Research Article

Dual Spaces of Multiparameter Local Hardy Spaces

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In this paper, we study the duality theory of the multiparameter local Hardy spaces $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, and we prove that $(h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}))^* = \text{cmo}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, where $\text{cmo}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ are defined by discrete Carleson measure. Moreover, we discuss the relationship among $\text{cmo}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, $\text{Lip}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, and rectangle $\text{cmo}_{\text{rect}}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$.

1. Introduction

The classical theory of one-parameter harmonic analysis may be considered as centering around the Hardy-Littlewood maximal operator and its relationship with certain singular integral operators which commute with the usual one-parameter dilations on $\mathbb{R}^m$, given by $\delta(x) = (\delta x_1, \dots, \delta x_m)$, $\delta > 0$. If this isotropic dilation is replaced by more general nonisotropic groups of dilations, then many nonisotropic variants of the classical theories can be produced, such as the strong maximal functions and multiparameter singular integral operators, corresponding to the multiparameter dilations $\delta : x \rightarrow (\delta_1 x_1, \delta_2 x_2)$, $x = (x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^m$, and $\delta = (\delta_1, \delta_2)$, $\delta_1, \delta_2 > 0$. Such a multiparameter theory has been developed extensively over the past decades. We refer the reader to the work in [1–18].

Since $H^p(\mathbb{R}^n)$ space is well suited only to the pure Fourier analysis and not stable under multiplication by test functions which associated with PDE, in [19], Goldberg introduced the class of localizable Hardy spaces $h^p(\mathbb{R}^n)$, $0 < p < \infty$. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with $\int \varphi \neq 0$ and

$$m_\varphi(f)(x) = \sup_{0 < t < 1} |\varphi_t * f(x)|. \quad (1)$$

Then, $h^p(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n), m_\varphi(f) \in L^p(\mathbb{R}^n)\}$, where the symbol $\mathcal{S}(\mathbb{R}^n)$ has its usual meaning, the Schwartz class

of rapidly decreasing test functions, and $\mathcal{S}'(\mathbb{R}^n)$ is the dual space of $\mathcal{S}(\mathbb{R}^n)$. Goldberg proved that $\mathcal{S}(\mathbb{R}^n)$ is dense in $h^p(\mathbb{R}^n)$ and $h^p(\mathbb{R}^n)$ is preserved by pseudodifferential operators. Similar as Hardy space $H^p(\mathbb{R}^n)$, local Hardy space $h^p(\mathbb{R}^n)$ can be characterized by Poisson maximal function, Poisson area integral integral [19], and local Riesz transforms [20]. In 2001, Rychkov in [21] pointed out that this space has its continuous Littlewood-Paley characterization. Recently, its discrete Littlewood-Paley characterization has also been obtained in [22].

Motivated by those, multiparameter local Hardy spaces $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ were discussed in [23, 24]. To be precise, let φ_0^i be functions defined on $\mathbb{R}^{n_i}$ satisfying

$$\begin{aligned} \varphi_0^i &\in \mathcal{D}(\mathbb{R}^{n_i}), \\ \int \varphi_0^i &= 1, \end{aligned} \quad (2)$$

where $\mathcal{D}(\mathbb{R}^n)$ denotes the set of all smooth functions with compact support on $\mathbb{R}^n$. Then, the multiparameter local Hardy spaces $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ are introduced in [23].

Definition 1. Let $0 < p < \infty$. For $i = 1, 2$, suppose that $\varphi_0^{(i)}$ satisfies condition (2), and $\varphi^{(i)}(x) = \varphi_0^{(i)}(x) - 2^{-n_i} \varphi_0^{(i)}(x/2)$. Then, the multiparameter local Hardy space $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ is defined to be the set of $f \in \mathcal{S}'(\mathbb{R}^{n_1+n_2})$ such that

$$\left\| \left(\sum_{j,k \in \mathbb{N}} |\varphi_{j,k} * f(\cdot)|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^{n_1+n_2})} < \infty. \quad (3)$$

In this paper, for $j, k \in \mathbb{N}, j \geq 1$, set $\psi_j^{(1)}(x) = 2^{jn_1} \psi^{(1)}(2^j x)$, $\psi_k^{(2)}(x) = 2^{kn_2} \psi^{(2)}(2^k x)$, $\psi_{j,k}(x, y) = \psi_j^{(1)}(x) \psi_k^{(2)}(y)$, $\psi_{0,k}(x, y) = \psi_0^{(1)}(x) \psi_k^{(2)}(y)$, $\psi_{j,0}(x, y) = \psi_j^{(1)}(x) \psi_0^{(2)}(y)$, and $\psi_{0,0}(x, y) = \psi_0^{(1)}(x) \psi_0^{(2)}(y)$. For $i = 1, 2$ and any $j \in \mathbb{Z}$, denote $\Pi_j^{n_i} = \{I : I \text{ are dyadic cubes in } \mathbb{R}^{n_i} \text{ with the side length } l(I) = 2^{-j}, \text{ and the left lower corners of } I \text{ are } x_I = 2^{-j} \ell, \ell \in \mathbb{Z}^{n_i}\}$, $\Pi_{j,k} = \Pi_j^{n_1} \times \Pi_k^{n_2}$, and $\Pi = \bigcup_{j,k \in \mathbb{N}} \Pi_{j,k}$, $\tilde{\Pi} = \bigcup_{j,k \in \mathbb{Z}} \Pi_{j,k}$.

Similar to the classical one-parameter local Hardy space theory, it is proved in [23] that $\mathcal{S}(\mathbb{R}^{n_1+n_2})$ is dense in $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ and $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) = L^p(\mathbb{R}^{n_1+n_2})$, if $p > 1$. Recently, we pointed out in [24] that this multiparameter local Hardy space can also be characterized by a discrete Littlewood-Paley norm. More precisely, for $i = 1, 2$, let $\psi_0^{(i)} \in \mathcal{S}(\mathbb{R}^{n_i})$ with

$$\text{supp } \widehat{\psi_0^{(i)}} \subseteq \{\xi \in \mathbb{R}^{n_i} : |\xi| \leq 2\}; \quad \widehat{\psi_0^{(i)}}(\xi) = 1, \quad \text{if } |\xi| \leq 1, \quad (4)$$

$$\text{supp } \widehat{\psi^{(i)}} \subseteq \left\{ \xi \in \mathbb{R}^{n_i} : \frac{1}{2} \leq |\xi| \leq 2 \right\}, \quad (5)$$

$$\left| \psi_0^{(i)} \wedge(\xi) \right|^2 + \sum_{j=1}^{\infty} \left| \psi^{(i)} \wedge(2^{-j} \xi) \right|^2 = 1, \quad \text{for all } \xi \in \mathbb{R}^{n_i}. \quad (6)$$

Then, the following discrete multiparameter local Calderón identity is proved in [24]. For some other discrete Calderón's identities, we refer the readers to [25–27].

Theorem 2. For $i = 1, 2$, suppose that $\psi_0^{(i)}$ and $\psi^{(i)} \in \mathcal{S}(\mathbb{R}^{n_i})$ are functions satisfying conditions (4)-(6), respectively. Then,

$$f(x, y) = \sum_{j=0}^{+\infty} \sum_{k=0}^{+\infty} \sum_{I \times J \in \Pi_j^{n_1} \times \Pi_k^{n_2}} |I||J| \left(\psi_{j,k} * f \right)(x_I, x_J) \times \psi_{j,k}(x - x_I, y - x_J), \quad (7)$$

where the series converges in $L^2(\mathbb{R}^{n_1+n_2})$, $\mathcal{S}(\mathbb{R}^{n_1+n_2})$, and $\mathcal{S}'(\mathbb{R}^{n_1+n_2})$.

With the above discrete multiparameter local Calderón identity, multiparameter local Hardy spaces $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ can also be defined by discrete Littlewood-Paley norm [24]; that is, for every $f \in \mathcal{S}'(\mathbb{R}^{n_1+n_2})$, the $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ norm is equivalent to

$$\left\| \left(\sum_{j,k \in \mathbb{N}} \sum_{I \times J \in \Pi_j^{n_1} \times \Pi_k^{n_2}} \left| \psi_{j,k} * f(x_I, x_J) \right|^2 \chi_I(\cdot) \chi_J(\cdot) \right)^{1/2} \right\|_{L^p(\mathbb{R}^{n_1+n_2})}. \quad (8)$$

The multiparameter Hardy space was introduced by Gundy and Stein in the 1970s in [11] and was developed by Chang and Fefferman in [1, 3]. Moreover, Chang and Fefferman in [1, 28] characterized duality of the product $H^1(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ by the product Carleson measure. The generalized Carleson measure space CMO^p was first introduced in [29]. It has been proved in [30] that the duality of multiparameter Hardy space $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ is $\text{CMO}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. We refer the readers to [26, 31] for the recent development of generalized Carleson measure spaces.

The main goal of this paper is to identify the duality of $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ with a new product Carleson measure space $\text{cmo}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$.

Definition 3. Let $0 < p \leq 1$. For $i = 1, 2$, suppose that $\psi_0^{(i)}$ and $\psi^{(i)} \in \mathcal{S}(\mathbb{R}^{n_i})$ are functions satisfying conditions (4)-(6), respectively. The multiparameter local $\text{cmo}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ is defined to be the set of $f \in \mathcal{S}'(\mathbb{R}^{n_1+n_2})$ such that

$$\|f\|_{\text{cmo}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} = \sup_{\Omega} \left(\frac{1}{|\Omega|^{(2/p)-1}} \int_{\Omega} \sum_{j,k \in \mathbb{N}} \sum_{I \times J \in \Pi_{j,k}, R \subseteq \Omega} \left| \psi_{j,k} * f(x_I, x_J) \right|^2 \chi_R(x) dx \right)^{1/2} < \infty, \quad (9)$$

where Ω ranges over all open sets in $\mathbb{R}^{n_1+n_2}$ with finite measures.

To see that the space $\text{cmo}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ is well defined, one needs to show the following theorem.

Theorem 4. Let $0 < p \leq 1$. Suppose that both $\psi_{j,k}$ and $\varphi_{j,k}$ satisfy the same conditions in Definition 3. Then, one has

$$\begin{aligned} & \sup_{\Omega} \left(\frac{1}{|\Omega|^{(2/p)-1}} \int_{\Omega} \sum_{j,k \in \mathbb{N}} \sum_{I \times J \in \Pi_{j,k}, R \subseteq \Omega} \left| \psi_{j,k} * f(x_I, x_J) \right|^2 \chi_R(x) dx \right)^{1/2} \\ & \approx \sup_{\Omega} \left(\frac{1}{|\Omega|^{(2/p)-1}} \int_{\Omega} \sum_{j,k \in \mathbb{N}} \sum_{I \times J \in \Pi_{j,k}, R \subseteq \Omega} \left| \varphi_{j,k} * f(x_I, x_J) \right|^2 \chi_R(x) dx \right)^{1/2} \end{aligned} \quad (10)$$

for all $f \in \mathcal{S}'(\mathbb{R}^{n_1+n_2})$.

This result can be obtained if we prove the following Theorem 5. We will give a direct proof in Section 2.

Theorem 5. Let $0 < p \leq 1$. Then,

$$(h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}))^* = \text{cmo}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}). \quad (11)$$

Namely, if $g \in \text{cmo}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, the map l_g , given by $l_g(f) = \langle f, g \rangle$, defined initially for $f \in \mathcal{S}(\mathbb{R}^{n_1+n_2})$, extends to a continuous linear functional on $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ with $\|l_g\| \leq \|g\|_{\text{cmo}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})}$. Conversely, every $l \in (h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}))^*$ satisfies $l = l_g$ for some $g \in \text{cmo}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ with $\|l_g\| \approx \|g\|_{\text{cmo}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})}$.

Certainly, the dual spaces of one-parameter local Hardy spaces $h^p(\mathbb{R}^n)$ can be also characterized by the above form, namely, $\text{cmo}^p(\mathbb{R}^n)$. Details can be seen in [22]. On the other hand, in [19], Goldberg showed that the duality of $h^1(\mathbb{R}^n)$ is $\text{bmo}(\mathbb{R}^n)$, which is defined as the set of $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ such that

$$\begin{aligned} S_1(f) &= \sup_{|Q| < 1} \frac{1}{|Q|} \int_Q |f(x) - f_Q| dx < \infty, \\ S_2(f) &= \sup_{|Q| \geq 1} \frac{1}{|Q|} \int_Q |f(x)| dx < \infty, \end{aligned} \quad (12)$$

equipped with the norm $\|f\|_{\text{bmo}(\mathbb{R}^n)} = \max \{S_1(f), S_2(f)\}$, where f_Q is the mean of f over Q , i.e., $f_Q = (1/|Q|) \int_Q f(x) dx$. For $0 < r < 1$, let $\Lambda_r = \{f \in L^\infty(\mathbb{R}^n) : \sup_{x \neq y} |f(x) - f(y)| / |x - y|^r < \infty\}$. Then, the duality of $h^p(\mathbb{R}^n)$ is $\Lambda_{n((1/p)-1)}$ for $n/(n+1) < p < 1$. We refer the reader to [19] for more details of Λ_r when $r \geq 1$. It is convenient to denote $\Lambda_0 = \text{bmo}(\mathbb{R}^n)$. Since $\text{cmo}^p(\mathbb{R}^n)$ and $\Lambda_{n((1/p)-1)}$ are all dualities of $h^p(\mathbb{R}^n)$ for $n/(n+1) < p \leq 1$, $\text{cmo}^p(\mathbb{R}^n)$ should be coincident with $\Lambda_{n((1/p)-1)}$. In [32], we give a direct proof to identify them with equivalent norms. Moreover, we proved that, for $0 < p < 1$, $\text{cmo}^p(\mathbb{R}^n)$ is coincident with the local Lipschitz space $\text{Lip}^p(\mathbb{R}^n)$ defined by

$$\text{Lip}^p(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{\text{Lip}^p(\mathbb{R}^n)} < \infty \right\}, \quad (13)$$

where

$$\|f\|_{\text{Lip}^p(\mathbb{R}^n)} = \sup_{j \geq 0, x \in \mathbb{R}^n} 2^{jn((1/p)-1)} \left| \psi_j * f(x) \right|. \quad (14)$$

A natural question, does this result hold in multiparameter setting? First, we introduce the following multiparameter local Lipschitz spaces.

Definition 6. Let $0 < p \leq 1$. Suppose that $\psi_{j,k}$ satisfy the conditions in Definition 3. The multiparameter local Lipschitz space $\text{Lip}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ is defined by

$$\text{Lip}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) = \left\{ f \in \mathcal{S}'(\mathbb{R}^{n_1+n_2}) : \|f\|_{\text{Lip}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} < \infty \right\}, \quad (15)$$

where

$$\|f\|_{\text{Lip}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} = \sup_{j,k \geq 0, x \in \mathbb{R}^{n_1+n_2}} 2^{jn_1((1/p)-1)} 2^{kn_2((1/p)-1)} \left| \psi_{j,k} * f(x) \right|. \quad (16)$$

Some results about Lipschitz spaces associated with mixed homogeneities can be seen in [33]. In this paper, one of our main results is to discuss the relationship among $\text{Lip}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, $\text{cmo}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, and the following multiparameter local rectangle $\text{cmo}^p_{\text{rect}}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$.

Definition 7. Let $0 < p \leq 1$. For $i = 1, 2$, suppose that $\psi_0^{(i)}$ and $\psi^{(i)} \in \mathcal{S}(\mathbb{R}^{n_i})$ are functions satisfying conditions (4)-(6), respectively. The multiparameter local rectangle $\text{cmo}^p_{\text{rect}}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ is defined as the set of $f \in \mathcal{S}'(\mathbb{R}^{n_1+n_2})$ such that

$$\|f\|_{\text{cmo}^p_{\text{rect}}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} = \sup_{R_0} \left(\frac{1}{|R_0|^{(2/p)-1}} \int_{R_0} \sum_{j,k \in \mathbb{N}} \sum_{R \in \Pi_{j,k}, R \subseteq R_0} \left(\left| \psi_{j,k} * f(x_I, x_J) \right|^2 \chi_R(x) \right) dx \right)^{1/2} < \infty, \quad (17)$$

where the supremum is taken over all possible dyadic rectangles $R_0 \in \tilde{\Pi}$.

It is easy to see that, for $0 < p \leq 1$, $\text{cmo}^p_{\text{rect}}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \supseteq \text{cmo}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. Moreover, due to Carleson, there is a well-known counterexample to show that this inclusion is true inclusion when $p = 1$ [4, 34].

Theorem 8. Let $0 < p < 1$. Then, for every $f \in \mathcal{S}'(\mathbb{R}^{n_1+n_2})$, one has

$$\|f\|_{\text{cmo}^p_{\text{rect}}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \lesssim \|f\|_{\text{Lip}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \lesssim \|f\|_{\text{cmo}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})}. \quad (18)$$

One may ask whether $\|f\|_{\text{cmo}^p_{\text{rect}}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \approx \|f\|_{\text{Lip}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})}$ or $\|f\|_{\text{cmo}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \approx \|f\|_{\text{Lip}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})}$ is true. It would be an interesting question.

The organization of this paper is as follows. In Section 2, we introduce the multiparameter ψ -transform S_ψ and its inverse ψ -transform T_ψ . These transforms correspond between $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ ($\text{cmo}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$) and sequence spaces. Using these ψ -transform S_ψ , we prove Theorem 4. In Section 3, we establish the duality of sequence spaces. The proof of Theorem 5 is placed in Section 4. In the last section, we prove Theorem 8.

Finally, we make some conventions. Throughout the paper, C denotes a positive constant that is independent of the main parameters involved, but whose value may vary from line to line. Constants with subscript, such as C_1 , do not change in different occurrences. We denote $f \leq Cg$ by $f \lesssim g$. If $f \lesssim g \lesssim f$, we write $f \approx g$.

2. Multiparameter ψ -Transform

In order to prove the duality theorems, following the idea of Frazier and Jawerth in [35], we first define and study the corresponding sequence spaces. For any $R \in \Pi_{j,k}$, setting $\psi_R(x_1, x_2) = |R|^{1/2} \psi_{j,k}(x_1 - x_I, x_2 - x_J)$, then by (7), it is easy to have

$$f(x) = \sum_{R \in \Pi} \langle f, \psi_R \rangle \psi_R(x). \quad (19)$$

Definition 9. The multiparameter ψ -transform S_ψ is the map taking $f \in \mathcal{S}'(\mathbb{R}^{n_1+n_2})$ to the sequence $S_\psi f = \{(\langle f, \psi_R \rangle)_R\}_{R \in \Pi}$, where $(S_\psi f)_R = \langle f, \psi_R \rangle$. Define the inverse multiparameter

ψ -transform T_ψ as the map taking a sequence $s = \{s_R\}_{R \in \Pi}$ to $T_\psi s = \sum_{R \in \Pi} s_R \psi_R(x)$.

By (19), for $f \in \mathcal{S}(\mathbb{R}^{n_1+n_2})$ and $g \in \mathcal{S}'(\mathbb{R}^{n_1+n_2})$, one has

$$\langle f, g \rangle = \left\langle \sum_{R \in \Pi} (S_\psi f)_R \psi_R(x), g \right\rangle = \langle S_\psi f, S_\psi g \rangle. \quad (20)$$

For a sequence $s = s_R$, one also has the following identity:

$$\langle S_\psi f, s \rangle = \sum_{R \in \Pi} \langle f, \psi_R \rangle s_R = \left\langle f, \sum_{R \in \Pi} s_R \psi_R \right\rangle = \langle f, T_\psi s \rangle. \quad (21)$$

Now, we define two discrete sequence spaces corresponding to $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ and $\text{cmo}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, respectively.

Definition 10. Let $0 < p \leq 1$. Sequence space $\mathfrak{h}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ is defined to be the collection of all complex-valued sequences $s = \{s_R\}_{R \in \Pi}$ such that

$$\|s\|_{\mathfrak{h}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} = \left\| \left(\sum_{R \in \Pi} (|s_R| \tilde{\chi}_R(\cdot))^2 \right)^{1/2} \right\|_{L^p} < \infty, \quad (22)$$

where $\tilde{\chi}_R(x) = |R|^{-1/2} \chi_R(x)$.

Definition 11. For $0 < p \leq 1$, define $\mathfrak{G}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ to be the collection of all complex-valued sequences $t = \{t_R\}_{R \in \Pi}$ such that

$$\|t\|_{\mathfrak{G}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} = \sup_{\Omega} \left(\frac{1}{|\Omega|^{(2/p)-1}} \int_{\Omega} \sum_{R \subseteq \Omega, R \in \Pi} (|t_R| \tilde{\chi}_R(x))^2 dx \right)^{1/2}, \quad (23)$$

where Ω ranges over all open sets in $\mathbb{R}^{n_1+n_2}$ with finite measures, and $\tilde{\chi}_R(x)$ is defined as in Definition 10.

Then, we have the following generalization of the fundamental result of Theorem 2.2 in [35].

Theorem 12. Let $0 < p < \infty$. For $i = 1, 2$, suppose that $\psi_0^{(i)}$ and $\psi^{(i)} \in \mathcal{S}(\mathbb{R}^{n_i})$ satisfy conditions (4)-(6), respectively. Then, the operators $S_\psi : h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \rightarrow \mathfrak{h}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ and $T_\psi : \mathfrak{h}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \rightarrow h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ are all bounded; moreover, $T_\psi \circ S_\psi$ is the identity on $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$.

Proof. The boundedness of S_ψ is obvious since

$$\|S_\psi(f)\|_{\mathfrak{h}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \approx \|f\|_{h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \quad (24)$$

by Definition 10 and (8).

We now prove the boundedness of T_ψ . Similar proofs can be seen in [25, 31]. For a sequence $s = \{s_R\}_{R \in \Pi}$, denote

$f(x) = T_\psi s = \sum_{R \in \Pi} s_R \psi_R(x)$. Then, by almost orthogonality estimates [36], for any positive integers N, L , there exists a constant C such that

$$\begin{aligned} & \left| \psi_{j',k'} * \psi_{j,k}(x_{I'} - x_I, x_{J'} - x_J) \right| \\ & \leq C \frac{2^{(j \wedge j')n_1} 2^{-|j-j'|N}}{\left(1 + 2^{(j \wedge j')} |x_{I'} - x_I|\right)^L} \frac{2^{(k \wedge k')n_2} 2^{-|k-k'|N}}{\left(1 + 2^{(k \wedge k')} |x_{J'} - x_J|\right)^L}, \end{aligned} \quad (25)$$

where the notation $j \wedge k$ means $\min\{j, k\}$. Hence, for any $u \in I', v \in J'$,

$$\begin{aligned} & \left| f * \psi_{j',k'}(x_{I'}, x_{J'}) \right| \\ & \leq \sum_{j,k \in \mathbb{N}} 2^{-|j-j'|L} 2^{-|k-k'|L} C_1 \\ & \quad \cdot \left\{ \mathcal{M}_s \left(\sum_{R \in \Pi_{j,k}} |R|^{-1} |s_R|^2 \chi_I \chi_J \right)^{\delta/2} (u, v) \right\}^{1/\delta}, \end{aligned} \quad (26)$$

where $\delta > 0$, which can be sufficiently small if L is large enough, and $\mathcal{M}_s$ is the strong maximal operator. Summing over j', k' and $R' \in \Pi_{j'}^{n_1} \times \Pi_{k'}^{n_2}$, one has

$$\begin{aligned} & \left(\sum_{j',k' \in \mathbb{N}} \sum_{R' \in \Pi_{j',k'}} \left| \psi_{j',k'} * f(x_{I'}, x_{J'}) \right|^2 \chi_{I'}(x'_1) \chi_{J'}(x'_2) \right)^{1/2} \\ & \leq C \left(\sum_{j',k' \in \mathbb{N}} \left[\sum_{j,k \in \mathbb{N}} 2^{-|j-j'|L} 2^{-|k-k'|L} C_1 \right. \right. \\ & \quad \cdot \left. \left. \left\{ \mathcal{M}_s \left(\sum_{R \in \Pi_{j,k}} |R|^{-1} |s_R|^2 \chi_I \chi_J \right)^{\delta/2} (u, v) \right\}^{1/\delta} \right]^2 \right)^{1/2}. \end{aligned} \quad (27)$$

Then, by Cauchy's inequality, we obtain

$$\begin{aligned} & \left(\sum_{j',k' \in \mathbb{N}} \sum_{R' \in \Pi_{j',k'}} \left| \psi_{j',k'} * f(x_{I'}, x_{J'}) \right|^2 \chi_{I'}(x'_1) \chi_{J'}(x'_2) \right)^{1/2} \\ & \leq \left(\sum_{j,k \in \mathbb{N}} \left\{ \mathcal{M}_s \left(\sum_{R \in \Pi_{j,k}} |R|^{-1} |s_R|^2 \chi_I \chi_J \right)^{\delta/2} (u, v) \right\}^{2/\delta} \right)^{1/2}. \end{aligned} \quad (28)$$

Applying Fefferman-Stein's vector-valued strong maximal inequality on $L^{p/\delta}(\ell^{2/\delta})$ provided $\delta < p$, we complete the proof. $\square$

In the next theorem, we discuss the actions of the multi-parameter ψ -transform S_ψ and its inverse ψ -transform T_ψ on the space $\text{cmo}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ and discrete sequence space $\mathfrak{C}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, respectively. We will obtain that operators $S_\psi : \text{cmo}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \rightarrow \mathfrak{C}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ and $T_\psi : \mathfrak{C}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \rightarrow \text{cmo}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ are bounded, and $T_\psi \circ S_\psi$ is the identity on $\text{cmo}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$.

Theorem 13. Let $0 < p < \infty$. For $i = 1, 2$, suppose that $\psi_0^{(i)}$ and $\psi^{(i)} \in \mathcal{S}(\mathbb{R}^{n_i})$ are functions satisfying conditions (4)–(6), respectively. Then, the operators $S_\psi : \text{cmo}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \rightarrow \mathfrak{C}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ and $T_\psi : \mathfrak{C}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \rightarrow \text{cmo}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ are bounded, and $T_\psi \circ S_\psi$ is the identity on $\text{cmo}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$.

Proof. We only need to prove that T_ψ is bounded. Let $t = \{t_{R'}\}_{R' \in \Pi} \in \mathfrak{C}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ and $f = \sum_{R' \in \Pi} t_{R'} \psi_{R'}^{(i)}$. For $i = 1, 2$, suppose that $\varphi_0^{(i)}$ and $\varphi^{(i)} \in \mathcal{S}(\mathbb{R}^{n_i})$ are functions satisfying conditions (4)–(6), respectively. We are going to prove

$$\begin{aligned} & \sup_{\Omega} \frac{1}{|\Omega|^{(1/p)-(1/2)}} \left(\sum_{R=I \times J \subseteq \Omega, R \in \Pi} |\varphi_{j,k} * f(x_I, x_J)|^2 |R| \right)^{1/2} \\ & \leq \sup_{\Omega} \frac{1}{|\Omega|^{(1/p)-(1/2)}} \left(\sum_{R'=I' \times J' \subseteq \Omega, R' \in \Pi} (|t_{R'}| |R'|^{-(1/2)})^2 |R'| \right)^{1/2}. \end{aligned} \quad (29)$$

Given any $R' \in \Pi_{j'}^{n_1} \times \Pi_{k'}^{n_2}$, by classical almost orthogonality estimates, for any positive integers L_1, L_2, M_1, M_2 , there exists a constant C such that

$$\begin{aligned} & |\varphi_{j,k} * \psi_{R'}(x_I, x_J)| \\ & \leq C |R'|^{1/2} \frac{2^{(j \wedge j')n_1} 2^{-|j-j'|L_1}}{(1 + 2^{j \wedge j'} |x_I - x_{I'}|)^{M_1}} \frac{2^{(k \wedge k')n_2} 2^{-|k-k'|L_2}}{(1 + 2^{k \wedge k'} |x_J - x_{J'}|)^{M_2}}, \end{aligned} \quad (30)$$

which implies that, for any positive integers L_1, L_2, M_1, M_2 ,

$$\begin{aligned} & |\varphi_{j,k} * \psi_{R'}(x_I, x_J)| \\ & \leq C |R'|^{-1/2} \frac{2^{-|j-j'|L_1}}{(1 + 2^{j'} |x_I - x_{I'}|)^{M_1}} \frac{2^{-|k-k'|L_2}}{(1 + 2^{k'} |x_J - x_{J'}|)^{M_2}}. \end{aligned} \quad (31)$$

On the other hand, by conditions (4) and (5), one has that $(\varphi_{j,k} * \psi_{j',k'})^\wedge(\xi) = 0$ if $|j' - j| > 4$ or $|k' - k| > 4$; it follows that

$$|\varphi_{j,k} * f(x_I, x_J)|^2 \lesssim \sum_{R' \in \Pi} |t_{R'}|^2 |\varphi_{j,k} * \psi_{R'}(x_I, x_J)|^2. \quad (32)$$

Combining the above two estimates, one has

$$\begin{aligned} & \sum_{R=I \times J \subseteq \Omega, R \in \Pi} |\varphi_{j,k} * f(x_I, x_J)|^2 |R| \\ & \leq \sum_{R=I \times J \subseteq \Omega, R \in \Pi} \sum_{R'=I' \times J' \in \Pi} |t_{R'}|^2 |R'|^{-1} |R| \left(\frac{|I|}{|I'|} \wedge \frac{|J|}{|J'|} \right)^{2L_1/n_1} \\ & \quad \cdot \frac{1}{(1 + (\text{dist}(I, I')/\ell(I')))^{2M_1}} \left(\frac{|J|}{|J'|} \wedge \frac{|I|}{|I'|} \right)^{2L_2/n_2} \\ & \quad \cdot \frac{1}{(1 + (\text{dist}(J, J')/\ell(J')))^{2M_2}} \\ & = \sum_{R=I \times J \subseteq \Omega, R \in \Pi} \sum_{R'=I' \times J' \in \Pi} \left(\frac{|I|}{|I'|} \wedge \frac{|J|}{|J'|} \right)^{2L_1/n_1} \left(\frac{|J|}{|J'|} \wedge \frac{|I|}{|I'|} \right)^{2L_2/n_2} \\ & \quad \cdot \frac{|R|}{|R'|} \frac{1}{(1 + (\text{dist}(I, I')/\ell(I')))^{2M_1}} \\ & \quad \cdot \frac{1}{(1 + (\text{dist}(J, J')/\ell(J')))^{2M_2}} (|t_{R'}| |R'|^{-(1/2)})^2 |R'|, \end{aligned} \quad (33)$$

which gives that

$$\begin{aligned} & \sum_{R=I \times J \subseteq \Omega, R \in \Pi} |\varphi_{j,k} * f(x_I, x_J)|^2 |R| \\ & \leq \sum_{R=I \times J \subseteq \Omega, R \in \Pi} \sum_{R'=I' \times J' \in \Pi} (|t_{R'}| |R'|^{-(1/2)})^2 |R'| \left(\frac{|I|}{|I'|} \wedge \frac{|J|}{|J'|} \right)^L \\ & \quad \cdot \left(\frac{|J|}{|J'|} \wedge \frac{|I|}{|I'|} \right)^L \frac{1}{(1 + (\text{dist}(I, I')/\ell(I')))^M} \\ & \quad \cdot \frac{1}{(1 + (\text{dist}(J, J')/\ell(J')))^M} \end{aligned} \quad (34)$$

for any positive integers L, M .

Thus,

$$\begin{aligned} & \sup_{\Omega} \frac{1}{|\Omega|^{(1/p)-(1/2)}} \left(\sum_{R=I \times J \subseteq \Omega, R \in \Pi} |\varphi_{j,k} * f(x_I, x_J)|^2 |R| \right)^{1/2} \\ & \leq \sup_{\Omega} \frac{1}{|\Omega|^{(1/p)-(1/2)}} \left(\sum_{R=I \times J \subseteq \Omega, R \in \Pi} \sum_{R'=I' \times J' \in \Pi} r(R, R') p \right. \\ & \quad \cdot (R, R') (|t_{R'}| |R'|^{-(1/2)})^2 |R'| \Big)^{1/2}, \end{aligned} \quad (35)$$

where

$$r(R, R') = \left(\frac{|I|}{|I'|} \wedge \frac{|I'|}{|I|} \right)^L \left(\frac{|J|}{|J'|} \wedge \frac{|J'|}{|J|} \right)^L,$$

$$p(R, R') = \frac{1}{\left(1 + \left(\text{dist}(I, I')/\ell(I')\right)\right)^M} \frac{1}{\left(1 + \left(\text{dist}(J, J')/\ell(J')\right)\right)^M}.$$
(36)

To obtain the inequality (29), by (35), we only need to prove

$$\sup_{\Omega} \frac{1}{|\Omega|^{(1/p)-(1/2)}} \left(\sum_{R=I \times J \subseteq \Omega, R \in \Pi} \sum_{R'=I' \times J' \in \Pi} r(R, R') p(R, R') \left(|t_{R'}| |R'|^{-(1/2)} \right)^2 |R'| \right)^{1/2}$$

$$\leq \sup_{\Omega} \frac{1}{|\Omega|^{(1/p)-(1/2)}} \left(\sum_{R'=I' \times J' \subseteq \Omega, R' \in \Pi} \left(|t_{R'}| |R'|^{-(1/2)} \right)^2 |R'| \right)^{1/2}.$$
(37)

Let

$$\Omega^{0,0} = \bigcup_{R=I \times J \subseteq \Omega, R \in \Pi} 3(I \times J).$$
(38)

Given any $R \subseteq \Omega$, let $A_{i,l}(R)$ be the set of dyadic rectangles R' defined as follows:

$$A_{0,0}(R) = \left\{ R' = I' \times J' \in \Pi : \text{dist}(I, I') \leq \ell(I) \vee \ell(I'), \right.$$

$$\left. \text{dist}(J, J') \leq \ell(J) \vee \ell(J') \right\},$$
(39)

and for $i \geq 1$,

$$A_{i,0}(R) = \left\{ R' = I' \times J' \in \Pi : \left(2^{i-1} \ell(I') \right) \vee \ell(I) < \text{dist}(I, I') \right.$$

$$\left. \leq \left(2^i \ell(I') \right) \vee \ell(I), \text{dist}(J, J') \leq \ell(J) \vee \ell(J') \right\},$$
(40)

and for $l \geq 1$,

$$A_{0,l}(R) = \left\{ R' = I' \times J' \in \Pi : \text{dist}(I, I') \right.$$

$$\leq \ell(I) \vee \ell(I'), \left(2^{l-1} \ell(J') \right) \vee \ell(J)$$

$$< \text{dist}(J, J') \leq \left(2^l \ell(J') \right) \vee \ell(J) \left. \right\},$$
(41)

and for $i, l \geq 1$,

$$A_{i,l}(R) = \left\{ R' = I' \times J' \in \Pi : \left(2^{i-1} \ell(I') \right) \vee \ell(I) < \text{dist}(I, I') \right.$$

$$\leq \left(2^i \ell(I') \right) \vee \ell(I), \left(2^{l-1} \ell(J') \right) \vee \ell(J) < \text{dist}(J, J')$$

$$\leq \left(2^l \ell(J') \right) \vee \ell(J) \left. \right\},$$
(42)

and $i, l \geq 0$,

$$A_{i,j} = \left\{ R' = I' \times J' \in \Pi : 3 \left(2^i I' \times 2^j J' \right) \cap \Omega^{0,0} \neq \emptyset \right\}. \quad (43)$$

Obviously, for any $R \subseteq \Omega, R \in \Pi$, one has that $\bigcup_{i,l \geq 0} A_{i,l}(R) = \Pi$; moreover, $A_{i,l}(R) \cap A_{i',l'}(R) = \emptyset$ if $(i, l) \neq (i', l')$ and $A_{i,l}(R) \subseteq A_{i,l}$. It is easy to see that, for $R' \in A_{i,j}(R), i, l \geq 0$,

$$1 + \frac{\text{dist}(I, I')}{\ell(I')} \gtrsim 2^i, 1 + \frac{\text{dist}(J, J')}{\ell(J')} \gtrsim 2^l, \quad (44)$$

which gives that

$$p(R, R) \lesssim 2^{-(i+l)M}. \quad (45)$$

Therefore,

$$\sup_{\Omega} \frac{1}{|\Omega|^{(1/p)-(1/2)}} \left(\sum_{R=I \times J \subseteq \Omega, R \in \Pi} \sum_{R'=I' \times J' \in \Pi} r(R, R') p(R, R') \left(|t_{R'}| |R'|^{-(1/2)} \right)^2 |R'| \right)^{1/2}$$

$$= \sup_{\Omega} \frac{1}{|\Omega|^{(1/p)-(1/2)}} \left(\sum_{R=I \times J \subseteq \Omega, R \in \Pi} \sum_{i,l \geq 0} \sum_{R' \in A_{i,l}(R)} r(R, R') p(R, R') \left(|t_{R'}| |R'|^{-(1/2)} \right)^2 |R'| \right)^{1/2}$$

$$= \sup_{\Omega} \frac{1}{|\Omega|^{(1/p)-(1/2)}} \left(\sum_{R=I \times J \subseteq \Omega, R \in \Pi} \sum_{i,l \geq 0} \sum_{R' \in A_{i,l}(R)} \chi_{R' \in A_{i,l}(R)} r(R, R') p(R, R') \left(|t_{R'}| |R'|^{-(1/2)} \right)^2 |R'| \right)^{1/2}$$

$$= \sup_{\Omega} \frac{1}{|\Omega|^{(1/p)-(1/2)}} \left(\sum_{i,l \geq 0} \sum_{R' \in A_{i,l}} \sum_{R=I \times J \subseteq \Omega, R \in \Pi} \chi_{R' \in A_{i,l}(R)} r(R, R') p(R, R') \left(|t_{R'}| |R'|^{-(1/2)} \right)^2 |R'| \right)^{1/2}$$

$$\leq \sup_{\Omega} \frac{1}{|\Omega|^{(1/p)-(1/2)}} \left(\sum_{i,l \geq 0} \sum_{R' \in A_{i,l}} \sum_{R=I \times J \subseteq \Omega, R \in \Pi} \chi_{R' \in A_{i,l}(R)} 2^{-(i+l)M} r(R, R') \left(|t_{R'}| |R'|^{-(1/2)} \right)^2 |R'| \right)^{1/2}$$

$$= \sup_{\Omega} \frac{1}{|\Omega|^{(1/p)-(1/2)}} \left(\left(\sum_{R' \in A_{0,0}} + \sum_{i \geq 1} \sum_{R' \in A_{i,0}} + \sum_{l \geq 1} \sum_{R' \in A_{0,l}} + \sum_{i,l \geq 1} \sum_{R' \in A_{i,l}} \right) 2^{-(i+l)M} \right.$$

$$\cdot \sum_{R=I \times J \subseteq \Omega, R \in \Pi} \chi_{R' \in A_{i,l}(R)} r(R, R') \left(|t_{R'}| |R'|^{-(1/2)} \right)^2 |R'| \left. \right)^{1/2}$$

$$= (I_1 + I_2 + I_3 + I_4)^{1/2}.$$
(46)

We only estimate I_4 since the other three terms can be considered by the same routine. Details can be seen in [26, 29].

For each integer $h \geq 1$, let $\mathcal{F}_h^{i,l} = \{R' = I' \times J' \in A_{i,l} : |3(2^i I' \times 2^j J') \cap \Omega^{0,0}| \geq (1/2^h) |2^i I' \times 2^j J'| \}$. Denote $\mathcal{F}_0^{i,l} = \emptyset$,

$$\mathcal{D}_h^{i,l} = \mathcal{F}_h^{i,l} \setminus \mathcal{F}_{h-1}^{i,l},$$

$$\Omega_h^{i,l} = \bigcup_{R' \in \mathcal{D}_h^{i,l}} R'. \quad (47)$$

Then, I_4 can be rewritten as

$$I_4 = \frac{1}{|\Omega|^{(2/p)-1}} \sum_{i,l \geq 1} \sum_{h \geq 1} \sum_{R' \in \mathcal{D}_h^{i,l}} 2^{-(i+l)M} \sum_{R \subseteq \Omega, R \in \Pi} \chi_{R' \in A_{i,l}(R)} r$$

$$\cdot (R, R') \left(|t_{R'}| |R'|^{-(1/2)} \right)^2 |R'|.$$
(48)

To estimate I_4 , we first estimate the following:

$$\sum_{R \subseteq \Omega, R \in \Pi} \chi_{R' \in A_{i,l}(R)} r(R, R'). \quad (49)$$

Note that if $R' \in A_{i,l}(R)$, one has $3R \cap 3(2^i I' \times 2^j J') \neq \emptyset$.

For $R \subseteq \Omega$, $R \in \Pi$, we consider the following four cases:

Case 1. $|2^i I'| \geq |I|$, $|2^j J'| \geq |J|$.

Case 2. $|2^i I'| \geq |I|$, $|2^j J'| \leq |J|$.

Case 3. $|2^i I'| \leq |I|$, $|2^j J'| \geq |J|$.

Case 4. $|2^i I'| \leq |I|$, $|2^j J'| \leq |J|$.

From the definition of $A_{i,l}(R)$, one can see that, if $R' \in$ Case 2, then

$$\begin{aligned} \ell(J) &= \left(2^{l-1} \ell(J')\right) \vee \ell(J) < \text{dist}(J, J') \\ &\leq \left(2^l \ell(J')\right) \vee \ell(J) = \ell(J), \end{aligned} \quad (50)$$

which implies Case 2 is an empty set. For the same reason, Case 3 and Case 4 are also empty. Hence,

$$\begin{aligned} I_4 &= \frac{1}{|\Omega|^{(2/p)-1}} \sum_{i,l \geq 1} \sum_{h \geq 1} \sum_{R' \in \mathcal{D}_h^{i,l}} 2^{-(i+l)M} \sum_{R \in \text{Case 1}} \chi_{R' \in A_{i,l}(R)} r \\ &\quad \cdot (R, R') \left(|t_{R'}| |R'|^{-(1/2)} \right)^2 |R'|. \end{aligned} \quad (51)$$

Since $R' \in A_{i,l}(R)$ and $R' \in \mathcal{D}_h^{i,l}$, one has

$$\begin{aligned} |R| &\leq \left| 3R \cap 3(2^i I' \times 2^j J') \right| \\ &\leq \left| 3(2^i I' \times 2^j J') \cap \Omega^{0,0} \right| \\ &\leq \frac{1}{2^{h-1}} \left| 3(2^i I' \times 2^j J') \right|. \end{aligned} \quad (52)$$

To estimate $r(R, R')$, we should compare the side length of R with the side length of R' . Hence, we divide $R \subseteq \Omega$ into four categories.

Category 1. $|I| \leq |I'|$, $|J| \leq |J'|$.

Category 2. $|I| \leq |I'|$, $|J| > |J'|$.

Category 3. $|I| > |I'|$, $|J| \leq |J'|$.

Category 4. $|I| > |I'|$, $|J| > |J'|$.

In Category 1, by (52), one has $2^{i n_1 + l n_2} |R'| = 2^{h-1-2(n_1+n_2)+\eta} |R|$ for some integer $\eta > 0$, since I, I' are all dyadic. Note that, for fixed i, l, h , these η should be bigger than some positive integer η_0 to ensure $|I| \leq |I'|$, $|J| \leq |J'|$, that is, satisfying $(2^{i n_1 + l n_2} / 2^{h-1-2(n_1+n_2)+\eta}) \leq 1$, $\eta \geq \eta_0$. Moreover, for each fixed $\eta > \eta_0$, the number of such R 's must be less than $7^{n_1+n_2} 2^{h-1-2(n_1+n_2)+\eta}$ since $R \subseteq 7(2^i I' \times 2^j J')$. There-

fore,

$$\begin{aligned} &\sum_{R \in \text{Case 1}, |I| \leq |I'|, |J| \leq |J'|} \chi_{R' \in A_{i,l}(R)} r(R, R') \\ &\leq \sum_{\eta \geq \eta_0} 7^{n_1+n_2} 2^{h-1-2(n_1+n_2)+\eta} \left(\frac{2^{i n_1 + l n_2}}{2^{h-1-2(n_1+n_2)+\eta}} \right)^L \\ &\leq 2^{-h(L-1)+(i+2)n_1 L + (l+2)n_2 L}. \end{aligned} \quad (53)$$

For Category 2, $|I| \leq |I'|$, $|J| > |J'|$, from (52), one has

$$\begin{aligned} |I| |J'| &\leq |R| \leq |3R \cap 3(2^i I' \times 2^j J')| \\ &\leq \left| 3(2^i I' \times 2^j J') \cap \Omega^{0,0} \right| \\ &\leq \frac{1}{2^{h-1}} \left| 3(2^i I' \times 2^j J') \right|, \end{aligned} \quad (54)$$

which implies that

$$|I| \leq \frac{1}{2^{h-1}} 3^{n_1+n_2} 2^{i n_1 + l n_2} |I'|. \quad (55)$$

Hence, $2^{i n_1 + l n_2} |I'| = 2^{h-1-2(n_1+n_2)+\theta} |I|$ for some integer $\theta > 0$. Note that for each fixed $\theta > 0$, the number of such I 's must be less than $7^{n_1} 2^{h-1-2(n_1+n_2)-l n_2+\theta}$ because $I \subseteq 7(2^i I')$. Moreover, from $|J'| < |J| \leq |2^j J'|$, one has $|2^\beta J'| = |J|$ for some integer β with $1 \leq \beta \leq l$. For each fixed $\beta > 0$, the number of such J 's must be less than $7^{n_2} 2^{(l-\beta)n_2}$ since $J \subseteq 7(2^l J')$. So

$$\begin{aligned} &\sum_{R \in \text{Case 1}, |I| \leq |I'|, |J| > |J'|} \chi_{R' \in A_{i,l}(R)} r(R, R') \\ &\leq \sum_{\theta > 0} \sum_{\beta=1}^l 7^{n_1} 2^{h-1-2(n_1+n_2)-l n_2+\theta} \\ &\quad \cdot \left(\frac{2^{i n_1 + l n_2}}{2^{h-1-2(n_1+n_2)+\theta}} \right)^L 7^{n_2} 2^{(l-\beta)n_2} \frac{1}{2^{L n_2 \beta}} \\ &\lesssim 2^{-h(L-1)+(i+2)n_1 L + (l+2)n_2 L} 2^{(l-L)n_2} \\ &\leq 2^{-h(L-1)+(i+2)n_1 L + (l+4)n_2 L}. \end{aligned} \quad (56)$$

Similar as Category 2, for Category 3, one can obtain the following estimate:

$$\sum_{R \in \text{Case 1}, |I| > |I'|, |J| \leq |J'|} \chi_{R' \in A_{i,l}(R)} r(R, R') \leq 2^{-h(L-1)+(l+2)n_2 L + (i+4)n_1 L}. \quad (57)$$

For Category 4, $|I| > |I'|$, $|J| > |J'|$, from (52), one has

$$|R'| \leq \frac{1}{2^{h-1}} \left| 3(2^i I' \times 2^j J') \right|, \quad (58)$$

which implies that $3^{n_1+n_2}2^{in_1+ln_2} \geq 2^{h-1}$. Furthermore, by $|R'| \leq |R| \leq |2^I I'| \times 2^J J'|$, one has $2^{in_1+ln_2} |R'| = 2^\lambda |R|$ for some integer $0 \leq \lambda \leq in_1 + ln_2$ and for each fixed $\lambda \geq 0$, the number of such R 's must be less than $7^{n_1+n_2}2^\lambda$ since $R \subseteq 7(2^I I' \times 2^J J')$. Hence,

$$\begin{aligned} & \sum_{R \in \text{Case1}, |I| > |I'|^{h/|J|} > |J'|} \chi_R'^{\in A_{il}(R)} r(R, R') \\ & \leq \sum_{\lambda=0}^{in_1+ln_2} 7^{n_1+n_2} 2^\lambda \left(\frac{2^\lambda}{2^{in_1+ln_2}} \right)^L \lesssim 2^{in_1+ln_2}. \end{aligned} \quad (59)$$

Then,

$$\begin{aligned} I_4 &= \frac{1}{|\Omega|^{(2/p)-1}} \sum_{i,j \geq 1} \sum_{h \geq 1} \sum_{R' \in \mathcal{D}_h^{i,j}} 2^{-(i+l)M} \sum_{R \in \text{Case1}} \chi_{R'}^{\in A_{il}(R)} r \\ & \cdot (R, R') \left(|t_{R'}| |R'|^{-(1/2)} \right)^2 |R'| \\ & \leq \frac{1}{|\Omega|^{(2/p)-1}} \sum_{i,j \geq 1} \left(\sum_{h \geq 1} 2^{-h(L-1)+(l+4)n_2L+(i+4)n_1L} + \sum_{3^{n_1+n_2}2^{in_1+ln_2} \geq 2^{h-1}} 2^{in_1+ln_2} \right) \\ & \cdot \left(2^{-(i+l)M} |\mathcal{D}_h^{i,j}|^{(2/p)-1} \frac{1}{|\Omega|^{(2/p)-1}} \sum_{R' \in \mathcal{D}_h^{i,j}} \left(|t_{R'}| |R'|^{-(1/2)} \right)^2 |R'| \right). \end{aligned} \quad (60)$$

From the definition of $\mathcal{D}_h^{i,j}$, it is easy to see that $\mathcal{D}_h^{i,j} \subseteq \{x \in \mathbb{R}^{n_1+n_2} : M_s(\chi_{\Omega^{0,0}})(x) \geq (1/2^{h-1})\}$, which implies $|\mathcal{D}_h^{i,j}| 2^{2h} |\Omega^{0,0}| \geq 2^{2h} |\Omega|$. Hence,

$$\begin{aligned} I_4 &\leq \sum_{i,j \geq 1} \left(\sum_{h \geq 1} 2^{-h(L-1)+(l+4)n_2L+(i+4)n_1L} + \sum_{3^{n_1+n_2}2^{in_1+ln_2} \geq 2^{h-1}} 2^{in_1+ln_2} \right) 2^{-(i+l)M} 2^{2h} \\ &\cdot \left(\sup_{\Omega} \frac{1}{|\Omega|^{(2/p)-1}} \sum_{R'=I' \times J' \subseteq \Omega, R' \in \Pi} \left(|t_{R'}| |R'|^{-(1/2)} \right)^2 |R'| \right) \\ &\leq \sum_{i,j \geq 1} 2^{-(i+l)M} 2^{2(l+4)n_2L+2(i+4)n_1L} \\ &\cdot \left(\sup_{\Omega} \frac{1}{|\Omega|^{(2/p)-1}} \sum_{R'=I' \times J' \subseteq \Omega, R' \in \Pi} \left(|t_{R'}| |R'|^{-(1/2)} \right)^2 |R'| \right) \\ &\leq \sup_{\Omega} \frac{1}{|\Omega|^{(2/p)-1}} \sum_{R'=I' \times J' \subseteq \Omega, R' \in \Pi} \left(|t_{R'}| |R'|^{-(1/2)} \right)^2 |R'|, \end{aligned} \quad (61)$$

by choosing proper M, L .

We complete the proof. $\square$

Proof of Theorem 4. Suppose that $\psi_{j,k}, \varphi_{j,k}$ are functions satisfying Definition 3. We may assume that

$$\sup_{\Omega} \left(\frac{1}{|\Omega|^{(2/p)-1}} \int_{\Omega} \sum_{j,k \in \mathbb{N}} \sum_{R \in \Pi_{j,k}, R \subseteq \Omega} \left(|\varphi_{j,k} * f(x_I, x_J)|^2 \chi_R(x) \right) dx \right)^{1/2} < \infty. \quad (62)$$

For $R \in \Pi_{j,k}$, set $\varphi_R(x) = |R|^{1/2} \varphi_{j,k}(x_1 - x_I, x_2 - x_J)$, and $t_R = \langle f, \varphi_R \rangle = |R|^{1/2} \varphi_{j,k} * f(x_I, x_J)$. Then,

$$\begin{aligned} & \sup_{\Omega} \left(\frac{1}{|\Omega|^{(2/p)-1}} \int_{\Omega} \sum_{j,k \in \mathbb{N}} \sum_{R \in \Pi_{j,k}, R \subseteq \Omega} \left(|\varphi_{j,k} * f(x_I, x_J)|^2 \chi_R(x) \right) dx \right)^{1/2} \\ &= \sup_{\Omega} \left(\frac{1}{|\Omega|^{(2/p)-1}} \int_{\Omega} \sum_{R \subseteq \Omega, R \in \Pi} (|t_R| \tilde{\chi}_R(x))^2 dx \right)^{1/2}, \end{aligned} \quad (63)$$

that is, $t = \{t_R\}_{R \in \Pi} \in \mathfrak{G}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$.

By (19), $f(x) = \sum_{R \in \Pi} t_R \varphi_R(x)$. Hence, by Theorem 13,

$$\begin{aligned} & \sup_{\Omega} \left(\frac{1}{|\Omega|^{(2/p)-1}} \int_{\Omega} \sum_{j,k \in \mathbb{N}} \sum_{R \in \Pi_{j,k}, R \subseteq \Omega} \left(|\psi_{j,k} * f(x_I, x_J)|^2 \chi_R(x) \right) dx \right)^{1/2} \\ & \cdot \left(\frac{1}{|\Omega|^{(2/p)-1}} \int_{\Omega} \sum_{R \subseteq \Omega, R \in \Pi} (|t_R| \tilde{\chi}_R(x))^2 dx \right)^{1/2}. \end{aligned} \quad (64)$$

Thus, we complete the proof. $\square$

3. Duality of Sequence Spaces

In this section, we establish the dual relationship between $\mathfrak{H}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ and $\mathfrak{G}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$.

Theorem 14. Let $0 < p \leq 1$. Then,

$$(\mathfrak{H}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}))^* = \mathfrak{G}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}). \quad (65)$$

Proof. Let $t \in \mathfrak{G}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. For any $s \in \mathfrak{H}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, denote

$$H(s)(x) = \left(\sum_{R \in \Pi} (|s_R| \tilde{\chi}_R(x))^2 \right)^{1/2}, \quad (66)$$

and for $k \in \mathbb{Z}$,

$$\Omega_k = \left\{ x \in \mathbb{R}^{n_1+n_2} : H(s)(x) > 2^k \right\},$$

$$\mathcal{B}_k = \left\{ R \in \Pi : |R \cap \Omega_k| > \frac{1}{2} |R|, |R \cap \Omega_{k+1}| \leq \frac{1}{2} |R| \right\}. \quad (67)$$

Then,

$$\begin{aligned} |\langle t, s \rangle| &= \left| \sum_{R \in \Pi} s_R t_R \right| \leq \sum_k \sum_{R \in \mathcal{B}_k} |t_R| |s_R| \\ &\leq \left\{ \sum_k \left[\sum_{R \in \mathcal{B}_k} |t_R|^2 \right]^{p/2} \left[\sum_{R \in \mathcal{B}_k} |s_R|^2 \right]^{p/2} \right\}^{1/p}. \end{aligned} \quad (68)$$

Let $\tilde{\Omega}_k = \{x \in \mathbb{R}^{n_1+n_2}, \mathcal{M}_s(\chi_{\Omega_k})(x) > 1/2\}$, where $\mathcal{M}_s(f)$ is the strong maximal function. Then, obviously, $|\tilde{\Omega}_k| |\Omega_k|$ by the boundedness of $\mathcal{M}_s$, and $\cup_{R \in \mathcal{B}_k} R \subseteq \tilde{\Omega}_k$. Hence,

$$\begin{aligned} \sum_{R \in \mathcal{B}_k} |t_R|^2 &= \left| \tilde{\Omega}_k \right|^{(2/p)-1} \frac{1}{\left| \tilde{\Omega}_k \right|^{(2/p)-1}} \sum_{R \in \mathcal{B}_k} |t_R|^2 \\ &\leq \|t\|_{\mathfrak{C}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})}^2 \left| \tilde{\Omega}_k \right|^{(2/p)-1}. \end{aligned} \quad (69)$$

On the other hand, $x \in R \in \mathcal{B}_k$ implies $M_s(\chi_{R \cap (\tilde{\Omega}_k \setminus \Omega_{k+1})})(x) > 1/2$. Therefore,

$$\begin{aligned} \sum_{R \in \mathcal{B}_k} |s_R|^2 &= \sum_{R \in \mathcal{B}_k} \int |s_R|^2 |R|^{-1} \chi_R(x) dx \\ &\leq \int \sum_{R \in \mathcal{B}_k} M_s^2(|s_R| |R|^{-(1/2)} \chi_{R \cap (\tilde{\Omega}_k \setminus \Omega_{k+1})})(x) dx \\ &\leq \int_{\tilde{\Omega}_k \setminus \Omega_{k+1}} \sum_{R \in \mathcal{B}_k} |s_R|^2 |R|^{-1} \chi_R(x) dx \\ &\leq \int_{\tilde{\Omega}_k \setminus \Omega_{k+1}} (H(s)(x))^2 dx 2^{2k} |\Omega_k|. \end{aligned} \quad (70)$$

The above two estimates yield that

$$\|\langle t, s \rangle\| \|t\|_{\mathfrak{C}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \left(\sum_k 2^{kp} |\Omega_k| \right)^{1/p} \|t\|_{\mathfrak{C}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \|s\|_{\mathfrak{H}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})}, \quad (71)$$

which implies

$$\mathfrak{C}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \subseteq (\mathfrak{H}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}))^*. \quad (72)$$

Conversely, let $\ell \in (\mathfrak{H}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}))^*$; then, there exists some $t = \{t_R\}_{R \in \Pi}$, such that for every $s = \{s_R\}_{R \in \Pi} \in \mathfrak{H}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, $\ell(s) = \sum_{R \in \Pi} s_R t_R$, and

$$\left| \sum_{R \in \Pi} s_R t_R \right| \leq \|\ell\| \|s\|_{\mathfrak{H}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})}. \quad (73)$$

To complete the proof, it suffices to show that $\|t\|_{\mathfrak{C}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \leq \|\ell\|$. To do this, for any open set $\Omega \subseteq \mathbb{R}^{n_1+n_2}$ with finite measure, denote $X = \{R \in \Pi : R \subseteq \Omega\}$, and let μ be a measure on X such that $\mu(R) = |R|$ if $R \in X$, and $\mu(R) = 0$ when $R \notin X$. Furthermore, let $l^2(\mu)$ be the sequence space such that if $\{s_R\}_{R \in \Pi} \in l^2(\mu)$, $\sum_{R \in \Pi} |s_R|^2 \mu(R) = \sum_{R \in X} |s_R|^2 |R| < \infty$.

Note that

$$\begin{aligned} \left(\sum_{R \subseteq \Omega, R \in \Pi} |t_R|^2 \right)^{1/2} &= \left\| \left\{ |t_R| |R|^{-(1/2)} \right\}_{R \in \Pi} \right\|_{l^2(\mu)} \\ &= \sup_{\|s\|_{l^2(\mu)} \leq 1} \left| \sum_{R \in \Pi} |t_R| |R|^{-(1/2)} s_R \mu(R) \right| \\ &\leq \|\ell\| \sup_{\|s\|_{l^2(\mu)} \leq 1} \left\| \left\{ |R|^{-(1/2)} s_R \mu(R) \right\}_{R \in \Pi} \right\|_{\mathfrak{H}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})}. \end{aligned} \quad (74)$$

On the other hand,

$$\begin{aligned} &\left\| \left\{ |R|^{-(1/2)} s_R \mu(R) \right\}_{R \in \Pi} \right\|_{\mathfrak{H}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \\ &= \left\| \left(\sum_{R \subseteq \Omega, R \in \Pi} \left(|R|^{-(1/2)} |s_R| |R| \chi_R(x) \right)^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^{n_1+n_2})} \\ &= \left\| \left(\sum_{R \subseteq \Omega, R \in \Pi} (|s_R| \chi_R(x))^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^{n_1+n_2})} \\ &\leq \left\{ \int_{\Omega} \sum_{R \subseteq \Omega, R \in \Pi} (|s_R| \chi_R(x))^2 dx \right\}^{1/2} |\Omega|^{(1/p)-(1/2)} \\ &= \|s\|_{l^2(\mu)} |\Omega|^{(1/p)-(1/2)}, \end{aligned} \quad (75)$$

by Hölder's inequality. It gives that

$$\left(\sum_{R \subseteq \Omega, R \in \Pi} |t_R|^2 \right)^{1/2} \leq \|\ell\| |\Omega|^{(1/p)-(1/2)}, \quad (76)$$

that is,

$$\frac{1}{|\Omega|^{(1/p)-(1/2)}} \left(\sum_{R \subseteq \Omega, R \in \Pi} |t_R|^2 \right)^{1/2} \leq \|\ell\|, \quad (77)$$

for any open set $\Omega \subseteq \mathbb{R}^{n_1+n_2}$ with finite measure.

We thus complete the proof. $\square$

4. Duality of $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$

Once the duality of sequence spaces are established, we can obtain the dual relationship between $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ and $\text{cmo}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ by multiparameter ψ -transform.

Proof of Theorem 5. It is known that $\mathcal{S}(\mathbb{R}^{n_1+n_2})$ is dense in $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ ([23]). Let $g \in \text{cmo}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, $f \in \mathcal{S}(\mathbb{R}^{n_1+n_2})$. Then, by (20), we have

$$\langle f, g \rangle = \langle S_\psi f, S_\psi g \rangle. \quad (78)$$

By Theorem 14, Theorem 12, and Theorem 13,

$$|\langle f, g \rangle| \leq \|S_\psi f\|_{\mathfrak{H}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \|S_\psi g\|_{\mathfrak{C}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \|f\|_{h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \|g\|_{\text{cmo}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})}. \quad (79)$$

This implies that for any $g \in \text{cmo}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, the map l_g , given by $l_g(f) = \langle f, g \rangle$, is a continuous linear functional on $\mathcal{S}(\mathbb{R}^{n_1+n_2})$, a dense subset of $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, and hence can be extended to a continuous linear functional on $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ with $\|l_g\| \|g\|_{\text{cmo}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})}$.

Conversely, suppose that $l \in (h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}))^*$. Then, $l_1 \equiv l \circ T_\psi \in (\mathfrak{h}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}))^*$ by Theorem 12. So there exists $t = \{t_R\}_{R \in \Pi} \in \mathfrak{G}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ by Theorem 14 such that

$$l_1(s) = \langle s, t \rangle = \sum_{R \in \Pi} s_R \bar{t}_R \quad (80)$$

for all $s = \{s_R\}_{R \in \Pi} \in \mathfrak{h}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, and $\|t\|_{\mathfrak{G}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \approx \|l_1\|$ for the boundedness of T_ψ . Obviously, $l_1 \circ S_\psi = l \circ T_\psi \circ S_\psi = l$ since $T_\psi \circ S_\psi$ is identified by Theorem 12. If we denote $g = T_\psi(t)$, then for $f \in \mathcal{S}(\mathbb{R}^{n_1+n_2})$, one has

$$l(f) = l_1(S_\psi(f)) = \langle S_\psi(f), t \rangle = \langle f, T_\psi(t) \rangle = \langle f, g \rangle \quad (81)$$

by (21), which implies that $l = l_g$. At last, by Theorem 13, we have

$$\|g\|_{\text{cmo}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} = \|T_\psi(t)\|_{\text{cmo}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \|t\|_{\mathfrak{G}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \|l\|. \quad (82)$$

Thus, we complete the proof of Theorem 5. $\square$

5. The Spaces $\text{Lip}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ and $\text{cmo}_{\text{rect}}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$

Proof of Theorem 8. Given a dyadic rectangles R_0 , obviously,

$$\begin{aligned} & \frac{1}{|R_0|^{(2/p)-1}} \int \sum_{j,k \in \mathbb{N}} \sum_{R \in \Pi_j^{n_1} \times \Pi_k^{n_2}, R \subseteq \Omega} \left(|\psi_{j,k} * f(x_I, x_J)|^2 \chi_R(x) \right) dx \\ &= \frac{1}{|R_0|^{(2/p)-1}} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{R \in \Pi_j^{n_1} \times \Pi_k^{n_2}, R \subseteq \Omega} |R| |\psi_{j,k} * f(x_I, x_J)|^2. \end{aligned} \quad (83)$$

Set $j_0, k_0 \in \mathbb{Z}$ such that $R_0 = I_0 \times J_0 \in \Pi_{j_0, k_0}$ and $|R_0| = 2^{-j_0 n_1} 2^{-k_0 n_2}$.

Hence,

$$\begin{aligned} & \frac{1}{|R_0|^{(2/p)-1}} \int \sum_{j,k \in \mathbb{N}} \sum_{R \in \Pi_j^{n_1} \times \Pi_k^{n_2}, R \subseteq R_0} \left(|\psi_{j,k} * f(x_I, x_J)|^2 \chi_R(x) \right) dx \\ & \leq \|f\|_{\text{Lip}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})}^2 \frac{1}{|R_0|^{(2/p)-1}} \sum_{j=j_0 \vee 0}^{\infty} \sum_{k=k_0 \vee 0}^{\infty} \sum_{R \in \Pi_j^{n_1} \times \Pi_k^{n_2}, R \subseteq R_0} \\ & \cdot |R| 2^{2j n_1 (1-(1/p))} 2^{2k n_2 (1-(1/p))} \\ & \leq \|f\|_{\text{Lip}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})}^2 |R_0|^{2-(2/p)} \sum_{j=j_0 \vee 0}^{\infty} \sum_{k=k_0 \vee 0}^{\infty} 2^{2j n_1 (1-(1/p))} 2^{2k n_2 (1-(1/p))} \\ & \leq \|f\|_{\text{Lip}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})}^2 |R_0|^{2-(2/p)} 2^{2(j_0 \vee 0) n_1 (1-(1/p))} 2^{2(k_0 \vee 0) n_2 (1-(1/p))} \\ & \leq \|f\|_{\text{Lip}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})}^2 |R_0|^{2-(2/p)} 2^{2j_0 n_1 (1-(1/p))} 2^{2k_0 n_2 (1-(1/p))} \\ & = \|f\|_{\text{Lip}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})}^2, \end{aligned} \quad (84)$$

since $0 < p \leq 1$, where the notation $j \vee k$ means $\max\{j, k\}$.

To prove the second inequality in Theorem 8, we first claim that

$$\left| \psi_{j',k'} * f(x) \right| \leq \|f\|_{\text{cmo}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \left\| \psi_{j'}^{',k'}(x - \cdot) \right\|_{h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})}. \quad (85)$$

Then, if we obtain

$$\left\| \psi_{j'}^{',k'}(x - \cdot) \right\|_{h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \leq 2^{j' n_1 (1-(1/p))} 2^{k' n_2 (1-(1/p))}, \quad (86)$$

one can complete the proof. Now, we prove (85). Set $h(t) = \psi_{j'}^{',k'}(x - t)$, and define

$$S(h)(x) = \left(\sum_{j,k \in \mathbb{N}} \sum_{I \times J \in \Pi_j^{n_1} \times \Pi_k^{n_2}} \left| \psi_{j,k} * h(x_I, x_J) \right|^2 \chi_I(x) \chi_J(y) \right)^{1/2}. \quad (87)$$

For any $i \in \mathbb{Z}$, denote

$$\begin{aligned} \Omega_i &= \{x \in \mathbb{R}^{n_1+n_2} : S(h)(x) > 2^i\}, \tilde{\Omega}_i \\ &= \left\{x \in \mathbb{R}^{n_1+n_2} : M_s(\chi_{\Omega_i}) > \frac{1}{10^{n_1+n_2}}\right\}, \end{aligned}$$

$$\mathcal{B}_i = \left\{R : R \in \Pi, |R \cap \Omega_i| > \frac{1}{2}|R|, |R \cap \Omega_{i+1}| \leq \frac{1}{2}|R|\right\}. \quad (88)$$

Then, $\cup_{R \in \mathcal{B}_i} R \subseteq \tilde{\Omega}_i$. By locally reproducing formula (7), one has

$$\begin{aligned} \psi_{j'}^{',k'} * f(x) &= \sum_{j=0}^{+\infty} \sum_{k=0}^{+\infty} \sum_{I \times J \in \Pi_j^{n_1} \times \Pi_k^{n_2}} |I||J| \\ & \cdot \left(\psi_{j,k} * f \right) (x_I, x_J) \psi_{j',k'} * \psi_{j,k}(x - x_I, y - x_J) \\ &= \sum_{j=0}^{+\infty} \sum_{k=0}^{+\infty} \sum_{I \times J \in \Pi_j^{n_1} \times \Pi_k^{n_2}} |I||J| \left(\psi_{j,k} * f \right) \\ & \cdot (x_I, x_J) h * \psi_{j,k}(x_I, x_J) \\ &= \sum_{i=-\infty}^{+\infty} \sum_{R \in \mathcal{B}_i} |R| |(\psi_R * f)(x_I, x_J)| \\ & \cdot |(\psi_R * h)(x_I, x_J)|, \end{aligned} \quad (89)$$

where $\psi_{j,k}$ is denoted by ψ_R if $R = I \times J \in \Pi_j^{n_1} \times \Pi_k^{n_2}$.

Hence,

$$\begin{aligned}
|\psi_{j',k'} * f(x)| &\leq \sum_{i=-\infty}^{+\infty} \sum_{R \in \mathcal{B}_i} |R| |(\psi_R * f)(x_I, x_J)| |(\psi_R * g)(x_I, x_J)| \\
&\leq \left\{ \sum_{i=-\infty}^{+\infty} \left(\sum_{R \in \mathcal{B}_i} |R| |(\psi_R * f)(x_I, x_J)|^2 \right)^{p/2} \right. \\
&\quad \cdot \left. \left(\sum_{R \in \mathcal{B}_i} |R| |(\psi_R * h)(x_I, x_J)|^2 \right)^{p/2} \right\}^{1/p} \\
&= \left\{ \sum_{i=-\infty}^{+\infty} \left(|\tilde{\Omega}_i|^{(2/p)-1} \sum_{R \in \mathcal{B}_i} |R| |(\psi_R * h)(x_I, x_J)|^2 \right)^{p/2} \right. \\
&\quad \cdot \left. \left(\frac{1}{|\tilde{\Omega}_i|^{(2/p)-1}} \sum_{R \in \mathcal{B}_i} |R| |(\psi_R * f)(x_I, x_J)|^2 \right)^{p/2} \right\}^{1/p} \\
&\leq \|f\|_{\text{cmo}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \left\{ \sum_{i=-\infty}^{+\infty} |\tilde{\Omega}_i|^{1-(p/2)} \right. \\
&\quad \cdot \left. \left(\sum_{R \in \mathcal{B}_i} |R| |(\psi_R * h)(x_I, x_J)|^2 \right)^{p/2} \right\}^{1/p}. \quad (90)
\end{aligned}$$

Note that $x \in R \in \mathcal{B}_i$ implies $M_s(\chi_{R \cap (\tilde{\Omega}_i \setminus \Omega_{i+1})})(x) > 1/2$; then,

$$\chi_R(x) = \chi_R^2(x) \leq 2^2 M_s^2(\chi_{R \cap (\tilde{\Omega}_i \setminus \Omega_{i+1})})(x), \quad (91)$$

which follows that

$$\begin{aligned}
&\left\| \left(\sum_{R \in \mathcal{B}_i} |\psi_R * h(x_I, x_J)| \chi_R(x) \right)^{1/2} \right\|_{L^2(\mathbb{R}^{n_1+n_2})}^2 \\
&\leq \left\| \left(\sum_{R \in \mathcal{B}_i} M_s^2(|\psi_R * h(x_I, x_J)| \chi_{R \cap (\tilde{\Omega}_i \setminus \Omega_{i+1})})(x) \right)^{1/2} \right\|_{L^2(\mathbb{R}^{n_1+n_2})}^2 \\
&\leq \int_{\tilde{\Omega}_i \setminus \Omega_{i+1}} \sum_{R \in \mathcal{B}_i} |\psi_R * h(x_I, x_J)|^2 \chi_R(x) dx \quad (92)
\end{aligned}$$

by the Fefferman-Stein vector-valued strong maximal inequality. Since $S(h)(x) \leq 2^{i+1}$ in the domain $\tilde{\Omega}_i \setminus \Omega_{i+1}$, one has

$$\left\| \left(\sum_{R \in \mathcal{B}_i} |\psi_R * h(x_I, x_J)| \chi_R(x) \right)^{1/2} \right\|_{L^2(\mathbb{R}^{n_1+n_2})}^2 \leq 2^{2i} |\tilde{\Omega}_i|. \quad (93)$$

Hence,

$$\begin{aligned}
|\psi_{j',k'} * f(x)| &\leq \|g\|_{\text{cmo}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \left\{ \sum_{i=-\infty}^{+\infty} |\tilde{\Omega}_i|^{1-(p/2)} (2^{2i} |\tilde{\Omega}_i|)^{(p/2)} \right\}^{1/p} \\
&\leq \|f\|_{\text{cmo}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \|h\|_{h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})}, \quad (94)
\end{aligned}$$

which implies (85).

At last, to prove (86), using the continuous-type norm of $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, by almost orthogonality estimates, one has

$$\begin{aligned}
\|\psi_{j',k'}(x - \cdot)\|_{h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})}^p &= \int \left(\sum_{j,k \in \mathbb{N}} |\psi_{j,k} * \psi_{j',k'}(x - \cdot)(y)|^2 \right)^{p/2} dy \\
&\leq \int \sum_{j,k \in \mathbb{N}} \frac{2^{(j \wedge j')n_1 p} 2^{-|j-j'|Np}}{(1 + 2^{(j \wedge j')}|y_1|)^{Lp}} \\
&\quad \cdot \frac{2^{(k \wedge k')n_2 p} 2^{-|k-k'|Np}}{(1 + 2^{(k \wedge k')}|y_2|)^{Lp}} dy_1 dy_2 \\
&\leq \sum_{j,k \in \mathbb{N}} 2^{(j \wedge j')(n_1 p - n_1)} \\
&\quad \cdot 2^{-|j-j'|Np} 2^{(k \wedge k')(n_2 p - n_2)} \cdot 2^{-|k-k'|Np} \\
&\leq 2^{j'n_1(p-1)} 2^{k'n_2(p-1)}. \quad (95)
\end{aligned}$$

Thus, we have completed the proof of Theorem 8. $\square$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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