In this paper, we investigate topological transitivity of operators on nonseparable Hilbert spaces which are similar to backward weighted shift operators. In particular, we show that abstract differential operators and dual operators to operators of multiplication in graded Hilbert spaces are similar to backward weighted shift operators.

1. Introduction

Let $X$ be a Hausdorff locally convex space. A continuous operator $T : X \to X$ is called topologically transitive if for each pair $U, V$ of nonempty open subsets of $X$, there is some $n \in \mathbb{N}$ with $T^n(U) \cap V \neq \emptyset$. If the underlying space is a separable Baire space, the transitivity is equivalent to the hypercyclicity by Birkhoff’s transitivity theorem (see [1], p. 2). A continuous linear operator $T : X \to X$ acting on a separable Fréchet space $X$ is called hypercyclic if there is a vector $x \in X$ for which the orbit under $T$, $\text{Orb}_T(x) = \{x, Tx, T^2x, \ldots\}$ is dense in $X$. Every such vector $x$ is called a hypercyclic vector of $T$.

The study of topological transitivity was started from Birkhoff’s result [2] in 1929 for nonlinear continuous functions. This result is important in the theory of chaos (see e.g., [3]). For the space of all entire functions, Birkhoff [2] also proved that the translation operator $T_a : f(x) \mapsto f(x + a)$, ($a \neq 0$) is hypercyclic. MacLane in 1952 showed that the differentiation operator $T(f) = f'$ is hypercyclic too (see [4]).

A lot of work on hypercyclicity has been based on the well-known so-called hypercyclicity criterion (the Kitai-Gethner-Shapiro theorem). In [5], Bermúdez and Kalton observed that similar criterion holds for topologically transitive operators on Banach spaces.

**Theorem 1** (topologically transitive criterion). Let $T$ be a bounded linear operator on a complex Banach space $X$ (not necessarily separable). Suppose that there exists a strictly increasing sequence $(n_k)$ of positive integers for which there are

(i) A dense subset $X_0 \subset X$ such that $T^{n_k}(x) \to 0$ for every $x \in X_0$ as $k \to \infty$

(ii) A dense subset $Y_0 \subset X$ and a sequence of mappings $S_k : Y_0 \to X$ such that $S_k(y) \to 0$ for every $y \in Y_0$ and $T^{n_k} \circ S_k(y) \to y$ for every $y \in Y_0$ as $k \to \infty$

Then, $T$ is topologically transitive.

More information on transitive and hypercyclic operators can be found in [1, 6]. The existence of an element with a dense orbit implies that $X$ must be separable. On the other hand, the transitivity of $T : X \to X$ does not require that the space $X$ is separable.

**Problem 2** (Bermúdez and Kalton [5]). Is there any characterization of nonseparable Banach spaces which support a topologically transitive operator?
Note that any hypercyclic operator \( T : X \to X \) always has an invariant, norm dense, linear subspace in which every nonzero vector is a hypercyclic vector for \( T \).

**Theorem 3** (see [7]). Let \( X \) be a complete metric space which has no isolated points. Let \( T : X \to X \) be topologically transitive operator and \( y \in X \). There exists a dense \( G_δ \) subset \( D \) of \( X \) such that for each \( z \in D \), there exists a \( T \)-invariant (separable) closed subspace \( Y_z \) of \( X \) with \( y, z \in Y_z \) such that the restriction of \( T \) to \( Y_z, T : Y_z \to Y_z \), is hypercyclic.

The hypercyclicity of composition operators and different operators on some function spaces of finite and infinite variables was studied in [8–14]. According to a classical theorem of Rolewicz [15], the weighted backward shifts on \( \ell_2 \)

\[
(x_1, x_2, \ldots, x_n, \ldots) \mapsto (\lambda x_2, \lambda x_3, \ldots, \lambda x_{n+1}, \ldots) \quad (2)
\]

is hypercyclic if \( |\lambda| > 1 \). This fact remains to be true if we replace \( \ell_2 \) into \( \ell_p \) for \( 1 \leq p < \infty \). Bermúdez and Kalton in [5] showed that spaces like \( \ell_\infty \) and \( X(\ell_2) \) do not support topologically transitive operators, where \( X(\ell_2) \) is the space of all bounded operators on \( \ell_2 \).

In [7] (Proposition 3.4), Manoussos proved that the backward weighted shift is topologically transitive on nonseparable Hilbert space. More detailed, let \( H \) be a Hilbert space (possibly nonseparable) and \( \ell_2(H) \) be the \( \ell_2 \)-sum of infinitely many copies of \( H \). That is, if \( x \in \ell_2(H) \), then

\[
x = (x_0, x_1, \ldots, x_n, \ldots), \quad x_n \in H, \quad \|x\|^2 = \sum_{n=0}^{\infty} \|x_n\|^2. \quad (3)
\]

Then, the backward weighted shift \( T \) with weight sequence \((\omega_n)\) is defined by the following:

\[
T : (x_0, x_1, \ldots, x_n, \ldots) \mapsto (\omega_1 x_1, \omega_2 x_2, \ldots, \omega_n x_n, \ldots). \quad (4)
\]

Unfortunately, Proposition 3.4 in [7] contains a slight inaccuracy which for us is essential. So, we propose the corrected version.

**Proposition 4.** Let \( H \) be a Hilbert space and \( T : \ell_2(H) \to \ell_2(H) \) be a backward weighted shift with positive weight sequence \((\omega_n)\). The following are equivalent:

(i) \( T \) is topologically transitive

(ii) There exists a nontrivial \( T \)-invariant (separable) closed subspace \( \mathcal{Y} \subset \ell_2(H) \) on which the restriction of \( T \) to \( \mathcal{Y}, T : \mathcal{Y} \to \mathcal{Y} \), is hypercyclic

(iii) The restriction \( T : \mathcal{Y} \to \mathcal{Y} \) to any \( T \)-invariant (separable) closed subspace \( \mathcal{Y} \subset \ell_2(H) \) which contains nonzero vectors of the form \((0, \ldots, 0, x_n, 0, \ldots)\) for every \( n \in \mathbb{Z}_+ \), is hypercyclic

(iv) \( \limsup_{n \to \infty} \prod_{k=1}^{n} \omega_k = +\infty \)

Note that in [7], item (iii) is written: “The restriction \( T : \mathcal{Y} \to \mathcal{Y} \) to any \( T \)-invariant (separable) closed subspace \( \mathcal{Y} \subset \ell_2(H) \) is hypercyclic.” But it is not correct because the subspace \( Y_0 \) consisting of vectors \((x_0, 0, 0, \ldots) \) is invariant but the restriction of \( T \) to any separable subspace of \( Y_0 \) is not hypercyclic since \( Y_0 \subset \ker T \).

In Section 2, we consider abstract shift similar operators on nonseparable function Hilbert spaces \( \ell_2(H) \). In particular, abstract differentiation operators and dual operators to abstract multiplication operators can be considered as abstract shift similar operators. In Section 3, we construct examples of topologically transitive operators.

### 2. Abstract Shift Similar Operators

Let \((H_n)_{n=0}^{\infty} \) be a sequence of Hilbert spaces. Throughout this paper, we assume that all \( H_n \) are nontrivial; that is, \( H_n \neq \{0\} \) and not necessary separable. Let us suppose that for every \( n \) and \( m, H_n \) is isomorphic to \( H_m \). We denote by \( \ell_2(H_n) = \ell_2(H_n, H_0) \) the Hilbert space consisting of elements \( x = (x_0, x_1, \ldots, x_n, \ldots) \) \( x_n \in H_k \) endowed with norm \( \|x\| = (\sum_{n=0}^{\infty} \|x_n\|^2)^{1/2} \). Let \((\omega_n)\) be a sequence of positive numbers (weights). Let us fix a sequence of isomorphisms \( J_m : H_m \to H_{m-1} \|J_m\| = 1, m \in \mathbb{N} \). An operator \( T : \ell_2(H_n) \to \ell_2(H_n) \) will be called a backward weighted shift (with respect to the family \((J_m)\)) with weight sequence \((\omega_n)\) if it is of the following form:

\[
T(x) = (\omega_1 J_1(x_1), \omega_2 J_2(x_2), \ldots, \omega_m J_m(x_m), \ldots). \quad (5)
\]

From Proposition 4, it follows the next corollary.

**Corollary 5.** Let \((H_n)_{n=0}^{\infty} \) be a sequence of Hilbert spaces and \( T : \ell_2(H_n) \to \ell_2(H_n) \) be a backward weighted shift with respect to \((J_m)\) and with positive weight sequence \((\omega_n)\). Let us suppose that

\[
\sup_{m \in \mathbb{Z}_+} \prod_{n=0}^{m} \| J_n^{-1} \| < \infty. \quad (6)
\]

Then, the following are equivalent:

(i) \( T \) is topologically transitive

(ii) There exists a nontrivial \( T \)-invariant (separable) closed subspace \( \mathcal{Y} \subset \ell_2(H_n) \) on which the restriction of \( T \) to \( \mathcal{Y}, T : \mathcal{Y} \to \mathcal{Y} \), is hypercyclic

(iii) The restriction \( T : \mathcal{Y} \to \mathcal{Y} \) to any \( T \)-invariant (separable) closed subspace \( \mathcal{Y} \subset \ell_2(H_n) \) which contains nonzero vectors of the form \((0, \ldots, 0, x_n, 0, \ldots)\) for every \( n \in \mathbb{Z}_+ \), is hypercyclic

(iv) \( \limsup_{n \to \infty} \prod_{k=1}^{n} \omega_k = \infty \)
Proof. We set $H = H_0$. Then,

$$J_1(H_1) = H_0 = H,$$

$$J_1 * J_2)(H_2) = H_0 = H,$$

$$J_1 * \cdots * J_n)(H_n) = H_0 = H,$$

and so on. For any $x = (x_1, \cdots, x_n, \cdots) \in \ell_2(H_n)$,

$$J : x \mapsto (x_0, J_1(x_1), \cdots, (J_1 * \cdots * J_n)(x_n), \cdots) \in \ell_2(H),$$

$$||J(x)|| \leq ||x||.$$

Because of Equation (6), the inverse operator

$$J^{-1} : z \mapsto (z_0, J^{-1}_1(z_1), \cdots, (J^{-1}_1 * \cdots * J^{-1}_n)(z_n), \cdots) \in \ell_2(H_n), \quad z \in \ell_2(H)$$

is well defined and bounded. So, $J$ is an isometric isomorphism from $\ell_2(H_n)$ to $\ell_2(H)$. For a closed subspace $\mathcal{Y} \subset \ell_2(H_n)$, the range $J(\mathcal{Y})$ is a closed subspace of $\ell_2(H)$. Let $T' = J * T * J^{-1}$. Then, $T'$ is a backward weighted shift on $\ell_2(H)$ with positive weight sequence $(\omega_n)$. Since $J$ is an isomorphism, the topological transitivity of $T$ or hypercyclicity of $T$ on $\mathcal{Y}$ is equivalent to the topological transitivity of $T'$ on $\ell_2(H)$, respectively. Hence, we can apply Theorem 3 for $T'$.

Note that the case when $||J|| = \nu_n > 0$ and $\sup_n \nu_n < \infty$ can be reduced to our case if we consider $J' = J/\nu_n'$ and $\omega_n' = \omega_n \nu_n$ instead of $J$ and $\omega_n$, respectively.

An operator $B : \ell_2(H_n) \rightarrow \ell_2(H_n)$ is a backward weighted shift similar operator if there exists an isomorphism $A : \ell_2(H_n) \rightarrow \ell_2(E_m)$ for some sequence of Hilbert spaces $(E_m)$ and a backward weighted shift $T : \ell_2(E_m) \rightarrow \ell_2(E_m)$ such that $B = A T A^{-1}$. It is clear that $B$ is topologically transitive if and only if $T$ is topologically transitive.

Let us suppose that $H_n = H_n' \oplus H_n''$, and the isomorphisms $J_n : H_n \rightarrow H_n'$ be such that $J_n$ is an isometric isomorphism of $H_n'$ onto $H_n'$. Then, we define a backward weighted shift (with respect to family $(J_m \circ \pi_m)$) with weight sequence $(\omega_n)$ on $\ell_2(H_n)$ by the following:

$$T(x) = (\omega_1(J_1 \circ \pi_1)(x_1), \omega_2(J_2 \circ \pi_2)(x_2), \cdots, \omega_m(J_m \circ \pi_m)(x_m), \cdots).$$

In other words, $\ell_2(H_n) = \ell_2(H_n') \oplus \ell_2(H_n'')$, and $T$ is a backward weighted shift on $\ell_2(H_n')$ and ker $T = \ell_2(H_n'').$

Let us recall that the density $d(X)$ of a metric space $X$ is the smallest cardinality of a dense subset of $X$. It is well known that two infinite dimensional Hilbert spaces $H$ and $E$ are isometrically isomorphic if and only if $d(H) = d(E).$

The next proposition shows that the backward weighted partial shift can be reduced to the weighted shift.

Proposition 6. Let $T$ be a backward weighted partial shift with respect to the family of operators $(I_m \circ \pi_m)_{m=1}^\infty$ with weight sequence $(\omega_n)$ on $\ell_2(H_n)$. Let us suppose that all $H_n$ are infinite dimensional for $n \geq 1$ and $d(H_n) \leq d(H_n')$. Then, $T$ is a backward weighted shift similar operator; that is, there is a sequence of Hilbert spaces $(E_m)_{m=0}^\infty$ and an isometric isomorphism $\mathcal{J} : \ell_2(H_n) \rightarrow \ell_2(E_m)$ such that $\mathcal{J} \circ T \circ \mathcal{J}^{-1}$ is a backward weighted shift.

Proof. Let us define spaces $E_n$ by the following way:

$$E_0 = H_0 \oplus H_1 \oplus H_2 \oplus \cdots \oplus H_m \oplus \cdots,$$

$$E_n = H_n' \oplus H_n'' , \quad n > 1.$$

Since spaces $H_n$ are isomorphic each to others, $d(H_n) = d(H_n')$ for all $n, m \in \mathbb{Z}_+$. Since $d(H_0) \leq d(H_n)$, we have $d(E_n) = d(E_m)$ and so $E_n$ and $E_m$ are isomorphic for all $n, m \in \mathbb{Z}_+$. It is clear that $\ell_2(H_n)$ and $\ell_2(E_n)$ are just different representations of the same space, and so $\mathcal{J}$ is the identical operator. Also, the restrictions of $I_n \circ \pi_n$ to $E_n$ on $H_n$ are isometric isomorphisms. So, $\mathcal{J} \circ T \circ \mathcal{J}^{-1}$ is a backward weighted shift with weight sequence $(\omega_n)$ on $\ell_2(E_n)$.

Corollary 7. Let $T : \ell_2(H_n) \rightarrow \ell_2(H_n)$ be a backward weighted partial shift with positive weight sequence $(\omega_n)$ which satisfies conditions of Proposition 6. Then, the following are equivalent:

(i) $T$ is topologically transitive

(ii) There exists a nontrivial $T$-invariant (separable) closed subspace $\mathcal{Y} \subset \ell_2((H_n)_{m=0}^\infty)$ on which the restriction of $T$ to $\mathcal{Y}$, $T : \mathcal{Y} \rightarrow \mathcal{Y}$, is hypercyclic

(iii) The restriction $T : \mathcal{Y} \rightarrow \mathcal{Y}$ to any $T$-invariant (separable) closed subspace $\mathcal{Y} \subset \ell_2((H_n)_{m=0}^\infty)$ which contains nonzero vectors of the form $(0, \cdots, 0, x_n, 0 \cdots )$, $x_n \in H_n'$, for every $n \in \mathbb{Z}_+$, is hypercyclic

(iv) $\limsup_{n \rightarrow \infty} \prod_{k=1}^n \omega_k = \infty.$

This approach can be generalised by the following way.

Theorem 8. Let $D$ be a surjective bounded operator on a $\ell_2$-sum $\ell_2(\mathcal{H}_n)$ of Hilbert spaces $\mathcal{H}_n$ such that $D : H_n \rightarrow H_{n+1}, n \geq 0$. Then, $D$ is a backward weighted shift similar operator with some weight sequence $(\omega_n)(0 < \omega_n < ||D_n||)$, where $D_n$ is the restriction of $D$ to $\mathcal{H}_n$.

Proof. Let us define the following spaces:

$$\mathcal{H}_0 = \mathcal{H},$$

$$\mathcal{H}_1 = \ker D_1, \quad \mathcal{H}_2 = \ker (D_1 \circ D_2) \oplus \mathcal{H}_2,$$

$$\mathcal{H}_2 = \ker (D_1 \circ D_2) \oplus \ker (D_1 \circ D_2),$$
Let us denote by the following:

$$H_k = \ell_2(\mathcal{H}_n)_{n \leq k} = \mathcal{H}_k \oplus \mathcal{H}_{k+1} \oplus \cdots, \quad k \in \mathbb{Z}_+.$$  

(13)

In other words, if \(0 \neq x \in H_1\), then \(0 \neq D_1 \circ \cdots \circ D_k(x) \in \mathcal{H}_k\). Let \(D_k^n\) be the restriction of \(D_n\) to \(\mathcal{H}_k^n\). Then, \(\|D_k^n\| \leq \|D_n\|\|k \leq n\). Thus, the action of \(D\) on \(H_k\) can be written by the following:

$$D_k^n = D_k^n \oplus D_k^n \oplus \cdots \oplus D_k^n, \quad k \in \mathbb{Z}_+.$$  

(14)

and \(\|D_k^n\| = \sup_n \|D_k^n\|\). So, \(\|D_k^n\| \leq \|D_k\| \leq \|D\|\|k \leq n\). It is clear that the spaces \(\ell_2(\mathcal{H})\) and \(\ell_2(H_n)\) consist of the same vectors and the identity map \(\mathcal{S} : \ell_2(\mathcal{H}_n) \to \ell_2(H_n)\) is an isometric isomorphism. Thus, \(D_k^n\) is the restriction of \(3D\mathcal{S}\) to \(H_k\). By the construction, \(D_k^n : H_k \to H_k, k > 0\) and \(D_k(0) = 0\). Moreover, since \(\ker 3D\mathcal{S} = H_0\ker D_k = 0\) if \(k > 0\). In addition, since \(D\) is onto, \(D_k^n\) must be onto for every \(k\), because the preimage of \(H_{k-1}\) under \(D_k^n\) is \(H_k\). Hence, every \(D_k^n, k > 0\) is an isomorphism.

Let us set the following:

$$J_k = \frac{D_k^n}{\|D_k^n\|}, \quad \omega_k = \|D_k^n\|, \quad k > 1.$$  

(15)

Then for every \(x = (x_0, x_1, \cdots, x_m, \cdots) \in \ell_2(H_n)\),

$$3D\mathcal{S}^{-1}(x) = (\omega_1 J_1(x_1), \omega_2 J_2(x_2), \cdots, \omega_n J_n(x_n), \cdots).$$  

(16)

That is, \(3D\mathcal{S}^{-1}\) is a backward weighted shift.

Let us show that \(\omega_k > 0\) for every \(k \in \mathbb{N}\). Since all spaces \(\mathcal{H}_k\) are nontrivial and \(D\) is onto, there is a vector \(a_i \in \mathcal{H}_k\) such that \(a_i \notin \ker D\). So, \(a_i \in H_1\) and \(D_1(a_i) = D_1(a_1) \neq 0\); that is, \(\omega_1 = \|D_1(a_1)\| \neq 0\). If already we have nonzero elements \(a_j \in H_j \cap \mathcal{H}_j, j = 1, 2, \cdots, k - 1\), then we denote by \(a_k\) a vector in \(\mathcal{H}_k\) in the preimage of \(a_{k-1}\) under \(D_k\). By the assumption \(a_{k-1} \notin \ker D_1 \circ \cdots \circ D_k\(, so \(a_k \notin \ker D_1 \circ \cdots \circ D_k\). Thus, \(a_k \in H_k\) and \(D_k(a_k) = D_k(a_1) = a_{k-1} \neq 0\). Hence, \(\omega_k = \|D_k^n\| \neq 0\).

An element \(x \in \ell_2(H_n)\) is a finite type vector if there is \(m \in \mathbb{N}\) such that \(x_k = 0\) for all \(k \geq m\).

Definition 9. Let \(\mathcal{H}_n, n \in \mathbb{Z}_+\) be the Hilbert spaces. We say that \(\ell_2(\mathcal{H}_n)\) is a graded Hilbert space, if there is a bilinear map \((x, y) \to xy\) (multiplication) defined for every finite type vector \(x\) and an arbitrary \(y \in \ell_2(\mathcal{H}_n)\) with values in \(\ell_2(\mathcal{H}_n)\) such that the multiplication is associative, nondegenerated (that is, \(xy = 0\) only if \(x = 0\) or \(y = 0\)) and \(x_n y_m \in \mathcal{H}_{nm}\) for every \(x_n \in \mathcal{H}_n\) and \(y_m \in \mathcal{H}_m\).

Definition 10. For a given graded Hilbert space \(\ell_2(\mathcal{H}_n)\) and a finite type vector \(a \in \ell_2(\mathcal{H}_n)\), we denote by \(M_a\) the multiplication operator.

$$M_a(x) = ax, \quad x \in \ell_2(\mathcal{H}_n).$$  

(17)

Theorem 11. Let \(a \in \mathcal{H}_m\) and \(M_a\) be continuous on \(\ell_2(\mathcal{H}_n)\). If the dual operator \(M_a^* : (\ell_2(\mathcal{H}_n))^* \to \ell_2(\mathcal{H}_n)^*\) is surjective, then it is a backward weighted shift similar operator.

Proof. Let

$$E_0 = \mathcal{H}_0 \oplus \cdots \oplus \mathcal{H}_{m-1}, \quad E_k = \mathcal{H}_{km} \oplus \cdots \oplus \mathcal{H}_{km+k-1}, \cdots$$  

(18)

and \(H_n = E_n^*\) be the Hermitian dual space for every \(n \in \mathbb{Z}_+\). Then, \(M_a\) maps \(E_n\) to \(E_{n+1}\), \(n \in \mathbb{Z}_+\), and \(M_a^*\) maps \(H_n\) to \(H_{n+1}\), \(n > 0\), and \(H_0 = \ker M_a^*\). Let \((M_a^*)_n\) be the restriction of \(M_a^*\) to \(H_n\). Since \(M_a\) is continuous, \(M_a^*\) is continuous on \(\ell_2(H_n)\) as well and \(\sup_n \|M_a^*_n\| < \infty\). We set the following:

$$J_n = \frac{(M_a^*)_n}{\|(M_a^*)_n\|}, \quad \omega_n = \|(M_a^*)_n\|.$$  

(19)

Since \(M_a^*\) is surjective, \((M_a^*)_n\) are surjective for all \(n \in \mathbb{Z}_+\) and since \(H_0 = \ker (M_a^*)_n\) are bijective for \(n > 0\). So,

$$M_a^*(x_0, x_1, \cdots, x_m, \cdots) = (\omega_1 J_1(x_1), \omega_2 J_2(x_2), \cdots, \omega_m J_m(x_m), \cdots)$$  

(20)

is a backward weighted shift on \(\ell_2(H_n)\).

Note that if \(a \in \mathcal{H}_1\), then \(E_k = \mathcal{H}_k\) and \(\omega_n = \|(M_a^*)_k\| = \|(M_a)_k\| = \|(M_a)_k\|\), where \((M_a)_k\) is the restriction of \(M_a\) to \(\mathcal{H}_k\), \(k \in \mathbb{Z}_+\).

Definition 12. Let \(\ell_2(\mathcal{H}_n)\) be a graded Hilbert space. A continuous linear operator \(D : \ell_2(\mathcal{H}_n) \to \ell_2(\mathcal{H}_n)\) is called a differential operator if \(\ell_0 \subset \ker D\), the restriction of \(D\) to \(\mathcal{H}_n\) maps \(\mathcal{H}_n\) onto \(\mathcal{H}_{n+1}\), \(n > 1\) and

$$D(xy) = D(x)y + xD(y)$$  

(21)

for every finite type vector \(x\) and for all \(y \in \ell_2(\mathcal{H}_n)\).

Corollary 13. Let \(D\) be a differential operator on a graded Hilbert space \(\ell_2(\mathcal{H}_n)\) such that \(D\) is onto and \(\|D\| \neq 0\). Then, there is a sequence of Hilbert spaces \((H_n)_{n \in \mathbb{N}}\) and an isometric isomorphism \(\mathcal{S} : \ell_2(\mathcal{H}_n) \to \ell_2(H_n)\) such that \(\mathcal{S} \circ D \circ \mathcal{S}^{-1}\) is a backward weighted shift with some weight sequence \((\omega_n)\), \(0 < \omega_n \leq \|D_n\|\).
Proof. Since \( \|D_k\| \neq 0 \), there is \( a \in \mathcal{H}_1 \) such that \( D(a) \neq 0 \). So, \( a \in H_1 \). Moreover, for every \( k \in \mathbb{N} \), \( a^k \neq a \neq 0 \) and \( D(a^k) = k a^{k-1} \). Thus, \( a^k \in H_k \) and we can apply Theorem 2.4. \( \square \)

**Corollary 14.** Let \( D \) be a continuous surjective differential operator on a graded Hilbert space \( \ell_2(\mathcal{H}_n) \). The following are equivalent:

(i) \( D \) is topologically transitive

(ii) There exist \( a \in \mathcal{H}_1 \), \( \epsilon > 0 \), and \( m \in \mathbb{N} \) such that \( D(a) \neq 0 \) and

\[
\limsup_{m \to \infty} \prod_{n=1}^{m} \left\| a^{n-1} \right\| = \infty. \tag{22}
\]

(iii)

\[
\limsup_{m \to \infty} \prod_{n=1}^{m} \left\| D(a^n) \right\| = \infty. \tag{23}
\]

Proof. As we observed, \( a^k \in H_k \) and so

\[
\omega_k = \left\| D_k \right\| \geq \frac{\left\| D(a^k) \left\| a^{k-1} \right\|}{\| a^k \|} = k \| a^{k-1} \|. \tag{24}
\]

By Theorem 8., \( \omega_k > 0, k \in \mathbb{N} \). So if \( \omega_k > 1 + \epsilon \) for all \( k > m \), then

\[
\prod_{n=1}^{m} \omega_n = \infty \tag{25}
\]

and \( D \) must be topologically transitive. \( \square \)

3. Examples of Topologically Transitive Operators

3.1. Partial Shifts. It is well known that any element \( x \) of the (complete) Hilbert tensor product \( \ell_2 \otimes_h \ell_2 \) can be represented by the following series

\[
x = \sum_{i,j=0}^{\infty} x_{ij} e_i \otimes e_j, \text{ where } (e_i)_{i \in \mathbb{Z}_+} \text{ is the standard basis in } \ell_2
\]

or by the following infinite matrix

\[
x = (x_{ij})_{i,j \in \mathbb{Z}_+},
\]

with \( \|x\|^2 = \sum_{i,j=0}^{\infty} |x_{ij}|^2 \). Note that \( \ell_2 \otimes_h \ell_2 \) can be considered as \( \ell_2(\ell_2) \) by the following representation:

\[
x = (x_0, x_1, \ldots, x_n, \ldots) \in \ell_2(\ell_2), \tag{28}
\]

where

\[
x_n = (x_{00}, x_{11}, \ldots, x_{nn}, \ldots) \in \ell_2, \ n \in \mathbb{Z}_+.
\]

Let \( \omega_{ij} \) be an infinite matrix of positive numbers. We consider the following operators on \( \ell_2(\ell_2) \).

Let

\[
T_\omega(x) = \left( \omega_{ij} x_{i+j} \right)_{i,j \in \mathbb{Z}_+} = \begin{pmatrix}
\omega_{10} x_{10} & \omega_{11} x_{11} & \ldots & \omega_{1n} x_{1n} & \ldots \\
\omega_{20} x_{20} & \omega_{21} x_{21} & \ldots & \omega_{2n} x_{2n} & \ldots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\omega_{m0} x_{m0} & \omega_{m1} x_{m1} & \ldots & \omega_{mn} x_{mn} & \ldots \\
\end{pmatrix}
\]

and \( T_\omega^* \) be the transposed operator, \( T_\omega^*(x) = (\omega_{ij} x_{j+i})_{i,j \in \mathbb{Z}_+} \). Let \( \omega_i = \sup_j \omega_{ij} \). From Theorem 8., it follows that if \( T\) is continuous and

\[
\limsup_{n \to \infty} \prod_{k=1}^{n} \omega_n = +\infty, \tag{30}
\]

then operators \( T_m T_n \) are topologically transitive for every \( m, n, m + 1 \geq 0 \). In particular, the operator

\[
T_\omega T_\omega^*(x) = \begin{pmatrix}
\omega_{11} x_{11} & \omega_{12} x_{12} & \ldots & \omega_{1n} x_{1n} & \ldots \\
\omega_{21} x_{21} & \omega_{22} x_{22} & \ldots & \omega_{2n} x_{2n} & \ldots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\omega_{n1} x_{n1} & \omega_{n2} x_{n2} & \ldots & \omega_{nn} x_{nn} & \ldots \\
\end{pmatrix}
\]

looks like weighted diagonal shift operator. It is clear that these examples can be generalised for any tensor degree \( \otimes^m \ell_2 \), \( m \in \mathbb{N} \). Next, we consider a nonseparable case.

Let \( \mathfrak{A} \) be the set of indexes (uncountable, in general) which can be represented as a disjoint union of subsets \( \mathfrak{A}_i \),

\[
\mathfrak{A} = \bigcup_{i=0}^{\infty} \mathfrak{A}_i, \tag{32}
\]
and there are a sequence of bijections \( \tau_n : \mathfrak{A}_n \to \mathfrak{A}_{n-1} \), \( n \in \mathbb{N} \). So, we can write \( \xi_j(\mathfrak{A}) = \xi_j(\mathfrak{A}_n) \) and define \( J_n : L_2(\mathfrak{A}_n) \to L_2(\mathfrak{A}_{n-1}) \), \( n \in \mathbb{N} \) by \( J_n(x_i) = x_{i(n)}, i \in \mathfrak{A}_n \). By Corollary 5, we have that

\[
T(x) = (\omega_j J_1(x_1), \omega_j J_2(x_2), \cdots, \omega_m J_m(x_m), \cdots)
\]  

(33)

is topologically transitive if and only if the weight \((\omega_m)\) satisfies Equation (31). Moreover, the Hilbert tensor product \( L_2(\mathfrak{A}) \otimes L_2(\mathfrak{A}) \) is isomorphic to

\[
L_2(\mathfrak{A} \times \mathfrak{A}) = L_2(\mathfrak{A}_n \times \mathfrak{A}_m).
\]

(34)

For a given weight matrix \( \omega = \omega_{i,j}, i, j \in \mathbb{N}, \omega_{i,j} > 0 \), we define operators \( T_\omega \) and \( T^*_\omega \) on \( L_2(\mathfrak{A}_n \times \mathfrak{A}_m) \) by the following:

\[
T_\omega(x_{nm}) = x_{(n)m} \text{ and } T^*_\omega(x_{nm}) = x_{nm(\omega)},
\]

(35)

where \( x_{nm} \in L_2(\mathfrak{A}_n \times \mathfrak{A}_m) \).

Corollary 15. Let

\[
\sup_{i,j\in\mathbb{N}} \omega_{i,j} < \infty,
\]

(36)

then for every \( k, j \in \mathbb{Z}^+ \), \( k + j > 0 \), operators \( T^k_\omega T^j_\omega \) are topologically transitive if and only if

\[
\limsup_{n \to \infty} \prod_{i=1}^{n} \sup_{m} \omega_{mi} = +\infty.
\]

(37)

3.2. Differential Operator on a Function Space. Examples of Hilbert spaces of analytic functions with continuous differential operators can be found among reproducing kernel Hilbert spaces.

Definition 16. Let \( Z \) be an abstract set and \( \mathfrak{H} \) a Hilbert space of complex valued functions \( f \) on \( Z \) equipped with inner product \( \langle \cdot | \cdot \rangle_\mathfrak{H} \). A function \( K(x | z) \) defined on \( Z \times Z \) is called reproducing kernel of a closed subspace \( \mathfrak{K} \subset \mathfrak{H} \) if

(i) For any fixed \( z \in Z, K(x | z) \) belongs to \( \mathfrak{K} \) as a function.

(ii) For any \( f \in \mathfrak{K} \) and any \( z \in Z, f(z) = \langle f | K(\cdot | z) \rangle_\mathfrak{H} \).

The space \( \mathfrak{K} \) is called a Hilbert space with reproducing kernel. A function \( h : Z \to \mathfrak{H} \) such that

\[
f(x) = (f(\cdot) | h(x))_\mathfrak{H}
\]

(38)

for every \( f \in \mathfrak{K} \) and \( x \in Z \) is called the kernel function of \( \mathfrak{K} \).

Let \( \mathfrak{H}(\gamma) \) be the Hilbert space of power series

\[
f(z) = \sum_{n=0}^{\infty} \gamma^n z^n, z \in \mathbb{C}
\]

(39)

deed with the norm

\[
||f||^2 = \sum_{n=0}^{\infty} \gamma^n |f(n)|^2 \leq \infty,
\]

(40)

where \( \gamma = \{\gamma_n\} \) is a positive sequence and \( \bar{f}(n) \in \mathbb{C} \). So, the inner product on \( \mathfrak{H}(\gamma) \) is defined by \( \langle f \mid g \rangle = \sum_{n=0}^{\infty} \bar{f}(n) \bar{g}(n) \) and \( \mathfrak{H}(\gamma) \) is a reproducing kernel space with the reproducing kernel.

\[
K(x | z) = \sum_{n=0}^{\infty} \gamma^n z^n.
\]

(41)

It is known [16] that if \( \gamma_{n+k}/\gamma_{n} \to 0 \) as \( n \to \infty \), \( \mathfrak{H}(\gamma) \) consists of entire functions on \( \mathbb{C} \).

Let \( D = dz/dz \) be the differential operator on \( \mathfrak{H}(\gamma) \) Then,

\[
D_n(\bar{f}(n) z^n) = \bar{f}(n)n z^{n-1}, n > 0,
\]

and \( ||D_n|| = n \gamma_{n-1} \). Thus, \( D \) is continuous (c. f. [9]) if and only if

\[
K(x | z) = \sum_{n=0}^{\infty} \gamma^n z^n.
\]

(42)

On the other hand, according to Corollary 14, \( D \) is topologically transitive and so hypercyclic if and only if

\[
\limsup_{m \to \infty} \prod_{n=1}^{m} \gamma^n \to 0.
\]

(43)

It is clear that \( \mathfrak{H}(\gamma) = \mathfrak{H}(1/\gamma) \). Let us consider the multiplication operator \( M_z \) on \( \mathfrak{H}(\gamma) \) \( M_z(f) = zf(z) \). Then, \( ||M_z|| = \gamma_{\gamma+1} \). According to Theorem 11, \( M_z \) is bounded if and only if \( \gamma_{n+1}/\gamma_{n} \to 0 \) and \( M_z \) is topologically transitive if only if

\[
\limsup_{m \to \infty} \prod_{n=1}^{m} \gamma^n \to 0.
\]

(44)

This result agrees with [8]. Various Hilbert spaces of analytic functions of infinite many variables were constructed in [14, 17, 18] using the Hilbertian symmetric tensor products of Hilbert spaces. Such constructions can be easily extended to nonseparable Hilbert spaces. In [14], the authors found some conditions under which the translation operator \( f(x) \to f(x + a) \) on a Hilbert space of entire analytic functions on \( \ell_2 \) is hypercyclic. Taking into account that the translation operator can be represented as an exponential function of a differential operator, we would like to ask the following question.

Question 17. Let \( D \) be a differential operator on a graded nonseparable Hilbert space. Suppose that \( D \) is topologically transitive. Does
\[ e^D = \sum_{n=0}^{\infty} \frac{D^n}{n!} \] (45)

topologically transitive?

4. Conclusion

We can see that dual to abstract multiplication operators \( M^*_a \) and abstract differentiation operators \( D \) on Hilbert spaces can be represented as backward weighted shift operators. Thus, we can use known results about topological transitivity of backward weighted shift operators to get conditions of topological transitivity of operators \( M^*_a \) and \( D \).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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