Research Article

Impulsive Fractional Semilinear Integrodifferential Equations with Nonlocal Conditions

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This paper is devoted to a class of impulsive fractional semilinear integrodifferential equations with nonlocal initial conditions. Based on the semigroup theory and some fixed point theorems, the existence theory of PC-mild solutions is established under the condition of compact resolvent operator. Furthermore, the uniqueness of PC-mild solutions is proved in the case of the noncompact resolvent operator.

1. Introduction

The fractional evolution equation has been applied to many fields, and scholars have obtained abundant research achievements [1–13]. Impulsive fractional integrodifferential equations can describe some phenomena which often occur in physics, geology, and economics, for instance, earthquake, the closing of the switch in the circuit, and so on. Many scholars are committed to this subject and have achieved plentiful results [1–7]. Based on the fact that nonlocal initial conditions are more effective than classical initial conditions in applied physics, the study of differential equations with nonlocal conditions has attracted more and more researchers' attention [8–13].

Ji and Li [14] studied the following impulsive differential evolution equations with nonlocal conditions:

$$\begin{cases} u'(t) = Au(t) + f(t, u(t)), & t \in [0, b], t \neq t_i, \\ \Delta u|_{t=t_i} = I_i(u(t_i)), & i = 1, 2, \dots, m, \\ u(0) = g(u), \end{cases} \quad (1)$$

where A is the generator of a strongly continuous semigroup $T_\beta(t)$; sufficient conditions for the existence of mild solutions have been established by the Hausdorff measure of noncompactness and fixed point theorems.

Zhu et al. [15] investigated the fractional semilinear integrodifferential equations of mixed type with nonlocal conditions:

$$\begin{cases} {}^c D_t^\beta u(t) = A(t)u(t) + f(t, u(t), \mathcal{G}u(t), \mathcal{S}u(t)), & t \in [0, T_0], \\ u(0) + g(u) = u_0, \end{cases} \quad (2)$$

where $0 < \beta \leq 1$, $A(t)$ is a closed linear operator with domain $D(A)$ defined on a Banach space E ; the existence and uniqueness of mild solutions have been established by k -set contraction and β -resolvent family.

Gou and Li [16] studied the fractional impulsive integrodifferential equations in Banach space E ; local and global existences of mild solutions have been proved by measure of noncompactness and Sadovskii's fixed point theorem:

$$\begin{cases} {}^c D_t^\beta u(t) + Au(t) = f(t, u(t)) + \int_0^t q(t-s)g(s, u(s))ds, & t \geq 0, t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k^-)), & k = 1, 2, \dots, m, \\ u(0) = u_0 \in E, \end{cases} \quad (3)$$

where $0 < \beta < 1$, $A : D(A) \subset E \longrightarrow E$ is a closed linear operator and $-A$ generates a uniformly bounded C_0 -semigroup $T(t)$.

Inspired by these contributions, we consider the following impulsive fractional semilinear integrodifferential equations with nonlocal initial conditions:

$$\begin{cases} {}^c D_t^\beta x(t) - A(t)x(t) = f(t, x(t), (\mathcal{G}x)(t), (\mathcal{H}x)(t)), & t \in [0, T], t \neq t_k, \\ x(0) + w(x) = x_0, \\ \Delta x(t_k) = I_k(x(t_k)), & k = 1, 2, \dots, m, \end{cases} \quad (4)$$

where ${}^c D_t^\beta$ is the Caputo's fractional derivative of order β , $\beta \in (0, 1]$, $A(t)$ is a closed linear operator with domain $D(A)$ defined on a Banach space E , and two integral operators $\mathcal{G}$ and $\mathcal{H}$ are defined by

$$\begin{aligned} \mathcal{G}x(t) &= \int_0^t g(t, s, x(s))ds, \\ \mathcal{H}x(t) &= \int_0^T h(t, s, x(s))ds, \end{aligned} \quad (5)$$

$g : B \times E \longrightarrow E$, $h : B_0 \times E \longrightarrow E$ are continuous and nonlinear functions, $B = \{(t, s) \mid 0 \leq s \leq t \leq T\}$, $B_0 = \{(t, s) \mid 0 \leq t, s \leq T\}$, f and w are to be specified later, $I_k : E \longrightarrow E$ ($k = 1, 2, \dots, m$) are continuous impulsive functions, the prefixed numbers t_k ($k = 1, 2, \dots, m$) satisfy $0 = t_0 < t_1 < t_2 < \dots < t_m < t_{m+1} = T$, $x(t_k) = x(t_k^-)$, and $x(t_k^-) = \lim_{h \rightarrow 0^+} x(t_k + h)$ represent the left limit of $x(t)$ at $t = t_k$.

In this paper, we demonstrate the existence of PC-mild solutions for problem (4) via the theory of semigroup and fixed point theorem under the condition of compact resolvent operator. Meanwhile, the uniqueness of PC-mild solutions is proved in the case of noncompact resolvent operator. The kernels g and h of the integral operators $\mathcal{G}$ and $\mathcal{H}$ are nonlinear functions; the function w of the nonlocal conditions is noncompact. In addition, the closed linear operator $A(t)$ is dependent on t . The rest of this paper is organized as follows. In Section 2, some basic definitions and lemmas are collected that will be needed throughout the remaining sections. The existence and uniqueness of PC-mild solutions are shown in Section 3 via the theories of resolvent operators and various fixed point theorems. Finally, the summary of our results comes in Section 4.

2. Preliminaries

Let $(E, \|\cdot\|)$ be a Banach space, $J = [0, T]$ and $0 < T < \infty$. The collection of all continuous functions from J into E , denoted $C(J, E)$, is a Banach space equipped with the norm $\|x\|_C = \max \{\|x(t)\|, t \in J\}$ for $x \in C(J, E)$. Let $PC(J, E) = \{x \mid x : J \longrightarrow E : x \in C((t_k, t_{k+1}], E), \text{ and there exist } x(t_k^-) \text{ and } x(t_k^+) \text{ with } x(t_k) = x(t_k^-), k = 1, \dots, m\}$ endowed with the PC-norm $\|x\|_{PC} = \sup \{\|x(t)\|, t \in J\}$, $J_0 = [0, t_1]$, $J_1 = (t_1, t_2]$, $\dots$, $J_m = (t_m, T]$.

Lemma 1 (nonlinear alternative for single-valued maps). *Let E be a Banach space, $C \subset E$ be a closed convex set, V be an open subset of C , and $0 \in V$. Suppose that $Q : \bar{V} \longrightarrow C$ is completely continuous, then either*

- (i) Q has a fixed point in $\bar{V}$ or
- (ii) there is a $u \in \partial V$ and $\lambda \in (0, 1)$ with $u = \lambda Q(u)$

Lemma 2 (see [17]). *Let $0 < \eta < 1$, $\gamma > 0$, denote*

$$S_n = \eta^n + C_n^1 \eta^{n-1} \gamma + \frac{C_n^2 \eta^{n-2}}{2!} \gamma^2 + \dots + \frac{\gamma^n}{n!}, \quad n \in N, \quad (6)$$

where $C_n^k = n!/(k!(n-k)!)$. Then, for any fixed constant $0 < \xi < 1$ and any real number $s > 1$, we get

$$S_n \leq O\left(\frac{\xi^n}{\sqrt{n}}\right) + o\left(\frac{1}{n^s}\right) = o\left(\frac{1}{n^s}\right), \text{ as } n \longrightarrow \infty. \quad (7)$$

Definition 3 (see [18, 19]). The Caputo fractional derivative of order β of a function $f : (0, \infty) \longrightarrow R$ is defined as

$${}^c D_t^\beta f(t) = \frac{1}{\Gamma(n-\beta)} \int_0^t (t-s)^{n-\beta-1} f^{(n)}(s)ds, \quad (8)$$

where $n-1 < \beta < n$, $n \in N$, $\Gamma(\cdot)$ denotes the Gamma function. The Laplace transform of the Caputo fractional derivative of order β is given as

$$\begin{aligned} \mathcal{L}\left({}^c D_t^\beta f(t)\right)(s) &= s^\beta (\mathcal{L}f)(s) - \sum_{j=1}^{n-1} s^{\beta-j-1} x^{(j)}(0), \\ n-1 &< \beta \leq n, \end{aligned} \quad (9)$$

where $(\mathcal{L}f)(s) = \int_0^\infty e^{-st} f(t)dt$ is the Laplace transform of the function $f(t)$.

Definition 4 (see [20, 21]). Let $A(t)$ be a closed and linear operator with domain $D(A)$ defined on a Banach space E and $\beta > 0$. Let $\rho[A(t)]$ be the resolvent set of $A(t)$; $A(t)$ is called the generator of a β -resolvent family if there exist $\omega \geq 0$ and a strongly continuous function $U_\beta : \mathbb{R}_+^2 \longrightarrow B(E)$ such that $\{\lambda^\beta : \operatorname{Re} \lambda > \omega\} \subset \rho(A)$ and

$$\left(\lambda^\beta I - A(s)\right)^{-1} x = \int_0^\infty e^{-\lambda(t-s)} U_\beta(t, s) x ds, \quad \operatorname{Re}(\lambda) > \omega, x \in E. \quad (10)$$

In this case, $U_\beta(t, s)$ is called the β -resolvent family generated by $A(t)$.

Lemma 5 (see [21, 22]). $U_\beta(t, s)$ satisfies the following properties:

- (i) $U_\beta(s, s) = I$, $U_\beta(t, s) = U_\beta(t, r)U_\beta(r, s)$, for $0 \leq s \leq r \leq t \leq a$
- (ii) $(t, s) \longrightarrow U_\beta(t, s)$ is strongly continuous for $0 \leq s \leq t \leq a$
- (iii) If $U_\beta(t, s)$ is compact for $t, s > 0$, then the $U_\beta(t, s)$ is continuous in the uniform operator topology

Definition 6. A function $x \in PC(J, E)$ is said to be a PC-mild solution of problem (4) if $x(t)$ satisfies the integral equation:

$$\begin{aligned} x(t) = & U_\beta(t, 0)(x_0 - \omega(x)) \\ & + \int_0^t U_\beta(t, s)f(s, x(s), (\mathcal{G}x)(s), (\mathcal{H}x)(s))ds \\ & + \sum_{0 < t_k < t} U_\beta(t, t_k)I_k(x(t_k)), \quad t \in J. \end{aligned} \quad (11)$$

3. Existence and Uniqueness of Mild Solution

Theorem 7. Assume that the conditions (H_1) – (H_3) hold true and the resolvent operator $U_\beta(t, s)$ ($t, s > 0$) is compact.

(H_1) The function $f : J \times E \times E \times E \longrightarrow E$ is continuous, and there exist nonnegative Lebesgue integrable functions $a, l_i \in L(J, \mathbb{R}_+)$ ($i = 1, 2, 3$), for every $t \in J$, $x_i \in E$, such that

$$\|f(t, x_1, x_2, x_3)\| \leq a(t) + l_1(t)\|x_1\| + l_2(t)\|x_2\| + l_3(t)\|x_3\|. \quad (12)$$

(H_2) There exist nonnegative Lebesgue integrable functions $b, c, l_i \in L(J, \mathbb{R}_+)$ ($i = 4, 5$), for all $x \in E$, such that

$$\begin{aligned} \|g(t, s, x)\| &\leq b(t) + l_4(t)\|x\|, \quad (t, s) \in B, \\ \|h(t, s, x)\| &\leq c(t) + l_5(t)\|x\|, \quad (t, s) \in B_0. \end{aligned} \quad (13)$$

(H_3) The functions $\omega : PC(J, E) \longrightarrow E$ and $I_k : E \longrightarrow E$ are continuous, and there exist constants $d_\omega, e_\omega, d_k, e_k > 0$, such that

$$\begin{aligned} \|\omega(x)\| &\leq d_\omega\|x\|_{PC} + e_\omega, \quad x \in PC(J, E), \\ \|I_k(x)\| &\leq d_k\|x\| + e_k, \quad x \in E, k = 1, 2 \dots m. \end{aligned} \quad (14)$$

Then, problem (4) has at least one PC-mild solution in $PC(J, E)$.

Proof. Let us consider the operator $Q : PC(J, E) \longrightarrow PC(J, E)$ as follows:

$$\begin{aligned} (Qx)(t) = & U_\beta(t, 0)(x_0 - \omega(x)) \\ & + \int_0^t U_\beta(t, s)f(s, x(s), (\mathcal{G}x)(s), (\mathcal{H}x)(s))ds \\ & + \sum_{0 < t_k < t} U_\beta(t, t_k)I_k(x(t_k)). \end{aligned} \quad (15)$$

It is easy to see that the operator Q is well defined in $PC(J, E)$.

At first, we claim that $Q : PC(J, E) \longrightarrow PC(J, E)$ is a continuous operator. Let $\{x_n\}_0^\infty \subset PC(J, E)$ be a sequence such that $x_n \longrightarrow x$ ($n \longrightarrow \infty$) in $PC(J, E)$. Since for all $t \in J$,

$$\begin{aligned} \|(Qx_n)(t) - (Qx)(t)\| &\leq \|U_\beta(t, 0)(\omega(x_n) - \omega(x))\| \\ &\quad + \left\| \int_0^t U_\beta(t, s)f(s, x_n(s), (\mathcal{G}x_n)(s), (\mathcal{H}x_n)(s)) \right. \\ &\quad \left. - f(s, x(s), (\mathcal{G}x)(s), (\mathcal{H}x)(s))ds \right\| \\ &\quad + \sum_{k=1}^m \|U_\beta(t, t_k)(I_k(x_n(t_k)) - I_k(x(t_k)))\| \\ &\leq M\|\omega(x_n) - \omega(x)\| + M \int_0^t \|f(s, x_n(s), (\mathcal{G}x_n)(s), \\ &\quad \cdot (\mathcal{H}x_n)(s)) - f(s, x(s), (\mathcal{G}x)(s), (\mathcal{H}x)(s))\|ds \\ &\quad + \sum_{k=1}^m M\|I_k(x_n(t_k)) - I_k(x(t_k))\|, \end{aligned} \quad (16)$$

where $M = \max_{0 \leq s \leq t \leq T} \|U_\beta(t, s)\|$. Using the fact that $f : J \times E \times E \times E \longrightarrow E$, $\omega : PC(J, E) \longrightarrow E$, and $I_k : E \longrightarrow E$ ($k = 1, 2 \dots m$) are continuous, we obtain

$$\|Qx_n - Qx\|_{PC} \longrightarrow 0, \text{ as } n \longrightarrow \infty. \quad (17)$$

Therefore, $Q : PC(J, E) \longrightarrow PC(J, E)$ is continuous.

Furthermore, for any $R > 0$, we prove that $Q(T_R)$ is equicontinuous in J_k ($k = 0, 1, 2 \dots m$). For all $x \in T_R = \{x \in PC(J, E) : \|x\|_{PC} \leq R\}$ and $\tau_1, \tau_2 \in J_k$ ($\tau_1 \leq \tau_2$), by the condition (H_3) , we have

$$\begin{aligned} \|(Qx)(\tau_2) - (Qx)(\tau_1)\| &\leq \|U_\beta(\tau_2, 0) - U_\beta(\tau_1, 0)\|\|x_0 - \omega(x)\| \\ &\quad + \left\| \int_{\tau_1}^{\tau_2} U_\beta(\tau_2, s)f(s, x(s), (\mathcal{G}x)(s), (\mathcal{H}x)(s))ds \right\| \\ &\quad + \left\| \int_0^{\tau_1} (U_\beta(\tau_2, s) - U_\beta(\tau_1, s))f(s, x(s), (\mathcal{G}x)(s), (\mathcal{H}x)(s))ds \right\| \\ &\quad + \left\| \sum_{0 < t_k < \tau_2} U_\beta(\tau_2, t_k)I_k(x(t_k)) - \sum_{0 < t_k < \tau_1} U_\beta(\tau_1, t_k)I_k(x(t_k)) \right\| \\ &\leq \|U_\beta(\tau_2, 0) - U_\beta(\tau_1, 0)\|(\|x_0\| + d_\omega R + e_\omega) \\ &\quad + \int_{\tau_1}^{\tau_2} M\|f(s, x(s), (\mathcal{G}x)(s), (\mathcal{H}x)(s))\|ds \\ &\quad + \sup_{s \in J_k} \|U_\beta(\tau_2, s) - U_\beta(\tau_1, s)\| \\ &\quad \cdot \int_0^{\tau_1} \|f(s, x(s), (\mathcal{G}x)(s), (\mathcal{H}x)(s))\|ds \\ &\quad + \sum_{k=1}^m \|U_\beta(\tau_2, t_k) - U_\beta(\tau_1, t_k)\|\|I_k(x(t_k))\| =: I_1 + I_2 + I_3 + I_4. \end{aligned} \quad (18)$$

For all $x \in T_R$, $s \in J$, we get $\|x(s)\| \leq R$, by the condition (H_2) ,

$$\begin{aligned} \|(\mathcal{G}x)(s)\| &\leq \int_0^s \|g(s, v, x(v))\| dv \leq \int_0^s (b(v) + l_4(v)\|x(v)\|) dv \\ &\leq \int_0^T b(v) dv + R \int_0^T l_4(v) dv, \end{aligned} \quad (19)$$

meanwhile,

$$\begin{aligned} \|(\mathcal{H}x)(s)\| &\leq \int_0^T \|h(s, v, x(v))\| dv \leq \int_0^T (c(v) + l_5(v)\|x(v)\|) dv \\ &\leq \int_0^T c(v) dv + R \int_0^T l_5(v) dv. \end{aligned} \quad (20)$$

According to the condition (H_1) and the above inequalities, for all $s \in J$, we get

$$\begin{aligned} &\|f(s, x(s), (\mathcal{G}x)(s), (\mathcal{H}x)(s))\| \\ &\leq a(s) + l_1(s)\|x(s)\| + l_2(s)\|(\mathcal{G}x)(s)\| + l_3(s)\|(\mathcal{H}x)(s)\| \\ &\leq a(s) + l_1(s)R + l_2(s) \left(\int_0^T b(v) dv + R \int_0^T l_4(v) dv \right) \\ &\quad + l_3(s) \left(\int_0^T c(v) dv + R \int_0^T l_5(v) dv \right) \leq a_1(s) + b_1(s)R, \end{aligned} \quad (21)$$

where

$$\begin{aligned} a_1(s) &= a(s) + l_2(s) \int_0^T b(v) dv + l_3(s) \int_0^T c(v) dv, \\ b_1(s) &= l_1(s) + l_2(s) \int_0^T l_4(v) dv + l_3(s) \int_0^T l_5(v) dv. \end{aligned} \quad (22)$$

Obviously, $a_1(s)$ and $b_1(s)$ are nonnegative Lebesgue integrable functions, then

$$\begin{aligned} I_2 &= \int_{\tau_1}^{\tau_2} M \|f(s, x(s), (\mathcal{G}x)(s), (\mathcal{H}x)(s))\| ds \\ &\leq M \int_{\tau_1}^{\tau_2} (a_1(s) + b_1(s)R) ds, \end{aligned}$$

$$\begin{aligned} I_3 &= \sup_{s \in J_k} \|U_\beta(\tau_2, s) - U_\beta(\tau_1, s)\| \\ &\quad \cdot \int_0^{\tau_1} \|f(s, x(s), (\mathcal{G}x)(s), (\mathcal{H}x)(s))\| ds \\ &\leq \sup_{s \in J_k} \|U_\beta(\tau_2, s) - U_\beta(\tau_1, s)\| \int_0^{\tau_1} (a_1(s) + b_1(s)R) ds, \end{aligned}$$

$$\begin{aligned} I_4 &= \sum_{k=1}^m \|U_\beta(\tau_2, t_k) - U_\beta(\tau_1, t_k)\| \|I_k(x(t_k))\| \\ &\leq \sum_{k=1}^m \|U_\beta(\tau_2, t_k) - U_\beta(\tau_1, t_k)\| (d_k R + e_k). \end{aligned} \quad (23)$$

In view of Lemma 5, the compactness of the resolvent operator $U_\beta(t, s)$ ($t, s > 0$) implies the continuity in the uniform operator topology. As a result, from the above inequalities, we deduce that $\|(Qx)(\tau_2) - (Qx)(\tau_1)\| \rightarrow 0$ independently of $x \in T_R$ as $\tau_2 - \tau_1 \rightarrow 0$. That is, $Q(T_R)$ is equicontinuous in J_k ($k = 0, 1, 2, \dots, m$).

In the end, we demonstrate that $Q(T_R) \subset PC(J, E)$ is precompact.

For any t ($0 < t \leq T$), $0 < \varepsilon < t$, and $x \in T_R$, the operator $Q_\varepsilon x$ is defined by

$$\begin{aligned} (Q_\varepsilon x)(t) &= U_\beta(t, 0)(x_0 - \omega(x)) \\ &\quad + \int_0^{t-\varepsilon} U_\beta(t, s)f(s, x(s), (\mathcal{G}x)(s), (\mathcal{H}x)(s)) ds \\ &\quad + \sum_{0 < t_k < t} U_\beta(t, t_k)I_k(x(t_k)), \quad t \in J. \end{aligned} \quad (24)$$

Since $U_\beta(t, s)$ is compact resolvent operator, the set $Y_\varepsilon(t) = \{(Q_\varepsilon x)(t) : x \in T_R\}$ is relatively compact in E for every ε ($0 < \varepsilon < t$).

Moreover, for any $x \in T_R$, $t \in J$, one can show that

$$\begin{aligned} &\|(Qx)(t) - (Q_\varepsilon x)(t)\| \\ &= \left\| \int_{t-\varepsilon}^t U_\beta(t, s)f(s, x(s), (\mathcal{G}x)(s), (\mathcal{H}x)(s)) ds \right\| \\ &\leq M \int_{t-\varepsilon}^t \|f(s, x(s), (\mathcal{G}x)(s), (\mathcal{H}x)(s))\| ds \\ &\leq M \int_{t-\varepsilon}^t (a_1(s) + b_1(s)R) ds. \end{aligned} \quad (25)$$

Thus, $Y(t) = \{(Qx)(t) : x \in T_R\}$ is totally bounded. Hence, $Y(t)$ is relatively compact in E , and so, with the help of the Arzelà-Ascoli theorem, $Q : PC(J, E) \rightarrow PC(J, E)$ is completely continuous.

For $0 < \lambda < 1$, let $x = \lambda(Qx)$, we get

$$\begin{aligned} x(t) &= \lambda U_\beta(t, 0)(x_0 - \omega(x)) \\ &\quad + \lambda \int_0^t U_\beta(t, s)f(s, x(s), (\mathcal{G}x)(s), (\mathcal{H}x)(s)) ds \\ &\quad + \lambda \sum_{0 < t_k < t} U_\beta(t, t_k)I_k(x(t_k)). \end{aligned} \quad (26)$$

Then, using the conditions (H_1) – (H_3) , it follows that

$$\begin{aligned}
\|x(t)\| &\leq M\|x_0 - \omega(x)\| + M \int_0^t \|f(s, x(s), \\
&\quad \cdot (\mathcal{G}x)(s), (\mathcal{H}x)(s))\| ds + \sum_{k=1}^m M\|I_k(x(t_k))\| \\
&\leq M(\|x_0\| + d_\omega R + e_\omega) + M \int_0^t (a_1(s) \\
&\quad + b_1(s)R) ds + \sum_{k=1}^m M(d_k R + e_k) \\
&\leq M(\|x_0\| + e_\omega) + M \int_0^T a_1(s) ds + \sum_{k=1}^m M e_k \\
&\quad + \left(M d_\omega + M \int_0^T b_1(s) ds + \sum_{k=1}^m M d_k \right) R =: \rho.
\end{aligned} \tag{27}$$

That is, $\|x(t)\| \leq \rho$ for $t \in J$, then there exists a constant $\rho_1 > \rho$ such that $\|x\|_{PC} \neq \rho_1$. Let $V = \{x \in PC(J, E) : \|x\|_{PC} < \rho_1\}$, obviously, there is no $x \in \partial V$ such that $x = \lambda(Qx)$ for $0 < \lambda < 1$. Therefore, thanks to Lemma 1, one gets that Q has at least one fixed point x in V , which is a PC-mild solution of problem (4). This completes the proof. $\square$

Remark 8. Theorem 7 is proved under the condition that $U_\beta(t, s)$ is compact for $t, s > 0$ and the functions f, g, h meet corresponding conditions; in the case that the resolvent operator $U_\beta(t, s)$ is noncompact, we would obtain Theorem 9 and Theorem 10.

Theorem 9. Suppose that the conditions (H_4) – (H_6) are satisfied, $M = \max_{0 \leq s < t \leq T} \|U_\beta(t, s)\|$, and $M(L_\omega + \int_0^T L_1(s) ds + \int_0^T L_2(s)$

$ds \int_0^T L_4(v) dv + \int_0^T L_3(s) ds \int_0^T L_5(v) dv + \sum_{k=1}^m L_{I_k}) < 1$.

(H_4) The function $f : J \times E \times E \times E \longrightarrow E$ is continuous, and there exist nonnegative Lebesgue integrable functions $L_i \in L(J, \mathbb{R}_+)$ ($i = 1, 2, 3$), for any $u_i, v_i \in E$, $t \in J$, such that

$$\begin{aligned}
&\|f(t, u_1, u_2, u_3) - f(t, v_1, v_2, v_3)\| \\
&\leq L_1(t)\|u_1 - v_1\| + L_2(t)\|u_2 - v_2\| + L_3(t)\|u_3 - v_3\|.
\end{aligned} \tag{28}$$

(H_5) There exist nonnegative Lebesgue integrable functions $L_4, L_5 \in L(J, \mathbb{R}_+)$, for each $u, v \in E$, such that

$$\begin{aligned}
&\|g(t, s, u) - g(t, s, v)\| \leq L_4(t)\|u - v\|, \quad (t, s) \in B, \\
&\|h(t, s, u) - h(t, s, v)\| \leq L_5(t)\|u - v\|, \quad (t, s) \in B_0.
\end{aligned} \tag{29}$$

(H_6) The functions $I_k : E \longrightarrow E$ and $\omega : PC(J, E) \longrightarrow E$ are continuous, and there exist nonnegative constants $L_\omega, L_{I_k} > 0$, such that

$$\|\omega(u) - \omega(v)\| \leq L_\omega \|u - v\|_{PC}, \quad u, v \in PC(J, E),$$

$$\|I_k(u) - I_k(v)\| \leq L_{I_k} \|u - v\|, \quad u, v \in E, k = 1, 2, \dots, m.$$

(30)

Then, problem (4) has a unique PC-mild solution x^* in $PC(J, E)$.

Proof. It follows from the conditions (H_4) – (H_6) , for any $u, v \in PC(J, E)$, $t \in J$, one can derive

$$\begin{aligned}
&\|(Qu)(t) - (Qv)(t)\| \\
&\leq \|U_\beta(t, 0)(\omega(u) - \omega(v))\| \\
&\quad + \sum_{0 < t_k < t} \|U_\beta(t, t_k)(I_k(u(t_k)) - I_k(v(t_k)))\| \\
&\quad + \left\| \int_0^t U_\beta(t, s) (f(s, u(s), (\mathcal{G}u)(s), (\mathcal{H}u)(s)) \right. \\
&\quad \left. - f(s, v(s), (\mathcal{G}v)(s), (\mathcal{H}v)(s))) ds \right\| \\
&\leq M\|\omega(u) - \omega(v)\| + \sum_{k=1}^m M\|I_k(u(t_k)) - I_k(v(t_k))\| \\
&\quad + M \int_0^t \|f(s, u(s), (\mathcal{G}u)(s), (\mathcal{H}u)(s)) \\
&\quad - f(s, v(s), (\mathcal{G}v)(s), (\mathcal{H}v)(s))\| ds \\
&\leq ML_\omega \|u - v\|_{PC} + \sum_{k=1}^m ML_{I_k} \|u(t_k) - v(t_k)\| \\
&\quad + M \int_0^t (L_1(s)\|u(s) - v(s)\| + L_2(s)\|(\mathcal{G}u)(s) - (\mathcal{G}v)(s)\| \\
&\quad + L_3(s)\|(\mathcal{H}u)(s) - (\mathcal{H}v)(s)\|) ds \\
&\leq ML_\omega \|u - v\|_{PC} + \sum_{k=1}^m ML_{I_k} \|u - v\|_{PC} \\
&\quad + M \int_0^t L_1(s) ds \|u - v\|_{PC} + \left(M \int_0^t L_2(s) \int_0^s L_4(v) dv ds \right. \\
&\quad \left. + M \int_0^t L_3(s) \int_0^T L_5(v) dv ds \right) \|u - v\|_{PC} \\
&\leq M \left(L_\omega + \sum_{k=1}^m L_{I_k} + \int_0^T L_1(s) ds + \int_0^T L_2(s) ds \int_0^T L_4(v) dv \right. \\
&\quad \left. + \int_0^T L_3(s) ds \int_0^T L_5(v) dv \right) \|u - v\|_{PC}.
\end{aligned} \tag{31}$$

Based on the assumption, we have $\|Qu - Qv\|_{PC} < \|u - v\|_{PC}$, which means that the operator Q is a contraction mapping. Hence, the operator Q has a unique fixed point $x^* \in PC(J, E)$, which implies that problem (4) has a unique PC-mild solution. This completes the proof. $\square$

Theorem 10. Assume that the conditions (H_7) and (H_8) hold, $\omega \equiv 0$, $M = \max_{0 \leq s < t \leq T} \|U_\beta(t, s)\|$, $a = \sum_{k=1}^m ML_{I_k} < 1$.

(H₇) The function $f : J \times E \times E \times E \longrightarrow E$ is continuous, and there exist nonnegative Lebesgue integrable functions $L, L' \in L(J, \mathbb{R}_+)$, for all $u_i, v_i \in E, t \in J$, such that

$$\begin{aligned} & \|f(t, u_1, u_2, u_3) - f(t, v_1, v_2, v_3)\| \\ & \leq L'_1(t)\|u_1 - v_1\| + L'_2(t)\|u_2 - v_2\|. \end{aligned} \quad (32)$$

(H₈) There exist constants $L_g, L_{I_k} > 0$, for all $u, v \in E$, satisfying

$$\begin{aligned} & \|g(t, s, u) - g(t, s, v)\| \leq L_g\|u - v\|, \quad (t, s) \in B, \\ & \|I_k(u) - I_k(v)\| \leq L_{I_k}\|u - v\|, \quad k = 1, 2, \dots, m. \end{aligned} \quad (33)$$

Then, problem (4) has a unique PC-mild solution x^* in $PC(J, E)$. For all $t \in J$, $x_0 \in PC(J, E)$, iterative sequence $x_n(t)$ are defined by

$$\begin{aligned} x_n(t) = & U_\beta(t, 0)x_0 + \int_0^t U_\beta(t, s)f(s, x_{n-1}(s), \\ & \cdot (\mathcal{G}x_{n-1})(s), (\mathcal{H}x_{n-1})(s))ds \\ & + \sum_{0 < t_k < t} U_\beta(t, t_k)I_k(x_{n-1}(t_k)), \quad n = 1, 2, \dots, \end{aligned} \quad (34)$$

uniformly converge to the unique PC-mild solution $x^*(t)$ in $t \in J$, and for any $s > 0$,

$$\|x_n - x^*\|_{PC} = o\left(\frac{1}{n^s}\right), \text{ as } n \longrightarrow \infty. \quad (35)$$

Proof. Combining the conditions (H₇) and (H₈), for all $t \in J$, $u, v \in PC(J, E)$, we get

$$\begin{aligned} & \|(Qu)(t) - (Qv)(t)\| \\ & \leq M \int_0^t \|f(s, u(s), (\mathcal{G}u)(s), (\mathcal{H}u)(s)) \\ & \quad - f(s, v(s), (\mathcal{G}v)(s), (\mathcal{H}v)(s))\| ds \\ & \quad + \sum_{0 < t_k < t} M \|I_k(u(t_k)) - I_k(v(t_k))\| \\ & \leq M \int_0^t (L'_1(s) + L'_2(s)L_g s) \|u(s) - v(s)\| ds \\ & \quad + \sum_{k=1}^m ML_{I_k} \|u(t_k) - v(t_k)\| \leq \int_0^t L(s) \|u(s) - v(s)\| ds \\ & \quad + \sum_{k=1}^m ML_{I_k} \|u - v\|_{PC} \leq \left(\int_0^t L(s) ds + \sum_{k=1}^m ML_{I_k} \right) \|u - v\|_{PC} \\ & \leq \left(\int_0^t L(s) ds + a \right) \|u - v\|_{PC}, \end{aligned} \quad (36)$$

where $L(s) = M(L'_1(s) + L'_2(s)L_g T)$. It is easy to see that $L \in L(J, \mathbb{R}_+)$. Notice that $a = \sum_{k=1}^m ML_{I_k} < 1$, then there is $\varepsilon > 0$ such that $0 < b = \varepsilon + a < 1$. For the above $\varepsilon > 0$, there exists a continuous function $\phi(s)$ such that

$$\int_0^T |L(s) - \phi(s)| ds < \varepsilon. \quad (37)$$

Consequently,

$$\begin{aligned} & \|(Qu)(t) - (Qv)(t)\| \\ & \leq \left(\int_0^t |L(s) - \phi(s)| ds + \int_0^t |\phi(s)| ds \right) \|u - v\|_{PC} + a \|u - v\|_{PC} \\ & \leq (\varepsilon + \Phi t) \|u - v\|_{PC} + a \|u - v\|_{PC} \leq (b + \Phi t) \|u - v\|_{PC} \\ & = \left(C_1^0 b + C_1^1 \frac{(\Phi t)}{1!} \right) \|u - v\|_{PC}, \end{aligned} \quad (38)$$

where $\Phi = \max \{|\phi(s)| : s \in J\}$. We next prove the following inequalities, for every positive integer n and $t \in J$,

$$\begin{aligned} & \|(Q^n u)(t) - (Q^n v)(t)\| \\ & \leq \left(C_n^0 b^n + C_n^1 \frac{b^{n-1}(\Phi t)}{1!} + C_n^2 \frac{b^{n-2}(\Phi t)^2}{2!} \right. \\ & \quad \left. + \dots + C_n^n \frac{b^0(\Phi t)^n}{n!} \right) \|u - v\|_{PC}, \end{aligned} \quad (39)$$

where $C_n^m = n!/(m!(n-m)!)$.

Assume that, for any positive integer k , we have

$$\begin{aligned} & \|(Q^k u)(t) - (Q^k v)(t)\| \\ & \leq \left(C_k^0 b^k + C_k^1 \frac{b^{k-1}(\Phi t)}{1!} + C_k^2 \frac{b^{k-2}(\Phi t)^2}{2!} \right. \\ & \quad \left. + \dots + C_k^k \frac{b^0(\Phi t)^k}{k!} \right) \|u - v\|_{PC}. \end{aligned} \quad (40)$$

By the formula $C_{k+1}^m = C_k^m + C_k^{m-1}$, for all $t \in J$,

$$\begin{aligned} & \|(Q^{k+1} u)(t) - (Q^{k+1} v)(t)\| \\ & = \|Q(Q^k u)(t) - Q(Q^k v)(t)\| \\ & \leq M \int_0^t \|f(s, (Q^k u)(s), \mathcal{G}(Q^k u)(s), \mathcal{H}(Q^k u)(s)) \\ & \quad - f(s, (Q^k v)(s), \mathcal{G}(Q^k v)(s), \mathcal{H}(Q^k v)(s))\| ds \\ & \quad + \sum_{0 < t_k < t} M \|I_k(Q^k u)(t_k) - I_k(Q^k v)(t_k)\| \\ & \leq M \int_0^t (L'_1(s) + L'_2(s)L_g s) \|(Q^k u)(s) - (Q^k v)(s)\| ds \\ & \quad + \sum_{k=1}^m ML_{I_k} \|(Q^k u)(t_k) - (Q^k v)(t_k)\| \end{aligned}$$

$$\begin{aligned}
& \leq \int_0^t L(s) \left\| (Q^k u)(s) - (Q^k v)(s) \right\| ds \\
& \quad + \sum_{k=1}^m ML_{I_k} \left\| (Q^k u)(t_k) - (Q^k v)(t_k) \right\| \\
& \leq \int_0^t |L(s) - \phi(s)| \left(C_k^0 b^k + C_k^1 \frac{b^{k-1}(\Phi s)}{1!} \right. \\
& \quad \left. + C_k^2 \frac{b^{k-2}(\Phi s)^2}{2!} + \dots + C_k^k \frac{b^0(\Phi s)^k}{k!} \right) ds \|u - v\|_{PC} \\
& \quad + \int_0^t |\phi(s)| \left(C_k^0 b^k + C_k^1 \frac{b^{k-1}(\Phi s)}{1!} \right. \\
& \quad \left. + C_k^2 \frac{b^{k-2}(\Phi s)^2}{2!} + \dots + C_k^k \frac{b^0(\Phi s)^k}{k!} \right) ds \|u - v\|_{PC} \\
& \quad + \sum_{k=1}^m ML_{I_k} \left(C_k^0 b^k + C_k^1 \frac{b^{k-1}(\Phi t_k)}{1!} \right. \\
& \quad \left. + C_k^2 \frac{b^{k-2}(\Phi t_k)^2}{2!} + \dots + C_k^k \frac{b^0(\Phi t_k)^k}{k!} \right) \|u - v\|_{PC} \\
& \leq \varepsilon \left(C_k^0 b^k + C_k^1 \frac{b^{k-1}(\Phi t)}{1!} + C_k^2 \frac{b^{k-2}(\Phi t)^2}{2!} \right. \\
& \quad \left. + \dots + C_k^k \frac{b^0(\Phi t)^k}{k!} \right) \|u - v\|_{PC} + \Phi \int_0^t \left(C_k^0 b^k \right. \\
& \quad \left. + C_k^1 \frac{b^{k-1}(\Phi s)}{1!} + C_k^2 \frac{b^{k-2}(\Phi s)^2}{2!} + \dots \right. \\
& \quad \left. + C_k^k \frac{b^0(\Phi s)^k}{k!} \right) ds \|u - v\|_{PC} + a \left(C_k^0 b^k + C_k^1 \frac{b^{k-1}(\Phi t)}{1!} \right. \\
& \quad \left. + C_k^2 \frac{b^{k-2}(\Phi t)^2}{2!} + \dots + C_k^k \frac{b^0(\Phi t)^k}{k!} \right) \|u - v\|_{PC} \\
& \leq b \left(C_k^0 b^k + C_k^1 \frac{b^{k-1}(\Phi t)}{1!} + C_k^2 \frac{b^{k-2}(\Phi t)^2}{2!} + \dots \right. \\
& \quad \left. + C_k^k \frac{b^0(\Phi t)^k}{k!} \right) \|u - v\|_{PC} + \Phi \int_0^t \left(C_k^0 b^k + C_k^1 \frac{b^{k-1}(\Phi s)}{1!} \right. \\
& \quad \left. + C_k^2 \frac{b^{k-2}(\Phi s)^2}{2!} + \dots + C_k^k \frac{b^0(\Phi s)^k}{k!} \right) ds \|u - v\|_{PC} \\
& \leq \left(C_{k+1}^0 b^{k+1} + C_{k+1}^1 \frac{b^k(\Phi t)}{1!} + C_{k+1}^2 \frac{b^{k-1}(\Phi t)^2}{2!} \right. \\
& \quad \left. + \dots + C_{k+1}^{k+1} \frac{b^0(\Phi t)^{k+1}}{(k+1)!} \right) \|u - v\|_{PC}.
\end{aligned} \tag{41}$$

By mathematical induction, for every positive integer n , we obtain

$$\begin{aligned}
\|Q^n u - Q^n v\|_{PC} & \leq \left(C_n^0 b^n + C_n^1 \frac{b^{n-1}(\Phi t)}{1!} + C_n^2 \frac{b^{n-2}(\Phi t)^2}{2!} \right. \\
& \quad \left. + \dots + C_n^n \frac{b^0(\Phi t)^n}{n!} \right) \|u - v\|_{PC}.
\end{aligned} \tag{42}$$

Using Lemma 2, it follows that

$$\|Q^n u - Q^n v\|_{PC} \leq o\left(\frac{1}{n^s}\right) \|u - v\|_{PC}, \text{ as } n \longrightarrow \infty. \tag{43}$$

Thus, for any fixed constant $s > 1$, we can find a positive integer n_0 such that, for any $u, v \in PC(J, E)$ and $n > n_0$, we have

$$\|Q^n u - Q^n v\|_{PC} \leq \frac{1}{n^s} \|u - v\|_{PC}. \tag{44}$$

Applying the general Banach contraction mapping principle, we deduce that the operator Q has a unique fixed point x^* in $PC(J, E)$, which means that problem (4) has a unique PC-mild solution x^* in $PC(J, E)$. This completes the proof. $\square$

4. Conclusion

In this paper, we demonstrate the existence theory of PC-mild solutions for the impulsive fractional semilinear integrodifferential equations with nonlocal initial conditions (4) via the theory of semigroup and fixed point theorem under the condition of compact resolvent operator. Meanwhile, the uniqueness of PC-mild solutions is proved under the condition of noncompact resolvent operator. The kernels g and h of the integral operators $\mathcal{S}$ and $\mathcal{H}$ are nonlinear functions, and the function ω of the nonlocal initial conditions is noncompact. In addition, the closed linear operator $A(t)$ is dependent on t . As a consequence, our main theorems improve and generalize many existing results on this topic.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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