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Research Article

Topological Transitivity of Shift Similar Operators on Nonseparable Hilbert Spaces

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In this paper, we investigate topological transitivity of operators on nonseparable Hilbert spaces which are similar to backward weighted shifts. In particular, we show that abstract differential operators and dual operators to operators of multiplication in graded Hilbert spaces are similar to backward weighted shift operators.

1. Introduction

Let X be a Hausdorff locally convex space. A continuous operator $T : X \longrightarrow X$ is called *topologically transitive* if for each pair U,V of nonempty open subsets of X, there is some $n \in \mathbb{N}$ with $T^n(U) \cap V \neq \varnothing$. If the underlying space is a separable Baire space, the transitivity is equivalent to the hypercyclicity by Birkhoff's transitivity theorem (see [1], p. 2). A continuous linear operator $T : X \longrightarrow X$ acting on a separable Fréchet space X is called *hypercyclic* if there is a vector $x \in X$ for which the orbit under T,

$$\mathrm{Orb}(T, x) = \{x, Tx, T^2x, \cdots\} \quad (1)$$

is dense in X. Every such vector x is called a *hypercyclic vector* of T.

The study of topological transitivity was started from Birkhoff's result [2] in 1929 for nonlinear continuous functions. This result is important in the theory of chaos (see e.g., [3]). For the space of all entire functions, Birkhoff [2] also proved that the translation operator $T_a : f(x) \mapsto f(x+a)$, $(a \neq 0)$ is hypercyclic. MacLane in 1952 showed that the differentiation operator $T(f) = f'$ is hypercyclic too (see [4]).

A lot of work on hypercyclicity has been based on the well-known so-called hypercyclicity criterion (the Kitai-Gethner-Shapiro theorem). In [5], Bermúdez and Kalton observed that similar criterion holds for topologically transitive operators on Banach spaces.

Theorem 1 (topologically transitive criterion). *Let T be a bounded linear operator on a complex Banach space X (not necessarily separable). Suppose that there exists a strictly increasing sequence $\{n_k\}$ of positive integers for which there are*

(i) *A dense subset $X_0 \subset X$ such that $T^{n_k}(x) \longrightarrow 0$ for every $x \in X_0$ as $k \longrightarrow \infty$*

(ii) *A dense subset $Y_0 \subset X$ and a sequence of mappings $S_k : Y_0 \longrightarrow X$ such that $S_k(y) \longrightarrow 0$ for every $y \in Y_0$ and $T^{n_k} \circ S_k(y) \longrightarrow y$ for every $y \in Y_0$ as $k \longrightarrow \infty$*

Then, T is topologically transitive.

More information on transitive and hypercyclic operators can be found in [1, 6]. The existence of an element with a dense orbit implies that X must be separable. On the other hand, the transitivity of $T : X \longrightarrow X$ does not require that the space X is separable.

Problem 2 (Bermúdez and Kalton [5]). Is there any characterization of nonseparable Banach spaces which support a topologically transitive operator?

Note that any hypercyclic operator $T : X \longrightarrow X$ always has an invariant, norm dense, linear subspace in which every nonzero vector is a hypercyclic vector for T.

Theorem 3 (see [7]). *Let X be a complete metric space which has no isolated points. Let $T : X \longrightarrow X$ be topologically transitive operator and $y \in X$. There exists a dense G_δ subset D of X such that for each $z \in D$, there exists a T-invariant (separable) closed subspace Y_z of X with $y, z \in Y_z$ such that the restriction of T to Y_z,$T : Y_z \longrightarrow Y_z$, is hypercyclic.*

The hypercyclisity of composition operators and differentiation operators on some function spaces of finite and infinite variables was studied in [8–14]. According to a classical theorem of Rolewicz [15], the weighted backward shifts on ℓ_2

$$(x_1, x_2, \cdots, x_n, \cdots) \mapsto (\lambda x_2, \lambda x_3, \cdots, \lambda x_{n+1}, \cdots) \qquad (2)$$

is hypercyclic if $|\lambda| > 1$. This fact remains to be true if we replace ℓ_2 into ℓ_p for $1 \le p < \infty$. Bermúdez and Kalton in [5] showed that spaces like ℓ_∞ and $\mathscr{L}(\ell_2)$) do not support topologically transitive operators, where $\mathscr{L}(\ell_2)$ is the space of all bounded operators on ℓ_2.

In [7] (Proposition 3.4), Manoussos proved that the backward weighted shift is topologically transitive on nonseparable Hilbert space. More detailed, let H be a Hilbert space (possibly nonseparable) and $\ell_2(H)$ be the ℓ_2-sum of infinitely many copies of H. That is, if $x \in \ell_2(H)$, then

$$x = (x_0, x_1, \cdots, x_n, \cdots), \quad x_n \in H, \|x\|^2 = \sum_{n=0}^{\infty} \|x_n\|^2. \qquad (3)$$

Then, the *backward weighted shift T with weight sequence* (ω_n) is defined by the following:

$$T : (x_0, x_1, \cdots, x_n, \cdots) \mapsto (\omega_1 x_1, \omega_2 x_2, \cdots, \omega_n x_n, \cdots). \qquad (4)$$

Unfortunately, Proposition 3.4 in [7] contains a slight inaccuracy which for us is essential. So, we propose the corrected version.

Proposition 4. *Let H be a Hilbert space and $T : \ell_2(H) \longrightarrow \ell_2(H)$ be a backward weighted shift with positive weight sequence (ω_n). The following are equivalent:*

(i) T is topologically transitive

(ii) There exists a nontrivial T-invariant (separable) closed subspace $\mathscr{Y} \subset \ell_2(H)$ on which the restriction of T to $\mathscr{Y}$,$T : \mathscr{Y} \longrightarrow \mathscr{Y}$, is hypercyclic

(iii) The restriction $T : \mathscr{Y} \longrightarrow \mathscr{Y}$ to any T-invariant (separable) closed subspace $\mathscr{Y} \subset \ell_2(H)$ which contains nonzero vectors of the form $(0, \cdots, 0, x_n, 0 \cdots)$ for every $n \in \mathbb{Z}_+$ is hypercyclic

(iv) $\limsup_{n\longrightarrow\infty} \prod_{k=1}^{n} \omega_n = +\infty$

Note that in [7], item (iii) is written: "The restriction $T : \mathscr{Y} \longrightarrow \mathscr{Y}$ to any T-invariant (separable) closed subspace $\mathscr{Y} \subset \ell_2(H)$ is hypercyclic." But it is not correct because the subspace Y_0 consisting of vectors $(x_0, 0, 0, \cdots)$,$x_0 \in H$ is invariant but the restriction of T to any separable subspace of Y_0 is not hypercyclic since $Y_0 \subset \ker T$.

In Section 2, we consider abstract shift similar operators on nonseparable function Hilbert spaces $\ell_2(H)$. In particular, abstract differentiation operators and dual operators to abstract multiplication operators can be considered as abstract shift similar operators. In Section 3, we construct examples of topologically transitive operators.

2. Abstract Shift Similar Operators

Let $(H_n)_{n=0}^{\infty}$ be a sequence of Hilbert spaces. Throughout this paper, we assume that all H_n are nontrivial; that is, $H_n \ne \{0\}$ and not necessary separable. Let us suppose that for every n and m,H_n is isomorphic to H_m. We denote by $\ell_2(H_n) = \ell_2((H_n)_{n=0}^{\infty})$ the Hilbert space consisting of elements $x = (x_0, x_1, \cdots, x_n, \cdots)$,$x_k \in H_k$ endowed with norm $\|x\| = (\sum_{i=0}^{\infty} \|x_i\|^2)^{1/2}$. Let (ω_n) be a sequence of positive numbers (weights). Let us fix a sequence of isomorphisms $J_m : H_m \longrightarrow H_{m-1}$,$\|J_m\| = 1$,$m \in \mathbb{N}$. An operator $T : \ell_2(H_n) \longrightarrow \ell_2(H_n)$ will be called a *backward weighted shift* (*with respect to the family* (J_m)) *with weight sequence* (ω_n) if it is of the following form:

$$T(x) = (\omega_1 J_1(x_1), \omega_2 J_2(x_2), \cdots, \omega_m J_m(x_m), \cdots). \qquad (5)$$

From Proposition 4, it follows the next corollary.

Corollary 5. *Let $(H_n)_{n=0}^{\infty}$ be a sequence of Hilbert spaces and $T : \ell_2(H_n) \longrightarrow \ell_2(H_n)$ be a backward weighted shift with respect to (J_m) and with positive weight sequence (ω_n). Let us suppose that*

$$\sup_{m\in\mathbb{Z}_+} \prod_{n=0}^{m} \|J_n^{-1}\| < \infty. \qquad (6)$$

Then, the following are equivalent:

(i) T is topologically transitive

(ii) There exists a nontrivial T-invariant (separable) closed subspace $\mathscr{Y} \subset \ell_2(H_n)$ on which the restriction of T to $\mathscr{Y}$,$T : \mathscr{Y} \longrightarrow \mathscr{Y}$, is hypercyclic

(iii) The restriction $T : \mathscr{Y} \longrightarrow \mathscr{Y}$ to any T-invariant (separable) closed subspace $\mathscr{Y} \subset \ell_2(H_n)$ which contains nonzero vectors of the form $(0, \cdots, 0, x_n, 0, \cdots)$,$x_n \in H_n$ for every $n \in \mathbb{Z}_+$, is hypercyclic

(iv) $\limsup_{n\longrightarrow\infty} \prod_{k=1}^{n} \omega_k = \infty$

Proof. We set $H = H_0$. Then,

$$\begin{aligned} J_1(H_1) &= H_0 = H, \\ (J_1 \circ J_2)(H_2) &= H_0 = H, \\ (J_1 \circ \cdots \circ J_n)(H_n) &= H_0 = H, \end{aligned} \tag{7}$$

and so on. For any $x = (x_1, \cdots, x_n, \cdots) \in \ell_2(H_n)$,

$$\begin{gathered} J : x \mapsto (x_0, J_1(x_1), \cdots, (J_1 \circ \cdots \circ J_n)(x_n), \cdots) \in \ell_2(H), \\ \|J(x)\| \le \|x\|. \end{gathered} \tag{8}$$

Because of Equation (6), the inverse operator

$$J^{-1} : z \mapsto \left(z_0, J_1^{-1}(z_1), \cdots, \left(J_n^{-1} \circ \cdots \circ J_1^{-1}\right)(z_n), \cdots\right) \in \ell_2(H_n), \quad z \in \ell_2(H) \tag{9}$$

is well defined and bounded. So, J is an isometric isomorphism from $\ell_2(H_n)$ to $\ell_2(H)$. For a closed subspace $\mathscr{Y} \subset \ell_2(H_n)$, the range $J(\mathscr{Y})$ is a closed subspace of $\ell_2(H)$. Let $T' = J \circ T \circ J^{-1}$. Then, T' is a backward weighted shift on $\ell_2(H)$ with positive weight sequence (ω_n). Since J is an isomorphism, the topological transitivity of T or hypercyclicity of T on $\mathscr{Y}$ is equivalent to the topological transitivity of T' on $\ell_2(H)$, or hypercyclicity of T' on $J(\mathscr{Y})$, respectively. Hence, we can apply Theorem 3 for T'.

Note that the case when $\|J_n\| = \nu_n > 0$ and $\sup_n \nu_n < \infty$ can be reduced to our case if we consider $J_n' = J_n/\nu_n$ and $\omega_n' = \omega_n \nu_n$ instead of J_n and $\omega_n, n \in \mathbb{N}$, respectively.

An operator $B : \ell_2(H_n) \longrightarrow \ell_2(H_n)$ is a *backward weighted shift similar operator* if there exists an isomorphism $A : \ell_2(H_n) \longrightarrow \ell_2(E_n)$ for some sequence of Hilbert spaces (E_n) and a backward weighted shift $T : \ell_2(E_n) \longrightarrow \ell_2(E_n)$ such that $B = ATA^{-1}$. It is clear that B is topologically transitive if and only if T is topologically transitive.

Let us suppose that $H_n = H_n{}' \oplus H_n^0, n \in \mathbb{Z}_+$ and the isomorphisms $J_n : H_n \longrightarrow H_{n-1}$, be such that J_n is an isometric isomorphism of $H_n{}'$ onto $H_{n-1}{}'$ and J_n maps H_n^0 onto H_{n-1}^0. Also, we denote by π_n the projection of H_n onto $H_n{}'$. Then, we define *a backward weighted partial shift* (with respect to family $(J_m \circ \pi_m)$) with weight sequence (ω_n) on $\ell_2(H_n)$ by the following:

$$T(x) = (\omega_1 (J_1 \circ \pi_1)(x_1), \omega_2 (J_2 \circ \pi_2)(x_2), \cdots, \omega_m (J_m \circ \pi_m)(x_m), \cdots). \tag{10}$$

In other words, $\ell_2(H_n) = \ell_2(H_n') \oplus \ell_2(H_n^0)$, and T is a backward weighted shift on $\ell_2(H_n')$ and $\ker T = \ell_2(H_n^0)$.

Let us recall that the *density* $d(X)$ of a metric space X is the smallest cardinality of a dense subset of X. It is well known that two infinite dimensional Hilbert spaces H and E are isometrically isomorphic if and only if $d(H) = d(E)$. The next proposition shows that the backward weighted partial shift can be reduced to the weighted shift. □

Proposition 6. *Let T be a backward weighted partial shift with respect to the family of operators $(J_m \circ \pi_m)_{m=1}^{\infty}$ with weight sequence (ω_n) on $\ell_2(H_n)$. Let us suppose that all H_n are infinite dimensional for $n > 1$ and $d(H_n^0) \le d(H_n)$. Then, T is a backward weighted shift similar operator; that is, there is a sequence of Hilbert spaces $(E_n)_{n=0}^{\infty}$ and an isometric isomorphism $\mathscr{J} : \ell_2(H_n) \longrightarrow \ell_2(E_n)$ such that $\mathscr{J} \circ T \circ \mathscr{J}^{-1}$ is a backward weighted shift.*

Proof. Let us define spaces E_n by the following way:

$$\begin{aligned} E_0 &= H_0 \oplus H_1^0 \oplus H_2^0 \oplus \cdots \oplus H_m^0 \oplus \cdots, \\ E_n &= H_n{}', \quad n > 1. \end{aligned} \tag{11}$$

Since spaces H_n are isomorphic each to others, $d(H_n) = d(H_m)$ for all $n, m \in \mathbb{Z}_+$. Since $d(H_n^0) \le d(H_n)$, we have $d(E_n) = d(E_m)$ and so E_n and E_m are isomorphic for all n, $m \in \mathbb{Z}_+$. It is clear that $\ell_2(H_n)$ and $\ell_2(E_n)$ are just different representations of the same space, and so $\mathscr{J}$ is the identical operator. Also, the restrictions of $J_m \circ \pi_m$ to $E_m, m > 1$ are isometric isomorphisms. So, $\mathscr{J} \circ T \circ \mathscr{J}^{-1}$ is a backward weighted shift with weight sequence (ω_n) on $\ell_2(E_n)$. □

Corollary 7. *Let $T : \ell_2(H_n) \longrightarrow \ell_2(H_n)$ be a backward weighted partial shift with positive weight sequence (ω_n) which satisfies conditions of Proposition 6. Then, the following are equivalent:*

(i) T is topologically transitive

(ii) There exists a nontrivial T-invariant (separable) closed subspace $\mathscr{Y} \subset \ell_2((H_n)_{n=0}^{\infty})$ on which the restriction of T to $\mathscr{Y}, T : \mathscr{Y} \longrightarrow \mathscr{Y}$, is hypercyclic

(iii) The restriction $T : \mathscr{Y} \longrightarrow \mathscr{Y}$ to any T-invariant (separable) closed subspace $\mathscr{Y} \subset \ell_2((H_n)_{n=0}^{\infty})$ which contains nonzero vectors of the form $(0, \cdots, 0, x_n, 0 \cdots), x_n \in H_n{}'$ for every $n \in \mathbb{Z}_+$, is hypercyclic

(iv) $\limsup_{n \longrightarrow \infty} \prod_{k=1}^{n} \omega_k = \infty$.

This approach can be generalised by the following way.

Theorem 8. *Let D be a surjective bounded operator on a ℓ_2-sum $\ell_2(\mathscr{H}_n)$ of Hilbert spaces $\mathscr{H}_n$ such that $D : H_n \longrightarrow H_{n-1}, n \ge 0$. Then, D is a backward weighted shift similar operator with some weight sequence $(\omega_n), 0 < \omega_n \le \|D_n\|$, where D_n is the restriction of D to $\mathscr{H}_n$.*

Proof. Let us define the following spaces:

$$\begin{aligned} &\mathscr{H}_0^0 = \mathscr{H}_0, \\ &\mathscr{H}_1^0 = \ker D_1, \mathscr{H}_1^1 = \mathscr{H}_1 \ominus \ker D_1, \\ &\mathscr{H}_2^0 = \ker D_2, \mathscr{H}_2^1 = \ker (D_1 \circ D_2) \ominus \mathscr{H}_2^0, \mathscr{H}_2^2 = \mathscr{H}_2 \ominus \ker (D_1 \circ D_2), \end{aligned}$$

$$\mathscr{H}_n^0 = \ker D_n, \cdots, \mathscr{H}_n^k = \ker (D_k \circ \cdots \circ D_{n-1} \circ D_n) \ominus \mathscr{H}_n^{k-1}, \cdots,$$

$$\mathscr{H}_n^n = \mathscr{H}_n \ominus \ker (D_1 \circ \cdots \circ D_{n-1} \circ D_n). \quad (12)$$

Let us denote by the following:

$$H_k = \ell_2\left(\mathscr{H}_n^k\right)_{n\geq k} = \mathscr{H}_k^k \oplus \mathscr{H}_{k+1}^k \oplus \cdots, \quad k \in \mathbb{Z}_+. \quad (13)$$

In other words, if $0 \neq x \in H_k$, then $0 \neq D_1 \circ \cdots \circ D_k(x) \in \mathscr{H}_0$. Let D_n^k be the restriction of D_n to $\mathscr{H}_n^k$. Then, $\|D_n^k\| \leq \|D_n\| \leq \|D\|, k \leq n$. Thus, the action of D on H_k can be written by the following:

$$D_{(k)} \coloneqq D_0^k \oplus D_1^k \oplus \cdots \oplus D_n^k \oplus \cdots, \quad k \in \mathbb{Z}_+ \quad (14)$$

and $\|D_{(k)}\| = \sup_n \|D_n^k\|$. So, $\|D_n^k\| \leq \|D_{(k)}\| \leq \|D\|, k \leq n$. It is clear that the spaces $\ell_2(\mathscr{H}_n)$ and $\ell_2(H_n)$ consist of the same vectors and the identity map $\mathfrak{J} : \ell_2(\mathscr{H}_n) \longrightarrow \ell_2(H_n)$ is an isometric isomorphism. Thus, $D_{(k)}$ is the restriction of $\mathfrak{J}D\mathfrak{J}^{-1}$ to H_k. By the construction, $D_{(k)} : H_k \longrightarrow H_{k-1}, k > 0$ and $D_{(0)} = 0$. Moreover, since $\ker \mathfrak{J}D\mathfrak{J}^{-1} = H_0$, $\ker D_{(k)} = 0$ if $k > 0$. In addition, since D is onto, $D_{(k)}$ must be onto for every k, because the preimage of H_{k-1} under $D_k, k > 0$ is H_k. Hence, every $D_{(k)}, k > 0$ is an isomorphism.

Let us set the following:

$$J_k \coloneqq \frac{D_k}{\|D_k\|}, \qquad \omega_k = \|D_k\|, \quad \text{for } k > 1. \quad (15)$$

Then for every $x = (x_0, x_1, \cdots, x_n, \cdots) \in \ell_2(H_n)$,

$$\mathfrak{J}D\mathfrak{J}^{-1}(x) = (\omega_1 J_1(x_1), \omega_2 J_2(x_2), \cdots, \omega_n J_n(x_n), \cdots), \quad (16)$$

that is, $\mathfrak{J}D\mathfrak{J}^{-1}$ is a backward weighted shift.

Let us show that $\omega_k > 0$ for every $k \in \mathbb{N}$. Since all spaces $\mathscr{H}_k$ are nontrivial and D is onto, there is a vector $a_1 \in \mathscr{H}_1$ such that $a_1 \notin \ker D$. So, $a_1 \in H_1$ and $D_{(1)}(a_1) = D_1(a_1) \neq 0$; that is, $\omega_1 = \|D_{(1)}\| \neq 0$. If already we have nonzero elements $a_j \in H_j \cap \mathscr{H}_j, j = 1, 2, \cdots, k-1$, then we denote by a_k a vector in $\mathscr{H}_k$ in the preimage of a_{k-1} under D_k. By the assumption $a_{k-1} \notin \ker (D_1 \circ \cdots \circ D_{k-1})$, so $a_k \notin \ker (D_1 \circ \cdots \circ D_k)$. Thus, $a_k \in H_k$ and $D_{(k)}(a_k) = D_k(a_k) = a_{k-1} \neq 0$. Hence, $\omega_k = \|D_{(k)}\| \neq 0$.

An element $x \in \ell_2(\mathscr{H}_n)$ is a *finite type vector* if there is $m \in \mathbb{N}$ such that $x_k = 0$ for all $k \geq m$. □

Definition 9. Let $\mathscr{H}_n, n \in \mathbb{Z}_+$ be the Hilbert spaces. We say that $\ell_2(\mathscr{H}_n)$ is a graded Hilbert space, if there is a bilinear map $(x, y) \mapsto xy$ (multiplication) defined for every finite type vector x and an arbitrary $y \in \ell_2(\mathscr{H}_n)$ with values in $\ell_2(\mathscr{H}_n)$ such that the multiplication is associative, nondegenerated (that is, $xy = 0$ only if $x = 0$ or $y = 0$) and $x_n y_m \in \mathscr{H}_{n+m}$ for every $x_n \in \mathscr{H}_n$ and $y_m \in \mathscr{H}_m$.

Definition 10. For a given graded Hilbert space $\ell_2(\mathscr{H}_n)$ and a finite type vector $a \in \ell_2(\mathscr{H}_n)$, we denote by M_a the multiplication operator.

$$M_a(x) = ax, \quad x \in \ell_2(\mathscr{H}_n). \quad (17)$$

Theorem 11. *Let $a \in \mathscr{H}_m$ and M_a be continuous on $\ell_2(\mathscr{H}_n)$. If the dual operator $M_a^* : (\ell_2(\mathscr{H}_n))^* = \ell_2(\mathscr{H}_n^*) \longrightarrow \ell_2(\mathscr{H}_n^*)$ is surjective, then it is a backward weighted shift similar operator.*

Proof. Let

$$E_0 = \mathscr{H}_0 \oplus \cdots \oplus \mathscr{H}_{m-1}, \cdots, E_k = \mathscr{H}_{km} \oplus \cdots \oplus \mathscr{H}_{km+k-1}, \cdots \quad (18)$$

and $H_n = E_n^*$ be the Hermitian dual space for every $n \in \mathbb{Z}_+$. Then, M_a maps E_n to $E_{n+1}, n \in \mathbb{Z}_+$ and M_a^* maps H_n to $H_{n-1}, n > 0$, and $H_0 = \ker M_a^*$. Let $(M_a^*)_n$ be the restriction of M_a^* to H_n. Since M_a is continuous, M_a^* is continuous on $\ell_2(H_n)$ as well and $\sup_n \|(M_a^*)_n\| < \infty$. We set the following:

$$J_n = \frac{(M_a^*)_n}{\|(M_a^*)_n\|}, \qquad \omega_n = \|(M_a^*)_n\|. \quad (19)$$

Since M_a^* is surjective, $(M_a^*)_n$ are surjective for all $n \in \mathbb{Z}_+$ and since $H_0 = \ker M_a^*$, $(M_a^*)_n$ are bijective for $n > 0$. So,

$$M_a^*(x_0, x_1, \cdots, x_k, \cdots) = (\omega_1 J_1(x_1), \omega_2 J_2(x_2), \cdots, \omega_k J_k(x_k), \cdots) \quad (20)$$

is a backward weighted shift on $\ell_2(H_n)$.

Note that if $a \in \mathscr{H}_1$, then $E_k = \mathscr{H}_k$ and $\omega_n = \|(M_a^*)_n\| = \|(M_a)_{n-1}\|$, where $(M_a)_k$ is the restriction of M_a to $\mathscr{H}_k$, $k \in \mathbb{Z}_+$. □

Definition 12. Let $\ell_2(\mathscr{H}_n)$ be a graded Hilbert space. A continuous linear operator $D : \ell_2(\mathscr{H}_n) \longrightarrow \ell_2(\mathscr{H}_n)$ is called a differential operator if $\mathscr{H}_0 \subset \ker D$, the restriction of D to $\mathscr{H}_n$ maps $\mathscr{H}_n$ onto $\mathscr{H}_{n-1}, n > 1$ and

$$D(xy) = D(x)y + xD(y) \quad (21)$$

for every finite type vector x and for all $y \in \ell_2(\mathscr{H}_n)$.

Corollary 13. *Let D be a differential operator on a graded Hilbert space $\ell_2(\mathscr{H}_n)$ such that D is onto and $\|D_1\| \neq 0$. Then, there is a sequence of Hilbert spaces $(H_n)_{n=0}^{\infty}$ and an isometric isomorphism $\mathfrak{J} : \ell_2(\mathscr{H}_n) \longrightarrow \ell_2(H_n)$ such that $\mathfrak{J} \circ D \circ \mathfrak{J}^{-1}$ is a backward weighted shift with some weight sequence (ω_n), $0 < \omega_n \leq \|D_n\|$.*

Proof. Since $\|D_1\| \neq 0$, there is $a \in \mathscr{H}_1$ such that $D(a) \neq 0$. So, $a \in H_1$. Moreover, for every $k \in \mathbb{N}, a^k = \underbrace{a \cdots a}_{k} \neq 0$ and $D(a^k) = ka^{k-1}$. Thus, $a^k \in H_k$ and we can apply Theorem 2.4. □

Corollary 14. *Let D be a continuous surjective differential operator on a graded Hilbert space $\ell_2(\mathscr{H}_n)$. The following are equivalent:*

(i) D is topologically transitive

(ii) There exist $a \in \mathscr{H}_1, \varepsilon > 0$, and $m \in \mathbb{N}$ such that $D(a) \neq 0$ and

$$\limsup_{m \longrightarrow \infty} \prod_{n=1}^{m} \frac{n\left\|a^{n-1}\right\|}{\|a^n\|} = \infty. \tag{22}$$

(iii)

$$\limsup_{m \longrightarrow \infty} \prod_{n=1}^{m} \left\|D_{(n)}\right\| = \infty. \tag{23}$$

Proof. As we observed, $a^k \in H_k$ and so

$$\omega_k = \left\|D_{(k)}\right\| \geq \frac{\left\|D_{(k)}\left(a^k\right)\right\|}{\left\|a^k\right\|} = \frac{k\left\|a^{k-1}\right\|}{\left\|a^k\right\|}. \tag{24}$$

By Theorem 8., $\omega_k > 0, k \in \mathbb{N}$. So if $\omega_k > 1 + \varepsilon$ for all $k > m$, then

$$\prod_{n=1}^{\infty} \omega_n = \infty \tag{25}$$

and D must be topologically transitive. □

3. Examples of Topologically Transitive Operators

3.1. Partial Shifts. It is well known that any element x of the (complete) Hilbert tensor product $\ell_2 \otimes_h \ell_2$ can be represented by the following series

$$x = \sum_{i,j=0}^{\infty} x_{ij} e_i \otimes e_j, \text{ where } (e_i)_{i \in \mathbb{Z}_+} \text{ is the standard basis in } \ell_2 \tag{26}$$

or by the following infinite matrix

$$x = \left(x_{ij}\right)_{i,j \in \mathbb{Z}_+} = \begin{pmatrix} x_{00} & x_{01} & \cdots & x_{0n} & \cdots \\ x_{10} & x_{11} & \cdots & x_{1n} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{n0} & x_{n1} & \cdots & x_{nn} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \tag{27}$$

with $\|x\|^2 = \sum_{i,j=0}^{\infty} |x_{ij}|^2$. Note that $\ell_2 \otimes_h \ell_2$ can be considered as $\ell_2(\ell_2)$ by the following representation:

$$x = (x_0, x_1, \cdots, x_n, \cdots) \in \ell_2(\ell_2), \tag{28}$$

where
$x_n = (x_{n0}, x_{n1}, \cdots, x_{nn}, \cdots) \in \ell_2, \quad n \in \mathbb{Z}_+$.

Let $\omega = (\omega_{ij})_{i,j}$ be an infinite matrix of positive numbers. We consider the following operators on $\ell_2(\ell_2)$. Let

$$T_\omega(x) = \left(\omega_{i+1,j} x_{i+1,j}\right)_{i,j \in \mathbb{Z}_+} = \begin{pmatrix} \omega_{10}x_{10} & \omega_{11}x_{11} & \cdots & \omega_{1n}x_{1n} & \cdots \\ \omega_{20}x_{20} & \omega_{21}x_{21} & \cdots & \omega_{2n}x_{2n} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \omega_{n0}x_{n0} & \omega_{n1}x_{n1} & \cdots & \omega_{nn}x_{nn} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \tag{29}$$

and T_ω^t be the transposed operator, $T_\omega(x) = (\omega_{i,j+1} x_{i,j+1})_{i,j \in \mathbb{Z}_+}$. Let $\omega_i = \sup_j \omega_{ij}$. From Theorem 8., it follows that if T_ω is continuous and

$$\limsup_{n \longrightarrow \infty} \prod_{i=1}^{n} \omega_i = +\infty, \tag{30}$$

then operators $T_\omega^m T_\omega^{tn}$ are topologically transitive for every $m, n, m+1 \geq 0$. In particular, the operator

$$T_\omega T_\omega^t(x) = \begin{pmatrix} \omega_{11}x_{11} & \omega_{12}x_{12} & \cdots & \omega_{1n}x_{1n} & \cdots \\ \omega_{21}x_{21} & \omega_{22}x_{22} & \cdots & \omega_{2n}x_{2n} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \omega_{n1}x_{n1} & \omega_{n2}x_{n2} & \cdots & \omega_{nn}x_{nn} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \tag{31}$$

looks like weighted diagonal shift operator. It is clear that these examples can be generalised for any tensor degree $\otimes_h^m \ell_2$, $m \in \mathbb{N}$. Next, we consider a nonseparable case.

Let $\mathfrak{A}$ be the set of indexes (uncountable, in general) which can be represented as a disjoint union of subsets $\mathfrak{A}_i$.

$$\mathfrak{A} = \coprod_{i=0}^{\infty} \mathfrak{A}_i, \tag{32}$$

and there are a sequence of bijections $\tau_n : \mathfrak{A}_n \longrightarrow \mathfrak{A}_{n-1}$, $n \in \mathbb{N}$. So, we can write $\ell_2(\mathfrak{A}) = \ell_2(\ell_2(\mathfrak{A}_n))$ and define $J_n : \ell_2(\mathfrak{A}_n) \longrightarrow \ell_2(\mathfrak{A}_{n-1}), n \in \mathbb{N}$ by $J_n(x_i) = x_{\tau_n(i)}, i \in \mathfrak{A}_n$. By Corollary 5, we have that

$$T(x) = (\omega_1 J_1(x_1), \omega_2 J_2(x_2), \cdots, \omega_m J_m(x_m), \cdots) \tag{33}$$

is topologically transitive if and only if the weight (ω_m) satisfies Equation (31). Moreover, the Hilbert tensor product $\ell_2(\mathfrak{A}) \otimes_h \ell_2(\mathfrak{A})$ is isomorphic to

$$\ell_2(\mathfrak{A} \times \mathfrak{A}) = \ell_2(\ell_2(\mathfrak{A}_n \times \mathfrak{A}_m)). \tag{34}$$

For a given weight matrix $\omega = \omega_{i,j}, i, j \in \mathbb{Z}_+, \omega_{i,j} > 0$, we define operators T_ω and T_ω^t on $\ell_2(\ell_2(\mathfrak{A}_n \times \mathfrak{A}_m))$ by the following:

$$T_\omega(x_{nm}) = x_{\tau(n)m} \text{ and } T_\omega^t(x_{nm}) = x_{n\tau(m)}, \tag{35}$$

where $x_{nm} \in \ell_2(\mathfrak{A}_n \times \mathfrak{A}_m)$.

Corollary 15. *Let*

$$\sup_{i,j \in \mathbb{Z}_+} \omega_{i,j} < \infty, \tag{36}$$

then for every $k, j \in \mathbb{Z}_+, k + j > 0$, operators $T_\omega^k T_\omega^{tj}$ are topologically transitive if and only if

$$\limsup_{n \longrightarrow \infty} \prod_{i=1}^{n} \sup_m \omega_{mi} = +\infty. \tag{37}$$

3.2. Differential Operator on a Function Space. Examples of Hilbert spaces of analytic functions with continuous differential operators can be found among reproducing kernel Hilbert spaces.

Definition 16. Let Z be an abstract set and $\mathscr{H}$ a Hilbert space of complex valued functions f on Z equipped with inner product $\langle \cdot | \cdot \rangle_{\mathscr{H}}$. A function $K(x | z)$ defined on $Z \times Z$ is called reproducing kernel of a closed subspace $\mathscr{H}_K \subset \mathscr{H}$ if

(i) For any fixed $z \in Z, K(x | z)$ belongs to $\mathscr{H}_K$ as a function in $x \in Z$

(ii) For any $f \in \mathscr{H}_K$ and for any $z \in Z, f(z) = \langle f | K(\cdot | z) \rangle_{\mathscr{H}}$

The space $\mathscr{H}_K$ is called a Hilbert space with reproducing kernel. A function $h : Z \longrightarrow \mathscr{H}$ such that

$$f(x) = \langle f(\cdot) | h(x) \rangle_{\mathscr{H}} \tag{38}$$

for every $f \in \mathscr{H}_K$ and $x \in Z$ is called the kernel function of $\mathscr{H}_K$.

Let $\mathscr{H}(\gamma)$ be the Hilbert space of power series

$$f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n, z \in \mathbb{C} \tag{39}$$

endowed with the norm

$$\|f\|^2 = \sum_{n=0}^{\infty} \gamma_n^{-2} \left| \widehat{f}(n) \right|^2 \leq \infty, \tag{40}$$

where $\gamma = \{\gamma_n\}$ is a positive sequence and $\widehat{f}(n) \in \mathbb{C}$. So, the inner product on $\mathscr{H}(\gamma)$ is defined by $\langle f | g \rangle = \sum_{n=0}^{\infty} \gamma_n^{-2} \widehat{f}(n) \widehat{g}(n)$ and $\mathscr{H}(\gamma)$ is a reproducing kernel space with the reproducing kernel.

$$K(x | z) = \sum_{n=0}^{\infty} \gamma_n^2 x^n \bar{z}^n. \tag{41}$$

It is known [16] that if $\gamma_{n+1}/\gamma_n \longrightarrow 0$ as $n \longrightarrow \infty, \mathscr{H}(\gamma)$ consists of entire functions on $\mathbb{C}$.

Let $D = d/dz$ be the differential operator on $\mathscr{H}(\gamma)$.Then, $D_n(\widehat{f}(n) z^n) = \widehat{f}(n) n z^{n-1}, n > 0$, and so $\|D_n\| = n\gamma_n/\gamma_{n-1}$. Thus, D is continuous (c. f. [9]) if and only if

$$\sup_n \|D_n\| = \sup_n \frac{n\gamma_n}{\gamma_{n-1}} < \infty. \tag{42}$$

On the other hand, according to Corollary 14, D is topologically transitive and so hypercyclic if and only if

$$\limsup_{m \longrightarrow \infty} \prod_{n=1}^{m} \frac{n\gamma_n}{\gamma_{n-1}} \longrightarrow \infty. \tag{43}$$

It is clear that $\mathscr{H}^*(\gamma) = \mathscr{H}(1/\gamma)$. Let us consider the multiplication operator M_z on $\mathscr{H}(\gamma), M_z(f) = z f(z)$. Then, $\|(M_z)_n\| = \gamma_n/\gamma_{n+1}$. According to Theorem 11, M_z is bounded if and only if $\sup_n \gamma_n/\gamma_{n+1} < \infty$ and M_z is topologically transitive if and only if

$$\limsup_{m \longrightarrow \infty} \prod_{n=1}^{m} \frac{\gamma_n}{\gamma_{n+1}} \longrightarrow \infty. \tag{44}$$

This result agrees with [8]. Various Hilbert spaces of analytic functions of infinite many variables were constructed in [14, 17, 18] using the Hilbertian symmetric tensor products of Hilbert spaces. Such constructions can be easily extended to nonseparable Hilbert spaces. In [14], the authors found some conditions under which the translation operator $f(x) \mapsto f(x + a)$ on a Hilbert space of entire analytic functions on ℓ_2 is hypercyclic. Taking into account that the translation operator can be represented as an exponential function of a differential operator, we would like to ask the following question.

Question 17. Let D be a differential operator on a graded nonseparable Hilbert space. Suppose that D is topologically transitive. Does

$$e^D = \sum_{n=0}^{\infty} \frac{D^n}{n!} \tag{45}$$

topologically transitive?

4. Conclusion

We can see that dual to abstract multiplication operators M_a^* and abstract differentiation operators D on Hilbert spaces can be represented as backward weighted shift operators. Thus, we can use known results about topological transitivity of backward weighted shift operators to get conditions of topological transitivity of operators M_a^* and D.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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