonlocal conditions," *AIMS Mathematics*, vol. 6, no. 9, pp. 9899–9910, 2021.

[31] A. Khanfer and L. Bougoffa, "On the nonlinear system of fourth-order beam equations with integral boundary conditions," *AIMS Mathematics*, vol. 6, no. 10, pp. 11467–11481, 2021.

[32] Y. Guo, Y. Fei, and Y. Liang, "Positive solutions for nonlocal fourth-order boundary value problems with all order derivatives," *Boundary Value Problems*, vol. 2012, no. 1, 2012.

[33] X. Zhang and W. Ge, "Positive solutions for a class of boundary-value problems with integral boundary conditions," *Computers and Mathematics with Applications*, vol. 58, no. 2, pp. 203–215, 2009.

[34] T. Jankowski, "Positive solutions for fourth-order differential equations with deviating arguments and integral boundary conditions," *Nonlinear Analysis*, vol. 73, no. 5, pp. 1289–1299, 2010.

[35] J.-P. Sun and H.-B. Li, "Monotone positive solution of nonlinear third-order BVP with integral boundary conditions," *Boundary Value Problems*, vol. 2010, no. 1, Article ID 874959, 12 pages, 2010.

[36] Y. Li and Y. Dong, "Multiple positive solutions for a fourth-order integral boundary value problem on time scales," *Boundary Value Problems*, vol. 2011, no. 59, 2011.

[37] M. Feng, "Existence of symmetric positive solutions for a boundary value problem with integral boundary conditions," *Applied Mathematics and Letters*, vol. 24, no. 8, pp. 1419–1427, 2011.

[38] H. Li, L. Wang, and M. Pei, "Solvability of a fourth-order boundary value problem with integral boundary conditions," *Journal of Applied Mathematics*, vol. 2013, Article ID 782363, 7 pages, 2013.

[39] X. Lv, L. Wang, and M. Pei, "Monotone positive solution of a fourth-order BVP with integral boundary conditions," *Boundary Value Problems*, vol. 2015, no. 1, 2015.

[40] Y. Ji, Y. Guo, and Y. Yao, "Positive solutions for higher order differential equations with integral boundary conditions," *Boundary Value Problems*, vol. 2015, no. 1, 2015.

[41] P. Kang, Z. Wei, and J. Xu, "Positive solutions to fourth-order singular boundary value problems with integral boundary conditions in abstract spaces," *Applied Mathematics and Computation*, vol. 206, no. 1, pp. 245–256, 2008.