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Research Article

Starlikeness of Normalized Bessel Functions with Symmetric Points

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Bessel functions are related with the known Bessel differential equation. In this paper, we determine the radius of starlikeness for starlike functions with symmetric points involving Bessel functions of the first kind for some kinds of normalized conditions. Our prime tool in these investigations is the Mittag-Leffler representation of Bessel functions of the first kind.

1. Introduction and Definitions

Let $\mathbb{E}(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$ and $\mathbb{E}(0,1) \subset \mathbb{E}(z_0, r)$ denote the interior of the unit circle with center at origin. Suppose that $\mathscr{A}$ represent functions f in $\mathbb{E}(0,1)$:

$$f(z) = z + a_2 z^2 + \cdots. \quad (1)$$

Obviously, $f(0) = 0$ along with $f'(0) = 1$. The subclass $\mathscr{S} \subset A$ only contains univalent (one-to-one) functions and $\mathscr{S}^* \subset \mathscr{S}$ represent the set of starlike functions. A function f for which $f(\mathbb{E}(0,1))$ is star-shaped is starlike if $\operatorname{Re}\{zf'(z)/f(z)\} > 0$. Also, $f \in \mathscr{S}_s$ if

$$\operatorname{Re}\left\{\frac{2zf'(z)}{f(z) - f(-z)}\right\} > 0, \quad z \in \mathbb{E}(0,1), \quad (2)$$

and $f \in \mathscr{S}_s(\eta)$ if

$$\operatorname{Re}\left\{\frac{2zf'(z)}{f(z) - f(-z)}\right\} > \eta, \quad 0 \le \eta < 1, z \in \mathbb{E}(0,1). \quad (3)$$

Let

$$r^*(f) = \sup\left\{r > 0 : \operatorname{Re}\left\{\frac{2zf'(z)}{f(z) - f(-z)}\right\} > 0, z \in \mathbb{E}(0,r)\right\},$$
$$r^*_\eta(f) = \sup\left\{r > 0 : \operatorname{Re}\left\{\frac{2zf'(z)}{f(z) - f(-z)}\right\} > \eta, z \in \mathbb{E}(0,r)\right\} \quad (4)$$

be the radii of the classes defined above. We note that r^* is the maximum value of the radius such that $f(\mathbb{E}(0, r^*(f))) \in \mathscr{S}^*$ and r^*_η is the maximum value of the radius such that $f(\mathbb{E}(0, r^*_\eta(f))) \in \mathscr{S}^*$ with symmetric points. Consider the following representation of the function b_μ as in [1], which satisfies the well-known Bessel differential equation:

$$b_\mu(z) = \sum_{j\ge 0} \frac{(-1)^j}{j!\Gamma(j+\mu+1)}\left(\frac{z}{2}\right)^{2j+\mu} = \sum_{j\ge 0 j!} \frac{(-1)^j}{(j+\mu)!}\left(\frac{z}{2}\right)^{2j+\mu}, \quad (5)$$

where $z, \mu \in \mathbb{C}$ such that $\mu \ne -1, -2, -3, \cdots$. Observe that $b_\mu(z) \notin A$. Thus, we consider the following normalizations:

$$f_\mu(z) = \left[2^\mu \Gamma(\mu+1) b_\mu(z)\right]^{1/\mu}, \quad \mu \ne 0, \quad (6)$$

$$k_\mu(z) = 2^\mu \Gamma(\mu+1) z^{1-\mu} b_\mu(z), \tag{7}$$

$$\ell_\mu(z) = 2^\mu \Gamma(\mu+1) z^{1-(\mu/2)} b_\mu(\sqrt{z}). \tag{8}$$

Clearly, the function $f_\mu, k_\mu, \ell_\mu \in \mathscr{A}$. We see that

$$f_\mu(z) = \exp\left[\frac{1}{\mu}\log\left(2^\mu \Gamma(\mu+1) b_\mu(z)\right)\right]. \tag{9}$$

The geometric behavior and properties of the functions f_μ, k_μ, and ℓ_μ were studied by Brown, Kreyszig, Robertson, and many others (for detail, see [2–4] and also the references therein). The related problems were also studied in [3, 5–8] with references therein. We study the radius problems for the functions f_μ, k_μ, and ℓ_μ starlike with symmetric points. Mittag-Leffler expansion for Bessel functions is used as a prime tool along with the conclusion that the specific positive roots of the Dini functions are always smaller than the related zeros $b_\mu(z)$, for reference, see [9].

2. Preliminaries

Lemma 1. *Let $f : \mathbb{E}(0,1) \longrightarrow \mathbb{C}$ be a transcendental function having the following expansion:*

$$f(z) = z\prod_{j\geq 1}\left(1 - \frac{z}{z_j}\right), \tag{10}$$

where $z_j : |z_j| > 1$ have the same argument. For a univalent function f in $\mathbb{E}(0,1)$, we have

$$\sum_{j\geq 1}\frac{1}{|z_j| - 1} \leq 1. \tag{11}$$

This result holds if and only if $f \in \mathcal{S}^*$, and each of its derivatives is close to convex in the open unit disk $\mathbb{E}(0,1)$. Furthermore, for z_j', the zeroes of the derivative of f, f, and f' are univalent in $\mathbb{E}(0,1)$ and for $\mathbb{E}(0,1), f(\mathbb{E}(0,1))$ is a convex-shaped if and only if

$$\sum_{j\geq 1}\frac{1}{|z_j'| - 1} \leq 1. \tag{12}$$

Lemma 2. *The function*

$$\ell_\mu(z) = 2^\mu \Gamma(\mu+1) z^{1-\mu/2} b_\mu(\sqrt{z}) \in \mathcal{S}^*(\eta) \tag{13}$$

and each of its derivative is close to convex in $\mathbb{E}(0,1)$ if and only if $\mu > \mu_0(\eta)$, where $\mu_0(\eta) \simeq 0.5623\cdots$ is a unique zero of $\ell_\mu(1) = 0$ on $(-1,\infty)$.

The proof of Lemma 1 and Lemma 2 is found in [10].

Lemma 3. *The function*

$$f_\mu(z) = \left(2^\mu \Gamma(\mu+1) b_\mu(z)\right)^{1/\mu} \in \mathcal{S}^*(\eta) \tag{14}$$

in $\mathbb{E}(0,1)$ if and only if $\mu > \mu_1(\eta)$, where $0 < \mu_1(\eta) < \infty$ is the unique solution of

$$(1-\eta)\mu b_\mu(1) = b_{\mu+1}(1). \tag{15}$$

In particular, $f_\mu \in \mathcal{S}^*$ in $\mathbb{E}(0,1)$ if and only if $\mu > \mu_1(0)$, where $\mu_1(0) \simeq 0.3908\cdots$ is a unique zero of

$$\mu b_\mu(1) = b_{\mu+1}(1). \tag{16}$$

Lemma 4. *The function*

$$k_\mu(z) = 2^\mu \Gamma(\mu+1) b_\mu(z) z^{1-\mu} \in \mathcal{S}^*(\eta), \quad 0 \leq \eta < 1, \tag{17}$$

in $\mathbb{E}(0,1)$ if and only if $\mu > \mu_2(\eta)$, where $\mu_2(\eta)$ is the unique zero of

$$(1-\eta) b_\mu(1) = b_{\mu+1}(1), \tag{18}$$

lies in $(\tilde{\mu},\infty)$, where $\tilde{\mu} \simeq -0.7745\cdots$ is the unique root of $b_{\mu,1} = 1$ and $b_{\mu,1}$ is the first positive zero of b_μ. In particular, $k_\mu \in \mathcal{S}^*$ in $\mathbb{E}(0,1)$ if and only if $\mu > \mu_2(0)$, where $\mu_2(0) \simeq -0.3397\cdots$ is a unique zero of

$$b_\mu(1) = b_{\mu+1}(1). \tag{19}$$

The proof of Lemma 3 and Lemma 4 can be seen in [11].

Lemma 5. *If $z \in \mathbb{C}$ and $\eta \in \mathbb{R} : \eta > |z|$, then*

$$\frac{|z|}{\eta - |z|} \geq \operatorname{Re}\left(\frac{z}{\eta - z}\right). \tag{20}$$

For the detail of the above Lemma 5, we refer to [3].

3. Main Results

Theorem 6. *Let $1 > \eta \geq 0$, and $\mu \in (-1,0)$. Then, $r_\eta^*(I_\mu)$, is a unique positive zero of*

$$z I_\mu'(z) - \eta\mu I_\mu(z) = 0, \tag{21}$$

where $I_\mu(z) = i^{-\mu} b_\mu(iz)$. Moreover, if $\mu > 0$, then $r_\eta^*(b_\mu)$ is the least positive zero of

$$z b_\mu'(z) - \eta\mu b_\mu(z) = 0. \tag{22}$$

Proof. Using Lemma 3, we see that the function $f_\mu(z) = (2^\mu \Gamma(\mu+1) b_\mu(z))^{1/\mu} \in \mathcal{S}^*(\eta)$ in $\mathbb{E}(0,1)$ with respect to z iff $\mu > \mu_1(\eta)$, where $\mu_1(\eta)$ is a unique zero of

$$(1-\eta)\mu b_\mu(1) = b_{\mu+1}(1), \tag{23}$$

lies in $(0,\infty)$. Suppose $b_{\mu,j}$ is the jth positive root of $b_\mu(z)$. By using infinite product representation,

$$b_\mu(z) = \frac{1}{\Gamma(1+\mu)}\left(\frac{z}{2}\right)^\mu \prod_{j\geq 1}\left(1-\frac{z^2}{b_{\mu,j}^2}\right). \tag{24}$$

Also, as given in [12], we see that f_μ has the following form:

$$\begin{aligned} f_\mu(z) &= z\prod_{j\geq 1}\left(1-\frac{z}{b_{\mu,j}^2}\right) \\ &= \left[\sum_{j\geq 0}\frac{(-1)^j\Gamma(1+\mu)}{j!\Gamma(\mu+j+1)}z^{\mu+2j}\right]^{1/\mu} \\ &= z-\frac{1}{4\mu(\mu+1)}z^3+\cdots, \quad \mu\neq 0. \end{aligned} \tag{25}$$

From Lemma 3, we see that for $\mu > \mu^* \simeq -0.7745\cdots$ the unique value of the root of $f_\mu(1)=0$ or $b_{\mu,1}=1$, we have $b_{\mu,1}>1$, and $b_{\mu,j}>1, j=1,2,\cdots$. The above result is immediate, if $b_{\mu,1}$ is increasing on $(-1,\infty)$. Using (24) and (25), we can write

$$\begin{aligned} -b_\mu(z) &= b_\mu(-z), \\ -b'_\mu(z) &= b'_\mu(-z). \end{aligned} \tag{26}$$

Also, from (6), we have

$$\begin{aligned} -f_\mu(z) &= f_\mu(-z), \\ f'_\mu(-z) &= (-1)f'_\mu(z), \end{aligned} \tag{27}$$

which in the context of $zb'_\mu(z)-\mu b_\mu(z) = -zb_{\mu+1}(z)$ is equivalent to the Mittag-Leffler representation:

$$\frac{1}{b_\mu(z)}b_{\mu+1}(z) = \sum_{j\geq 1}\frac{2z}{b_{\mu,j}^2-z^2}. \tag{28}$$

Consequently,

$$\frac{b_{\mu+1}(-z)}{b_\mu(-z)} = -\frac{b_{\mu+1}(z)}{b_\mu(z)}. \tag{29}$$

In view of (6), (25), and (27), we can write

$$\frac{zf'_\mu(-z)}{f_\mu(-z)} = -1+\frac{1}{\mu}\sum_{j\geq 1}\frac{2z^2}{b_{\mu,j}^2-z^2} = -=-\frac{1}{\mu}\frac{zb'_\mu(z)}{b_\mu(z)}. \tag{30}$$

Using Lemma 1, we find that for $j\in\mathbb{N}, \mu=-1$, and $z\in\mathbb{E}(0,b_{\mu,1})$, the following inequality

$$\frac{|z|^2}{b_{\mu,j}^2-|z|^2} \geq \operatorname{Re}\left(\frac{z^2}{b_{\mu,j}^2-z^2}\right) \tag{31}$$

implies that

$$\operatorname{Re}\frac{zf'_\mu(-z)}{f_\mu(-z)} \leq -1+\frac{1}{\mu}\sum_{j\geq 1}\frac{2|z|^2}{b_{\mu,j}^2-|z|^2} = -\frac{|z|f'_\mu(|z|)}{f_\mu(|z|)}. \tag{32}$$

When $|z|<b_{\mu,1}$, we observe that

$$\begin{aligned} \operatorname{Re}\left\{\frac{zf'_\mu(-z)}{f_\mu(-z)}\right\} &\leq -1+\frac{1}{\mu}\sum_{j\geq 1}\frac{2r^2}{b_{\mu,j}^2-r^2} \\ &\leq -1+\frac{1}{\mu}\sum_{j\geq 1}\frac{2}{b_{\mu,j}^2-1} \\ &= -\frac{f'_\mu(1)}{f_\mu(1)}. \end{aligned} \tag{33}$$

As in [12], we see that $b'_{\mu,j}>0$ on $(0,\infty)$ for a fixed $j\in N$. Thus, $f'_\mu(1)/f_\mu(1)$ is increasing on $(0,\infty)$, and $-(f'_\mu(1)/f_\mu(1))$ is decreasing on $(0,\infty)$. Also,

$$-\frac{f'_\mu(1)}{f_\mu(1)} < -\eta \Leftrightarrow \mu<\mu_1(\eta), \tag{34}$$

where $\mu_1(\eta)$ is the unique zero of

$$f'_\mu(1) = \eta f_\mu(1) \text{ or } \eta\mu b_\mu(1) = b'_\mu(1) \text{ or } (1-\eta)\mu b_\mu(1) = b_{\mu+1}(1). \tag{35}$$

We also note that

$$\frac{zf'_\mu(-z)}{f_\mu(-z)} = 1-\frac{b_{\mu+1}(z)}{\mu b_\mu(z)}, \tag{36}$$

when $\mu\in(-1,0)$. For $-1<\mu<-\infty$, the Dini function $zb'_\mu(z)+\eta b_\mu(z)$ has real roots except a pair of complex conjugate roots (for detail, see [1]). Thus,

$$f_\mu(-z)\in\mathcal{S}^*(\eta), \quad \eta\in(-1,0), \tag{37}$$

in $\mathbb{E}(0,1)$ if and only if $\mu<\mu_1(\eta)$. Considering (5), (30), (33), and (36), we have

$$\begin{aligned} \frac{2zf'_\mu(z)}{f_\mu(z)-f_\mu(-z)} &= \frac{2zf_\mu(z)b'_\mu(z)}{\mu\left(f_\mu(z)+f_\mu(z)\right)b_\mu(z)} \\ &= \frac{1}{\mu}\frac{zb'_\mu(z)}{b_\mu(z)} \\ &= 1-\frac{1}{\mu}\sum_{j\geq 1}\frac{2z^2}{b_{\mu,j}^2-z^2}. \end{aligned} \tag{38}$$

Also, from (38), it is obvious that

$$\begin{aligned}\operatorname{Re}\frac{2zf_\mu'(z)}{f_\mu(z)-f_\mu(-z)} &\geq 1-\frac{1}{\mu}\sum_{j\geq 1}\frac{2|z|^2}{b_{\mu,j}^2-|z|^2}\\ &= \frac{|z|f_\mu'(|z|)}{f_\mu(|z|)} \\ &= \operatorname{Re}\frac{zf_\mu'(z)}{f_\mu(z)}.\end{aligned} \tag{39}$$

As in [1], for $1>\eta\geq 0$ and $\mu\in(-1,0)$, $r_\eta^*(I_\mu)$ is the unique value of positive zero of $zI_\mu'(z)-\eta\mu I_\mu(z)=0$. Moreover, if $\mu>0$, then we have $r_\eta^*(b_\mu)$ which is the least positive zero of $zb_\mu'(z)-\eta\mu b_\mu(z)=0$. □

Theorem 7. *If $\mu>-1$, then $r_\eta^*(k_\mu)>0$ is the smallest zero of*

$$zb_\mu'(z)+(1-\eta-\mu)b_\mu(z)=0, \tag{40}$$

where $k_\mu(z)=2^\mu\Gamma(\mu+1)b_\mu(z)(z)^{1-\mu}$.

Proof. By Lemma 4, the function

$$k_\mu(z)=2^\mu\Gamma(\mu+1)\left[b_\mu(z)\right]^{1-\mu}\in\mathcal{S}^*(\eta), \tag{41}$$

for $\eta\in(-1,0)$in the open unit disk $\mathbb{E}(0,1)$ if and only if $\mu<\mu_2(\eta)$, where $\mu_2(\eta)$ is the unique value of the zero of the following equation:

$$(1-\eta)b_\mu(1)=b_{\mu+1}(1). \tag{42}$$

Suppose that $b_{\mu,j}$ is the jth positive zero of $b_\mu(z)$ given by (24) and (25). Consider the normalization (7) such that

$$k_\mu(-z)=(-1)k_\mu(z). \tag{43}$$

We write

$$\frac{zk_\mu'(-z)}{k_\mu(-z)}=-\frac{zk_\mu'(z)}{k_\mu(z)}=-1+\mu-\partial\left(b_\mu\right)(z)=-1+\frac{zb_{\mu+1}(z)}{b_\mu(z)}. \tag{44}$$

For $\mu>-1$ and $r=|z|<b_{\mu,1}$, we see that

$$\operatorname{Re}\frac{zk_\mu'(-z)}{k_\mu(-z)}\leq-\frac{zk_\mu'(\mathrm{r})}{k_\mu(r)}=-1+\sum_{j\geq 1}\frac{2|z|^2}{b_{\mu,j}^2-|z|^2}=-\frac{k_\mu'(1)}{k_\mu(1)}. \tag{45}$$

For detail, we refer to [12]. Since the function $b_{\mu,j}'>0$ on $(-1,\infty)$ for fixed $j\in\mathbb{N}$, thus $k_\mu'(1)/k_\mu(1)$ is increasing on $(\tilde{\mu},\infty)$, and $-(k_\mu'(1)/k_\mu(1))$ is decreasing on $(\tilde{\mu},\infty)$ and $-(k_\mu'(1)/k_\mu(1))<-\eta$ if and only if $\mu<\mu_2(\eta)$, where $\mu_2(\eta)$ is the unique value of the root of

$$\begin{aligned}k_\mu'(1)=\eta k_\mu(1)\text{or}(1-\mu-\eta)b_\mu(1)+b_\mu'(1)\\ =0\text{ or}(1-\eta)b_\mu(1)\\ =b_{\mu+1}(1).\end{aligned} \tag{46}$$

Thus,

$$\begin{aligned}\frac{zb_\mu'(-z)}{b_\mu(-z)} &= -1+\operatorname{Re}\sum_{j\geq 1}\frac{2z^2}{b_{\mu,j}^2-z^2}\\ &\leq -1+\sum_{j\geq 1}\frac{2|z|^2}{b_{\mu,j}^2-|z|^2}\\ &= -\frac{|z|k_\mu'(|z|)}{k_\mu(|z|)}\\ &= -\operatorname{Re}\frac{zk_\mu'(z)}{k_\mu(z)},\end{aligned} \tag{47}$$

and equality holds for $|z|=r$. The above inequality implies that the function $k_\mu(-z)\in\mathcal{S}^*(\eta),\eta\in(-1,0)$, in $\mathbb{E}(0,1)$ if and only if $\mu<\mu_2(\eta)$. Considering normalization in (7), we can write

$$\begin{aligned}k_\mu(-z)=(-1)k_\mu(z)\frac{2zk_\mu'(z)}{k_\mu(z)-k_\mu(-z)}\\ =\frac{zb_\mu'(z)}{b_\mu(z)}+(1-\mu)\\ =1-\sum_{j\geq 1}\frac{2z^2}{b_{\mu,j}^2-z^2}.\end{aligned} \tag{48}$$

From (48), it is known that

$$\begin{aligned}\operatorname{Re}\left\{\frac{2zk_\mu'(z)}{k_\mu(z)-k_\mu(-z)}\right\} &= 1-\operatorname{Re}\sum_{j\geq 1}\frac{2z^2}{b_{\mu,j}^2-z^2}\\ &\geq 1-\sum_{j\geq 1}\frac{2|z|^2}{b_{\mu,j}^2-|z|^2}\\ &= \frac{|z|k_\mu'(|z|)}{k_\mu(|z|)}\\ &= \operatorname{Re}\frac{zk_\mu'(z)}{k_\mu(z)}.\end{aligned} \tag{49}$$

For1 $>\eta\geq 0$ and $\mu>-1$, we see that $r_\eta^*(k_\mu)$ is the least positive zero of the following equation:

$$zb_\mu'(z)+(1-\eta-\mu)b_\mu(z)=0, \tag{50}$$

where $k_\mu(z)=2^\mu\Gamma(\mu+1)b_\mu(z)(z)^{1-\mu}$. □

Theorem 8. *If $\mu > -1$ and $1 > \eta \geq 0$, then $r_\eta^*(\ell_\mu)$ is the least positive zero of the differential equation $zb_\mu'(z) + (2 - 2\eta - \mu) b_\mu(z) = 0$, where $\ell_\mu(z) = 2^\mu \Gamma(\mu+1) b_\mu(\sqrt{z})(z)^{1-\mu/2}$.*

Proof. Assume that $b_{\mu,j}$ is the jth positive zero of $b_\mu(z)$ given by (24) and (25). Considering the normalization given in (7), we write

$$\ell_\mu(-z) = 2^\mu \Gamma(\mu+1) b_\mu\left(\sqrt{-z}\right)(-z)^{1-(\mu/2)}. \tag{51}$$

Since

$$\begin{aligned}\ell_\mu(z) &= 2^\mu \Gamma(\mu+1) b_\mu\left(\sqrt{z}\right)(z)^{1-(\mu/2)} \\ &= \sum_{j\geq 0} \frac{(-1)^j \Gamma(1+\mu)}{4^j j! \Gamma(\mu+j+1)} z^{1+j} \\ &= z \prod_{j\geq 1}\left(1 - \frac{z}{b_{\mu,j}^2}\right),\end{aligned} \tag{52}$$

so we can write

$$\operatorname{Re} \frac{z\ell_\mu'(z)}{\ell_\mu(z)} = 1 - \frac{\mu}{2} + \operatorname{Re} \frac{1}{2\sqrt{z}} \frac{zb_\mu'\left(\sqrt{z}\right)}{b_\mu\left(\sqrt{z}\right)} = 1 - \operatorname{Re} \sum_{j\geq 1} \frac{z}{b_{\mu,j}^2 - z}. \tag{53}$$

This result shows that

$$\begin{aligned}\operatorname{Re} \frac{z\ell_\mu'(-z)}{\ell_\mu(-z)} &= -1 + \frac{\mu}{2} - \frac{1}{2\sqrt{-z}} \frac{-zb_\mu'\left(\sqrt{-z}\right)}{b_\mu\left(\sqrt{-z}\right)} \\ &= -1 + \sum_{j\geq 1} \frac{-z}{b_{\mu,j}^2 + z},\end{aligned} \tag{54}$$

or

$$\begin{aligned}\operatorname{Re} \frac{z\ell_\mu'(-z)}{\ell_\mu(-z)} &= -1 - \operatorname{Re} \sum_{j\geq 1} \frac{z}{b_{\mu,j}^2 + z} \\ &\leq -1 + \sum_{j\geq 1} \frac{|z|}{b_{\mu,j}^2 - |z|} \\ &= -\frac{|z|\ell_\mu'(|z|)}{\ell_\mu(|z|)} \\ &= -\operatorname{Re} \frac{z\ell_\mu'(z)}{\ell_\mu(z)}.\end{aligned} \tag{55}$$

The equality holds for $r = |z| = z$. The principle of minimum value for harmonic functions along with (7) shows that

$$\operatorname{Re} \frac{z\ell_\mu'(z)}{\ell_\mu(z)} \geq 1 - \sum_{j\geq 1} \frac{|z|}{b_{\mu,j}^2 - |z|} \tag{56}$$

is valid if and only if $|z| < b_{\mu,1}$, and $b_{\mu,1}$ is the minimum positive root of the equation

$$rb_\mu'(r) + (2-\mu) b_\mu(r) = 0. \tag{57}$$

Thus, we have

$$\begin{aligned}\frac{\ell_\mu'(-1)}{\ell_\mu(-1)} &= -1 + \sum_{j\geq 1} \frac{1}{1 - \left((-1)/b_{\mu,j}^2\right)} \frac{1}{b_{\mu,j}^2} \\ &= -1 - \sum_{j\geq 1} \frac{1}{b_{\mu,j}^2 + 1} \leq 0,\end{aligned} \tag{58}$$

$$\begin{aligned}\frac{\ell_\mu'(-z)}{\ell_\mu(-z)} &= -\frac{1}{z} - \sum_{j\geq 1} \frac{1}{b_{\mu,j}^2 + z} \text{ or } \frac{\partial}{\partial \mu}\left(\frac{\ell_\mu'(-z)}{\ell_\mu(-z)}\right) \\ &= -\sum_{j\geq 1} \frac{2b_{\mu,j}(\partial/\partial\mu) b_{\mu,j}}{\left(z + b_{\mu,j}^2\right)^2},\end{aligned} \tag{59}$$

or

$$\frac{\partial}{\partial \mu}\left(\frac{\ell_\mu'(-1)}{\ell_\mu(-1)}\right) = -\sum_{j\geq 1} \frac{2b_{\mu,j}(\partial/\partial\mu) b_{\mu,j}}{\left(1 + b_{\mu,j}^2\right)^2} \leq 0, \tag{60}$$

since $b_{\mu,j}' > 0$ on $(-1, \infty)$ for a fixed $j \in \mathbb{N}$. Thus, by using (58) and (60), we see that ℓ_μ satisfies (7) and by applying Lemma 1 and Lemma 2, we obtain that $\ell_\mu(-z) \in \mathcal{S}^*$ and decreasing on $(-1, \infty)$ and by considering (8), we can write

$$\begin{aligned}z\ell_\mu'(z) &= 2^\mu \Gamma(\mu+1)\left[\frac{zb_\mu'\left(\sqrt{z}\right)}{2\sqrt{z} z^{-1+(\mu/2)}} + \left(1 - \frac{\mu}{2}\right) \frac{b_\mu\left(\sqrt{z}\right)}{z^{-1+(\mu/2)}}\right] \\ &= \sum_{j\geq 0} \frac{(-1)^j (j+1) \Gamma(1+\mu)}{4^j j! \Gamma(\mu+j+1) z^{-j-1}}.\end{aligned} \tag{61}$$

We also write

$$\ell_\mu(z) - \ell_\mu(-z) = \sum_{j\geq 0}\left(1 - (-1)^{1+j}\right) \frac{(-1)^j \Gamma(1+\mu)}{4^j j! \Gamma(j+\mu+1)} z^{1+j}. \tag{62}$$

We can write

$$\begin{aligned}\frac{2z\ell_\mu'(z)}{\ell_\mu(z) - \ell_\mu(-z)} &= \frac{\sum_{j\geq 0}\left((j+1)(-1)^j \Gamma(1+\mu) z^{j+1}\right)/\left(4^j j! \Gamma(\mu+j+1)\right)}{\sum_{j\geq 0}\left(1 - (-1)^{1+j}\right)\left((-1)^j \Gamma(1+\mu) z^{j+1}\right)/\left(4^j j! \Gamma(1+j+\mu)\right)} \\ &= \frac{2}{1 + (-i)^\mu b_\mu\left(i\sqrt{z}\right)/b_\mu\left(\sqrt{z}\right)} \frac{z\ell_\mu'(z)}{\ell_\mu(z)}.\end{aligned} \tag{63}$$

Since

$$b_\mu\left(\sqrt{-z}\right) = \sum_{j=0}^{\infty} \frac{(-1)^j\left(\sqrt{-z}/2\right)^{\mu+2j}}{\Gamma(1+j)\Gamma(1+\mu+j)} = \sum_{j=0}^{\infty} \frac{(i)^\mu(-1)^{2j}}{\Gamma(j+1)\Gamma(1+\mu+j)}\left(\frac{\sqrt{z}}{2}\right)^{\mu+2j}, \tag{64}$$

so we have

$$1+\frac{b_\mu\left(i\sqrt{z}\right)}{b_\mu\left(\sqrt{z}\right)}(-i)^\mu = \frac{\sum_{j=0}^{\infty}\left((-1)^j\left(1+(-1)^j\right)/\Gamma(j+1)\Gamma(1+\mu+j)\right)\left(\sqrt{z}/2\right)^{\mu+2j}}{\sum_{j=0}^{\infty}\left((-1)^j/\Gamma(1+j)\Gamma(\mu+j+1)\right)\left(\sqrt{z}/2\right)^{\mu+2j}}. \tag{65}$$

From (63) along with (65), we see that

$$\begin{aligned}
\operatorname{Re}\frac{2z\ell_\mu'(z)}{\ell_\mu(z)-\ell_\mu(-z)} &= \operatorname{Re}\frac{z\ell_\mu'(z)}{\ell_\mu(z)}\left[1+\frac{b_\mu\left(i\sqrt{z}\right)}{b_\mu\left(\sqrt{z}\right)}(-i)^\mu\right]^{-1} \\
&= \operatorname{Re}\frac{z\ell_\mu'(z)}{\ell_\mu(z)}\left[1-\frac{b_\mu\left(i\sqrt{z}\right)(-i)^\mu}{b_\mu\left(\sqrt{z}\right)}+\cdots\right] \\
&= \operatorname{Re}\frac{z\ell_\mu'(z)}{\ell_\mu(z)}\frac{\sum_{j=0}^{\infty}\left((-1)^j/\Gamma(j+1)\Gamma(1+\mu+j)\right)\left(\sqrt{z}/2\right)^{\mu+2j}\left(1+(-1)^j\right)}{\sum_{j=0}^{\infty}\left((-1)^j/\Gamma(j+1)\Gamma(1+\mu+j)\right)\left(\sqrt{z}/2\right)^{\mu+2j}} \\
&\geq 1-\sum_{j\geq 1}\frac{2|z|}{b_{\mu,j}^2-|z|} = \frac{|z|\ell_\mu'(|z|)}{\ell_\mu(|z|)}.
\end{aligned} \tag{66}$$

As in [11], we observe that

$$\operatorname{Re}\frac{2z\ell_\mu'(z)}{\ell_\mu(z)-\ell_\mu(-z)} \geq \operatorname{Re}\frac{z\ell_\mu'(z)}{\ell_\mu(z)} \geq 1-\sum_{j\geq 1}\frac{2|z|}{b_{\mu,j}^2-|z|} = \frac{|z|\ell_\mu'(|z|)}{\ell_\mu(|z|)}. \tag{67}$$

For $1>\eta\geq 0$ and $\mu>-1$, $r_\eta^*(\ell_\mu)$ is the least positive zero of

$$zb_\mu'(z)+(2-2\eta-\mu)b_\mu(z)=0, \tag{68}$$

where $\ell_\mu(z)=2^\mu\Gamma(\mu+1)b_\mu\left(\sqrt{z}\right)(z)^{1-\mu/2}$. □

4. Conclusion

The class of Bessel functions is originated as a solution of the well-known Bessel differential equation. We studied the radius problems of starlike functions with symmetric points involving Bessel functions under some kind of normalized conditions. We used the Mittag-Leffler representation of Bessel functions and derived our main results.

Data Availability

There is no data available.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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