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Research Article

Quantitative Fourth Moment Theorem of Functions on the Markov Triple and Orthogonal Polynomials

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In this paper, we consider a quantitative fourth moment theorem for functions (random variables) defined on the Markov triple (E, μ, Γ), where μ is a probability measure and Γ is the carré du champ operator. A new technique is developed to derive the fourth moment bound in a normal approximation on the random variable of a general Markov diffusion generator, not necessarily belonging to a fixed eigenspace, while previous works deal with only random variables to belong to a fixed eigenspace. As this technique will be applied to the works studied by Bourguin et al. (2019), we obtain the new result in the case where the chaos grade of an eigenfunction of Markov diffusion generator is greater than two. Also, we introduce the chaos grade of a new notion, called the *lower chaos grade*, to find a better estimate than the previous one.

1. Introduction

The aim of this paper is to find the fourth moment bound in the normal approximation of a random variable related to a general Markov diffusion generator. A central limit theorem, known as the fourth moment theorem, was first discovered in [1] by Nualart and Peccati, where the authors found a necessary and sufficient condition such that a sequence of random variables, belonging to a fixed Winer chaos, converges in distribution to a Gaussian random variable.

Throughout this paper, we use the mathematical expectation for the integral on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, so that, for example, the integral $\int_\Omega F d\mathbb{P}$ is denoted by $\mathbb{E}[F]$, and a real-valued measurable function defined on a probability space will be called a random variable. Also, we define the variance of a random variable F in $\mathbb{L}^2(\Omega, \mathbb{P})$ as

$$\mathbb{V}\mathrm{ar}(\mathrm{F}) = \int_\Omega F^2 d\mathbb{P} - \left(\int_\Omega F d\mathbb{P}\right)^2. \quad (1)$$

Theorem 1 (fourth moment theorem). *Fix an integer $q \geq 2$, and consider a sequence of random variables $\{F_n, n \geq 1\}$ belonging to the qth a Wiener chaos with $\mathbb{E}[F_n^2] = 1$ for all $n \geq 1$. Then, $F_n \xrightarrow{\mathscr{L}} Z$ if and only if $\mathbb{E}[F_n^4] \longrightarrow 3$, where Z is a standard Gaussian random variable and the notation $\xrightarrow{\mathscr{L}}$ denotes the convergence in distribution.*

Such a result constitutes a dramatic simplification of the method of moments and cumulants. In the paper [2], the fourth moment theorem is expressed in terms of Malliavin derivative. However, the results given in [1, 2] do not provide any estimates, whereas the authors in [3] prove that Theorem 1 can be recovered from the estimate of the Kolmogorov (or total variation, Wasserstein) distance obtained by using the techniques based on the combination between Malliavin calculus (see, e.g., [4–6]) and Stein's method for the normal approximation (see, e.g., [7–9]). Also, we refer to the papers [3, 4, 10–13] for an explanation of these techniques.

For estimates for a normal approximation, we consider the Kolmogorov distances of the type

$$d_{Kol}(X, Z) = \sup_{z \in \mathbb{R}} |\mathbb{P}(X \leq z) - \mathbb{P}(Z \leq z)|, \quad (2)$$

where Z is a standard Gaussian random variable. The following statement is the remarkable achievement of Nourdin-

Peccati [3] approach (see Theorem 3.1 in [3]). Let $F \in \mathbb{D}^{1,2}$ be such that $\mathbb{E}[F]=0$ and $\mathbb{E}[F^2]=1$. Then, the following bound holds:

$$d_{Kol}(F,Z) \le \sqrt{\mathbb{E}\left[(1-\Gamma_1(F))^2\right]}, \tag{3}$$

where $\Gamma_1(F)=\langle DF,-DL^{-1}F\rangle_{\mathfrak{H}}$. The notations $\mathbb{D}^{1,2}$, D, and L^{-1}, related to Malliavin calculus, are explained in [5] or [6]. In the particular case where F is an element in the qth Wiener chaos of X with $\mathbb{E}[F^2]=1$, the upper bound in (3) is given by

$$d_{Kol}(F,Z) \le \sqrt{\frac{q-1}{3q}}\sqrt{\mathbb{E}[F^4]-3}. \tag{4}$$

Here, $(1/6)(\mathbb{E}[F^4]-3)$ is just the fourth cumulant $\kappa_4(F)$ of F.

Recently, the author in [14] proves, from a purely spectral point of view, that the fourth moment theorem also holds in the general framework of Markov diffusion generators. Precisely, under a certain spectral condition on a Markov diffusion generator, a sequence of eigenfunctions of such a generator satisfies the bound given in (4) with a different constant. In particular, this new technique avoids the use of complicated product formula. The authors in [15] introduce a Markov chaos of eigenfunctions being less restrictive than the notion of Markov chaos defined in [14] and also obtain the quantitative four moment theorem for convergence of the eigenfunctions towards Gaussian, gamma, and beta distributions. Furthermore, Bourguin et al. in [16] prove that convergence of the elements of a Markov chaos to a Pearson distribution can be still bounded with just the first four moments of the form

$$d(G,Z) \le c\sqrt{\xi_1\int_E P(G)d\mu + \xi_2\int_E Q(G)d\mu}, \tag{5}$$

where d is a suitable distance, Z is a random variable with the law belonging to the Pearson family, and G is a chaotic random variable defined on (E,μ). Here, P and Q in (5) are polynomials of degree four, and the constants ξ_1 and ξ_2 are determined in terms of a *chaos grade* defined in Definition 3.5 of [16].

In this paper, we find a bound of the form

$$d_{\mathrm{Kol}}(F,Z) \le \xi\left(\mathbb{E}\left[F^4\right]-3\mathbb{E}\left[F^2\right]\right)^2\right) \tag{6}$$

for the Kolmogorov distance between a stand Gaussian random variable Z and a random variable F defined on (E,μ,Γ) related to a Markov diffusion generator L with an invariant measure μ. Since usually the central limit theorem (normal approximation) is a main topic of convergence in distribution, we confine our interest in a normal approximation. In the line of this research, the motivations and contributions of our work in comparison with other works will be summarized below:

(i) Compared to previous works [14–16], our studies are not limited to an element belonging to a fixed eigenspace of Markov chaos. Our result is a remarkable extension in comparison with other works dealing with only random variables in a fixed eigenspace. To achieve our goal, the starting point is the following bound

$$d_{\mathrm{Kol}}(F,Z) \le \mathbb{V}\mathrm{ar}\left(\Gamma\left(\mathrm{F},-L^{-1}F\right)\right), \tag{7}$$

where L^{-1} is the pseudo-inverse of the underlying Markov generator$L$ and Γ is the carré du champ operator, which is the result of the study in [16]. However, in order to find the fourth moment bound (6), we introduce a new technique relying on two types of the operators given in [17, 18]. First, we prove that the right-hand side of (7) can be represented as the sum of two integrals with the operators mentioned above, and use this representation to prove that the fourth moment bound (6) holds. This is the innovation point of this work

(ii) If the *upper chaos grade* of F is strictly greater than two, then, the constant ξ_2 and $Q(G)$ in the bound (5) are given as follows: $\xi_2 > 0$ and $Q(G)=\mathbb{V}\mathrm{ar}(G^2)$. This fact means that the fourth moment theorem in Theorem 1 is not working. However, applying the technique developed in this paper can eliminate the second term in (5), which means that, in such a random variable G, the fourth moment theorem holds unlike the previous result in [16], where the upper bound (5) for a sequence $\{F_n\}$ of chaotic random variables is given in (111) of Remark 12

(iii) In this paper, another notion of chaos grade, called a *lower chaos grade*, is introduced and used to provide a better estimate than the previous one obtained from (5) in [16]. Throughout this paper, the existing chaos grade in Definition 3.5 of [16] will be called an upper chaos grade

The rest of the paper is organized as follows: Section 2 introduces some basic notations and reviews the results of Markov diffusion generator. In Section 3, a new notion of chaos grade in a finite sum of Markov chaos is defined, and the orthogonal polynomials will be considered in order to illustrate the concept on chaos grades. In Section 4, our main result is covered in Theorem 8. Finally, as an application of our main result, in Section 5, we derive upper bounds in the Kolmogorov distance for an eigenfunction belonging to a fixed Markov chaos.

2. Preliminaries

In this section, we recall some basic facts about Markov diffusion generator. The reader is referred to [19] for a more detailed explanation. We begin by the definition of Markov triple (E, μ, Γ) in the sense of [19]. For the infinitesimal generator L of a Markov semigroup $P = (P_t)_{t\geq 0}$ with $\mathbb{L}^2(\mu)$-domain $\mathscr{D}(L)$, we associate a bilinear form Γ. Assume that we are given a vector space $\mathscr{A}_0$ of $\mathscr{D}(L)$ such that for every (F, G) of random variables defined on a probability space $(E, \mathfrak{F}, \mu)$, the product FG is in $\mathscr{D}(L)$ ($\mathscr{A}_0$ is an algebra). On this algebra $\mathscr{A}_0$, the bilinear map (carré du champ operator) Γ is defined by

$$\Gamma(F, G) = \frac{1}{2}(L(FG) - FLG - GLF), \tag{8}$$

for every $(F, G) \in \mathscr{A}_0 \times \mathscr{A}_0$. As the carré du champ operator Γ and the measure μ completely determine the symmetric Markov generator L, we will work throughout this paper with Markov triple (E, μ, Γ) equipped with a probability measure μ on a state space $(E, \mathfrak{F})$ and a symmetric bilinear map $\Gamma : \mathscr{A}_0 \times \mathscr{A}_0$ such that $\Gamma(F, F) \geq 0$.

Next, we construct the domain $\mathscr{D}(\mathscr{E})$ of the Dirichlet form $\mathscr{E}$ by completion of $\mathscr{A}_0$ and then obtain, from this Dirchlet domain, the domain $\mathscr{D}(L)$ of L. Recall the Dirchlet form $\mathscr{E}$ as

$$\mathscr{E}(F, G) = \mathbb{E}[\Gamma(F, G)] \text{ for } (F, G) \in \mathscr{A}_0 \times \mathscr{A}_0. \tag{9}$$

If $\mathscr{A}_0$ is endowed with the norm

$$\|F\|_{\mathscr{E}} = \left[\mathbb{E}\left[F^2\right] + \mathscr{E}(F, F)\right]^{1/2}, \tag{10}$$

the completion of $\mathscr{A}_0$ with respect to this norm turns it into a Hilbert space embedded in $\mathbb{L}^2(\mu)$. Once the Dirchlet domian $\mathscr{D}(\mathscr{E})$ is constructed, the domaion $\mathscr{D}(L) \subseteq \mathscr{D}(\mathscr{E})$ is defined as all elements $F \in \mathscr{D}(\mathscr{E})$ such that, for all $G \in \mathscr{D}(\mathscr{E})$,

$$|\mathscr{E}(F, G)| \leq c_F \mathbb{E}\left[G^2\right], \tag{11}$$

where c_F is a finite constant only depending on F. On these domains, a relation of L and Γ holds, namely, the *integration by parts* formula

$$\mathbb{E}[\Gamma(F, G)] = -\mathbb{E}[FLG] = -\mathbb{E}[GLF]. \tag{12}$$

By the integration by parts formula (12) and $\Gamma(F, F) \geq 0$, the operator $-L$ is nonnegative and symmetric, and therefore, the spectrum of $-L$ is contained $\mathcal{S} \subseteq [0,\infty)$.

A full Markov triple is a standard Markov triple for which there is an extended algebra $\mathscr{A}_0 \subset \mathscr{A}$, with no requirements of integrability for elements of $\mathscr{A}$, satisfying the requirements given in Section 3.4.3 of [19]. In particular, the diffusion property holds: for any $\mathscr{C}^\infty$ function $\Psi ; \mathbb{R}^k \longrightarrow \mathbb{R}$ and $F_1, \cdots, F_k, G \in \mathscr{A}$,

$$\begin{aligned} \Gamma(\Psi(F_1, \cdots F_k), G) &= \sum_{i=1}^k \partial_i \Psi(F_1, \cdots F_k)\Gamma(F_i, G), \\ L(\Psi(F_1, \cdots F_k)) &= \sum_{i=1}^k \partial_i \Psi(F_1, \cdots F_k) L F_i \\ &\quad + \sum_{i,j=1}^k \partial_{ij}\Psi(F_1, \cdots F_k)\Gamma\left(F_i, F_j\right). \end{aligned} \tag{13}$$

We also define the operator L^{-1}, called the pseudoinverse of L, satisfying for any $F \in \mathscr{D}(L)$,

$$LL^{-1}F = L^{-1}LF = F - \mathbb{E}[F]. \tag{14}$$

3. Chaos Grade and Orthogonal Polynomials

3.1. Chaos Grade. Fix a probability space $(E, \mathfrak{F}, \mu)$. We assume that $-L$ has a discrete spectrum $\Lambda = \{0 = \lambda_0 < \lambda_1 < \cdots < \lambda_k < \cdots\}$. Obviously, the zero is always an eigenfunction such that $-L(1) = 0$. By the assumption on the spectrum of L, one has that

$$L^2(E, \mathfrak{F}, \mu) = \bigoplus_{\ell=0}^{\infty} \text{Ker}\,(L + \lambda_\ell Id). \tag{15}$$

Now, we define chaotic random variables as follows.

Definition 2. Suppose that $F \in \oplus_{i=1}^M \text{Ker}\,(L + \lambda_{\ell_i} Id) \oplus \text{Ker}\,(L)$, where $0 < \lambda_{\ell_1} < \lambda_{\ell_2} < \cdots < \lambda_{\ell_M}$ and $F_{\ell_i} \in \text{Ker}\,(L + \lambda_{\ell_i} Id)$ for $i = 1, \cdots, M$. The random variable F is called chaotic if there exist $\mathfrak{u} > 1$ and $\mathfrak{g} < 1$ such that $\mathfrak{u}\lambda_{\ell_M} \in \Lambda$ and $\mathfrak{g}\lambda_{\ell_M} \in \Lambda$ satisfy

$$F^2 \in \text{Ker}\,\left(L + \mathfrak{g}\lambda_{\ell_M} Id\right) \oplus \cdots \oplus \text{Ker}\,\left(L + \mathfrak{u}\lambda_{\ell_M} Id\right) \oplus \text{Ker}\,(L). \tag{16}$$

In this case, the largest number $\mathfrak{g}$ satisfying (16) is called the lower chaos grade of F. On the other hand, the smallest number $\mathfrak{u}$ satisfying (16) is called the upper chaos grade of F.

Remark 3.

(1) The authors in [16] define the chaos grade of F, corresponding to the upper chaos grade in Definition 2, in the case where F is an eigenfunction with respect to an eigenvalue of the generator L. In this paper, we introduce the lower chaos grade of F, which will be used to obtain a better estimate for the four moments theorem than the estimate given in Theorem 3.9 of [16] in the particular case where the target distribution is a standard Gaussian distribution

(2) Let $F_{\ell_k} \in \text{Ker}\,(L + \lambda_{\ell_k} Id)$ for $k \in \{1, \cdots, M\}$. If F is a chaotic random variable, then $F_{\ell_i} F_{\ell_j}$, $i, j \in \{1, \cdots, M\}$, can be expanded as a sum of eigenfunctions with the

eigenvalue of the largest magnitude $\lambda_{\max}^{\ell_i,\ell_j}$ and the eigenvalue of the smallest magnitude $\lambda_{\min}^{\ell_i,\ell_j} > 0$, i.e.,

$$F_{\ell_i}F_{\ell_j} \in \mathrm{Ker}\left(L + \lambda_{\min}^{\ell_i,\ell_j} Id\right) \oplus \cdots \oplus \mathrm{Ker}\left(L + \lambda_{\max}^{\ell_i,\ell_j} Id\right) \oplus \mathrm{Ker}\,(L) \tag{17}$$

From (17), it follows that

$$F^2 \in \mathrm{Ker}\,(L + \lambda_{\min} Id) \oplus \cdots \oplus \mathrm{Ker}\,(L + \lambda_{\max} Id) \oplus \mathrm{Ker}\,(L), \tag{18}$$

where $\lambda_{\min} = \min_{1\le i,j\le M} \lambda_{\min}^{\ell_i,\ell_j}$ and $\lambda_{\max} = \max_{1\le i,j\le M} \lambda_{\max}^{\ell_i,\ell_j}$.

(3) In the paper [16], the authors describe how the a chaos grade, corresponding to the upper chaos grade in our works, behaves under tensorization. Let L_i be a generator with invariant measure μ_i. Define a generator $L = \otimes_{i=1}^M L_{\ell_i}$ by

$$L\left(\bigotimes_{i=1}^{M} F_{\ell_i}\right) = \sum_{i=1}^{M} \left(F_{\ell_1} \times \cdots \times F_{\ell_{i-1}} \times \left(L_{\ell_i} F_{\ell_i}\right) \times F_{\ell_{i+1}} \times \cdots \times F_{\ell_M}\right) \tag{19}$$

If $L_i F_i = \lambda_i F_i$, then $F = \otimes_{i=1}^M F_{\ell_i}$ is an eigenfunction of L with eigenvalue $\lambda = \sum_{i=1}^M \lambda_{\ell_i}$. Suppose that F_{ℓ_i}, $1 \le i \le M$, has the lower chaos grades $\mathfrak{g}_{\ell i}$. Then, the lower chaos grade $\mathfrak{g}$ of $F = \otimes_{i=1}^M F_{\ell_i}$ is given by

$$\mathfrak{g} = \frac{\sum_{i=1}^M \mathfrak{g}_{\ell_i} \lambda_{\ell_i}}{\sum_{i=1}^M \lambda_{\ell_i}}. \tag{20}$$

See Corollary 4.1 in [16] for the upper chaos grade of F.

Next, we consider a finite dimensional eigenfunction and a finite sum of eigenfunctions to illustrate the concept on chaos grades.

3.2. Ornstein-Uhlenbeck Operator. We consider the d-dimensional Ornstein-Uhlenbeck generator L, defined for any test function f by

$$Lf(x) = \Delta f(X) - \sum_{i=1}^{d} x_i \partial_i f(x) \text{ for } x \in \mathbb{R}^d, \tag{21}$$

action on $\mathbb{L}^2(\mathbb{R}^d, \otimes_{i=1}^d \mu_i(dx_i))$, where

$$\mu_i(dx_i) = \frac{1}{\sqrt{2\pi}} e^{-(x_i^2/2)} dx_i. \tag{22}$$

For a multi-index $m = (m_1, \cdots, m_d)$, we define a d-dimensional Hermite polynomial by

$$H_m(x) = \prod_{i=1}^{d} H_{m_i}(x_i). \tag{23}$$

We write $|m| = \sum_{i=1}^d m_i$, $m! = \prod_{i=1}^d m_i!$, and $m \le n$ if $m_i \le n_i$ for all $i = 1, \cdots, d$. Let us set

$$F = \sum_{\substack{m\ge 0 \\ |m|=q}} a_m H_m(x), q \ge 2, \tag{24}$$

where H_{m_i}, $m_i = 0, 1, \cdots$, denotes the Hermite polynomial of order m_i. Then, we have that $F \in \mathrm{Ker}\,(L + qId)$. By the well-known product formula of Hermite polynomials and a change of variables, we have that

$$F^2 = \sum_{\substack{m\ge 0 \\ |m|=q}} \sum_{\substack{n\ge 0 \\ |n|=q}} a_m a_n \sum_{r \le m\wedge n} r! \binom{m}{r} \binom{n}{r} H_{m+n-2r}(x) = \sum_{\substack{m\ge 0 \\ |m|=q}} \sum_{\substack{n\ge 0 \\ |n|=q}} \left\{ C(m,n,m\wedge n) H_{m+n-2(m\wedge n)}(x) + \cdots\cdots + C(m,n,0) H_{m+n}(x) \right\}, \tag{25}$$

where

$$C(m,n,r) = a_m a_n r! \binom{m}{r} \binom{n}{r}. \tag{26}$$

Since $m + n - 2(m\wedge n) = (|m_1 - n_1|, \cdots, |m_d - n_d|)$, F^2 can be expanded as a sum of eigenfunctions with the smallest eigenvalue, among positive eigenvalues, being given by

$$\lambda_{\min} = \min_{m\ge 0, n\ge 0} \left\{ \sum_{i=1}^{d} |m_i - n_i| > 0 : |m| = |n| = q \right\}. \tag{27}$$

Obviously, $\lambda_{\min} = 2$, so that the lower chaos grade is $\mathfrak{g} = 2/q$. On the other hand, the largest eigenvalue in the expansion of F^2, as a sum of eigenfunctions, is given by $|m + n| = |m| + |n| = 2q$. Therefore, the upper chaos grade is given by $\mathfrak{u} = 2$.

If $F = \sum_{i=1}^{M} H_{q_i}(x)$, where $0 < q_1 < \cdots < q_d$, then the lower and upper chaos grades are given, respectively, by

$$\mathfrak{g} = \frac{2 \wedge \min_{\substack{1\le i,j\le M \\ i\ne j}} |q_i - q_j|}{q_M}, \quad \mathfrak{u} = 2. \tag{28}$$

3.3. Jacobi Operator. For $\alpha, \beta > -1$, we consider the d-dimensional Jacobi generator L, defined for any test function f by

$$Lf(x) = \sum_{i=1}^{d} x_i(1-x_i)\partial_{ii}^2 f(x) - \sum_{i=1}^{d} ((\alpha+\beta)x_i - \alpha)\partial_i f(x) \quad \text{for } x \in \mathbb{R}^d, \tag{29}$$

action on $\mathbb{L}^2(\mathbb{R}^d, \otimes_{i=1}^{d} \mu_i(dx_i))$, where

$$\mu_i(dx_i) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x_i^{\alpha-1}(1-x_i)^{\beta-1} 1_{[0,1]}(x_i), \quad i = 1, \cdots, d. \tag{30}$$

Its spectrum Λ is of the form

$$\Lambda = \left\{ -\sum_{i=1}^{d} \lambda_{m_i} : m_1, \cdots, m_d \in \mathbb{N}_0 \right\}, \tag{31}$$

where $\lambda_k = k(k + \alpha + \beta - 1)$. Let us set

$$F = \sum_{\substack{m\ge 0 \\ \lambda_{|m|}=\lambda}} a_m P_m^{\alpha-1,\beta-1}(1-2x), \quad \lambda \ge 2, \tag{32}$$

where $P_{m_i}^{\alpha,\beta}(x)$, $m_i = 0, 1, \cdots$, denotes the m_ith Jacobi polynomial being given by

$$P_{m_i}^{(\alpha,\beta)}(x) = \frac{(-1)^{m_i}}{2^{m_i} m_i!}(1-x)^{-\alpha}(1+x)^{-\beta} \frac{d^{m_i}}{dx^{m_i}}\left((1-x)^{\alpha+m_i}(1-x)^{\beta+m_i}\right). \tag{33}$$

Recall that ${}_pF_q$ denotes the generalized hypergeometric function with p numerator and q denominator, given by

$${}_pF_q\left(\begin{matrix} (a_p) \\ \\ (b_q) \end{matrix} \middle| x \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_p)_k} \frac{x^k}{k!}, \tag{34}$$

where

$$(\alpha)_k = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}. \tag{35}$$

Then the Jacobi polynomials can be expressed as

$$P_{m_i}^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_{m_i}}{m_i!} {}_2F_1\left(\begin{matrix} -m_i, \alpha+\beta+m_i+1 \\ \alpha+1 \end{matrix} \middle| \frac{1-x}{2} \right). \tag{36}$$

The well-known product formula of Jacobi polynomials yields that

$$F^2 = \sum_{\substack{m\ge 0 \\ \sum_{i=1}^{d}\lambda_{m_i}=\lambda}} \sum_{\substack{n\ge 0 \\ \sum_{i=1}^{d}\lambda_{n_i}=\lambda}} a_m a_n \sum_{r\le m\wedge n} C(m,n,r) P_{m+n-2r}^{\alpha-1,\beta-1}(1-2x) = \sum_{\substack{m\ge 0 \\ \sum_{i=1}^{d}\lambda_{m_i}=\lambda}} \sum_{\substack{n\ge 0 \\ \sum_{i=1}^{d}\lambda_{n_i}=\lambda}} \left\{ C(m,n,m\wedge n) P_{m+n-2(m\wedge n)}^{\alpha-1,\beta-1}(1-2x) + \cdots\cdots + C(m,n,0) P_{m+n}^{\alpha-1,\beta-1}(1-2x) \right\}, \tag{37}$$

where

$$C(m,n,r) = \int_{[0,1]^d} P_{m+n-2r}^{\alpha-1,\beta-1}(1-2x) P_m^{\alpha-1,\beta-1}(1-2x) P_n^{\alpha-1,\beta-1}(1-2x) \bigotimes_{i=1}^{d} \mu_i(dx_i). \tag{38}$$

First, observe that

$$\lambda_{m_i+n_i} = (m_i+n_i)(m_i+n_i+\alpha+\beta-1) = \lambda_{m_i} + \lambda_{n_i} + 2m_i n_i. \tag{39}$$

It follows, from (39), that for any indices m and x such that $\sum_{i=1}^{d} \lambda_{m_i} = \sum_{i=1}^{d} \lambda_{x_i} = \lambda$ and $|m| = \deg(F)$,

$$\max_{\substack{x\ge 0 \\ |x|=\deg(F)}} = 2\lambda + 2 \max_{\substack{x\ge 0 \\ |x|=\deg(F)}} \sum_{i=1}^{d} m_i x_i, \tag{40}$$

where the notation $\deg(F)$ denotes the degree of F.

Successive applications of the arguments for (39) yield that

$$\lambda_{|m|} = \sum_{i=1}^{d} \lambda_{m_i} + 2 \sum_{1\le i<j\le d} m_i m_j. \tag{41}$$

For any another index x such that $|x| = \deg(F)$ and $\lambda_{|x|} = \lambda$, we have, from (41), that

$$\sum_{1\le i<j\le d} m_i m_j = \sum_{1\le i<j\le d} x_i x_j, \tag{42}$$

so that

$$\sum_{i=1}^{d} x_i^2 = \sum_{i=1}^{d} m_i^2. \tag{43}$$

Let $f(x) = \sum_{i=1}^{d} m_i x_i$. Now, we find a point yielding the maximum value of $f(x)$ under the restriction given by (43). Obviously, Lagrange's method shows that

$$m_i - 2\delta x_i = 0 \quad \text{for all } i = 1, \cdots, d, \tag{44}$$

where δ is a Lagrange multiplier. Plugging $x_i = m_i/2\delta$ into (43) yields that $4\delta^2 = 1$, so that $x_i = m_i$ for all $i = 1, \cdots, d$. Therefore, it follows, from (40), that

$$\max_{\substack{x\ge 0\\ |x|=\deg(F)}} \sum_{i=1}^{d} \lambda_{m_i+x_i} = \sum_{i=1}^{d} \lambda_{2m_i}. \tag{45}$$

Hence, the upper chaos grade is given by

$$\mathfrak{u} = \frac{\max_{\substack{m\ge 0\\ \sum_{i=1}^{d}\lambda_{m_i}=\lambda}} \sum_{i=1}^{d} \lambda_{2m_i}}{\lambda}. \tag{46}$$

Next, we find the lower chaos grade of F. From (37), the square of F can be expanded as a sum of eigenfunctions with the smallest eigenvalue, among positive eigenvalues, being given by

$$\lambda_{\min} = \min_{\substack{m\ge 0, n\ge 0\\ \sum_{i=1}^{d}\lambda_{m_i}=\sum_{i=1}^{d}\lambda_{n_i}=\lambda}} \left\{ \sum_{i=1}^{d} \lambda_{|m_i-n_i|} : |m| = |n| = \deg(F) \text{ and } m_i \neq n_i \text{ for some } i = 1, \cdots, d \right\}. \tag{47}$$

When $|m_i - n_i| = 0$ for $i \neq k, \ell \in \{1, \cdots, d\}$ and $|m_i - n_i| = 1$ for $i = k, \ell \in \{1, \cdots, d\}$, the sum $\sum_{i=1}^{d} \lambda_{|m_i-n_i|}$ has the minimum value. Hence, $\lambda_{\min} = 2\lambda_1$, so that the lower chaos grade is given by $\mathfrak{g} = (2(\alpha+\beta))/\lambda$.

4. Fourth Moment Theorem

In this section, we derive an upper bound on Kolmogorov distance $d_{\mathrm{Kol}}(F, Z)$, where F, not necessarily belonging to a fixed eigenspace, is a random variable related to Markov diffusion generator L, and Z is a standard Gaussian random variable.

4.1. Lemmas. We begin with stating a useful lemma, which is going to be frequently used in this section. Lemmas, which appeared in this section, are well-known in the particular case where F is a functionals of Gaussian random fields.

Lemma 4. *Let $F \in \mathscr{D}(\mathscr{E})$ and $G \in \mathscr{D}(L)$. Then, we have*

$$\mathbb{E}[FG] = \mathbb{E}[F]\mathbb{E}[G] + \mathbb{E}\left[\Gamma\left(F, -L^{-1}G\right)\right]. \tag{48}$$

Proof. Since $\mathscr{D}(L) \subseteq \mathscr{D}(\mathscr{E}) \subseteq \mathbb{L}^2(E,\mu)$, we have that $\mathbb{E}[|FG|] \le \|F\|_2\|G\|_2 < \infty$ and $\Gamma(F, -L^{-1}G) \in \mathbb{L}^1(E,\mu)$. Hence, all expectations in equation (48) are well defined. By the integration by parts formula (12) and (14), we have that

$$\begin{aligned}\mathbb{E}[FG] &= \mathbb{E}[F]\mathbb{E}[G] + \mathbb{E}[F(G - \mathbb{E}[G])] \\ &= \mathbb{E}[F]\mathbb{E}[G] - \mathbb{E}\left[FL\left(-L^{-1}G\right)\right] \\ &= \mathbb{E}[F]\mathbb{E}[G] + \mathbb{E}\left[\Gamma\left(F, -L^{-1}G\right)\right].\end{aligned} \tag{49}$$

This gives the desired result. □

Now, we extend the techniques developed in [20] in the case of a functional of Gaussian fields to a random variable belonging to $\mathbb{L}^2(E,\mu)$. Let $F \in \mathscr{D}(L)$. Define $\Gamma_0(F) = F$ and $\Gamma_1(F) = \Gamma(F, -L^{-1}F)$. If $\Gamma_j(F)$, $j \ge 1$, is a well-defined element in $\mathscr{D}(L)$, we set

$$\Gamma_{j+1}(F) = \Gamma\left(F, -L^{-1}\Gamma_j(F)\right) \quad \text{for } j = 0, 1, \cdots. \tag{50}$$

Similarly, let $F \in \mathscr{D}(L)$; we define $\Gamma_0^*(F) = F$ and $\Gamma_1^*(F) = \Gamma_1(F)$. If $\Gamma_j^*(F) \in \mathscr{D}(\mathscr{E})$ for fixed $j \le 1$,

$$\Gamma_{j+1}^*(F) = \left(-L^{-1}F, \Gamma_j^*(F)\right). \tag{51}$$

Lemma 5. *Suppose that $\Gamma_j(F) \in \mathscr{D}(L)$, $j = 0, 1, 2$, with $\mathbb{E}[F] = 0$. Then, we have*

$$\mathbb{E}[\Gamma_3(F)] = \frac{1}{6}\left(\mathbb{E}\left[F^4\right] - 3\left(\mathbb{E}\left[F^2\right]\right)^2\right). \tag{52}$$

Proof. Observe that $\Gamma_3(F) \in \mathbb{L}^1(E, \mu)$. Using the definition of Γ_3 and Lemma 4, we have that

$$\mathbb{E}[\Gamma_3(F)] = \mathbb{E}[F\Gamma_2(F)]. \tag{53}$$

The diffusion property and Lemma 4 yield, from (53), that

$$\mathbb{E}[\Gamma_3(F)] = \mathbb{E}\left[\Gamma\left(\frac{1}{2}F^2, -L^{-1}\Gamma_1(F)\right)\right] = \frac{1}{2}\mathbb{E}\left[F^2\Gamma\left(F, -L^{-1}F\right)\right] - \frac{1}{2}\left(\mathbb{E}\left[F^2\right]\right)^2. \tag{54}$$

The carré champ operator Γ and the integration by parts formula prove that the right-hand side of (54) can be computed as

$$\begin{aligned}\mathbb{E}[\Gamma_3(F)] &= \frac{1}{4}\mathbb{E}\left[F^2\left(L\left(F\left(-L^{-1}\right)F\right) - FL\left(\left(-L^{-1}F\right)\right.\right.\right.\\ &\quad \left.\left.\left. - \left(-L^{-1}F\right)LF\right)\right] - \frac{1}{2}\left(\mathbb{E}\left[F^2\right]\right)^2\\ &= -\frac{1}{4}\mathbb{E}\left[\Gamma\left(F^2, F\left(-L^{-1}F\right)\right)\right] + \frac{1}{4}\mathbb{E}\left[F^4\right]\\ &\quad + \frac{1}{4}\mathbb{E}\left[\Gamma\left(F^2\left(-L^{-1}F\right), F\right)\right] - \frac{1}{2}\left(\mathbb{E}\left[F^2\right]\right)^2.\end{aligned} \tag{55}$$

Again, using diffusion property, the first and third expectations in (55) can be represented as

$$\begin{aligned}-\frac{1}{4}\mathbb{E}\left[\Gamma\left(F^2, F\left(-L^{-1}F\right)\right)\right] &= -\frac{1}{4}\mathbb{E}\left[\Gamma\left(F^2, F\right)\left(-L^{-1}F\right)\right]\\ -\frac{1}{6}\mathbb{E}\left[\Gamma\left(F^3, -L^{-1}F\right)\right] &= -\frac{1}{4}\mathbb{E}\left[\Gamma\left(F^2, F\right)\left(-L^{-1}F\right)\right] - \frac{1}{6}\mathbb{E}\left[F^4\right],\end{aligned} \tag{56}$$

and similarly,

$$\frac{1}{4}\mathbb{E}\left[\Gamma\left(F^2\left(-L^{-1}F\right), F\right)\right] = \frac{1}{4}\mathbb{E}\left[\left(-L^{-1}F\right)\Gamma\left(F^2, F\right)\right] + \frac{1}{12}\mathbb{E}\left[F^4\right]. \tag{57}$$

Plugging (56) and (57) into (55) yields that the equality (52) holds. □

Next, we investigate the relation between Γ_3 and Γ_3^*.

Lemma 6. *Suppose that $\Gamma_j(F) \in \mathscr{D}(L)$, $j = 0, 1, 2$, with $\mathbb{E}[F] = 0$. We have*

$$\mathbb{V}ar(\Gamma_1(F)) = 2\mathbb{E}[\Gamma_3(F)] - \mathbb{E}[\Gamma_3^*(F)]. \tag{58}$$

Proof. By the definition of the operator Γ_1^* and the carré du champ operator Γ, we have that

$$\begin{aligned}\mathbb{E}\left[\Gamma\left(F, -L^{-1}F\right)^2\right] &= \frac{1}{2}\mathbb{E}\left[\Gamma_1^*(F)\left(L\left(F\left(-L^{-1}F\right)\right) - FL\left(-L^{-1}F\right) - \left(-L^{-1}F\right)LF\right)\right]\\ &= \frac{1}{2}\left\{-\mathbb{E}\left[\Gamma\left(F\left(-L^{-1}F\right), \Gamma_1^*(F)\right)\right] + \mathbb{E}\left[F^2\Gamma_1^*(F)\right]\right.\\ &\quad \left.+ \mathbb{E}\left[\Gamma\left(F, \Gamma_1^*(F)\left(-L^{-1}F\right)\right)\right]\right\}.\end{aligned} \tag{59}$$

Using the diffusion property, the first expectation in (59) can be written as

$$\begin{aligned}-\frac{1}{2}\mathbb{E}\left[\Gamma\left(F\left(-L^{-1}F\right), \Gamma_1^*(F)\right)\right] &= -\frac{1}{2}\mathbb{E}\left[F\Gamma\left(-L^{-1}F, \Gamma_1^*(F)\right)\right]\\ &\quad - \frac{1}{2}\mathbb{E}\left[\left(-L^{-1}F\right)\Gamma\left(F, \Gamma_1^*(F)\right)\right]\\ &= -\frac{1}{2}\mathbb{E}\left[F\Gamma_2^*(F)\right] - \frac{1}{2}\mathbb{E}\left[\left(-L^{-1}F\right)\Gamma\left(F, \Gamma_1^*(F)\right)\right].\end{aligned} \tag{60}$$

Lemma 4 shows that the second expectation in (59) can be computed as

$$\frac{1}{2}\mathbb{E}\left[F^2\Gamma_1^*(F)\right] = \frac{1}{6}\mathbb{E}\left[\Gamma\left(F^3, -L^{-1}F\right)\right] = \frac{1}{6}\mathbb{E}\left[F^4\right]. \tag{61}$$

The diffusion property shows that the third expectation in (59) can be represented as follows:

$$\begin{aligned}-\frac{1}{2}\mathbb{E}\left[\Gamma\left(F, \Gamma_1^*(F)\left(-L^{-1}F\right)\right)\right] &= \frac{1}{2}\mathbb{E}\left[\Gamma_1^*(F)\Gamma\left(F, -L^{-1}F\right)\right]\\ &\quad + \frac{1}{2}\mathbb{E}\left[\left(-L^{-1}F\right)\Gamma\left(F, \Gamma_1^*(F)\right)\right]\\ &= \frac{1}{2}\mathbb{E}\left[\Gamma_1^*(F)^2\right] + \frac{1}{2}\mathbb{E}\left[\left(-L^{-1}F\right)\Gamma\left(F, \Gamma_1^*(F)\right)\right].\end{aligned} \tag{62}$$

Plugging (60), (61), and (62) into (59) yields that

$$\mathbb{E}\left[\Gamma_1^*(F)^2\right] = -\frac{1}{2}\mathbb{E}\left[F\Gamma_2^*(F)\right] + \frac{1}{6}\mathbb{E}\left[F^4\right] + \frac{1}{2}\mathbb{E}\left[\Gamma_1^*(F)^2\right]. \tag{63}$$

From Lemma 4 and (63), it follows that

$$\mathbb{E}\left[\Gamma_1^*(F)^2\right] = \frac{1}{3}\mathbb{E}\left[F^4\right] - \mathbb{E}\left[\Gamma_3^*(F)\right]. \tag{64}$$

The above result (64) deduces that

$$\begin{aligned}\mathbb{V}\mathrm{ar}(\Gamma_1^*(F)) &= \frac{1}{3}\left(\mathbb{E}\left[F^4\right] - 3\left(\mathbb{E}\left[F^2\right]\right)^2\right) - \mathbb{E}[\Gamma_3^*(F)]\\ &= 2\mathbb{E}[\Gamma_3(F)] - \mathbb{E}[\Gamma_3^*(F)].\end{aligned} \tag{65}$$

□

Hence, the desired result follows.

4.2. Fourth Moment Theorem. Let us define a set

$$\mathfrak{C}(F) = \{\mathfrak{c} \in \mathbb{R} : \mathbb{E}[\Gamma_3^*(F)] > \mathfrak{c}\mathbb{E}[\Gamma_3(F)]\}. \tag{66}$$

Lemma 7. *For any random variable F related to a Markov diffusion generator L such that $\mathbb{V}ar(\Gamma_1(F)) > 0$, one has that $\mathfrak{C}(F) \neq \varnothing$.*

Proof. Define $\varphi(x) = \mathbb{E}[\Gamma_3(F)]x - \mathbb{E}[\Gamma_3^*(F)]$. Assumption and Lemma 6 yield that $\varphi(2) > 0$. If $\mathbb{E}[\Gamma_3^*(F)] \geq 0$, then $\varphi(0) \leq 0$ and $\mathbb{E}[\Gamma_3(F)] > 0$ by Lemma 6. Hence, there exists $\mathfrak{c} \in (0, 2)$ such that $\varphi(\mathfrak{c}) \leq 0$. If $\mathbb{E}[\Gamma_3^*(F)] < 0$ and $\mathbb{E}[\Gamma_3(F)] > 0$, then $\varphi(2) > \varphi(0) > 0$, so that there exists a constant $\mathfrak{c} \in (-\infty,0)$ such that $\varphi(\mathfrak{c}) \leq 0$. On the other hand, if $\mathbb{E}[\Gamma_3^*(F)] < 0$ and $\mathbb{E}[\Gamma_3(F)] < 0$, then $0 < \varphi(2) < \varphi(0)$, which implies that we can find a constant $\mathfrak{c} \in (2,\infty)$ such that $\varphi(\mathfrak{c}) \leq 0$. Obviously, combining the above results proves that $\mathfrak{C}(F) \neq \varnothing$. □

Theorem 8 (fourth moment bound). *Suppose that $\Gamma_j(F) \in \mathscr{D}(L)$, $j = 0, 1, 2$, with $\mathbb{E}[F] = 0$ and $\mathbb{V}ar(\Gamma_1(F)) > 0$. If the law of F is absolutely continuous with respect to the Lebesgue measure, there exists a constant $\mathfrak{c} \neq 2$ such that*

$$d_{Kol}(F, Z) \leq \sqrt{(2-\mathfrak{c})\mathbb{E}[\Gamma_3(F)]}. \tag{67}$$

Proof. By Stein's equation, we have that, for $z \in \mathbb{R}$,

$$\mathbb{P}(F \leq z) - \mathbb{P}(Z \leq z) = \mathbb{E}\left[f_z'(F) - Ff_z(F)\right], \tag{68}$$

where f_z is a solution of Stein's equation. Since $\mathbb{E}[F] = 0$, we have that $F = LL^{-1}F$. Therefore, by the integration by parts formula (12) and the derivation of Γ, the right-hand side of (68) can be written as

$$\begin{aligned} \mathbb{E}\left[f_z'(F) - Ff_z(F)\right] &= \mathbb{E}\left[f_z'(F) - LL^{-1}Ff_z(F)\right] \\ &= \mathbb{E}\left[f_z'(F) - \Gamma\left(f_z(F), -L^{-1}F\right)\right] \\ &= \mathbb{E}\left[f_z'(F)\left\{1 - \Gamma\left(F, -L^{-1}F\right)\right\}\right] \\ &\leq \|f_z'\|_\infty \sqrt{\mathbb{V}\mathrm{ar}(\Gamma_1^*(F))}. \end{aligned} \tag{69}$$

Since $\|f_z\|_\infty \leq 1$, we have, from Lemma 5 and Lemma 7 together with (69), that

$$d_{\mathrm{Kol}}(F, Z) \leq \sqrt{2\mathbb{E}[\Gamma_3(F)] - \mathbb{E}[\Gamma_3^*(F)]} \leq \sqrt{(2-\mathfrak{c})\mathbb{E}[\Gamma_3(F)]}. \tag{70}$$

□

Remark 9.

(1) If $\mathbb{E}[\Gamma_3(F)] \geq 0$, then we see, from the proof of Lemma 7, that $\mathfrak{c} \in (-\infty,2)$, and hence, the value in the square root in (67) is of positive. On the other hand, if $\mathbb{E}[\Gamma_3(F)] < 0$, then $\mathfrak{c} \in (2,\infty)$. This means that the value in the squre root of upper bound also is of positive

(2) If $2\mathbb{E}[\Gamma_3(F)] = \mathbb{E}[\Gamma_3^*(F)]$, then, by (70), F is a random variable having a standard Gaussian distribution. Conversely, suppose that F is a random variable having a standard Gaussian distribution. Then, we have that $-LF = F$ and $\Gamma(F, -L^{-1}F) = \Gamma(F) = 1$. Hence, $2\Gamma_3(F) = \Gamma_3^*(F) = 0$

As far as we know, the following is the first result of the quantitative fourth moment theorem for a random variable F belonging to a sum of Wiener chaoses.

Corollary 10. *Let $F = H_p(x) + H_q(x)$ for $p > q \geq 2$, where H_p and H_q are Hermite polynomials of order p and q, respectively. Then, one has that*

$$d_{Kol}(F, Z) \leq \sqrt{\frac{1}{6}(2-\mathfrak{g})\left(\mathbb{E}\left[F^4\right] - 3\left(\mathbb{E}\left[F^4\right]\right)^2\right)}, \tag{71}$$

where the lower chaos grade $\mathfrak{g}$ of F is given by

$$\mathfrak{g} = \frac{2 \wedge (p-q)}{p}. \tag{72}$$

Proof. We compute the expectation $\mathbb{E}[\Gamma_3^*(F)]$. First, note that

$$\begin{aligned} \Gamma_1^*(F) = {} & \frac{1}{p}\Gamma\left(H_p(x), H_p(x)\right) + \left(\frac{1}{p} + \frac{1}{q}\right)\Gamma\left(H_p(x), H_q(x)\right) \\ & + \frac{1}{q}\Gamma\left(H_q(x), H_q(x)\right). \end{aligned} \tag{73}$$

Let us set

$$c_r(p, q) = r!\binom{p}{r}\binom{q}{r}. \tag{74}$$

Obviously, the well-known product formula and the definition of carré du champ operator prove that for $i, j = p, q$,

$$\Gamma\left(H_i(x), H_j(x)\right) = \sum_{r=0}^{i\wedge j} c_r(i, j) r H_{i+j-2r}(x). \tag{75}$$

Direct computations yield, from (73) and (75), together with $\Gamma(c, H_i(x)) = 0$, that

$$\begin{aligned}\Gamma(H_p(x), \Gamma_1^*(F)) = &\frac{1}{p}\sum_{r=0}^{p-1} c_r(p,p)r \sum_{s=0}^{p\wedge 2(p-r)} c_s(p, 2(p-r))sH_{3p-2r-2s}(x)\\&+\left(\frac{1}{p}+\frac{1}{q}\right)\sum_{r=0}^{q} c_r(p,q)r\sum_{s=0}^{p\wedge 2(p-r)} c_s(p,p+q-2r)sH_{2p+q-2r-2s}(x)\\&+\frac{1}{q}\sum_{r=0}^{q-1} c_r(q,q)r\sum_{s=0}^{p\wedge 2(q-r)} c_s(p,2(q-r))sH_{p+2q-2r-2s}(x).\end{aligned} \quad (76)$$

From the right-hand side of (76), it follows that

$$\begin{aligned}\frac{1}{p}\mathbb{E}\left[H_p(x)\Gamma(H_p(x), \Gamma_1^*(F))\right] = &\frac{1}{p^2}\sum_{r=0}^{p-1} c_r(p,p)rc_{p-r}(p,2(p-r))(p-r)\mathbb{E}\left[H_p(x)^2\right]\\&+\frac{1}{p}\left(\frac{1}{p}+\frac{1}{q}\right)\sum_{r=0}^{q} c_r(p,q)r\sum_{s=0}^{p\wedge 2(p-r)} c_s(p,p+q-2r)s\times 1_{[2s=p+q-2r]}\mathbb{E}\left[H_p(x)^2\right]\\&+\frac{1}{pq}\sum_{r=0}^{q-1} c_r(q,q)rc_{q-r}(p,2(q-r))(q-r)\mathbb{E}\left[H_p(x)^2\right].\end{aligned} \quad (77)$$

On the other hand, since $L^{-1}c = 0$, we get

$$\begin{aligned}\Gamma(H_p(x), -L^{-1}\Gamma_1(F))] = &\frac{1}{p}\sum_{r=0}^{p-1}\frac{c_r(p,p)r}{2(p-r)}\sum_{s=0}^{p\wedge 2(p-r)} c_s(p,2(p-r))sH_{3p-2r-2s}(x)\\&+\left(\frac{1}{p}+\frac{1}{q}\right)\sum_{r=0}^{q}\frac{c_r(p,q)r}{p+q-2r}\sum_{s=0}^{p\wedge 2(p-r)} c_s(p,p+q-2r)sH_{2p+q-2r-2s}(x)\\&+\frac{1}{q}\sum_{r=0}^{q-1}\frac{c_r(q,q)r}{2(q-r)}\sum_{s=0}^{p\wedge 2(q-r)} c_s(p,2(q-r))sH_{p+2q-2r-2s}(x),\end{aligned} \quad (78)$$

so that

$$\begin{aligned}\mathbb{E}\left[H_p(x)\Gamma(H_p(x), -L^{-1}\Gamma_1(F)\right) = &\frac{1}{2p}\sum_{r=0}^{p-1} c_r(p,p)rc_{p-r}(p,2(p-r))\mathbb{E}\left[H_p(x)^2\right]\\&+\frac{1}{2}\left(\frac{1}{p}+\frac{1}{q}\right)\sum_{r=0}^{q} c_r(p,p)r\sum_{s=0}^{p\wedge 2(p-r)} c_s(p,p+q-2r)1_{[2s=p+q-2r]}\mathbb{E}\left[H_p(x)^2\right]\\&+\frac{1}{2q}\sum_{r=0}^{q-1} c_r(q,q)rc_{q-r}(p,2(q-r))\mathbb{E}\left[H_p(x)^2\right].\end{aligned} \quad (79)$$

From (77) and (79), we have that

$$\begin{aligned}&\frac{1}{p}\mathbb{E}\left[H_p(x)\Gamma(H_p(x), \Gamma_1^*(F))\right] - \mathfrak{c}\mathbb{E}\left[H_p(x)\Gamma(H_p(x), -L^{-1}\Gamma_1(F))\right]\\&= \frac{1}{2p^2}\sum_{r=0}^{p-1} c_r(p,p)c_{p-r}(p,2(p-r))r[2(p-r)-\mathfrak{c}p]\mathbb{E}\left[H_p(x)^2\right]\\&\quad+\frac{1}{2p}\left(\frac{1}{p}+\frac{1}{q}\right)\sum_{r=0}^{q} c_r(p,q)r\sum_{s=0}^{p\wedge 2(p-r)} c_s(p,p+q-2r)[2s-\mathfrak{c}p]\\&\quad\times 1_{[2s=p+q-2r]}\mathbb{E}\left[H_p(x)^2\right]\\&\quad+\frac{1}{2pq}\sum_{r=0}^{q-1} c_r(q,q)rc_{q-r}(p,2(q-r))[2(q-r)-\mathfrak{c}p]\mathbb{E}\left[H_p(x)^2\right]\\&\coloneqq A_1(\mathfrak{c}) + A_2(\mathfrak{c}) + A_3(\mathfrak{c}).\end{aligned} \quad (80)$$

Obviously, when $\mathfrak{c} = 2/p$, we have

$$\begin{aligned}A_1(\mathfrak{c}) &\geq (2-\mathfrak{c}p)\frac{1}{2p^2}\sum_{r=0}^{p-1} c_r(p,p)c_{p-r}(p,2(p-r))r\mathbb{E}\left[H_p(x)^2\right] = 0,\\A_3(\mathfrak{c}) &\geq (2-\mathfrak{c}p)\frac{1}{2pq}\sum_{r=0}^{q-1} c_r(q,q)rc_{q-r}(p,2(q-r))\mathbb{E}\left[H_p(x)^2\right] = 0.\end{aligned} \quad (81)$$

Similarly, if $\mathfrak{c} = (p-q)/p$, then

$$\begin{aligned}A_2(\mathfrak{c}) \geq &(p-q-\mathfrak{c}p)\frac{1}{2p}\left(\frac{1}{p}+\frac{1}{q}\right)\sum_{r=0}^{q} c_r(p,q)r\sum_{s=0}^{p\wedge 2(p-r)} c_s(p,p+q-2r)\\&\times 1_{[2s=p+q-2r]}\mathbb{E}\left[H_p(x)^2\right] = 0.\end{aligned} \quad (82)$$

Therefore, if $\mathfrak{c} = (2\wedge(p-q))/p$, then $A_i(\mathfrak{c}) \geq 0$ for $i = 1, 2, 3$, so that

$$\frac{1}{p}\mathbb{E}\left[H_p(x)\Gamma(H_p(x), \Gamma_1^*(F))\right] \geq \frac{2\wedge(p-q)}{p}\mathbb{E}\left[H_p(x)\Gamma(H_p(x), -L^{-1}\Gamma_1(F))\right]. \quad (83)$$

By a similar computation as for (77), one has that

$$\begin{aligned}\frac{1}{p}\mathbb{E}\left[H_q(x)\Gamma(H_p(x), \Gamma_1^*(F))\right] = &\frac{1}{p^2}\sum_{r=0}^{p-1} c_r(p,p)r\\&\cdot\sum_{s=0}^{p\wedge 2(p-r)} c_s(p,2(p-r))s1_{[2s=3p-q-2r]}\mathbb{E}\left[H_q(x)^2\right]\\&+\frac{1}{p}\left(\frac{1}{p}+\frac{1}{q}\right)\sum_{r=0}^{q} c_r(p,q)rc_{p-r}(p,p+q-2r)(p-r)\mathbb{E}\left[H_q(x)^2\right]\\&+\frac{1}{pq}\sum_{r=0}^{q-1} c_r(q,q)r\sum_{s=0}^{p\wedge 2(q-r)} c_s(p,2(q-r))s1_{[2s=p+q-2r]}\mathbb{E}\left[H_q(x)^2\right].\end{aligned} \quad (84)$$

On the other hand,

$$\begin{aligned}\mathbb{E}\left[H_q(x)\Gamma(H_p(x), -L^{-1}\Gamma_1(F))\right] = &\frac{1}{2p}\sum_{r=0}^{p-1}\frac{c_r(p,p)r}{p-r}\\&\cdot\sum_{s=0}^{p\wedge 2(p-r)} c_s(p,2(p-r))s1_{[2s=3p-q-2r]}\mathbb{E}\left[H_q(x)^2\right]\\&+\left(\frac{1}{p}+\frac{1}{q}\right)\sum_{r=0}^{q}\frac{c_r(p,q)r}{p+q-2r}c_{p-r}(p,p+q-2r)(p-r)\mathbb{E}\left[H_q(x)^2\right]\\&+\frac{1}{2q}\sum_{r=0}^{q-1}\frac{c_r(q,q)r}{q-r}\sum_{s=0}^{p\wedge 2(q-r)} c_s(p,2(q-r))s1_{[2s=p+q-2r]}\mathbb{E}\left[H_q(x)^2\right].\end{aligned} \quad (85)$$

Therefore, one has, from (84) and (85), that

$$
\begin{aligned}
&\frac{1}{p}\mathbb{E}\left[H_q(x)\Gamma\left(H_p(x),\Gamma_1^*(F)\right)\right]-\mathfrak{c}\mathbb{E}\left[H_q(x)\Gamma\left(H_p(x),-L^{-1}\Gamma_1(F)\right)\right]\\
&=\frac{1}{2p^2}\sum_{r=0}^{p-1}\frac{c_r(p,p)}{p-r}c_{p-r}(p,2(p-r))r\sum_{s=0}^{p\wedge 2(p-r)}c_s(p,2(p-r))s\\
&\quad\times 1_{[2s=3p-q-2r]}[2(p-r)-\mathfrak{c}p]\mathbb{E}\left[H_p(x)^2\right]\\
&\quad+\frac{1}{p}\left(\frac{1}{p}+\frac{1}{q}\right)\sum_{r=0}^{q}\frac{c_r(p,q)}{p+q-2r}c_s(p,p+q-2r)r(p-r)[p+q-2r-\mathfrak{c}p]\\
&\quad\times\mathbb{E}\left[H_q(x)^2\right]+\frac{1}{2pq}\sum_{r=0}^{q-1}\frac{c_r(q,q)}{q-r}rc_{q-r}(p,2(q-r))[2(q-r)-\mathfrak{c}p]\\
&\quad\sum_{s=0}^{p\wedge 2(q-r)}c_s(p,2(q-r))s\times 1_{[2s=p+q-2r]}\mathbb{E}\left[H_q(x)^2\right]\\
&\coloneqq B_1(\mathfrak{c})+B_2(\mathfrak{c})+B_3(\mathfrak{c}).
\end{aligned}
\tag{86}
$$

Using the same arguments as for the case of $A_i(\mathfrak{c})$, $i=1$, $2, 3$, yields that $B_i(\mathfrak{c})\geq 0$, $i=1,2,3$, for $\mathfrak{c}=(2\wedge(p-q))/p$. This implies that

$$
\frac{1}{p}\mathbb{E}\left[H_q(x)\Gamma\left(H_p(x),\Gamma_1^*(F)\right)\right]\geq\frac{2\wedge(p-q)}{p}\mathbb{E}\left[H_q(x)\Gamma\left(H_p(x),-L^{-1}\Gamma_1(F)\right)\right].
\tag{87}
$$

Similarly,

$$
\frac{1}{q}\mathbb{E}\left[H_p(x)\Gamma\left(H_q(x),\Gamma_1^*(F)\right)\right]\geq\frac{2\wedge(p-q)}{p}\mathbb{E}\left[H_p(x)\Gamma\left(H_q(x),-L^{-1}\Gamma_1(F)\right)\right],
\tag{88}
$$

$$
\frac{1}{q}\mathbb{E}\left[H_q(x)\Gamma\left(H_q(x),\Gamma_1^*(F)\right)\right]\geq\frac{2\wedge(p-q)}{p}\mathbb{E}\left[H_q(x)\Gamma\left(H_q(x),-L^{-1}\Gamma_1(F)\right)\right].
\tag{89}
$$

Combining the above results (83), (87), (88), and (89), we can show that

$$
\mathbb{E}[\Gamma_3^*(F)]\geq\frac{2\wedge(p-q)}{p}\mathbb{E}[\Gamma_3(F)].
\tag{90}
$$

Hence, the proof of this corollary is completed. □

5. Markov Chaos

In this section, as an application of Theorem 8, chaotic random variables such that $F\in\text{Ker}\,(L+\lambda Id)$ will be considered.

Theorem 11. *Let F be a chaotic eigenfunction of $-L$ with respect to eigenvalue λ with $\mathbb{E}[F]=0$ and $\mathbb{V}ar(\Gamma_1(F))>0$. Suppose that F has an upper chaos grade $\mathfrak{u}$ and a lower chaos grade $\mathfrak{g}$.*

(a) If $\mathfrak{u}\leq 2$, one has that

$$
d_{Kol}(F,Z)\leq\sqrt{\frac{1}{6}(2-\mathfrak{g})\left(\mathbb{E}\left[F^4\right]-3\left(\mathbb{E}\left[F^2\right]\right)^2\right)}
\tag{91}
$$

(b) If $\mathfrak{u}>2$ and $\mathbb{E}[\Gamma_3^*(F)]>0$, then there exist $\vartheta^*>1$ and $\vartheta>1$ such that

$$
d_{Kol}(F,Z)\leq\sqrt{\frac{1}{6}\left(2-\mathfrak{g}\lambda\frac{\vartheta^*-1}{\vartheta-1}\right)\left(\mathbb{E}\left[F^4\right]-3\left(\mathbb{E}\left[F^2\right]\right)^2\right)}
\tag{92}
$$

(c) If $\mathfrak{u}>2$, $\mathbb{E}[\Gamma_3^*(F)]<0$ and $\mathbb{E}[\Gamma_3^*(F)]>0$, then there exist $\vartheta^*<1$ and $\vartheta>1$ such that

$$
d_{Kol}(F,Z)\leq\sqrt{\frac{1}{6}\left(2+\mathfrak{u}\lambda\frac{1-\vartheta^*}{\vartheta-1}\right)\left(\mathbb{E}\left[F^4\right]-3\left(\mathbb{E}\left[F^2\right]\right)^2\right)}.
\tag{93}
$$

(d) If $\mathfrak{u}>2$, $\mathbb{E}[\Gamma_3^*(F)]<0$ and $\mathbb{E}[\Gamma_3(F)]<0$, then there exist $\vartheta^*<1$ and $\vartheta<1$ such that

$$
d_{Kol}(F,Z)\leq\sqrt{\frac{1}{6}\left(2-\mathfrak{u}\lambda\frac{1-\vartheta^*}{1-\vartheta}\right)\left(\mathbb{E}\left[F^4\right]-3\left(\mathbb{E}\left[F^2\right]\right)^2\right)}
\tag{94}
$$

Proof. We compute $\mathbb{E}[\Gamma_3^*(F)]$ and $\mathbb{E}[\Gamma_3(F)]$. By the definition of Γ_3^*, we have

$$
\begin{aligned}
\mathbb{E}[\Gamma_3^*(F)]&=\frac{1}{2\lambda}\mathbb{E}\left[\Gamma\left(F^2,\Gamma_1(F)\right)\right]\\
&=\frac{1}{2\lambda}\left(\mathbb{E}\left[-L\left(F^2\right)\Gamma_1(F)\right]-\mathbb{E}\left[-LF^2\right]\mathbb{E}[\Gamma_1(F)]\right)\\
&=\frac{1}{2\lambda}\mathbb{E}\left[-L\left(F^2\right)\left(\Gamma_1(F)-\mathbb{E}[\Gamma_1(F)]\right)\right].
\end{aligned}
\tag{95}
$$

On the other hand, by using Lemma 4, we have that

$$
\begin{aligned}
\mathbb{E}[\Gamma_3(F)]&=\frac{1}{2}\mathbb{E}\left[\Gamma\left(F^2,-L^{-1}\Gamma\left(F,-L^{-1}F\right)\right)\right]\\
&=\frac{1}{2}\left(\mathbb{E}\left[F^2\Gamma_1(F)\right]-\mathbb{E}\left[F^2\right]\mathbb{E}[\Gamma_1(F)]\right)\\
&=\frac{1}{2}\mathbb{E}\left[F^2\left(\Gamma_1(F)-\mathbb{E}[\Gamma_1(F)]\right)\right].
\end{aligned}
\tag{96}
$$

We denote by $J_{\lambda_\ell}(F^2)$ the projection of F^2 on Ker $(L+$

$\lambda_{\mathfrak{l}} Id$). From (16) in Definition 2, we know that F^2 has a chaos decomposition of the form

$$F^2 = \sum_{i=1}^{M} J_{\lambda_{\mathfrak{l}_i}}(F^2) + J_0(F^2), \tag{97}$$

where $\lambda_{\mathfrak{l}_1} = \mathfrak{g}\lambda$ and $\lambda_{\mathfrak{l}_M} = \mathfrak{u}\lambda$. By orthogonality, it follows, from (95) and (96), that

$$\begin{aligned}\mathbb{E}[\Gamma_3^*(F)] &= \frac{1}{4\lambda^2}\sum_{i=1}^{M}\sum_{j=1}^{M}\lambda_{\mathfrak{l}_i}\left(2\lambda-\lambda_{\mathfrak{l}_j}\right)\mathbb{E}\left[J_{\mathfrak{l}_i}(F^2)J_{\mathfrak{l}_j}(F^2)\right]\\ &= \frac{1}{4\lambda^2}\sum_{i=1}^{M}\lambda_{\mathfrak{l}_i}(2\lambda-\lambda_{\mathfrak{l}_i})\mathbb{E}\left[J_{\mathfrak{l}_i}(F^2)^2\right],\end{aligned} \tag{98}$$

$$\begin{aligned}\mathbb{E}[\Gamma_3(F)] &= \frac{1}{4\lambda}\sum_{i=1}^{M}\sum_{j=1}^{M}\left(2\lambda-\lambda_{\mathfrak{l}_j}\right)\mathbb{E}\left[J_{\mathfrak{l}_i}(F^2)J_{\mathfrak{l}_j}(F^2)\right]\\ &= \frac{1}{4\lambda}\sum_{i=1}^{M}(2\lambda-\lambda_{\mathfrak{l}_i})\mathbb{E}\left[J_{\mathfrak{l}_i}(F^2)^2\right].\end{aligned} \tag{99}$$

The proof of (a): since $\mathfrak{u} \le 2$, it is obvious, from (98) and (99), that

$$\begin{aligned}\mathbb{E}[\Gamma_3^*(F)] - \mathfrak{c}\mathbb{E}[\Gamma_3(F)] &= \frac{1}{4\lambda^2}\sum_{i=1}^{M}\left(\lambda_{\mathfrak{l}_i}-\mathfrak{c}\lambda\right)\left(2\lambda-\lambda_{\mathfrak{l}_i}\right)\mathbb{E}\left[J_{\mathfrak{l}_i}(F^2)^2\right]\\ &\ge \frac{1}{4\lambda}\left(\lambda_{\mathfrak{l}_1}-\mathfrak{c}\lambda\right)(2-\mathfrak{u})\mathbb{V}\mathrm{ar}(F^2).\end{aligned} \tag{100}$$

If $\mathfrak{c} \le (\lambda_{\mathfrak{l}_1}/\lambda) = \mathfrak{g}$, then we have, from (100), that $\mathbb{E}[\Gamma_3^*(F)] \ge \mathfrak{c}\mathbb{E}[\Gamma_3(F)] \ge 0$. Obviously, $\mathfrak{c} \le \mathfrak{g} < 1$. This fact implies that there exists $\mathfrak{c} \in (0,1)$ such that $\mathfrak{c} \in \mathfrak{C}(F)$. Application of Theorem 8 gives the desired result (91).

The proof of (b): first note that since $\mathfrak{u} > 2$,

$$(\mathfrak{u}-2)\lambda\sum_{i=1}^{M}\lambda_{\mathfrak{l}_i}\mathbb{E}\left[J_{\mathfrak{l}_i}(F^2)^2\right] > 0, \tag{101}$$

$$(\mathfrak{u}-2)\lambda\sum_{i=1}^{M}\mathbb{E}\left[J_{\mathfrak{l}_i}(F^2)^2\right] > 0. \tag{102}$$

If $\mathbb{E}[\Gamma_3^*(F)] > 0$, there exist $\vartheta^* > 1$ and $\vartheta > 1$, from (102), such that

$$\begin{aligned}\mathbb{E}[\Gamma_3^*(F)] - \mathfrak{c}\mathbb{E}[\Gamma_3(F)] &= \frac{1}{4\lambda^2}\Bigg\{\sum_{i=1}^{M}\lambda_{\mathfrak{l}_i}(\mathfrak{u}\lambda-\lambda_{\mathfrak{l}_i})\mathbb{E}\left[J_{\mathfrak{l}_i}(F^2)^2\right]\\ &\quad-(\mathfrak{u}-2)\lambda\sum_{i=1}^{M}\lambda_{\mathfrak{l}_i}\mathbb{E}\left[J_{\mathfrak{l}_i}(F^2)^2\right]-\mathfrak{c}\lambda\sum_{i=1}^{M}(\mathfrak{u}\lambda-\lambda_{\mathfrak{l}_i})\mathbb{E}\left[J_{\mathfrak{l}_i}(F^2)^2\right]\\ &\quad+\mathfrak{c}(\mathfrak{u}-2)\lambda\sum_{i=1}^{M}\mathbb{E}\left[J_{\mathfrak{l}_i}(F^2)^2\right]\Bigg\}\\ &= \frac{\mathfrak{u}-2}{4\lambda}\left\{(\vartheta^*-1)\sum_{i=1}^{M}\lambda_{\mathfrak{l}_i}\mathbb{E}\left[J_{\mathfrak{l}_i}(F^2)^2\right]-\mathfrak{c}(\vartheta-1)\sum_{i=1}^{M}\mathbb{E}\left[J_{\mathfrak{l}_i}(F^2)^2\right]\right\}\\ &\ge \frac{\mathfrak{u}-2}{4\lambda}\left[(\vartheta^*-1)\mathfrak{g}\lambda-\mathfrak{c}(\vartheta-1)\right]\mathbb{E}\left[F^4\right].\end{aligned} \tag{103}$$

Now, we take

$$\mathfrak{c} = \mathfrak{g}\lambda\frac{(\vartheta^*-1)}{\vartheta-1}, \tag{104}$$

so that the right-hand side of (103) is equal to 0. Since $2\mathbb{E}[\Gamma_3(F)] > \mathbb{E}[\Gamma_3^*(F)]$, we have that

$$\frac{\vartheta^*-1}{\vartheta-1} < \frac{2\sum_{i=1}^{M}\mathbb{E}\left[J_{\mathfrak{l}_i}(F^2)^2\right]}{\sum_{i=1}^{M}\lambda_{\mathfrak{l}_i}\mathbb{E}\left[J_{\mathfrak{l}_i}(F^2)^2\right]} < \frac{2}{\mathfrak{g}\lambda}. \tag{105}$$

This inequality (105) shows that $\mathfrak{c} \in (0,2)$, where $\mathfrak{c}$ is given by (104). Hence, applying Theorem 8 yields the bound (92).

The proof of (c): since $\mathbb{E}[\Gamma_3^*(F)] < 0$ and $\mathbb{E}[\Gamma_3(F)] > 0$, there exist $\vartheta^* < 1$ and $\vartheta > 1$, by the similar estimate as for the case of (b), such that

$$\begin{aligned}\mathbb{E}[\Gamma_3^*(F)] - \mathfrak{c}\mathbb{E}[\Gamma_3(F)] &= \frac{\mathfrak{u}-2}{4\lambda}\Bigg\{(\vartheta^*-1)\sum_{i=1}^{M}\lambda_{\mathfrak{l}_i}\mathbb{E}\left[J_{\mathfrak{l}_i}(F^2)^2\right]\\ &\quad-\mathfrak{c}(\vartheta-1)\sum_{i=1}^{M}\mathbb{E}\left[J_{\mathfrak{l}_i}(F^2)^2\right]\Bigg\}\\ &\ge \frac{\mathfrak{u}-2}{4\lambda}\left[(\vartheta^*-1)\mathfrak{u}\lambda-\mathfrak{c}(\vartheta-1)\right]\mathbb{E}\left[F^4\right].\end{aligned} \tag{106}$$

Taking

$$\mathfrak{c} = \mathfrak{u}\lambda\frac{(\vartheta^*-1)}{\vartheta-1} \tag{107}$$

yields that the right-hand side of (106) is equal to 0. Since $\vartheta^* < 1$ and $\vartheta > 1$, the constant $\mathfrak{c}$ given in (107) belongs to $(-\infty, 0)$. Application of Theorem 8 proves the bound (93).

The proof of (d): since $\mathbb{E}[\Gamma_3^*(F)] < 0$ and $\mathbb{E}[\Gamma_3(F)] < 0$, there exist $\vartheta^* < 1$ and $\vartheta < 1$ such that

$$\mathbb{E}[\Gamma_3^*(F)] - \mathfrak{c}\mathbb{E}[\Gamma_3(F)] \ge \frac{\mathfrak{u}-2}{4\lambda}\left[(\vartheta^*-1)\mathfrak{u}\lambda-\mathfrak{c}(\vartheta-1)\right]\mathbb{E}\left[F^4\right]. \tag{108}$$

The right-hand side of (108) is equal to 0 when one takes

$$\mathfrak{c} = \mathfrak{u}\lambda\left(\frac{\vartheta^* - 1}{\vartheta - 1}\right). \tag{109}$$

Note that

$$\frac{1-\vartheta^*}{1-\vartheta} > \frac{2\sum_{i=1}^{M}\mathbb{E}\left[J_{\ell_i}\left(F^2\right)^2\right]}{\sum_{i=1}^{M}\lambda_i\mathbb{E}\left[J_{\ell_i}\left(F^2\right)^2\right]} > \frac{2}{\mathfrak{u}\lambda}. \tag{110}$$

This estimate (110) shows that $\mathfrak{c}$ given in (109) belongs to $(2, \infty)$ from which we obtain the bound (93). □

Remark 12. Suppose that the target distribution ν in Theorem 4.9 of [16] is a standard Gaussian measure. If a chaotic random variable F_n, with $\mathbb{E}[F_n] = 0$ and $\mathbb{E}[F_n] = 1$, satisfies $\sup_n \mathfrak{u}_n > 2$, then the bound (3.7) in [16] becomes

$$d_{\mathrm{Kol}}(F_n, Z) \le \sqrt{\frac{2}{3}\left(1 - \frac{\mathfrak{u}_n}{4}\right)\left(\mathbb{E}\left[F_n^4\right] - 3\right) + \frac{\mathfrak{u}_n - 2}{2}\mathbb{V}\mathrm{ar}\left(F_n^2\right)}. \tag{111}$$

Even when the fourth moment of F_n in the first term of (111) converges to 3, the sequence $\{F_n\}$ does not converge, in distribution sense, to a standard Gaussian random variable Z, due to the second term in (111). It means that the fourth moment theorem of Theorem 1 is not working. The bounds in (b), (c), and (d) of Theorem 11 show that the fourth moment theorem holds by removing the second term in (111) even if an upper chaos grade is strictly greater than 2.

Remark 13. The Ornstein-Uhlenbeck operator L on $E = \mathbb{R}^d$ is then given by

$$Lf = \Delta f - x \cdot \nabla f. \tag{112}$$

The carré du champ operator Γ is the usual gradient operator $\Gamma(f, f) = |\nabla f|^2$. In infinite-dimensional setting, the infinite-dimensional Ornstein-Uhlenbeck generator on Wiener space can be obtained with Wiener measure as an invariant distribution. If F is an element belonging to a fixed Wiener chaos with order q, i.e, $F = I_q(f)$, $f \in \mathfrak{H}^{\otimes q}$, the well-known product formula of multiple stochastic integrals gives that

$$F^2 = \sum_{r=0}^{q} r! \binom{q}{r}^2 I_{2(q-r)}(f \otimes_r f), \tag{113}$$

where $f \otimes_r f$ is the contraction of the kernel f. This expansion of the square of F shows, from the definition of the chaos grade, that $\mathfrak{u} = 2$ and $\mathfrak{g} = 2/q$. In this case $(\lambda_q = q \text{ and } \mathfrak{g} = 2/q)$, it follows, from the above bound (91), that

$$d_{\mathrm{Kol}}(F, Z) \le \sqrt{\left(\frac{q-1}{3q}\right)\left(\mathbb{E}[F^4] - 3(\mathbb{E}[F^2])^2\right)}. \tag{114}$$

In this specific case, the above bound (114) has been applied to obtain the Berry–Esséen bound for parameter estimation of fractional Ornstein-Uhlenbeck processes (see [21]). Furthermore, the authors of [20] derive a bound for the form F/G to obtain an optimal Berry-Esséen bound for parameter estimation of stochastic partial differential equation.

6. Conclusions

In this paper, we derive the fourth moment bound in a normal approximation on the random variable of a general Markov diffusion generator. Significant features of our work are that (a) it provides a better estimate than the previous one; (b) it contains square integrable random variables unlike all previous works in this research line, where the authors deal with only random variables to belong to a fixed eigenspace (or Wiener chaos); and (c) unlike the previous result, the fourth moment theorem holds even if the upper chaos grade is strictly greater than 2. Future research plans will study whether our methods are applicable in other (non-Gaussian) target measures (for example, a bound for gamma or beta approximation).

Data Availability

There is no data used for this research.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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