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Research Article

A Strong Convergence Theorem for a Finite Family of Bregman Demimetric Mappings in a Banach Space under a New Shrinking Projection Method

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In this paper, using a new shrinking projection method and new generalized k-demimetric mappings, we consider the strong convergence for finding a common point of the sets of zero points of maximal monotone mappings, common fixed points of a finite family of Bregman k-demimetric mappings, and common zero points of a finite family of Bregman inverse strongly monotone mappings in a reflexive Banach space. To the best of our knowledge, such a theorem for Bregman k-demimetric mapping is the first of its kind in a Banach space. This manuscript is online on arXiv by the link http://arxiv.org/abs/2107.13254.

1. Introduction

Let H be a Hilbert space and let C be a nonempty, closed, and convex subset of H. Let $T : C \longrightarrow H$ be a mapping. Then, we denote by $F(T)$ the set of fixed points of T. For a real number t with $0 \le t \le 1$, a mapping $U : C \longrightarrow H$ is said to be a t-strict pseudocontraction [1] if

$$\|Ux - Uy\|^2 \le \|x - y\|^2 + t\|x - Ux - (y - Uy)\|^2, \quad (1)$$

for all $x, y \in C$. In particular, if $t = 0$, then U is nonexpansive, i.e.,

$$\|Ux - Uy\| \le \|x - y\|, \quad \forall x, y \in C. \quad (2)$$

If U is a t-strict pseudocontraction and $F(U) \ne \varnothing$, then we get that, for $x \in C$ and $p \in F(U)$,

$$\|Ux - p\|^2 \le \|x - p\|^2 + t\|x - Ux\|^2. \quad (3)$$

From this inequality, we get that

$$\|Ux - x\|^2 + \|x - p\|^2 + 2\langle Ux - x, x - p\rangle \le \|x - p\|^2 + t\|x - Ux\|^2. \quad (4)$$

Then, we get that

$$2\langle x - Ux, x - p\rangle \ge (1 - t)\|x - Ux\|^2. \quad (5)$$

A mapping $U : C \longrightarrow H$ is said to be a generalized hybrid [2] if there exist real numbers α, β such that

$$\alpha\|Ux - Uy\|^2 + (1 - \alpha)\|x - Uy\|^2 \le \beta\|Ux - y\|^2 + (1 - \beta)\|x - y\|^2, \quad (6)$$

for all $x, y \in C$. Such a mapping U is said to be a (α, β)-generalized hybrid. The class of generalized hybrid mappings covers several well-known mappings. A $(1, 0)$-generalized hybrid mapping is nonexpansive. For $\alpha = 2$ and $\beta = 1$, it is nonspreading [3, 4], i.e.,

$$2\|Ux - Uy\|^2 \le \|Ux - y\|^2 + \|Uy - x\|^2, \quad \forall x, y \in C. \quad (7)$$

For $\alpha = 3/2$ and $\beta = 1/2$, it is also a hybrid [5], i.e.,

$$3\|Ux - Uy\|^2 \le \|x - y\|^2 + \|Ux - y\|^2 + \|Uy - x\|^2, \quad \forall x, y \in C. \quad (8)$$

In general, nonspreading mappings and hybrid mappings are not continuous (see [6]). If U is a generalized hybrid and $F(U) \neq \emptyset$, then we get that, for $x \in C$ and $p \in F(U)$,

$$\alpha\|p - Ux\|^2 + (1-\alpha)\|p - Ux\|^2 \le \beta\|p - x\|^2 + (1-\beta)\|p - x\|^2, \quad (9)$$

and hence, $\|Ux - p\|^2 \le \|x - p\|^2$. From this, we have that

$$2\langle x - p, x - Ux\rangle \ge \|x - Ux\|^2. \quad (10)$$

Let E be a smooth Banach space and let G be a maximal monotone mapping with $G^{-1}0 \neq \emptyset$. Then, for the metric resolvent J_λ of G for a positive number $\lambda > 0$, we obtain from [7, 8] that, for $x \in E$ and $p \in G^{-1}0 = F(J_\lambda)$,

$$\langle J_\lambda x - p, J(x - J_\lambda x)\rangle \ge 0. \quad (11)$$

Then, we get

$$\langle J_\lambda x - x + x - p, J(x - J_\lambda x)\rangle \ge 0, \quad (12)$$

and hence,

$$\langle x - p, J(x - J_\lambda x)\rangle \ge \|x - J_\lambda x\|^2, \quad (13)$$

where J is the duality mapping on E. Motivated by (5), (10), and (13), Takahashi [9] introduced a nonlinear mapping in a Banach space as follows: let C be a nonempty, closed, and convex subset of a smooth Banach space E and let η be a real number with $\eta \in (-\infty, 1)$. A mapping $U : C \longrightarrow E$ with $F(U) \neq \emptyset$ is said to be η-demimetric if, for $x \in C$ and $p \in F(U)$,

$$2\langle x - p, J(x - Ux)\rangle \ge (1-\eta)\|x - Ux\|^2. \quad (14)$$

According to this definition, we have that a t-strict pseudocontraction U with $F(U) \neq \emptyset$ is t-demimetric, an (α, β)-generalized hybrid mapping U with $F(U) \neq \emptyset$ is 0-demimetric, and the metric resolvent J_λ with $G^{-1}0 \neq \emptyset$ is (-1)-demimetric.

On the other hand, in 1967, Bregman [10] discovered an effective technique using the so-called Bregman distance function D_f in the process of designing and analyzing feasibility and optimization algorithms. This led to a growing area of research in which Bregman's technique is applied in various ways in order to design and analyze iterative algorithms for solving feasibility problems, equilibrium problems, fixed point problems for nonlinear mappings, and so on (see, e.g., [11, 12] and the references therein).

In 2010, Reich and Sabach [11] using the Bregman distance function D_f introduced the concept of Bregman strongly nonexpansive mappings and studied the convergence of two iterative algorithms for finding common fixed points of finitely many Bregman strongly nonexpansive operators in reflexive Banach spaces.

In this paper, motivated by Takahashi [13], we generalize k-demimetric mappings by the Bregman distance, and using a new shrinking projection method, we deal with the strong convergence for finding a common point of the sets of zero points of maximal monotone mappings, common fixed points of a finite family of Bregman k-demimetric mappings, and common zero points of a finite family of Bregman inverse strongly monotone mappings in a reflexive Banach space (see [14]).

2. Preliminaries

Let E be a reflexive real Banach space and C be a nonempty, closed, and convex subset of E. Throughout this paper, the dual space of E is denoted by E^*. The norm and duality pairing between E and E^* are, respectively, denoted by $\|.\|$ and $\langle .,. \rangle$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in E, and we denote the strong convergence of $\{x_n\}_{n\in\mathbb{N}}$ to $x \in E$ as $n \longrightarrow \infty$ by $x_n \longrightarrow x$ and the weak convergence by $x_n \rightharpoonup x$.

Throughout this paper, $f : E \longrightarrow (-\infty, +\infty]$ is a proper, lower semicontinuous, and convex function. We denote by $\mathrm{dom}\, f \coloneqq \{x \in E; f(x) < \infty\}$, the domain of f. The function f is said to be strongly coercive if $\lim_{\|x\| \longrightarrow \infty} f(x)/\|x\| = +\infty$. Let $x \in \mathrm{int}\ \mathrm{dom}\, f$, and the subdifferential of f at x is the convex mapping set $\partial f : E \longrightarrow 2^{E^*}$ defined by

$$\partial f(x) = \{\xi \in E^* : f(x) + \langle y - x, \xi\rangle \le f(y), \forall y \in E\}, \quad \forall x \in E, \quad (15)$$

and $f^* : E^* \longrightarrow (-\infty, +\infty]$ is the Fenchel conjugate of f defined by

$$f^*(\xi) = \sup\{\langle \xi, x\rangle - f(x) : x \in E\}. \quad (16)$$

It is well known that $\xi \in \partial f(x)$ is equivalent to

$$f(x) + f^*(\xi) = \langle x, \xi\rangle. \quad (17)$$

For any $x \in \mathrm{int}\ \mathrm{dom}\, f$ and $y \in E$, we denote by $f^\circ(x, y)$ the right-hand derivative of f at x in the direction y, that is,

$$f^\circ(x, y) \coloneqq \lim_{t \longrightarrow 0^+} \frac{f(x + ty) - f(x)}{t}. \quad (18)$$

The function f is called Gâteaux differentiable at x, if the limit in (18) exists for any $y \in E$. In this case, the gradient of f at x is the linear function ∇f which is defined by $\langle y, \nabla f(x)\rangle \coloneqq f^\circ(x, y)$ for any $y \in E$. The function f is said to be Gâteaux differentiable if it is Gâteaux differentiable at each $x \in \mathrm{int}\ \mathrm{dom}\, f$. The function f is said to be Fréchet differentiable at x, if the limit in (18) is attained uniformly in $\|y\| = 1$, for any $y \in E$. Finally, f is said to be uniformly

Fréchet differentiable on a subset C of E, if the limit in (18) is attained uniformly for $x \in C$ and $\|y\| = 1$.

Lemma 1 (see [11]). *If $f : E \longrightarrow \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of E, then f is uniformly continuous on bounded subsets of E and ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.*

Proposition 2 (see [15]). *Let $f : E \longrightarrow \mathbb{R}$ be a convex function which is bounded on bounded subsets of E. Then, the following assertions are equivalent:*

(i) f is strongly coercive and uniformly convex on bounded subsets of E

(ii) f^* is Fréchet differentiable, and ∇f^* is uniformly norm-to-norm continuous on bounded subsets of $\operatorname{dom} f^* = E^*$

Definition 3. The function f is said to be "Legendre" if it satisfies the following two conditions:

(L1): $\operatorname{int} \ \operatorname{dom} f \neq \emptyset$, and ∂f is single-valued on its domain.

(L2): $\operatorname{int} \ \operatorname{dom} f^* \neq \emptyset$, and ∂f^* is single-valued on its domain.

Because here the space E is assumed to be reflexive, we always have $(\partial f)^{-1} = \partial f^*$ ([16], p. 83). This fact, when combined with the conditions (L1) and (L2), implies the following equalities:

$$\begin{aligned} \nabla f &= (\nabla f^*)^{-1}, \\ \operatorname{ran} \nabla f &= \operatorname{dom} \nabla f^* = \operatorname{int} \ \operatorname{dom} f^*, \\ \operatorname{ran} \nabla f^* &= \operatorname{dom} \nabla f = \operatorname{int} \ \operatorname{dom} f. \end{aligned} \tag{19}$$

In addition, the conditions (L1) and (L2), in conjunction with Theorem 5.4 of [17], imply that the functions f and f^* are strictly convex on the interior of their respective domains and f is Legendre if and only if f^* is Legendre.

One important and interesting Legendre function is $(1/p)\|.\|^p, p \in (1, 2]$. When E is a uniformly convex and p-uniformly smooth Banach space with $p \in (1, 2]$, the generalized duality mapping $J_p : E \longrightarrow 2^{E^*}$ is defined by

$$J_p(x) = \left\{ j_p(x) \in E^* : \left\langle j_p(x), x \right\rangle = \|x\| . \left\| j_p(x) \right\|, \left\| j_p(x) \right\| = \|x\|^{p-1} \right\}. \tag{20}$$

In this case, the gradient ∇f of f coincides with the generalized duality mapping J_p of E, $\nabla f = J_p, p \in (1, 2]$. Several interesting examples of Legendre functions are presented in [17–19].

From now on, we always assume that the convex function $f : E \longrightarrow (0,+\infty]$ is Legendre.

Definition 4 (see [20]). Let $f : E \longrightarrow (-\infty,+\infty$ be a convex and Gâteaux differentiable function. The bifunction $D_f : \operatorname{dom} f \times \operatorname{int} \ \operatorname{dom} f \longrightarrow [0,+\infty)$ defined by

$$D_f(y, x) := f(y) - f(x) - \langle \nabla f(x), y - x \rangle, \tag{21}$$

is called the Bregman distance with respect to f.

It should be noted that D_f is not a distance in the usual sense of the term. Clearly, $D_f(x, x) = 0$, but $D_f(y, x) = 0$ may not imply $x = y$. In our case, when f is Legendre, this indeed holds ([17], Theorem 7.3 (vi), p. 642). In general, D_f satisfies the three-point identity

$$D_f(x, y) + D_f(y, z) - D_f(x, z) = \langle x - y, \nabla f(z) - \nabla f(y) \rangle, \tag{22}$$

and the four-point identity

$$D_f(x, y) + D_f(\omega, z) - D_f(x, z) - D_f(\omega, y) = \langle x - \omega, \nabla f(z) - \nabla f(y) \rangle, \tag{23}$$

for any $x, \omega \in \operatorname{dom} f$ and $y, z \in \operatorname{int} \ \operatorname{dom} f$. Over the last 30 years, Bregman distances have been studied by many researchers (see [17, 21–23]).

Let $f : E \longrightarrow (-\infty,+\infty]$ be a convex function on E which is Gâteaux differentiable on $\operatorname{int} \ \operatorname{dom} f$. The function f is said to be totally convex at a point $x \in \operatorname{int} \ \operatorname{dom} f$ if its modulus of total convexity at $x, \nu_f(x,.)$: $[0,+\infty) \longrightarrow [0,+\infty]$, defined by

$$\nu_f(x, t) = \inf \left\{ D_f(y, x) \colon y \in \operatorname{dom} f, \|y - x\| = t \right\}, \tag{24}$$

is positive whenever $t > 0$. The function f is said to be totally convex when it is totally convex at every point of $\operatorname{int} \ \operatorname{dom} f$. The function f is said to be totally convex on bounded sets, if for any nonempty bounded set $B \subseteq E$, the modulus of total convexity of f on $B, \nu_f(B, t)$ is positive for any $t > 0$, where $\nu_f(B,.)$: $[0,+\infty) \longrightarrow [0,+\infty]$ is defined by

$$\nu_f(B, t) = \inf \left\{ \nu_f(x, t) \colon x \in B \cap \operatorname{int} \ \operatorname{dom} f \right\}. \tag{25}$$

We remark in passing that f is totally convex on bounded sets if and only if f is uniformly convex on bounded sets (see [24, 25]).

Proposition 5 (see [24]). *Let $f : E \longrightarrow (-\infty,+\infty]$ be a convex function that its domain contains at least two points. If f is lower semicontinuous, then f is totally convex on bounded sets if and only if f is uniformly convex on bounded sets.*

Lemma 6 (see [11]). *If $x \in \operatorname{int} \ \operatorname{dom} f$, then the following statements are equivalent:*

(i) The function f is totally convex at x

(ii) For any sequence $\{y_n\} \subset \operatorname{dom} f$,

$$\lim_{n\longrightarrow+\infty} D_f(y_n, x) = 0 \Rightarrow \lim_{n\longrightarrow+\infty} \|y_n - x\| = 0. \quad (26)$$

Recall that the function f is called sequentially consistent [25], if for any two sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ in E such that $\{x_n\}_{n\in\mathbb{N}}$ is bounded,

$$\lim_{n\longrightarrow+\infty} D_f(y_n, x_n) = 0 \Rightarrow \lim_{n\longrightarrow+\infty} \|y_n - x_n\| = 0. \quad (27)$$

Lemma 7 (see [14]). *If* dom f *contains at least two points, then the function* f *is totally convex on bounded sets if and only if the function* f *is sequentially consistent.*

Lemma 8 (see [26]). *Let* $f : E \longrightarrow \mathbb{R}$ *be a Gâteaux differentiable and totally convex function. If* $x_1 \in E$ *and the sequence* $\{D_f(x_n, x_1)\}$ *is bounded, then the sequence* $\{x_n\}$ *is also bounded.*

Lemma 9 (see [12]). *Let* $f : E \longrightarrow \mathbb{R}$ *be a Legendre function such that* ∇f^* *is bounded on bounded subsets of* int dom f^*. *Let* $x_1 \in E$, *and if* $\{D_f(x_1, x_n)\}$ *is bounded, then the sequence* $\{x_n\}$ *is bounded too.*

Recall that the Bregman projection [10] with respect to f of $x \in \text{int dom} f$ onto a nonempty, closed, and convex set $C \subseteq \text{int dom} f$ is the unique vector $\text{Proj}_C^f(x) \in C$ satisfying

$$D_f\left(\text{Proj}_C^f(x), x\right) = \inf\left\{D_f(y, x) : y \in C\right\}. \quad (28)$$

Similar to the metric projection in Hilbert spaces, the Bregman projection with respect to totally convex and Gâteaux differentiable functions has a variational characterization ([25], corollary 4.4, p. 23).

Lemma 10 (see [25]). *Suppose that* f *is Gâteaux differentiable and totally convex on* int dom f. *Let* $x \in \text{int dom} f$ *and* $C \subseteq \text{int dom} f$ *be a nonempty, closed, and convex set. Then, the following Bregman projection conditions are equivalent:*

(i) $z_0 = \text{Proj}_C^f(x)$

(ii) $z = z_0$ *is the unique solution of the following variational inequality:*

$$\langle z - y, \nabla f(x) - \nabla f(z)\rangle \geq 0, \quad \forall y \in C. \quad (29)$$

(iii) $z = z_0$ *is the unique solution of the following variational inequality:*

$$D_f(y, z) + D_f(z, x) \leq D_f(y, x), \quad \forall y \in C. \quad (30)$$

Let E be a real Banach space and C be a nonempty subset of E. An element $p \in C$ is called a fixed point of a single-valued mapping $T : C \longrightarrow C$, if $p = Tp$. The set of fixed points of T is denoted by $F(T)$.

A point $x \in C$ is called an asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges weakly to x and $\lim_{n\longrightarrow+\infty}\|x_n - Tx_n\| = 0$. We denote the asymptotic fixed points of T by $\tilde{F}(T)$.

Let C be a nonempty, closed, and convex subset of int dom f and $T : C \longrightarrow C$ be a mapping. Now, T is said to be Bregman quasi-nonexpansive, if $F(T) \neq \emptyset$ and

$$D_f(p, Tx) \leq D_f(p, x), \quad \forall x \in C, p \in F(T). \quad (31)$$

Let C be a nonempty, closed, and convex subset of int dom f. An operator $T : C \longrightarrow \text{int dom} f$ is said to be Bregman strongly nonexpansive with respect to a nonempty $\tilde{F}(T)$, if

$$D_f(y, Tx) \leq D_f(y, x), \quad \forall x \in C, y \in \tilde{F}(T), \quad (32)$$

and for any bounded sequence $\{x_n\} \subseteq C$ with

$$\lim_{n\longrightarrow\infty}\left(D_f(y, x_n) - D_f(y, Tx_n)\right) = 0, \quad (33)$$

it follows that

$$\lim_{n\longrightarrow\infty}\left(D_f Tx_n, x_n\right) = 0. \quad (34)$$

A mapping $B : E \longrightarrow 2^{E^*}$ is called Bregman inverse strongly monotone on the set C, if $C \cap (\text{int dom} f) \neq \emptyset$, and for any $x, y \in C \cap (\text{int dom} f)$, $\xi \in Bx$, and $\eta \in By$, we have that

$$\langle \xi - \eta, \nabla f^*(\nabla f(x) - \xi) - \nabla f^*(\nabla f(y) - \eta)\rangle \geq 0. \quad (35)$$

Let $B : E \longrightarrow 2^{E^*}$ be a mapping. Then, the mapping defined by

$$B_\lambda^f := \nabla f^* \circ (\nabla f - \lambda B) : E \longrightarrow E, \quad (36)$$

is called an antiresolvent associated with B and λ for any $\lambda > 0$.

Suppose that A is a mapping of E into 2^{E^*} for the real reflexive Banach space E. The effective domain of A is denoted by dom (A), that is, dom $(A) = \{x \in E : Ax \neq \emptyset\}$. A multivalued mapping A on E is said to be monotone if $\langle x - y, u^* - v^*\rangle \geq 0$ for all $x, y \in \text{dom}(A)$, $u^* \in Ax$, and $v^* \in Ay$. A monotone operator A on E is said to be maximal if graph A, the graph of A, is not a proper subset of the graph of any monotone operator on E.

Let E be a real reflexive Banach space, $f : E \longrightarrow (-\infty, +\infty]$ uniformly Fréchet differentiable and bounded on bounded subsets of E. Then for any $\lambda > 0$, the resolvent of A defined by

$$\text{Res}_A^f(x) = (\nabla f + \lambda A)^{-1} \circ \nabla f(x), \quad (37)$$

is a single-valued Bregman quasi-nonexpansive mapping from E onto dom (A) and $F(\text{Res}_A^f) = A^{-1}0$. We denote by

$A_\lambda = (1/\lambda)(\nabla f - \nabla f(\mathrm{Res}_A^f))$ the Yosida approximation of A for any $\lambda > 0$. We get from [26] (prop. 2.7, p. 10) that

$$A_\lambda(x) \in A\left(\mathrm{Res}_A^f(x)\right), \quad \forall x \in E, \lambda > 0. \tag{38}$$

See [11], too.

Lemma 11 (see [27]). *Let E be a real reflexive Banach space and $f : E \longrightarrow (-\infty,+\infty]$ be a Legendre function which is totally convex on bounded subsets of E. Also, let C be a nonempty, closed, and convex subset of* int dom f *and $T : C \longrightarrow 2^C$ be a multivalued Bregman quasi-nonexpansive mapping. Then, the fixed point set $F(T)$ of T is a closed and convex subset of C.*

Lemma 12 (see[28]). *Assume that $f : E \longrightarrow \mathbb{R}$ is a Legendre function which is uniformly Fréchet differentiable and bounded on bounded subsets of E. Let C be a nonempty, closed, and convex subset of E. Also, let $\{T_i : i = 1, \cdots, N\}$ be N Bregman strongly nonexpansive mappings which satisfy $\tilde{F}(T_i) = F(T_i)$ for each $1 \le i \le N$ and let $T = T_N T_{N-1} \cdots T_1$. If $F(T)$ and $\bigcap_{i=1}^N F(T_i)$ are nonempty, then T is also Bregman strongly nonexpansive with $F(T) = \tilde{F}(T)$.*

Lemma 13 (see [29]). *Let $G : E \longrightarrow 2^{E^*}$ be a maximal monotone operator and $B : E \longrightarrow E^*$ be a Bregman inverse strongly monotone mapping such that $(G+B)^{-1}(0^*) \neq \emptyset$. Also, let $f : E \longrightarrow \mathbb{R}$ be a Legendre function which is uniformly Fréchet differentiable and bounded on bounded subsets of E. Then,*

(i) $(G+B)^{-1}(0^*) = F(Res_{\lambda G}^f \circ B_\lambda^f)$

(ii) $Res_{\lambda G}^f \circ B_\lambda^f$ *is a Bregman strongly nonexpansive mapping such that*

$$F\left(Res_{\lambda G}^f \circ B_\lambda^f\right) = \tilde{F}\left(Res_{\lambda G}^f \circ B_\lambda^f\right). \tag{39}$$

(iii) $D_f(u, Res_{\lambda G}^f \circ B_\lambda^f(x)) + D_f(Res_{\lambda G}^f \circ B_\lambda^f(x), x) \le D_f(u, x)$, $\forall u \in (G+B)^{-1}(0^*)$, $x \in E$, *and* $\lambda > 0$

Lemma 14 (see [30]). *Let $f : E \longrightarrow (-\infty,+\infty]$ be a proper convex and lower semicontinuous Legendre function. Then, for any $z \in E$, for any $\{x_n\} \subseteq E$ and $\{t_i\}_{i=1}^N \subseteq (0,1)$ with $\sum_{i=1}^N t_i = 1$, the following holds*

$$D_f\left(z, \nabla f^*\left(\sum_{i=1}^N t_i \nabla f(x_i)\right)\right) \le \sum_{i=1}^N t_i D_f(z, x_i). \tag{40}$$

Proposition 15 (see [26], prop. 2.8, p. 10). *Let f be Gâteaux differentiable and $A : E \longrightarrow 2^{E^*}$ be a maximal monotone operator such that $A^{-1}0 \neq \emptyset$. Then*

$$D_f(q,x) \ge D_f\left(q, Res_{rA}^f(x)\right) + D_f\left(Res_{rA}^f(x), x\right), \tag{41}$$

for all $r > 0$, $q \in A^{-1}0$, and $x \in E$.

Next, we generalize the k-demimetric notation introduced in [15].

Definition 16. Let E be a reflexive Banach space, $f : E \longrightarrow (-\infty,+\infty]$ be a Legendre function which is Gâteaux differentiable, C be a nonempty, closed and convex subset of int dom f and let $k \in (-\infty,1)$. A mapping $T : C \longrightarrow$ int dom f with $F(T) \neq \emptyset$ is said to be Bregman k-demimetric, if for $x \in C$ and $q \in F(T)$,

$$\langle x - q, \nabla f(x) - \nabla f T(x) \rangle \ge (1-k) D_f(x, T(x)). \tag{42}$$

Example 1. Every Bregman quasi-nonexpansive mapping with the required conditions in Definition 16 is a Bregman 0-demimetric mapping. Let $p \in F(T) \neq \emptyset$ and $x \in C$, and we have

$$\begin{aligned} D_f(p, Tx) \le D_f(p, x) \Rightarrow 0 &\le D_f(p, x) - D_f(p, Tx) \\ &= D_f(p, x) + D_f(x, Tx) - D_f(p, Tx) - D_f(x, Tx) \\ &= \langle p - x, \nabla f T(x) - \nabla f(x) \rangle - D_f(x, Tx). \end{aligned} \tag{43}$$

Therefore,

$$\langle x - p, \nabla f(x) - \nabla f T(x) \rangle \ge D_f(x, Tx). \tag{44}$$

Example 2. From [31] (Lemma 2.1), every Bregman quasi-strictly pseudocontractive mapping with the required conditions is a Bregman k-demimetric mapping for $k \in [0,1)$.

Example 3. Let E be a reflexive Banach space, $f : E \longrightarrow (-\infty,+\infty]$ be a Legendre function which is Gâteaux differentiable, and $A : E \longrightarrow 2^{E^*}$ be a maximal monotone operator with $A^{-1}0 \neq \emptyset$ and $r > 0$. Then, the f-resolvent Res_{rA}^f is Bregman 0-demimetric. In fact, from (22) and Proposition 15, we have that

$$\begin{aligned} D_f(q,x) &\ge D_f\left(q, \mathrm{Res}_{rA}^f(x)\right) + D_f\left(\mathrm{Res}_{rA}^f(x), x\right) \\ &= D_f(q,x) + \left\langle q - \mathrm{Res}_{rA}^f(x), \nabla f(x) - \nabla f\left(\mathrm{Res}_{rA}^f(x)\right)\right\rangle, \end{aligned} \tag{45}$$

for any $x \in E$ and $q \in A^{-1}0$. Then, we obtain

$$\left\langle \mathrm{Res}_{rA}^f(x) - q, \nabla f(x) - \nabla f\left(\mathrm{Res}_{rA}^f(x)\right)\right\rangle \ge 0. \tag{46}$$

Therefore,

$$\left\langle \mathrm{Res}_{rA}^f(x) - x + x - q, \nabla f(x) - \nabla f\left(\mathrm{Res}_{rA}^f(x)\right)\right\rangle \ge 0, \tag{47}$$

and hence, from (22), we have that

$$\left\langle x-q,\nabla f(x)-\nabla f\left(\mathrm{Res}_{rA}^{f}(x)\right)\right\rangle \geq \left\langle x-\mathrm{Res}_{rA}^{f}(x),\nabla f(x)-\nabla f\left(\mathrm{Res}_{rA}^{f}(x)\right)\right\rangle = D_f\left(x,\mathrm{Res}_{rA}^{f}(x)\right)+D_f\left(\mathrm{Res}_{rA}^{f}(x),x\right). \tag{48}$$

Thus,

$$\left\langle x-q,\nabla f(x)-\nabla f\left(\mathrm{Res}_{rA}^{f}(x)\right)\right\rangle \geq D_f\left(x,\mathrm{Res}_{rA}^{f}(x)\right), \tag{49}$$

and then, we get that Res_{rA}^{f} is Bregman 0-demimetric.

3. Main Results

The following lemma is important and crucial in the proof of Theorem 18.

Lemma 17. *Let E be a reflexive Banach space and $f : E \longrightarrow (-\infty,+\infty]$ be a Legendre function which is Gâteaux differentiable, and let C be a nonempty, closed, and convex subset of* int dom f *and let k be a real number with $k\in(-\infty,1)$ and let T be a Bregman k-demimetric mapping of C into* int dom f. *Then, $F(T)$ is closed and convex.*

Proof. First, we show that $F(T)$ is closed. Consider a sequence $\{q_n\}$ such that $q_n\longrightarrow q$ and $q_n\in F(T)$. We conclude from the definition of T that

$$\langle q-q_n,\nabla f(q)-\nabla f\mathrm{T}(q)\rangle \geq (1-k)D_f(q,T(q)). \tag{50}$$

Since $q_n\longrightarrow q$, we have $0\geq(1-k)D_f(q,T(q))$. Then, from $1-k>0$, we have that $D_f(q,T(q))=0$, and hence, $q=T(q)$, and therefore, $q\in F(T)$. This implies that $F(T)$ is closed.

Next, we show that $F(T)$ is convex. Suppose $p,q\in F(T)$ and set $z=\alpha p+(1-\alpha)q$, where $\alpha\in[0,1]$. Then, we have that

$$\begin{aligned}\langle z-p,\nabla f(z)-\nabla f\mathrm{T}(z)\rangle &\geq (1-k)D_f(z,T(z)),\\ \langle z-q,\nabla f(z)-\nabla f\mathrm{T}(z)\rangle &\geq (1-k)D_f(z,T(z)).\end{aligned} \tag{51}$$

Thus, from $\alpha\geq 0$ and $1-\alpha\geq 0$, we have that

$$\langle \alpha z-\alpha p,\nabla f(z)-\nabla f\mathrm{T}(z)\rangle \geq \alpha(1-k)D_f(z,T(z)),$$

$$\langle (1-\alpha)z-(1-\alpha)q,\nabla f(z)-\nabla f\mathrm{T}(z)\rangle \geq (1-\alpha)(1-k)D_f(z,T(z)). \tag{52}$$

From these inequalities, we get that

$$0=\langle z-z,\nabla f(z)-\nabla f\mathrm{T}(z)\rangle \geq (1-k)D_f(z,T(z)). \tag{53}$$

Thus, we get that $D_f(z,T(z))=0$; hence, $z=Tz$; therefore, $z\in F(T)$. We conclude that $F(T)$ is convex. □

Theorem 18. *Let E be a real reflexive Banach space. Suppose that $f:E\longrightarrow\mathbb{R}$ is a proper, convex, lower semicontinuous, strongly coercive, Legendre function which is bounded on bounded subsets of E, uniformly Fréchet differentiable, and totally convex on bounded subsets of E. Let C be a nonempty, closed, and convex subset of* int dom f. *Let $\{k_1,k_2,\cdots,k_M\}\subseteq(-\infty,1)$ and $\{T_j\}_{j=1}^{M}$ be a finite family of Bregman k_j-demimetric and Bregman quasi-nonexpansive and demiclosed mappings of C into itself. Suppose that $\{B_i\}_{i=1}^{N}$ is a finite family of Bregman inverse strongly monotone mappings of C into E and $\{B_{i,\eta_n}^{f}\}_{i=1}^{N}$ is the family of antiresolvent mappings of $\{B_i\}_{i=1}^{N}$. Let $A:E\longrightarrow 2^{E^*}$ and $G:E\longrightarrow 2^{E^*}$ be maximal monotone mappings on E and let $Q_\eta=Res_{\eta G}^{f}=(\nabla f+\eta G)^{-1}\nabla f$ and $J_r=Res_{rA}^{f}=(\nabla f+rA)^{-1}\nabla f$ be the resolvents of G and A for $\eta>0$ and $r>0$, respectively. Assume that*

$$\Omega=A^{-1}0\cap\left(\cap_{j=1}^{M}F(T_j)\right)\cap\left(\cap_{i=1}^{N}(B_i+G)^{-1}0^*\right)\neq\varnothing. \tag{54}$$

For $x_1\in C$ and $C_1=Q_1=C$, let $\{x_n\}$ be a sequence defined by

$$\begin{cases} y_n=\nabla f^*\left(\sum_{j=1}^{M}\xi_j\left((1-\lambda_n)\nabla f+\lambda_n\nabla fT_j\right)x_n\right),\\ z_n=\nabla f^*\sum_{i=1}^{N}\sigma_i\nabla fQ_{\eta_n}B_{i,\eta_n}^{f}(y_n),\\ u_n=J_{r_n}z_n,\\ C_{n+1}=\left\{z\in C_n : D_f(z,y_n)\leq D_f(z,x_n), D_f(z,z_n)\leq D_f(z,y_n) \text{ and } \langle z_n-z,\nabla f(z_n)-\nabla f(u_n)\rangle\geq D_f(z_n,u_n)\right\},\\ x_{n+1}=\mathrm{Proj}_{C_{n+1}\cap Q_n}^{f}(x_1),\quad \forall n\in\mathbb{N},\\ Q_{n+1}=\{z\in Q_n : \langle x_{n+1}-z,\nabla f(x_1)-\nabla f(x_{n+1})\rangle\geq 0\},\end{cases} \tag{55}$$

where $\{\lambda_n\} \subseteq (0,1)$, $\{\eta_n\}$, $\{r_n\} \subseteq (0,+\infty)$, $\{\xi_1, \xi_2, \cdots, \xi_M\}$, $\{\sigma_1, \sigma_2, \cdots, \sigma_N\} \subseteq (0,1)$, *and* $a, b, c \in \mathbb{R}$ *satisfy the following:*

(1) $0 < a \leq \lambda_n$, $\forall n \in \mathbb{N}$

(2) $0 < c \leq r_n$, $\forall n \in \mathbb{N}$

(3) $\sum_{j=1}^{M} \xi_j = 1$ *and* $\sum_{i=1}^{N} \sigma_i = 1$

Then, $\{x_n\}$ *converges strongly to a point* $\omega_0 \in \Omega$ *where* $\omega_0 = Proj_{\Omega}^{f} x_1$.

Proof. We divide the proof into several steps:

(Step 1) First, we prove that Ω is a closed and convex subset of C.

Since $\{T_j\}_{j=1}^{M}$ is a finite family of k_j-Bregman demimetric mappings, by Lemma 17 and the condition $\Omega \neq \varnothing$, $F(T_j)$ is nonempty, closed, and convex for $1 \leq j \leq M$. Also, it follows from Lemma 13 (i)–(ii) that $(B_i + G)^{-1} 0^* = F(Q_{\eta_n} B_{i,\eta_n}^f)$ and $Q_{\eta_n} B_{i,\eta_n}^f$ is a Bregman strongly nonexpansive mapping, and therefore, from Lemma 12, we have that

$$F\left(Q_{\eta_n} B_{i,\eta_n}^f\right) = \tilde{F}\left(Q_{\eta_n} B_{i,\eta_n}^f\right). \tag{56}$$

Thus, $\{Q_{\eta_n} B_{i,\eta_n}^f\}$ is a family of Bregman quasi-nonexpansive mappings. Using $\Omega \neq \varnothing$ and Lemma 11, we see that $(B_i + G)^{-1} 0^* = F(Q_{\eta_n} B_{i,\eta_n}^f)$ is a nonempty, closed, and convex set. We also know that $A^{-1}0$ is closed and convex. Then, Ω is nonempty, closed, and convex. Therefore, $\mathrm{Proj}_{\Omega}^{f}$ is well defined.

(Step 2) We prove that C_n and Q_n are closed and convex subsets of C and $\Omega \subseteq C_{n+1} \cap Q_n$, $\forall n \in \mathbb{N}$.

In fact, it is clear that $C_1 = C$ is closed and convex. Suppose that C_k is closed and convex for some $k \geq 1$. Note that

$$\begin{aligned} D_f(z, y_k) \leq D_f(z, x_k) &\Leftrightarrow f(z) - f(y_k) - \langle \nabla f(y_k), z - y_k \rangle \\ &\leq f(z) - f(x_k) - \langle \nabla f(x_k), z - x_k \rangle \\ &\Leftrightarrow \langle \nabla f(x_k), z - x_k \rangle - \langle \nabla f(y_k), z - y_k \rangle \\ &\leq f(y_k) - f(x_k). \end{aligned} \tag{57}$$

Similarly, we have that

$$\begin{aligned} D_f(z, z_k) \leq D_f(z, y_k) &\Leftrightarrow \langle \nabla f(y_k), z - y_k \rangle - \langle \nabla f(z_k), z - z_k \rangle \\ &\leq f(z_k) - f(y_k). \end{aligned} \tag{58}$$

Thus, from the fact that $D_f(.,x)$ is continuous for each fixed x and using the above inequalities $\{z \in C_k : D_f(z, y_k) \leq D_f(z, x_k)\}$ and $\{z \in C_k : D_f(z, z_k) \leq D_f(z, y_k)\}$which are closed and convex. We have also $\{z \in C_k : \langle z_k - z, \nabla f(z_k) - \nabla f(u_k) \rangle \geq D_f(z_k, u_k)\}$ which is closed and convex. Therefore, C_{k+1} is closed and convex. From mathematical induction, we have that C_n is a closed and convex subset in C with $C_{n+1} \subseteq C_n$ for all $n \in \mathbb{N}$.

Also, it is clear that $Q_1 = C$ is closed and convex. Suppose that Q_k is closed and convex for some $k \geq 1$. Hence, $\{z \in Q_k : \langle x_{k+1} - z, \nabla f(x_1) - \nabla f(x_{k+1}) \rangle \geq 0\}$ is closed and convex; i.e., Q_{k+1} is a closed and convex subset of Q_k. Therefore, by mathematical induction, Q_n is a closed and convex subset of C with $Q_{n+1} \subseteq Q_n$ for all $n \in \mathbb{N}$. Next, we show that $\Omega \subseteq C_n$ for all $n \geq 1$. Clearly, $\Omega \subseteq C_1 = C$. Assume that $\Omega \subseteq C_k$ for some $k \in \mathbb{N}$. Note from Lemma 14 that

$$\begin{aligned} D_f(z, y_k) &= D_f\left(z, \nabla f^* \sum_{j=1}^{M} \xi_j \left((1 - \lambda_k) \nabla f + \lambda_k \nabla f T_j\right) x_k\right) \\ &= D_f\left(z, \nabla f^* \sum_{j=1}^{M} \xi_j \nabla f \nabla f^* \left((1 - \lambda_k) \nabla f + \lambda_k \nabla f T_j\right) x_k\right) \\ &\leq \sum_{j=1}^{M} \xi_j D_f\left(z, \nabla f^* \left((1 - \lambda_k) \nabla f + \lambda_k \nabla f T_j\right) x_k\right) \\ &\leq \sum_{j=1}^{M} \xi_j \left((1 - \lambda_k) D_f(z, x_k) + \lambda_k D_f\left(z, T_j x_k\right)\right) \\ &\leq \sum_{j=1}^{M} \xi_j \left((1 - \lambda_k) D_f(z, x_k) + \lambda_k D_f(z, x_k)\right) = D_f(z, x_k), \end{aligned} \tag{59}$$

for all $z \in \Omega$. Furthermore, since B_i is a Bregman inverse strongly monotone mapping for all $1 \leq i \leq N$ and hence from Lemmas 13 and 14, we have that

$$\begin{aligned} D_f(z, z_k) &= D_f\left(z, \nabla f^* \sum_{i=1}^{N} \sigma_i \nabla f Q_{\eta_k} B_{i,\eta_k}^f y_k\right) \\ &\leq \sum_{i=1}^{N} \sigma_i D_f\left(z, Q_{\eta_k} B_{i,\eta_k}^f y_k\right) \leq \sum_{i=1}^{N} \sigma_i D_f(z, y_k) = D_f(z, y_k), \end{aligned} \tag{60}$$

for all $z \in \Omega$. Also, since J_{r_k} is the resolvent of A and $u_k = J_{r_k} z_k$, we have from (22) and Proposition 15 that

$$\begin{aligned} \langle z_k - z, \nabla f(z_k) - \nabla f(u_k) \rangle &= D_f(z, z_k) + D_f(z_k, u_k) - D_f(z, u_k) \\ &= D_f(z, z_k) + D_f(z_k, u_k) - D_f\left(z, J_{r_k} z_k\right) \\ &\geq D_f(z, z_k) + D_f(z_k, u_k) - D_f(z, z_k) \\ &= D_f(z_k, u_k), \end{aligned} \tag{61}$$

for all $z \in \Omega$. From (59)–(61), we have that $\Omega \subseteq C_{k+1}$. Therefore, we have by mathematical induction that $\Omega \subseteq C_n$ for all $n \in \mathbb{N}$.

Now, we shall show that $\Omega \subseteq Q_n$ for all $n \in \mathbb{N}$. Note that $\Omega \subseteq Q_1 = C$. Assume that $\Omega \subseteq Q_k$ for some $k \in \mathbb{N}$. Thus, Ω

$\subseteq C_{k+1} \cap Q_k$ for some $k \in \mathbb{N}$. From $x_{k+1} = \mathrm{Proj}^f_{C_{k+1}\cap Q_k}(x_1)$ and Lemma 10, we have that

$$\langle x_{k+1} - z, \nabla f(x_1) - \nabla f(x_{k+1}) \rangle \geq 0, \quad \forall z \in C_{k+1} \cap Q_k. \tag{62}$$

Since $\Omega \subseteq C_{k+1} \cap Q_k$, we have that

$$\langle x_{k+1} - z, \nabla f(x_1) - \nabla f(x_{k+1}) \rangle \geq 0, \quad \forall z \in \Omega. \tag{63}$$

Then, we get $\Omega \subseteq Q_{k+1}$. By mathematical induction, we have that $\Omega \subseteq Q_n$ for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well defined.

(Step 3) We show that $\lim_{n\longrightarrow\infty} D_f(x_n, x_1)$ exists.

Since Ω is nonempty, closed, and convex, there exists a $\omega_0 \in \Omega$ such that $\omega_0 = \mathrm{Proj}^f_{\Omega}(x_1)$. From $x_{n+1} = \mathrm{Proj}^f_{C_{n+1}\cap Q_n}(x_1)$, we get that

$$D_f(x_{n+1}, x_1) \leq D_f(z, x_1), \tag{64}$$

for all $z \in C_{n+1} \cap Q_n$. From $\omega_0 \in \Omega \subseteq C_{n+1} \cap Q_n$, we obtain that

$$D_f(x_{n+1}, x_1) \leq D_f(\omega_0, x_1). \tag{65}$$

This shows that $\{D_f(x_n, x_1)\}$ is a bounded sequence. By Lemma 10 (iii), we have that

$$\begin{aligned} 0 \leq D_f(x_{n+1}, x_n) &= D_f\left(x_{n+1}, \mathrm{Proj}^f_{C_n\cap Q_{n-1}}(x_1)\right) \\ &\leq D_f(x_{n+1}, x_1) - D_f\left(\mathrm{Proj}^f_{C_n\cap Q_{n-1}}(x_1), x_1\right) \\ &= D_f(x_{n+1}, x_1) - D_f(x_n, x_1), \end{aligned} \tag{66}$$

for all $n > 1$. Therefore,

$$D_f(x_n, x_1) \leq D_f(x_{n+1}, x_1). \tag{67}$$

This implies that $\{D_f(x_n, x_1)\}$ is bounded and nondecreasing. Then, $\lim_{n\longrightarrow\infty} D_f(x_n, x_1)$ exists. In view of Lemma 8, we deduce that the sequence $\{x_n\}$ is bounded. Also, from (66), we have that

$$\lim_{n\longrightarrow\infty} D_f(x_{n+1}, x_n) = 0. \tag{68}$$

Since the function f is totally convex on bounded sets, by Lemma 7, we have that

$$\lim_{n\longrightarrow\infty} \|x_{n+1} - x_n\| = 0. \tag{69}$$

(Step 4) We prove that $\{x_n\}$ is a Cauchy sequence in C.

We have $C_m \subseteq C_n$ and $Q_m \subseteq Q_n$ for any $m, n \in \mathbb{N}$ with $m \geq n$. From $x_n = \mathrm{Proj}^f_{C_n\cap Q_{n-1}}(x_1) \in C_n \cap Q_{n-1}$ and Lemma 10, we have that

$$\begin{aligned} D_f(x_m, x_n) &= D_f\left(x_m, \mathrm{Proj}^f_{C_n\cap Q_{n-1}}(x_1)\right) \\ &\leq D_f(x_m, x_1) - D_f\left(\mathrm{Proj}^f_{C_n\cap Q_{n-1}}(x_1), x_1\right) \\ &= D_f(x_m, x_1) - D_f(x_n, x_1). \end{aligned} \tag{70}$$

Letting $m, n \longrightarrow \infty$ in (70), we deduce that $D_f(x_m, x_n) \longrightarrow 0$. In view of Lemma 7, since the function f is totally convex on bounded sets, we get that $\|x_m - x_n\| \longrightarrow 0$ as m, $n \longrightarrow \infty$. Thus, $\{x_n\}$ is a Cauchy sequence. Since E is a Banach space and C is closed and convex, we conclude that there exists $\bar{x} \in C$ such that

$$\lim_{n\longrightarrow\infty} \|x_n - \bar{x}\| = 0. \tag{71}$$

(Step 5) We prove that $\lim_{n\longrightarrow\infty} \|\nabla f(x_n) - \nabla f(y_n)\| = 0$.

Using (68) and from $x_{n+1} \in C_{n+1}$, we get that

$$D_f(x_{n+1}, y_n) \leq D_f(x_{n+1}, x_n) \longrightarrow 0 \quad (\text{as } n \longrightarrow \infty). \tag{72}$$

Then, $\lim_{n\longrightarrow\infty} D_f(x_{n+1}, y_n) = 0$. Since the function f is totally convex on bounded sets and $\{y_n\}$ is bounded, by Lemma 7, we have that

$$\lim_{n\longrightarrow\infty} \|x_{n+1} - y_n\| = 0. \tag{73}$$

Using (69) and (73), we have that

$$\|x_n - y_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \longrightarrow 0 \quad (\text{as } n \longrightarrow \infty). \tag{74}$$

Then,

$$\lim_{n\longrightarrow\infty} \|x_n - y_n\| = 0. \tag{75}$$

Since from Lemma 1, ∇f is uniformly continuous, we have that

$$\lim_{n\longrightarrow\infty} \|\nabla f(x_n) - \nabla f(y_n)\| = 0. \tag{76}$$

(Step 6) We prove that $\lim_{n\longrightarrow\infty} \|\nabla f(y_n) - \nabla f(z_n)\| = 0$.

Using (72) and from $x_{n+1} \in C_{n+1}$, we get that

$$D_f(x_{n+1}, z_n) \leq D_f(x_{n+1}, y_n) \longrightarrow 0 \quad (as\, n \longrightarrow \infty). \tag{77}$$

Then, $\lim_{n\longrightarrow\infty} D_f(x_{n+1}, z_n) = 0$. Since function f is totally convex on bounded sets and $\{z_n\}$ is bounded, by Lemma 7, we have that

$$\lim_{n\longrightarrow\infty}\|x_{n+1}-z_n\|=0. \quad (78)$$

Using (69) and (78), we have that

$$\lim_{n\longrightarrow\infty}\|x_n-z_n\|=0. \quad (79)$$

Applying (75) and (79), we get that

$$\lim_{n\longrightarrow\infty}\|y_n-z_n\|=0. \quad (80)$$

Since ∇f is uniformly continuous, we have that

$$\lim_{n\longrightarrow\infty}\|\nabla f(y_n)-\nabla f(z_n)\|=0. \quad (81)$$

(Step 7) We prove that $\lim_{n\longrightarrow\infty}\|\nabla f(z_n)-\nabla f(u_n)\|=0$.

Using $x_{n+1}\in C_{n+1}$ and from (22), we get that

$$\begin{aligned}D_f(z_n,u_n)&\le\langle z_n-x_{n+1},\nabla f(z_n)-\nabla f(u_n)\rangle\\&=D_f(x_{n+1},z_n)+D_f(z_n,u_n)-D_f(x_{n+1},u_n).\end{aligned} \quad (82)$$

Then,

$$0\le D_f(x_{n+1},z_n)-D_f(x_{n+1},u_n). \quad (83)$$

Therefore, from (77), we have that

$$D_f(x_{n+1},u_n)\le D_f(x_{n+1},z_n)\longrightarrow 0\quad(\text{as } n\longrightarrow\infty), \quad (84)$$

and then, $\lim_{n\longrightarrow\infty}D_f(x_{n+1},u_n)=0$. Since the function f is totally convex on bounded sets and from Proposition 15 and Lemma 9, $\{u_n\}$ is bounded; hence, by Lemma 7, we have that

$$\lim_{n\longrightarrow\infty}\|x_{n+1}-u_n\|=0. \quad (85)$$

Using (69) and (85), we conclude that

$$\lim_{n\longrightarrow\infty}\|x_n-u_n\|=0. \quad (86)$$

Now, by (79) and (86), we get that

$$\lim_{n\longrightarrow\infty}\|z_n-u_n\|=0. \quad (87)$$

Since from Lemma 1, ∇f is uniformly continuous, we have that

$$\lim_{n\longrightarrow\infty}\|\nabla f(z_n)-\nabla f(u_n)\|=0. \quad (88)$$

(Step 8) We prove that $\bar{x}\in\Omega$.

Since T_j is Bregman k_j-demimetric for all $1\le j\le M$, we get that

$$\begin{aligned}&\langle x_n-z,\nabla f(x_n)-\nabla f(y_n)\rangle\\&=\left\langle x_n-z,\nabla f(x_n)-\nabla f\nabla f^*\left(\sum_{j=1}^{M}\xi_j\left((1-\lambda_n)\nabla f+\lambda_n\nabla fT_j\right)x_n\right)\right\rangle\\&=\sum_{j=1}^{M}\xi_j\langle x_n-z,\nabla f(x_n)-\left((1-\lambda_n)\nabla f+\lambda_n\nabla fT_j\right)x_n\rangle\\&=\sum_{j=1}^{M}\xi_j\lambda_n\langle x_n-z,\nabla f(x_n)-\nabla fT_j(x_n)\rangle\\&\ge\sum_{j=1}^{M}\xi_j\lambda_n(1-k)D_f\left(x_n,T_jx_n\right)\ge\sum_{j=1}^{M}\xi_ja(1-k)D_f\left(x_n,T_jx_n\right),\end{aligned} \quad (89)$$

for all $z\in\cap_{j=1}^{M}F(T_j)$. We have from (76) that

$$\lim_{n\longrightarrow\infty}D_f\left(x_n,T_jx_n\right)=0, \quad (90)$$

for all $1\le j\le M$. Since the function f is totally convex on bounded sets, by Lemma 7, we have that

$$\lim_{n\longrightarrow\infty}\left\|x_n-T_jx_n\right\|=0,\quad\forall j\in\{1,\cdots,M\}. \quad (91)$$

Since T_j is demiclosed for all $1\le j\le M$ and from (71) where$x_n\longrightarrow\bar{x}$ as $n\longrightarrow\infty$, we have that $\bar{x}\in\cap_{j=1}^{M}F(T_j)$. We now show that $\bar{x}\in\cap_{i=1}^{N}(B_i+G)^{-1}0^*$. From (22) and Lemma 13 (iii), we get that

$$\begin{aligned}&\langle y_n-z,\nabla f(y_n)-\nabla f(z_n)\rangle\\&=\left\langle y_n-z,\nabla f(y_n)-\nabla f\nabla f^*\sum_{i=1}^{N}\sigma_i\nabla fQ_{\eta_n}B^f_{i,\eta_n}(y_n)\right\rangle\\&=\sum_{i=1}^{N}\sigma_i\left\langle y_n-z,\nabla f(y_n)-\nabla fQ_{\eta_n}B^f_{i,\eta_n}(y_n)\right\rangle\\&=\sum_{i=1}^{N}\sigma_i\left(D_f(z,y_n)+D_f\left(y_n,Q_{\eta_n}B^f_{i,\eta_n}y_n\right)-D_f\left(z,Q_{\eta_n}B^f_{i,\eta_n}y_n\right)\right)\\&\ge\sum_{i=1}^{N}\sigma_i\left(D_f(z,y_n)+D_f\left(y_n,Q_{\eta_n}B^f_{i,\eta_n}y_n\right)-D_f(z,y_n)\right)\\&=\sum_{i=1}^{N}\sigma_i\left(D_f\left(y_n,Q_{\eta_n}B^f_{i,\eta_n}y_n\right)\right),\end{aligned} \quad (92)$$

for all $z\in\cap_{i=1}^{N}(B_i+G)^{-1}0^*$ and $i\in\{1,\cdots,N\}$. Using (81), from the above, we have that

$$\lim_{n\longrightarrow\infty}D_f\left(y_n,Q_{\eta_n}B^f_{i,\eta_n}y_n\right)=0, \quad (93)$$

for all $1\le i\le N$. Since the function f is totally convex on bounded sets, by Lemma 7, we conclude that

$$\lim_{n\longrightarrow\infty}\left\|y_n - Q_{\eta_n}B^f_{i,\eta_n}y_n\right\| = 0. \tag{94}$$

On the other hand, it follows from (71) and (75) that

$$\lim_{n\longrightarrow\infty}\|y_n - \bar{x}\| = 0. \tag{95}$$

Now, from (94) and (95), we have that $\bar{x}\in\tilde{F}(Q_{\eta_n}B_{i,\eta_n})$ for all $i\in\{1,\cdots,N\}$. From Lemma 13, we conclude that

$$\tilde{F}\left(Q_{\eta_n}B_{i,\eta_n}\right) = F\left(Q_{\eta_n}B_{i,\eta_n}\right) = (B_i + G)^{-1}0^*, \quad \forall i\in\{1,\cdots,N\}, \tag{96}$$

and therefore, $\bar{x}\in\cap_{i=1}^{N}(B_i+G)^{-1}0^*$.

We now show that $\bar{x}\in A^{-1}0$. Using $r_n\geq c$, we have from (88) that

$$\lim_{n\longrightarrow\infty}\frac{1}{r_n}\|\nabla f(z_n)-\nabla f(u_n)\| = 0. \tag{97}$$

So applying A_{r_n}, the Yosida approximation of A, we have

$$\lim_{n\longrightarrow\infty}\left\|A_{r_n}z_n\right\| = \lim_{n\longrightarrow\infty}\frac{1}{r_n}\|\nabla f(z_n)-\nabla f(u_n)\| = 0. \tag{98}$$

Since $A_{r_n}z_n\in Au_n$, for $(p,p^*)\in A$, we have from the monotonicity of A that $\langle p-u_n, p^*-A_{r_n}z_n\rangle\geq 0$ for all $n\in\mathbb{N}$. By (71) and (79), we have that

$$\lim_{n\longrightarrow\infty}\|z_n - \bar{x}\| = 0. \tag{99}$$

From (87) and the above, we have $\|u_n-\bar{x}\|\longrightarrow 0$. Thus, we get $\langle p-\bar{x}, p^*\rangle\geq 0$. From the maximality of A, we have that $\bar{x}\in A^{-1}0$. Therefore, $\bar{x}\in\Omega$.

Since $\omega_0 = \mathrm{Proj}^f_{\Omega}(x_1)$, $\bar{x}\in\Omega$, and $\|x_n-\bar{x}\|\longrightarrow 0$, we have from (65) that

$$\begin{aligned} D_f(\omega_0, x_1) &\leq D_f(\bar{x}, x_1) = \lim_{n\longrightarrow\infty} D_f(x_n, x_1) \\ &\leq \lim_{n\longrightarrow\infty} D_f(\omega_0, x_1) = D_f(\omega_0, x_1), \end{aligned} \tag{100}$$

which implies that

$$\lim_{n\longrightarrow\infty} D_f(x_n, x_1) = D_f(\omega_0, x_1). \tag{101}$$

Therefore,

$$D_f(\bar{x}, x_1) = D_f(\omega_0, x_1). \tag{102}$$

From (55), (71), and $\omega_0\in\Omega\subseteq Q_{n+1}$ for all $n\in\mathbb{N}$ and also since ∇f is uniformly continuous on bounded subsets and is bounded on bounded subsets, we have that

$$\begin{aligned} &\langle x_{n+1}-\omega_0, \nabla f(x_1)-\nabla f(x_{n+1})\rangle \\ &\quad \geq 0 \Rightarrow \lim_{n\longrightarrow\infty}\langle x_{n+1}-\omega_0, \nabla f(x_1)-\nabla f(x_{n+1})\rangle \\ &\quad \geq 0 \Rightarrow \langle \bar{x}-\omega_0, \nabla f(x_1)-\nabla f(\bar{x})\rangle \geq 0. \end{aligned} \tag{103}$$

Now, from (22) and (102), we get that

$$\begin{aligned} D_f(\omega_0, \bar{x}) &= D_f(\omega_0, \bar{x}) + D_f(\bar{x}, x_1) - D_f(\omega_0, x_1) \\ &= \langle \omega_0 - \bar{x}, \nabla f(x_1)-\nabla f(\bar{x})\rangle. \end{aligned} \tag{104}$$

From (103) and the above, we conclude that $D_f(\omega_0,\bar{x})\leq 0$, and therefore, $D_f(\omega_0,\bar{x}) = 0$. Thus, $\omega_0 = \bar{x}$, and hence, $x_n\longrightarrow\omega_0$. This completes the proof. □

Next, we prove a proposition to extend Theorem 18.

Proposition 19. *Let E be a real reflexive Banach space. Suppose that $f: E\longrightarrow\mathbb{R}$ is a proper, convex, lower semicontinuous, Legendre function and is Gâteaux differentiable of E. Let C be a nonempty, closed, and convex subset of* int dom f. *Suppose $k\in(-\infty,0]$ and a mapping $T: C\longrightarrow C$ with $F(T)\neq\varnothing$ is Bregman k-demimetric. Then, T is a Bregman quasi-nonexpansive mapping.*

Proof. Let $p\in F(T)$ and $x\in C$. Using (22) and Definition 16, we have

$$\begin{aligned} D_f(p, Tx) - D_f(p, x) &= D_f(x, Tx) - D_f(p, x) - D_f(x, Tx) + D_f(p, Tx) \\ &= D_f(x, Tx) - \langle p - x, \nabla f(Tx)-\nabla f(x)\rangle \\ &\leq D_f(x, Tx) - (1-k)D_f(x, Tx) \\ &= kD_f(x, Tx)\leq 0. \end{aligned} \tag{105}$$

Then, T is a Bregman quasi-nonexpansive mapping. □

Remark 20. Using Proposition 19, for each $1\leq j\leq M$, we may remove the condition that T_j is Bregman quasi-nonexpansive in Theorem 18 when $k_j\leq 0$.

Open Problem 1. Can one generalize Proposition 19 to Bregman k-demimetric mappings with $k<1$?

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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