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Research Article

Midpoint Inequalities via Strong Convexity Using Positive Weighted Symmetry Kernels

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In the present research, we generalize the midpoint inequalities for strongly convex functions in weighted fractional integral settings. Our results generalize many existing results and can be considered as extension of existing results.

1. Introduction

One of the most interesting research areas of classical analysis is the study of functions and operators, especially convex functions, due to its applications in both integration and differentiation. In the last few years, a great effort has been put to develop new inequalities in convex analysis to deal with the various new applications, since modern problems are modeled by fractional calculus and new applications. So, the classical convexity [1, 2] and its related [3, 4] inequalities are not enough to tackle these ones. The new fractional integral inequalities in convexity are always appreciable. Moreover, the generalized and new mode of convexity is the area of interest for most of researcher of convex analysis [5, 6].

A function $f : I \longrightarrow \mathfrak{R}$ is said to be convex on I, if the inequality holds for all r, s $\in I$ and $\vartheta \in [0, 1]$,

$$f(\vartheta r + (1-\vartheta)s) \leq \leqslant \vartheta f(r) + (1-\vartheta)f(s). \tag{1}$$

Many problems may discuss in convexity of sets and functions. In recent year, convexity of sets and functions has been main object of study [7–9]. Some new generalized ideas in this point of view are pseudoconvex function, strongly convex function, quasiconvex, generalized convex function, preinvex functions [10], B-convex function, and invex functions. There are many different fundamental books of convex analysis optimization [11, 12].

Fractional calculus [13, 14] is not a new concept in mathematics, and similar discussion and controversy are observe in history by famous mathematician like Jensen, Hermite, H older, and Stolz. However, the subject of fractional calculus from an applied point of view got rapid development last years. Like other fields of mathematics, this also influences the integral inequalities and convex analysis ([15]). As a result, various trends in the result are settled recently. The famous fractional integral operators involve Riemann-Liouville [16], Caputo [17, 18], Hadamard [19], and Caputo Fabrizio [20, 21]. For more details about fractional integral operators, we refer [22–24].

The classical Hermite–Hadamard inequality is one of the most well-established inequalities in the theory of convex functions with geometrical interpretation, and it has many applications [25–27]. Recall Hermite-Hadamard-type inequality (simply H-H type inequality) which is given as:

Suppose function $s : [c, d] \subset \mathfrak{R}$ is convex, and then the inequality

$$s\left(\frac{c+d}{2}\right) \leq \frac{1}{d-c}\int_c^d s(\alpha_1)d\alpha_1 \leq \frac{s(c)+S(d)}{2} \tag{2}$$

is called the Hermite-Hadamard Inequality.

In the present research, we generalize the midpoint inequalities for strongly convex functions in weighted fractional integral settings. Our results generalize many existing results and can be considered as extension of existing results.

2. Definitions and Basic Results

Definition 1. Assume that $j \subseteq \mathfrak{R}$ is an interval and that "a" is a positive integer. If a function $s : j \subset \mathfrak{R} \longrightarrow \mathfrak{R}$ is strongly convex with modulus a, it is called strongly convex with modulus a.

$$s(l\beta_1 + (1-l)\beta_2) \leq ls(\beta_1) + 1(1-l)s(\beta_2) - al(1-l)\|\beta_1 - \beta_2\|^2 \tag{3}$$

for all $\beta_1, \beta_2 \in I$ and $l \in [0, 1]$.

Adamek expanded on the idea of a strongly convex function. They replaced the nonnegative term with a real-valued nonnegative function and defined it as follows:

If a function is strongly convex, it is defined as such.

$$s(l\beta_1 + (1-l)\beta_2) \leq ls(\beta_1) + 1(1-l)s(\beta_2) - l(1-l)M(\beta_1 - \beta_2) \tag{4}$$

for all $\beta_1, \beta_2 \in I$ and $l \in [0, 1]$. See [5, 28, 29] and references therein for more detail about strongly convex functionality.

Definition 2 [30]. Let $t : [d_1, d_2] \longrightarrow [0,\infty)$ be a function, Then, we say t is symmetric with respect to $d_1 + d_2/2$ if

$$t(d_1 + d_2 - \beta_1) = t(\beta_1) \forall \beta_1 \in [d_1, d_2]. \tag{5}$$

With the help of above definition, in [31], Fejér gave, namely, the Hermite-Hadamard-Fejér type inequality.

$$\begin{aligned} s\left(\frac{d_1+d_2}{2}\right)\int_{d_1}^{d_2} t(\beta_1)d\beta_1 &\leqslant \int_{d_1}^{d_2} s(\beta_1)t(\beta_1)d\beta_1 \\ &\leqslant \frac{s(d_1)+s(d_2)}{2}\int_{d_1}^{d_2} t(\beta_1)d\beta_1, \end{aligned} \tag{6}$$

where t is the integrable function.

Definition 3. Suppose $I_{l+}^a s(\beta_1)$ and $I_{m-}^a s(\beta_1)$ are the left- and rigt-sided RL fractional integrals of order $a > 0$ defined by [32]

$$\begin{aligned} {}^{RL}I_{d_1+}^{\nu} s(\beta 1) &= \frac{1}{\Gamma(\nu)}\int_{d_1}^{\beta_1} (\beta 1 - l)^{\nu-1} s(l)dl, \beta_1 > d_1, \\ {}^{RL}I_{d_2-}^{\nu} s(\beta_1) &= \frac{1}{\Gamma(\nu)}\int_{\beta 1}^{d2} (l - \beta_1)^{a-1} s(l)dl, \beta_1 < d_2, \end{aligned} \tag{7}$$

Endpoint inequalities were found, namely, the generalized and reformulated forms of H-H and H-H-F inequalities in terms of RL fractional integrals, respectively, in [22, 24].

$$\begin{aligned} s\left(\frac{d_1+d_2}{2}\right) &\leqslant \frac{2^{\nu-1}\Gamma(\nu+1)}{(d_2-d_1)^{\nu}}\left[I_{d_1+d_2/2+}^{\nu} s(d_2) + I_{\left(\frac{d_1+d_2}{2}\right)^-}^{\nu} s(d_2)\right] \\ &\leqslant \frac{s(d_1)+s(d_2)}{2}, \end{aligned}$$

$$\begin{aligned} &s\left(\frac{d_1+d_2}{2}\right)\left[I_{d_1+}^{\nu} t(d_2) + I_{d_2-}^{\nu} t(d_1)\right] \\ &\leqslant \left[I_{d_1+}^{\nu}(st)(d_2) + I_{d_2-}^{\nu}(st)(\nu)\right] \\ &\leqslant \frac{s(d_1)+s(d_2)}{2}\left[I_{d_1+}^{\nu} t(d_2) + I_{d_2-}^{\nu} t(l)\right], \end{aligned} \tag{8}$$

where s is the positive convex function, continuous on the closed interval $[d_1, d_2]$ when $s(\beta_1) \in L^1[d_1, d_2]$ and $d_1 < d_2$.

Definition 4. Let $(d_1, d_2) \subseteq \mathfrak{R}$ and $\sigma(\beta_1)$ be an increasing positive and monotone function on the interval $(d_1, d_2]$ with a continuous derivative $\sigma'(\beta_1)$ on the open interval (d_1, d_2). Then, left- and right-sided weighted fractional integral of a function s, according to another function $\sigma(\beta_1)$ on $[d_1, d_2]$, is defined by [25]:

$$\left({}_{d_1+}I_{\psi}^{\nu;\sigma}(s)\right)(\beta_1) = \frac{[\psi(\beta_1)]^{-1}}{\Gamma(\nu)}\int_{d_1}^{\beta_1} \sigma'(l)(\sigma(\beta_1) - \sigma(l))^{\nu-1} s(l)\psi(l)dl$$

$$\left({}_{\psi}I_{d_2-}^{\nu;\sigma}(s)\right)(\beta_1) = \frac{[\psi(\beta_1)]^{-1}}{\Gamma(\nu)}\int_{\beta_1}^{m} \sigma'(l)(\sigma(l) - \sigma(\beta_1))^{a-1} s(l)\psi(l)dl, a > 0, \tag{9}$$

where $[\psi(\beta_1)]^{-1} \coloneqq 1/\psi(\beta_1)$ such that $\psi(\beta_1) \neq 0$.

Midpoint inequalities were found, namely, the generalized and reformulated forms of H-H and H-H-F inequalities in terms of RL fractional integrals and weighted fractional integrals with positive weighted symmetric function in a kernel, due to using the midpoint $d_1 + m/2$ of the interval given by, respectively, in [33, 34].

$$\begin{aligned} s\left(\frac{d_1+d_2}{2}\right) &\leqslant \frac{2^{a-1}\Gamma(\nu+1)}{(d_2-d_1)^{\nu}}\left[I_{\left(\frac{d_1+d_2}{2}\right)+}^{\nu} s(d_2) + I_{\left(\frac{d_1+d_2}{2}\right)-}^{\nu} s(d_1)\right] \\ &\leqslant \frac{s(d_1)+s(d_2)}{2}, \end{aligned} \tag{10}$$

where $s : [d_1, d_2] \longrightarrow \mathfrak{R}$ is convex and continuous, and

$$\begin{aligned} s\left(\frac{d_1+d_2}{2}\right)&\left[\left({}_{\sigma^{-1}(d1)+}I^{v;\sigma}(\psi\circ\sigma)\right)\left(\sigma^{-1}(d_2)\right)\right.\\ &\left.+\left(I^{v;\sigma}_{\sigma^{-1}(d_2)-}(\psi\circ\sigma)\right)\left(\sigma^{-1}(d_1)\right)\right]\\ \leqslant\ &\psi(d_2)\left({}_{\sigma^{-1}(d1)+}I^{v;\sigma}_{\psi\circ\sigma}(s\circ\sigma)\right)\left(\sigma^{-1}(d_2)\right)\\ &+\psi(d_1)\left({}_{\psi\circ\sigma}I^{v;\sigma}_{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)-}(s\circ\sigma)\right)\left(\sigma^{-1}(d_1)\right)\\ \leqslant\ &\frac{s(d_1)+s(d_2)}{2}\left[\left({}_{\sigma^{-1}(d1)+}I^{v;\sigma}(\psi\circ\sigma)\right)\left(\sigma^{-1}(d_2)\right)\right.\\ &\left.+\left(I^{v;\sigma}_{\sigma^{-1}(d_2)-}(\psi\circ\sigma)\right)\left(\sigma^{-1}(d_1)\right)\right], \end{aligned} \tag{11}$$

where $s : [d_1, d_2] \longrightarrow \mathfrak{R}$ is convex and continuous, $\sigma(\beta_1)$ is the monotone increasing function from $(d_1, d_2]$ onto itself with σ' being continuous on (d_1, d_2), and $\psi : [d_1, d_2] \longrightarrow (0,\infty)$ is an integrable function, which is symmetric with respect to $d_1 + d_2/2$, where $d_1 < d_2$.

The fractional integral operators induced a new diversity in inequality theory. There are many fractional integral operators introduced by different mathematicians having their own characteristics.

Lemma 5. *Assume that $\psi : [d_1, d_2] \longrightarrow (0,\infty)$ is an integrable function and symmetric with respect to $(d_1 + d_2)/2$, $d_1 < d_2$, then*

(i) For each $l \in [0, 1]$, we have

$$\psi\left(\frac{l}{2}d_1+\frac{2-l}{2}d_2\right)=\psi\left(\frac{2-l}{2}d_1+\frac{l}{2}d_2\right). \tag{12}$$

(ii) For $v > 0$, we have

$$\begin{aligned} &\left({}_{\sigma^{-1}(d1+d22)^+}I^{v:\sigma}(\psi\circ\sigma)\right)\left(\sigma^{-1}(d_2)\right)\\ &=\left(I^{v:\sigma}_{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)^-}(\psi\circ\sigma)\right)\left(\sigma^{-1}(d_1)\right)\\ &=\frac{1}{2}\left[\left({}_{\sigma^{-1}(d1+d22)^+}I^{v:\sigma}(\psi\circ\sigma)\right)\left(\sigma^{-1}(d_2)\right)\right.\\ &\quad\left.+\left(I^{v:\sigma}_{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)^-}(\psi\circ\sigma)\right)\left(\sigma^{-1}(d_1)\right)\right]. \end{aligned} \tag{13}$$

Theorem 6. *Let $d_2 > d_1 \geq 0$, and let $s : [d_1, d_2] \longrightarrow \mathfrak{R}$ be an L^1 strong convex function and $\psi : [d_1, d_2] \longrightarrow \mathfrak{R}$ be an integrable, positive, and weighted symmetric function w.r.t $d_1 + d_2/2$. If, in addition, σ is an increasing and positive function from $[d_1, d_2)$ onto itself such that its derivative $\sigma'(\beta_1)$ is continuous on (d_1, d_2), then for $n > 0$, the following inequalities are valid:*

$$\begin{aligned} s\left(\frac{d_1+d_2}{2}\right)&\left[\left({}_{\sigma^{-1}(d1+d22)^+}I^{v:\sigma}(\psi\circ\sigma)\right)\left(\sigma^{-1}(d_2)\right)\right.\\ &\left.+\left(I^{v:\sigma}_{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)^-}(\psi\circ\sigma)\right)\left(\sigma^{-1}(d_1)\right)\right]\\ \leq\ &\psi(d_2)\left({}_{\sigma^{-1}(d1+d22)^+}I^{v:\sigma}_{\psi\circ\sigma}(s\circ\sigma)\right)\left(\sigma^{-1}(d_2)\right)\\ &+\psi(d_1)\left({}_{\psi\circ\sigma}I^{v:\sigma}_{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)^-}(s\circ\sigma)\right)\left(\sigma^{-1}(d_1)\right)\\ &-\frac{a}{2}M(d_1-d_2)\left[\left({}_{\sigma^{-1}(d1+d22)^+}I^{v:\sigma}(\psi\circ\sigma)\right)\left(\sigma^{-1}(d_2)\right)\right.\\ &\left.+\left(I^{v:\sigma}_{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)^-}(\psi\circ\sigma)\right)\left(\sigma^{-1}(d_1)\right)\right]\\ \leq\ &\left(\frac{s(d_1)+s(d_2)}{2}-\frac{a}{2}M(d_1-d_2)\right)\\ &\cdot\left[\left({}_{\sigma^{-1}(d1+d22)^+}I^{v:\sigma}(\psi\circ\sigma)\right)\left(\sigma^{-1}(d_2)\right)\right.\\ &\left.+\left(I^{v:\sigma}_{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)^-}(\psi\circ\sigma)\right)\left(\sigma^{-1}(d_1)\right)\right]. \end{aligned} \tag{14}$$

Proof. The strong convexity of "s" on $[d_1, d_2]$ gives

$$\begin{aligned} &s\left(\frac{\beta_1+\beta_2}{2}\right)\leq\frac{s(\beta_1)+s(\beta_2)}{2}-\frac{b}{2}M(\beta_1-\beta_2)\forall\beta_1,\beta_2\in[d_1,d_2],\\ &2s\left(\frac{\beta_1+\beta_2}{2}\right)\leq s(\beta_1)+s(\beta_2)-aM(\beta_1-\beta_2)\forall\beta_1,\beta_2\in[d_1,d_2]. \end{aligned} \tag{15}$$

So, for $\beta_1 = (l/2)d_1 + (2 - l/2)d_2$ and $\beta_2 = (2 - l/2)d_1 + (l/2)d_2$, $l \in [0, 1]$, it follows that

$$\begin{aligned} 2s\left(\frac{d_1+d_2}{2}\right)\leq\ &s\left(\frac{l}{2}d_1+\frac{2-l}{2}d_2\right)+s\left(\frac{2-l}{2}d_1+\frac{l}{2}d_2\right)\\ &-aM[(l-1)(d_1-d_2)]. \end{aligned} \tag{16}$$

Multiplying both sides of above equation (16) by $l^{v-1}\psi((l/2)d_1 + (2 - l/2)d_2)$ and integrating the resulting inequality with respect to l over [0, 1] yield that

$$\begin{aligned} 2s\left(\frac{d_1+d_2}{2}\right)&\int_0^1 l^{v-1}\psi\left(\frac{l}{2}d_1+\frac{2-l}{2}d_2\right)dl\\ \leq\ &\int_0^1 l^{v-1}s\left(\frac{l}{2}d_1+\frac{2-l}{2}d_2\right)\psi\left(\frac{l}{2}d_1+\frac{2-l}{2}d_2\right)dl\\ &+\int_0^1 l^{v-1}s\left(\frac{2-l}{2}d_1+\frac{l}{2}d_2\right)w\left(\frac{l}{2}d_1+\frac{2-l}{2}d_2\right)dl\\ &-\int_0^1 l^{v-1}aM[(l-1)(d_1-d_2)]\psi\left(\frac{l}{2}d_1+\frac{2-l}{2}d_2\right)dl, \end{aligned} \tag{17}$$

$$
\begin{aligned}
&2s\left(\frac{d_1+d_2}{2}\right)\int_0^1 l^{\nu-1}\psi\left(\frac{l}{2}d_1+\frac{2-l}{2}d_2\right)dl\\
&\quad\le \int_0^1 l^{\nu-1}s\left(\frac{l}{2}d_1+\frac{2-l}{2}d_2\right)\psi\left(\frac{l}{2}d_1+\frac{2-l}{2}d_2\right)dl\\
&\qquad+\int_0^1 l^{\nu-1}s\left(\frac{2-l}{2}d_1+\frac{l}{2}d_2\right)\psi\left(\frac{l}{2}d_1+\frac{2-l}{2}d_2\right)dl\\
&\qquad+aM(d_1-d_2)\left[\int_0^1 l^{\nu}\psi\left(\frac{l}{2}d_1+\frac{2-l}{2}d_2\right)dl\right.\\
&\qquad\left.-\int_0^1 l^{\nu-1}\psi\left(\frac{l}{2}d_1+\frac{2-l}{2}d_2\right)dl\right].
\end{aligned}
\tag{18}
$$

From the left hand side of inequality in equation (16), we use (13) to obtain

$$
\begin{aligned}
&\frac{2^{\nu-1}\Gamma(\nu)}{d_2-d_1}\left[\left({}_{\sigma^{-1}(d1+d22)^+}I^{\nu;\sigma}(\psi\circ\sigma)\right)\left(\sigma^{-1}(d_2)\right)\right.\\
&\qquad\left.+\left(I^{\nu;\sigma}_{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)^-}(\psi\circ\sigma)\right)\left(\sigma^{-1}(d_1)\right)\right]\\
&\quad=\int_0^1 l^{\nu-1}\psi\left(\frac{l}{2}d_1+\frac{2-l}{2}d_2\right)dl\left[l:=\frac{2(d_2-\sigma(\beta_1))}{d_2-d_1}\right],
\end{aligned}
\tag{19}
$$

which follows that

$$
\begin{aligned}
&2s\left(\frac{d_1+d_2}{2}\right)\int_0^1 l^{\nu-1}\psi\left(\frac{l}{2}d_1+\frac{2-l}{2}d_2\right)dl\\
&\quad=\frac{2^{\nu}\Gamma(\nu)}{(d_2-d_1)^{\nu}}s\left(\frac{d_1+d_2}{2}\right)\\
&\qquad\times\left[\left({}_{\sigma^{-1}(d1+d22)^+}I^{\nu;\sigma}(\psi\circ\sigma)\right)\left(\sigma^{-1}(d_2)\right)\right.\\
&\qquad\left.+\left(I^{\nu;\sigma}_{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)^-}(\psi\circ\sigma)\right)\left(\sigma^{-1}(d_1)\right)\right].
\end{aligned}
\tag{20}
$$

From the right hand side of inequality in (16), we use (13) to obtain

$$
\begin{aligned}
&\int_0^1 l^{\nu-1}s\left(\frac{l}{2}d_1+\frac{2-l}{2}d_2\right)\psi\left(\frac{l}{2}d_1+\frac{2-l}{2}d_2\right)dl\\
&\qquad+\int_0^1 l^{\nu-1}s\left(\frac{2-l}{2}d_1+\frac{l}{2}d_2\right)\psi\left(\frac{l}{2}d_1+\frac{2-l}{2}d_2\right)dl\\
&\qquad+aM(d_1-d_2)\left[\int_0^1 l^{\nu}\psi\left(\frac{l}{2}d_1+\frac{2-l}{2}d_2\right)dl\right.\\
&\qquad\left.-\int_0^1 l^{\nu-1}\psi\left(\frac{l}{2}d_1+\frac{2-l}{2}d_2\right)dl\right]\\
&\quad=\frac{2^{\nu}\Gamma(\nu)}{(d_2-d_1)^{\nu}}\left[\psi(d_2)\left({}_{\sigma^{-1}(d1+d22)^+}I^{\nu;\sigma}(\psi\circ\sigma)\right)\left(\sigma^{-1}(d_2)\right)\right.\\
&\qquad+\psi(d_1)\left(I^{\nu;\sigma}_{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)^-}(\psi\circ\sigma)\right)\left(\sigma^{-1}(d_1)\right)-\frac{a}{2}M(d_1-d_2)\\
&\qquad\cdot\left[\left({}_{\sigma^{-1}(d1+d22)^+}I^{\nu;\sigma}(\psi\circ\sigma)\right)\left(\sigma^{-1}(d_2)\right)\right.\\
&\qquad\left.\left.+\left(I^{\nu;\sigma}_{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)^-}(\psi\circ\sigma)\right)\left(\sigma^{-1}(d_1)\right)\right]\right].
\end{aligned}
\tag{21}
$$

By making use of (20) and (21) in (18), we get the desired result.

On the other hand, we can prove the second inequality of Theorem 6 by making use of the strong convexity of "s" to get

$$
\begin{aligned}
&s\left(\frac{l}{2}d_1+\frac{2-l}{2}d_2\right)+s\left(\frac{2-l}{2}d_1+\frac{l}{2}d_2\right)\\
&\qquad\le s(d_1)-s(d_2)-\left(2l-l^2\right)M(d_1-d_2).
\end{aligned}
\tag{22}
$$

Multiplying both sides of above equation (16) by $l^{\nu-1}\psi($ $(l/2)d_1+(2-l/2)d_2)$ and integrating the resulting inequality with respect to l over [0, 1] get

$$
\begin{aligned}
&\int_0^1 l^{\nu-1}s\left(\frac{l}{2}d_1+\frac{2-l}{2}d_2\right)\psi\left(\frac{l}{2}d_1+\frac{2-l}{2}d_2\right)\\
&\qquad+\int_0^1 l^{\nu-1}\psi\left(\frac{l}{2}d_1+\frac{2-l}{2}d_2\right)s\left(\frac{2-l}{2}+\frac{l}{2}\right)\\
&\quad\le\left[s(d_1)-s(d_2)-\left(2l-l^2\right)M(d_1-d_2)\right]\\
&\qquad\cdot\int_0^1 l^{\nu-1}\psi\left(\frac{l}{2}d_1+\frac{2-l}{2}d_2\right).
\end{aligned}
\tag{23}
$$

Now, by using $\psi((l/2)d_1+(2-l/2)d_2)=\psi((2-l/2)d_1$ $+(l/2)d_2)$ and (21) in (23), we get

$$
\begin{aligned}
&\frac{2^{\nu}\Gamma(\nu)}{(d_2-d_1)^{\nu}}\left({}_{\sigma^{-1}(d1+d22)^+}I^{\nu;\sigma}(\psi\circ\sigma)\right)\left(\sigma^{-1}(d_2)\right)\\
&\qquad+\left(I^{\nu;\sigma}_{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)^-}(\psi\circ\sigma)\right)\left(\sigma^{-1}(d_1)\right)-\frac{a}{2}M(d_1-d_2)\\
&\qquad\cdot\left[\left({}_{\sigma^{-1}(d1+d22)^+}I^{\nu;\sigma}(\psi\circ\sigma)\right)\left(\sigma^{-1}(d_2)\right)\right.\\
&\qquad\left.+\left(I^{\nu;\sigma}_{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)^-}(\psi\circ\sigma)\right)\left(\sigma^{-1}(d_1)\right)\right]\\
&\quad\le\left(\frac{s(d_1)+s(d_2)}{2}-\frac{a}{2}M(d_1-d_2)\right)\\
&\qquad\cdot\left[\left({}_{\sigma^{-1}(d1+d22)^+}I^{\nu;\sigma}(\psi\circ\sigma)\right)\left(\sigma^{-1}(d_2)\right)\right.\\
&\qquad\left.+\left(I^{\nu;\sigma}_{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)^-}(\psi\circ\sigma)\right)\left(\sigma^{-1}(d_1)\right)\right]
\end{aligned}
\tag{24}
$$

This ends our proof. □

Remark 7. From Theorem 6, we can obtain some special cases as follows:

(1) If $a=0$, $\sigma(\beta 1)=x$, then inequality (14) becomes

$$
\begin{aligned}
&s\left(\frac{d_1+d_2}{2}\right)\left[{}_{(d1+d22)+}I^{\nu}\psi(d_2)+I^{\nu}_{d_1+d_2/2-}\psi(d_2)\right]\\
&\quad\le\psi(d_2)\left({}_{(d1+d22)+}I^{\nu}_{\psi}s\right)(d_2)+\psi(d_1)\left({}_{\psi-}I^{\nu}_{\left(\frac{d_1+d_2}{2}\right)-}s\right)(d_1)\\
&\quad\le\frac{s(d_1)+s(d_2)}{2}\left[{}_{(d1+d22)+}I^{\nu}\psi(d_2)+I^{\nu}_{\left(\frac{d_1+d_2}{2}\right)-}\psi(d_1)\right],
\end{aligned}
\tag{25}
$$

where

$$\left({}_{d_1^+}I_{\psi}^{\nu}s\right)(x)=\frac{\psi^{-1}(x)}{\Gamma(\nu)}\int_{d_1}^{x}(x-l)^{\nu-1}s(l)\psi(l)dl,$$
$$\left({}_{\psi^-}I_{d_2}^{\nu}s\right)(x)=\frac{\psi^{-1}(x)}{\Gamma(\nu)}\int_{x}^{d_2}(l-x)^{\nu-1}s(l)\psi(l)dl\nu>0. \tag{26}$$

Lemma 8. *Let $d_2\geq d_1>0$, let $s:[d_1,d_2]\longrightarrow\mathfrak{R}$ be a continuous with a derivative $s'\in L^1[d_1,d_2]$ such that $s(\beta_1)=s(d_1)+\int_{d_1}^{\beta_1}s'(l)dl$, and let $\psi:[d_1,d_2]\longrightarrow R$ be an integrable, positive, and weighted symmetric function with respect to $d_1+d_2/2$. If σ is a continuous increasing mapping from the interval $[d_1,d_2)$ onto itself with a derivative $\sigma'(\beta_1)$ which is continuous on (d_1,d_2), then for $n>0$, the following equality is valid:*

$$\begin{aligned}
&s\left(\frac{d_1+d_2}{2}\right)\left[\left({}_{\sigma^{-1}(d1+d22)^+}I^{\nu;\sigma}(\psi\circ\sigma)\right)\left(\sigma^{-1}(d_2)\right)\right.\\
&\quad\left.+\left(I^{\nu;\sigma}_{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)^-}(\psi\circ\sigma)\right)\left(\sigma^{-1}(d_1)\right)\right]\\
&\quad-\left[\psi(d_2)\left({}_{\sigma^{-1}(d1+d22)^+}I^{\nu;\sigma}_{\psi\circ\sigma}(s\circ\sigma)\right)\left(\sigma^{-1}(d_2)\right)\right.\\
&\quad\left.+\psi(d_1)\left({}_{\psi\circ\sigma}I^{\nu;\sigma}_{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)^-}(s\circ\sigma)\right)\left(\sigma^{-1}(d_1)\right)\right]\\
&=\frac{1}{\Gamma(\nu)}\int_{\sigma^{-1}(d_1)}^{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)}\left[\int_{\sigma^{-1}(d_1)}^{l}\sigma'(\beta_1)(\sigma(\beta_1)-d_1)^{\nu-1}(\psi\circ\sigma)(\beta_1)d\beta_1\right]\\
&\quad\cdot\left(s'\circ\sigma\right)(l)\sigma'(l)dl-\frac{1}{\Gamma(\nu)}\int_{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)}^{\sigma^{-1}(d_2)}\\
&\quad\cdot\left[\int_{\sigma^{-1}(d_1)}^{l}\sigma'(\beta_1)(d_2-\sigma(\beta_1))^{\nu-1}(\psi\circ\sigma)(\beta_1)d\beta_1\right]\left(s'\circ\sigma\right)(l)\sigma'(l)dl.
\end{aligned}\tag{27}$$

3. Main Results

In this section, by using Lemma 8, one can extend to some new H-H-F type inequalities for strong convex functions.

Theorem 9. *Let $0\leq d_1<d_2$, let $s:[d_1,d_2]\subset[0,\infty)\longrightarrow\mathfrak{R}$ be a (continuously) differentiable mapping on $[d_1,d_2]$ such that $s(\beta_1)=s(d_1)+\int_{d_{d_1}}^{\beta_1}s'(l)dl$, and let $\psi:[d_1,d_2]\longrightarrow\mathfrak{R}$ be an integrable, positive, and weighted symmetric function with respect to $d_1+d_2/2$. If in addition, $|s'|$ is strong convex on $[d_1,d_2]$, and σ is an increasing and positive function from $[d_1,d_2]$ onto itself such that its derivative $\sigma'(\beta_1)$ is continuous on (d_1,d_2); then for $\nu>0$, the following inequalities hold:*

$$\begin{aligned}
|\Xi_1+\Xi_2|&=\left|\frac{1}{\Gamma(\nu)}\int_{\sigma^{-1}(d_1)}^{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)}\right.\\
&\quad\cdot\left[\int_{\sigma^{-1}d_1}^{l}\sigma'(\beta_1)(\sigma(\beta_1)-d_1)^{\nu-1}(\psi\circ\sigma)(\beta_1)d\beta_1\right]\\
&\quad\cdot\left(s'\circ\sigma\right)(l)\sigma'(l)dl-\frac{1}{\Gamma(\nu)}\int_{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)}^{\sigma^{-2}(d_2)}\\
&\quad\cdot\left[\int_{\sigma^{-1}d_1}^{l}\sigma'(\beta_1)(d_2-\sigma(\beta_1))^{\nu-1}(\psi\circ\sigma)(\beta_1)d\beta_1\right]\\
&\quad\cdot\left.\left(s'\circ\sigma\right)(l)\sigma'(l)dl\right|\\
&\leq\frac{(d_2-d_1)^{\nu+1}}{2^{\nu+2}\Gamma(\nu+3)}\left\{\left\|\psi\right\|_{\left[d_1,\frac{d_1+d_2}{2}\right],\infty}\right.\\
&\quad\cdot\left[(\nu+3)\left|s'(d_1)\right|+(\nu+1)\left|s'(d_2)\right|-\frac{a(\nu+4)(d_2-d_1)^2}{2}\right]\\
&\quad+\|\psi\|_{\left[\frac{d_1+d_2}{2},d_2\right],\infty}\left[(\nu+1)\left|s'(d_1)\right|\right.\\
&\quad\left.\left.+(\nu+3)\left|s'(d_2)\right|-\frac{a(\nu+4)(d_2-d_1)^2}{2\nu}\right\}\right.\\
&\leq\frac{(d_2-d_1)^{\nu+1}\|\psi\|_{[d_1,d_2],\infty}}{2^{\nu+1}\Gamma(\nu+2)}\left[\left|s'(d_1)\right|+\left|s'(d_2)\right|-a\|d_1-d_2\|^2\right].
\end{aligned}\tag{28}$$

Proof. By making use of Lemma 8 and properties of the modulus, we obtain

$$\begin{aligned}
|\Xi_1+\Xi_2|&=\left|\frac{1}{\Gamma(\nu)}\int_{\sigma^{-1}(d_1)}^{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)}\right.\\
&\quad\cdot\left[\int_{\sigma^{-1}d_1}^{l}\sigma'(\beta_1)(\sigma(\beta_1)-d_1)^{\nu-1}(\psi\circ\sigma)(\beta_1)d\beta_1\right]\\
&\quad\cdot\left(s'\circ\sigma\right)(l)\sigma'(l)dl-\frac{1}{\Gamma(\nu)}\int_{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)}^{\sigma^{-2}(d_2)}\\
&\quad\cdot\left[\int_{\sigma^{-1}d_1}^{l}\sigma'(\beta_1)(d_2-\sigma(\beta_1))^{\nu-1}(\psi\circ\sigma)(\beta_1)d\beta_1\right]\\
&\quad\cdot\left.\left(s'\circ\sigma\right)(l)\sigma'(l)dl\right|\\
&\leq\frac{1}{\Gamma(\nu)}\int_{\sigma^{-1}(d_1)}^{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)}\\
&\quad\cdot\left|\int_{\sigma^{-1}d_1}^{l}\sigma'(\beta_1)(\sigma(\beta_1)-d_1)^{\nu-1}(\psi\circ\sigma)(\beta_1)d\beta_1\right|\\
&\quad\cdot\left|\left(s'\circ\sigma\right)(l)\right|\left|\sigma'(l)dl-\frac{1}{\Gamma(\nu)}\int_{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)}^{\sigma^{-2}(d_2)}\right.\\
&\quad\cdot\left|\int_{\sigma^{-1}d_1}^{l}\sigma'(\beta_1)(d_2-\sigma(\beta_1))^{\nu-1}(\psi\circ\sigma)(\beta_1)d\beta_1\right|\\
&\quad\cdot\left|\left(s'\circ\sigma\right)(l)\right|\left|\sigma'(l)dl.\right.
\end{aligned}\tag{29}$$

Since $|s'|$ is strongly convex on $[d_1,d_2]$, we get for $l\in[\sigma^{-1}(d_1),\sigma^{-1}(d_2)]$:

$$\begin{aligned}\left|\left(s'\circ\sigma\right)(l)\right| &= \left|s'\left(\frac{d_2-\sigma(l)}{d_2-d_1}d_1+\frac{\sigma(l)-d_2}{d_2-d_1}d_2\right)\right| \\ &\le \frac{d_2-\sigma(l)}{d_2-d_1}\left|s'(d_1)\right|+\frac{\sigma(l)-d_2}{d_2-d_1}\left|s'(d_2)\right| \\ &\quad -a(d_2-\sigma(l))(\sigma(l)-d_2).\end{aligned} \tag{30}$$

Hence, we obtain

$$\begin{aligned}|\Xi_1+\Xi_2| \le{}& \frac{\|\psi\|_{[d_1,(d_1+d_2/2)],\infty}}{(d_2-d_1)\Gamma(\nu)}\int_{\sigma^{-1}(d_1)}^{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)} \\ &\cdot\left|\int_{\sigma^{-1}(d_1)}^{l}\sigma'(\beta_1)(\sigma(\beta_1)-d_1)^{\nu-1}d\beta_1\right| \\ &\times\left[(d_2-\sigma(l)\left|s'(d_1)\right|+(\sigma(l)-d_1)\left|s'(d_2)\right|\right. \\ &\left.-a(d_2-\sigma(l))(\sigma(l)-d_2)\right]\sigma'(l)dl \\ &+\frac{\|\psi\|_{[(d_1+d_2/2),d_1],\infty}}{(d_2-d_1)\Gamma(\nu)}\int_{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)}^{\sigma^{-1}(d_2)} \\ &\cdot\left|\int_{l}^{\sigma^{-1}(d_2)}\sigma'(\beta_1)(d_2-\sigma(\beta_1))^{\nu-1}d\beta_1\right| \\ &\times\left[(d_2-\sigma(l)\left|s'(d_1)\right|+(\sigma(l)-d_1)\left|s'(d_2)\right|\right. \\ &\left.-a(d_2-\sigma(l))(\sigma(l)-d_2)\right]\sigma'(l)dl\end{aligned} \tag{31}$$

By putting value of integration in above inequalities,

$$\begin{aligned}&\int_{\sigma^{-1}(d_1)}^{l}\sigma'(\beta_1)(\sigma(\beta_1)-d_1)^{\nu-1}d\beta_1=\frac{(\sigma(l)-d_1)^{\nu}}{\nu}, \\ &\int_{l}^{\sigma^{-1}(d_2)}\sigma'(\beta_1)(d_2-\sigma(\nu))^{\nu-1}d\beta_1=\frac{(d_2-\sigma(l))^{\nu}}{\nu}, \\ &\int_{\sigma^{-1}(d_1)}^{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)}(\sigma(l)-d_1)^{\nu+1}\sigma'(l)dl \\ &\quad=\int_{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)}^{\sigma^{-1}(d_2)}(d_2-\sigma(l))^{\nu+1}\sigma'(l)dl \\ &\quad=\frac{(d_2-d_1)^{\nu+2}}{2^{\nu+2}(\nu+2)}, \\ &\int_{\sigma^{-1}(d_1)}^{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)}(\sigma(l)-d_1)^{\nu}(d_2-\sigma(l))\sigma'(l)dl \\ &\quad=\int_{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)}^{\sigma^{-1}(d_2)}(d_2-\sigma(l))^{\nu}(\sigma(l)-d_1)\sigma'(l)dl \\ &\quad=\frac{(d_2-d_1)^{\nu+2}(\nu+3)}{2^{\nu+2}(\nu+1)(\nu+2)}, \\ &\int_{\sigma^{-1}(d_1)}^{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)}\left(\frac{\sigma(l)-d_1}{\nu}\right)^{\nu+1}(d_2-\sigma(l))\sigma'(l)dl \\ &\quad=\frac{(d_2-d_1)^{\nu+3}(\nu+4)}{2^{\nu+3}(\nu+2)(\nu+3)\nu},\end{aligned} \tag{32}$$

and simple computations yield

$$\begin{aligned}&=\frac{(d_2-d_1)^{\nu+1}}{2^{\nu+2}\Gamma(\nu+3)}\left\{\|\psi\|_{\left[d_1,\frac{d_1+d_2}{2}\right],\infty}\left[(\nu+3)\left|s'(d_1)\right|+(\nu+1)\right.\right. \\ &\quad\left.\cdot\left|s'(d_2)-\frac{a(\nu+4)(d_2-d_1)^2}{2}\right|\right]\Big\} \\ &\quad+\left\{\|\psi\|_{\left[\frac{d_1+d_2}{2},d_2\right],\infty}\left[(\nu+1)\left|s'(d_1)\right|\right.\right. \\ &\quad\left.\left.+(\nu+3)\left|s'(d_2)\right|-\frac{a(\nu+4)(d_2-d_1)^2}{2\nu}\right]\right\} \\ &\le\frac{(d_2-d_1)^{\nu+1}\|\psi\|_{[d_1,d_2],\infty}}{2^{\nu+1}\Gamma(\nu+2)}\left[\left|s'(d_1)\right|+\left|s'(d_2)\right|-a\|d_1-d_2\|^2\right],\end{aligned} \tag{33}$$

which complete our proof. □

Remark 10. From Theorem 9, we can obtain some special cases as follows:

(1) If $\sigma(\beta_1)=x$, $a=0$, then inequlaity (28) 3.1 becomes

$$\begin{aligned}&\left|s\left(\frac{d_1+d_2}{2}\right)\left[{}_{(d1+d22)^{+}}I^{\nu}\psi(d_2)+I^{\nu}_{\left(\frac{d_1+d_2}{2}\right)_{-}}\psi(d_1)\right]\right. \\ &\quad\left.-\left[\psi(d_2)\left({}_{(d1+d22)^{+}}I^{\nu}_{\psi}s\right)(d_1)+\psi(d_2)\left({}_{w}I^{\nu}_{\left(\frac{d_1+d_2}{2}\right)_{-}}s\right)(d_2)\right|\right. \\ &\le\frac{(d_2-d_1)^{\nu+1}}{2^{\nu+2}\Gamma(\nu+3)}\left\{\|\psi\|_{\left[d_1,\frac{d_1+d_2}{2}\right],\infty}\left[(\nu+3)\left|s'(d_1)\right|+(\nu+1)\left|s'(d_2)\right|\right]\right. \\ &\quad\left.+\|\psi\|_{\left[\frac{d_1+d_2}{2},d_2\right],\infty}\left[(\nu+1)\left|s'(d_1)\right|+(\nu+3)\left|s'(d_2)\right|\right]\right\} \\ &\le\frac{(d_2-d_1)^{\nu+1}\|\psi\|_{[d_1,d_2],\infty}}{2^{\nu+1}\Gamma(\nu+2)}\left[\left|s'(d_1)\right|+\left|s'(d_2)\right|\right].\end{aligned} \tag{34}$$

(2) If $a=0$, $\sigma(\beta_1)=x$, and $\psi(\beta_1)=1$,then inequality 3.1 becomes

$$\begin{aligned}&\left|\frac{2^{\nu-1}\Gamma(\nu+1)}{(d_2-d_1)^{\nu}}\left[{}_{(d1+d22)^{+}}I^{\nu}\psi(d_2)+I^{\nu}_{\left(\frac{d_1+d_2}{2}\right)_{-}}\psi(d_1)\right]-s\left(\frac{d_1+d_2}{2}\right)\right| \\ &\quad\le\frac{(d_2-d_1)^{\nu+1}}{2^{\nu+2}\Gamma(\nu+3)}\left\{\left[(\nu+3)\left|s'(d_1)+(\nu+1)\left|s'(d_2)\right|\right]\right.\right. \\ &\quad\left.\left.+\left[(\nu+1)\left|s'(d_1)\right|+(\nu+3)\left|s'(d_2)\right|\right]\right\} \\ &\quad\le\frac{(d_2-d_1)^{\nu+1}}{2^{\nu+2}\Gamma(\nu+2)}\left[\left|s'(d_1)+\left|s'(d_2)\right|\right]\right.,\end{aligned} \tag{35}$$

(3) If $a=0$, $\sigma(\beta_1)=x$, $\psi(\beta_1)=1$, and v=1, then inequality 3.1 becomes

$$\left|\frac{1}{d_2-d_1}\int_{d_1}^{d_2}s(\beta_1)d\beta_1-s\left(\frac{d_1+d_2}{2}\right)\right| \leq \frac{c_2-c_1}{8}\left[\left|s'(d_1)\right|+\left|s'(d_2)\right|\right], \tag{36}$$

Theorem 10. *Let $0 \leqslant d_1 < d_2$, let $s:[d_1,d_2]\subset[0,\infty)\longrightarrow\mathfrak{R}$ be a (continuously) differentiable mapping on $[d_1,d_2]$ such that $s(\beta_1)=s(d_1)+\int_{d_1}^{\beta_1}s'(l)dl$, and let $\psi:[d_1,d_2]\longrightarrow\mathfrak{R}$ be an integrable, positive, and weighted symmetric function with respect to $d_1+d_2/2$. If, in addition, $|s'|^q$ is strong convex on $[d_1,d_2]$ with $q\geq 1$, and "s" is an increasing and positive function from $[d_1,d_2]$ onto itself such that its derivative $s'(\beta_1)$ is continuous on (d_1,d_2), then for $\nu>0$, the following inequalities hold:*

$$\begin{aligned}
|\Xi_1+\Xi_2| &\leq \frac{(d_2-d_1)^{\nu+1}}{2^{\nu+1+(1/q)}(\nu+2)^{1/q}\Gamma(\nu+2)}\\
&\quad\times\left\{\|\psi\|_{\left[d_1,\frac{d_1+d_2}{2}\right],\infty}\left[(\nu+3)\left|s'(d_1)\right|^q+(\nu+1)\left|s'(d_2)\right|^q-\frac{a(\nu+4)(d_2-d_1)^2}{2}\right]^{1/q}\right\}\\
&\quad+\|\psi\|_{\left[\frac{d_1+d_2}{2},d_2\right],\infty}\left[(\nu+1)\left|s'(d_1)\right|^q+(\nu+3)\left|s'(d_2)\right|^q-\frac{a(\nu+4)(d_2-d_1)^2}{2\nu}\right]^{1/q}\Bigg\}\\
&\leq \frac{(d_2-d_1)^{\nu+1}\|\psi\|_{[d_1,d_2],\infty}}{2^{\nu+1+(1/q)}(\nu+2)^{1/q}\Gamma(\nu+2)}\\
&\quad\cdot\left\{\left[(\nu+3)\left|s'(d_1)\right|^q+(\nu+1)\left|s'(d_2)\right|^q-\frac{a(\nu+4)(d_2-d_1)^2}{2}\right]^{1/q}\right.\\
&\quad\left.+\left[(\nu+1)\left|s'(d_1)\right|^q+(\nu+3)\left|s'(d_2)\right|^q-\frac{a(\nu+4)(d_2-d_1)^2}{2}\right]^{1/q}\right\}.
\end{aligned} \tag{37}$$

Proof. Since $|s'|^q$ is strong convex on $[d_1,d_2]$, we get for $l\in[\sigma^{-1}(d_1),\sigma^{-1}(d_2)]$

$$\begin{aligned}
\left|\left(s'\circ\sigma\right)(l)\right|^q &= \left|s'\left(\frac{d_2-e(l)}{d_2-d_1}d_1+\frac{\sigma(l)-d_1}{d_2-d_1}d_2\right)\right|^q\\
&\leq \frac{d_2-\sigma(l)}{d_2-d_1}\left|s'(d_1)\right|^q+\frac{\sigma(l)-d_1}{d_2-d_1}\\
&\quad\cdot\left|s'(d_2)\right|^q-a(d_2-\sigma(l))(\sigma(l)-d_1).
\end{aligned} \tag{38}$$

By making use Lemma 8, power mean inequality, and strong convexity of $|s'|^q$, we get

$$\begin{aligned}
|\Xi_1+\Xi_2|\nu &\leq \frac{1}{\Gamma(\nu)}\int_{\sigma^{-1}(d_1)}^{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)}\\
&\quad\cdot\left|\int_{\sigma^{-1}(d_1)}^{l}\sigma'(\beta_1)(\sigma(\beta_1)-d_1)^{\nu-1}(\psi\circ\sigma)(\beta_1)d\beta_1\right|\\
&\quad\cdot\left|\left(s'\circ\sigma\right)(l)\right|\sigma'(l)dl+\frac{1}{\Gamma(\nu)}\int_{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)}^{\sigma^{-1}(d_2)}\\
&\quad\cdot\left|\int_{l}^{\sigma^{-1}(d_2)}\sigma'(\beta_1)(d_2-\sigma(\beta_1))^{\nu-1}(\psi\circ\sigma)(\beta_1)d\beta_1\right|\\
&\quad\cdot\left|\left(s'\circ\sigma\right)(l)\right|\sigma'(l)dl\\
&\leq \frac{1}{\Gamma(\nu)}\left(\int_{\sigma^{-1}(d_1)}^{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)}\left|\int_{\sigma^{-1}(d_1)}^{l}\sigma'(\beta_1)(\sigma(\nu)-d_1)^{\nu-1}(\psi\circ\sigma)(\beta_1)d\beta_1\right|\sigma'(l)dl\right)^{1-\frac{1}{q}}\\
&\quad\times\frac{1}{\Gamma(\nu)}\left(\int_{\sigma^{-1}(d_1)}^{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)}\left|\int_{\sigma^{-1}(d_1)}^{l}\sigma'(\beta_1)(\sigma(\beta_1)-d_1)^{\nu-1}(\psi\circ\sigma)(\nu)d\beta_1\right|\right.\\
&\quad\left.\cdot\left|\left(s'\circ\sigma\right)(l)\right|^q\sigma'(l)dl\right)^{\frac{1}{q}}+\frac{1}{\Gamma(\nu)}\\
&\quad\cdot\left(\int_{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)}^{\sigma^{-1}(d_2)}\left|\int_{l}^{\sigma^{-1}(d_2)}\sigma'(\beta_1)(d_2-\sigma(\beta_1))^{\nu-1}(\psi\circ\sigma)(\beta_1)d\nu\right|\sigma'(l)dl\right)^{1-\frac{1}{q}}\\
&\quad\times\frac{1}{\Gamma(\nu)}\left(\int_{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)}^{\sigma^{-1}(d_2)}\left|\int_{l}^{\sigma^{-1}(d_2)}\sigma'(\beta_1)(d_2-\sigma(\beta_1))^{\nu-1}(\psi\circ\sigma)(\beta_1)d\beta_1\right|\right.\\
&\quad\left.\cdot\left|\left(s'\circ\sigma\right)(l)\right|^q\sigma'(l)dl\right)^{\frac{1}{q}}\\
&\leq \frac{\|\psi\|_{[d_1,(d_1+d_2/2)],\infty}}{\Gamma(\nu)}\left(\int_{\sigma^{-1}(d_1)}^{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)}\left|\int_{\sigma^{-1}(d_1)}^{l}\sigma'(\beta_1)(\sigma(\beta_1)-d_1)^{\nu-1}d\beta_1\right|\sigma'(l)dl\right)^{1-1/q}\\
&\quad\times\left(\int_{\sigma^{-1}(d_1)}^{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)}\left|\int_{\nu^{-1}(d_1)}^{l}\sigma'(\nu)(\sigma(\beta_1)-d_1)^{\nu-1}d\beta_1\right|\left|\left(s'\circ\sigma\right)(l)\right|^q\sigma'(l)dl\right)^{1/q}\\
&\quad+\frac{\|\psi\|_{[(d_1+d_2/2),\nu],\infty}}{\Gamma(\nu)}\left(\int_{\sigma^{-1}\left(\frac{d_1+\nu}{2}\right)}^{\sigma^{-1}(d_2)}\left|\int_{l}^{\sigma^{-1}(d_2)}\sigma'(\nu)(d_2-\sigma(\beta_1))^{\nu-1}d\beta_1\right|\sigma'(l)dl\right)^{1-\frac{1}{q}}\\
&\quad\times\left(\int_{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)}^{\sigma^{-1}(d_2)}\left|\int_{l}^{\sigma^{-1}(d_2)}\sigma'(\beta_1)(d_2-\sigma(\beta_1))^{\nu-1}d\beta_1\right|\left|\left(s'\circ\sigma\right)(l)\right|^q\sigma'(l)dl\right)\\
&\leq \frac{\|\psi\|_{[d_1,(d_1+d_2/2)],\infty}}{\Gamma(\nu)}\left(\int_{\sigma^{-1}(d_1)}^{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)}\left|\int_{\sigma^{-1}(d_1)}^{l}\sigma'(\beta_1)(\sigma(\beta_1)-d_1)^{\nu-1}d\beta_1\right|\sigma'(l)dl\right)^{1-1/q}\\
&\quad\times\left(\int_{\nu^{-1}(d_1)}^{\sigma^{-1}\left(\frac{d_1+\nu}{2}\right)}\left|\int_{\sigma^{-1}(d_1)}^{l}\sigma'(\beta_1)(\sigma(\beta_1)-d_1)^{\nu-1}d\beta_1\right|\right.\\
&\quad\left.\times\left(\frac{d_2-\sigma(l)}{d_2-d_1}\left|s'(d_1)\right|^q+\frac{\sigma(l)-d_2}{d_2-d_1}\left|s'(d_2)\right|^q-a(d_2-\sigma(l))(\sigma(l)-d_2)\right)\sigma'(l)dl\right)^{1/q}\\
&\quad+\frac{\|\psi\|_{[(d_1+\nu/2),\nu],\infty}}{\Gamma(\nu)}\left(\int_{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)}^{\nu^{-1}(d_2)}\left|\int_{l}^{\sigma^{-1}(\nu)}\sigma'(\beta_1)(\nu-\sigma(\beta_1))^{\nu-1}d\beta_1\right|\sigma'(l)dl\right)^{1-1/q}\\
&\quad\times\left(\int_{\sigma^{-1}\left(\frac{d_1+\nu}{2}\right)}^{\sigma^{-1}(\nu)}\left|\int_{l}^{\sigma^{-1}(\nu)}\sigma'(\beta_1)(\nu-\sigma(\nu))^{\nu-1}d\beta_1\right|\left(\frac{d_2-\sigma(l)}{d_2-d_1}\left|s'(d_1)\right|^q\right.\right.\\
&\quad\left.\left.+\frac{\sigma(l)-d_2}{d_2-d_1}\left|s'(d_2)\right|^q-a(d_2-\sigma(l))(\sigma(l)-d_2)\right)\sigma'(l)dl\right)\\
&= \frac{(d_2-d_1)^{\nu+1}}{2^{\nu+1+(1/q)}(\nu+2)^{1/q}\Gamma(\nu+2)}\\
&\quad\cdot\left\{\|\psi\|_{\left[d_1,\frac{d_1+\nu}{2}\right],\infty}\left[(\nu+3)\left|s'(d_1)\right|^q+(\nu+1)\left|s'^{(\nu)}\right|^q-\frac{a(\nu+4)(d_2-d_1)^2}{2}\right]^{1/q}\right\}\\
&\quad+\|\psi\|_{\left[\frac{d_1+d_2}{2},d_2\right],\infty}\left[(\nu+1)\left|s'(d_1)\right|^q+(\nu+3)\left|s'(d_2)\right|^q-\frac{a(\nu+4)(d_2-d_1)^2}{2\nu}\right]^{1/q}\Bigg\}\\
&\leq \frac{(d_2-d_1)^{\nu+1}\|\psi\|_{[d_1,d_2],\infty}}{2^{\nu+1+(1/q)}(\nu+2)^{1/q}\Gamma(\nu+2)}\\
&\quad\cdot\left\{\left[(\nu+3)\left|s'(d_1)\right|^q+(\nu+1)\left|s'(d_2)\right|^q-\frac{a(\nu+4)(d_2-d_1)^2}{2}\right]^{1/q}\right.\\
&\quad\left.+\left[(\nu+1)\left|s'(d_1)\right|^q+(\nu+3)\left|s'(d_2)\right|^q-\frac{a(\nu+4)(d_2-d_1)^2}{2}\right]^{1/q}\right\}
\end{aligned} \tag{39}$$

Hence, the proof is completed. □

Remark 11. From Theorem 10, we can obtain some special cases as follows:

(1) Theorem 4 of [32] is obtained if we take $a = 0$, $\omega = \psi$, and $\rho = \sigma$ in Theorem 10

(2) If $a = 0$, $\sigma(\beta_1) = x$, then inequality (37) becomes

$$\begin{aligned}
&\left| s\left(\frac{d_1+d_2}{2}\right)\left[{}_{(d1+d22)^+}I^{\nu}\psi(d_2) + I^{\nu}_{\left(\frac{d_1+d_2}{2}\right)_-}\psi(d_1)\right]\right. \\
&\quad \left. - \left[\psi(d_2)\left({}_{\left(\frac{d_1+d_2}{2}\right)}I^{\nu}_{\psi}s\right)(d_1) + \psi(d_2)\left({}_{w}I^{\nu}_{\left(\frac{d_1+d_2}{2}\right)_-}s\right)(d_2)\right]\right| \\
&\le \frac{(d_2-d_1)^{\nu+1}}{2^{\nu+1+(1/q)}(\nu+2)^{1/p}\Gamma(\nu+2)} \\
&\quad \cdot\left\{ \|\psi\|_{\left[d_1,\frac{d_1+d_2}{2}\right],\infty}\left[(\nu+3)\left|s'(d_1)\right|^q + (\nu+1)\left|s'(d_2)\right|^q\right]^{1/q}\right. \\
&\quad \left. + \|\psi\|_{\left[\frac{d_1+d_2}{2},d_2\right],\infty}\left[(\nu+1)\left|s'(d_1)\right|^q + (\nu+3)\left|s'(d_2)\right|^q\right]^{1/q}\right\} \\
&\le \frac{(d_2-d_1)^{\nu+1}\|\psi\|_{[d_1,d_2],\infty}}{2^{\nu+1+(1/q)}(\nu+2)^{1/q}\Gamma(\nu+2)} \\
&\quad \cdot\left\{ \left[(\nu+3)\left|s'(d_1)\right|^q + (\nu+1)\left|s'(d_2)\right|^q\right]^{1/q}\right. \\
&\quad \left. + \left[(\nu+1)\left|s'(d_1)\right|^q + (\nu+3)\left|s'(d_2)\right|^q\right\}\right.
\end{aligned} \tag{40}$$

(3) If $a = 0$, $\sigma(\beta_1) = x$, and $\psi(\beta_1) = 1$, then inequality (37) becomes

$$\begin{aligned}
&\left|\frac{2^{\nu-1}\Gamma(\nu+1)}{(d_2-d_1)^{\nu}}\left[{}_{\left(\frac{d_1+d_2}{2}\right)^+}I^{\nu}\psi(d_2) + I^{\nu}_{\left(\frac{d_1+d_2}{2}\right)_-}\psi(d_1)\right] - s\left(\frac{d_1+d_2}{2}\right)\right| \\
&\le \frac{(d_2-d_1)^{\nu+1}\|\psi\|_{[d_1,d_2]},\infty}{2^{\nu+1+(2/q)}(p\nu+1)^{1/p}\Gamma(\nu+1)} \\
&\quad \cdot\left\{\left[3\left|s'(d_1)\right|^q + \left|s'(d_2)\right|^q\right]^{1/q} + \left|s'(d_1)\right|^q + \left[3\left|s'(d_2)\right|^q\right]^{1/q}\right\}
\end{aligned} \tag{41}$$

which is already obtained in ((39) [Theorem 5])

(4) If $a = 0$, $\sigma(\beta_1) = x$, $\psi(\beta_1) = 1$ and v=1, then inequality (37) 3.10 become

$$\begin{aligned}
&\left|\frac{1}{d_2-d_1}\int_{d_1}^{d_2} s(\beta_1)d\beta_1 - s\left(\frac{d_1+d_2}{2}\right)\right| \\
&\le \frac{c_2-c_1}{8\sqrt[q]{3}}\left\{\left[\left|s'(d_1)\right|^q + 2\left|s'(d_2)\right|^q\right]^{1/q}\right. \\
&\quad \left. + \left[2\left|s'(d_1)\right|^q + \left|s'(d_2)\right|^q\right]^{1/q}\right\}
\end{aligned} \tag{42}$$

Theorem 12. *Let $0 \le d_1 < d_2$ and $s : [d_1, d_2] \subset [0,\infty) \longrightarrow \Re$ be a (continuously) differentiable mapping on $[d_1, d_2]$ such that $s(\beta_1) = s(d_1) + \int_{d_1}^{\beta_1} s'(l)dl$, and let $h : [d_1, d_2] \longrightarrow \Re$ be an integrable, positive, and weighted symmetric function with respect to $d_1 + d_2/2$. If, in addition, $|s'|^r$ is strong convex on $[d_1, d_2]$with $(1/v) + (1/r) = 1$ and $r > 1$, and "s" is an increasing and positive function from $[d_1, d_2]$ onto itself such that its derivative $s'(\beta_1)$ is continuous on (d_1, d_2), then for $a > 0$, the following inequalities hold:*

$$\begin{aligned}
|\Xi_1 + \Xi_2| &\le \frac{(d_2-d_1)^{\nu+1}}{2^{\nu+1+(2/q)}(p\nu+1)^{1/p}\Gamma(\nu+1)} \\
&\quad \times\left\{\|\psi\|_{\left[d_1,\frac{d_1+d_2}{2}\right],\infty}\left[3\left|s'(d_1)\right|^q + \left|s'(d_2)\right|^q\right.\right. \\
&\quad \left.\left. - \frac{a(\nu+4)(d_2-d_1)^2}{2}\right]^{1/q}\right\} \\
&\quad + \|v\|_{\left[\frac{d_1+d_2}{2},d_2\right],\infty}\left[(\nu+1)\left|s'(d_1)\right|^q + (\nu+3)\left|s'(d_2)\right|^q\right. \\
&\quad \left. - \frac{a(\nu+4)(d_2-d_1)^2}{2\nu}\right]^{1/q}\Big\} \\
&\le \frac{(d_2-d_1)^{\nu+1}\|\psi\|_{[d_1,d_2],\infty}}{2^{\nu+1+(1/q)}(\nu+2)^{1/q}\Gamma(\nu+2)} \\
&\quad \cdot\left\{\left[(\nu+3)\left|s'(d_1)\right|^q + (\nu+1)\left|s'(d_2)\right|^q - \frac{a(\nu+4)(d_2-d_1)^2}{2}\right]^{1/q}\right. \\
&\quad \left. + \left[(\nu+1)\left|s'(d_1)\right|^q + (\nu+3)\left|s'(d_2)\right|^q - \frac{a(\nu+4)(d_2-d_1)^2}{2}\right]^{\frac{1}{q}}\right\}.
\end{aligned} \tag{43}$$

Proof. Sinces $|s'|^q$ is strong convex on $[d_1, d_2]$, we get for $l \in [\sigma^{-1}(d_1), \sigma^{-1}(d_2)]$

$$\begin{aligned}
\left|\left(s'\circ\sigma\right)(l)\right|^q &= \left|s'\left(\frac{d_2-\sigma(l)}{d_2-d_1}d_1 + \frac{\sigma(l)-d_1}{d_2-d_1}d_2\right)\right|^q \\
&\le \frac{d_2-\sigma(l)}{d_2-d_1}\left|s'(d_1)\right|^q + \frac{\sigma(l)-d_1}{d_2-d_1}\left|s'(d_2)\right|^q \\
&\quad - a(d_2-\sigma(l))(\sigma(l)-d_1).
\end{aligned} \tag{44}$$

By making use Lemma 8, Hölder inequality, and strong convexity of $|s'|^q$ and properties of modulus, we have

$$\begin{aligned}
|\Xi_1 + \Xi_2| &\le \frac{1}{\Gamma(\nu)}\int_{\sigma^{-1}(d_1)}^{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)} \\
&\quad \cdot\left|\int_{\sigma^{-1}(d_1)}^{l}\sigma'(\beta_1)(\sigma(\beta_1)-d_1)^{\nu-1}(\psi\circ\sigma)(\beta_1)d\beta_1\right| \\
&\quad \cdot\left|\left(s'\circ\sigma\right)(l)\right|\sigma'(l)dl + \frac{1}{\Gamma(\nu)}\int_{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)}^{\sigma^{-1}(d_2)} \\
&\quad \cdot\left|\int_{l}^{\sigma^{-1}(d_2)}\sigma'(\beta_1)(d_2-\sigma(\beta_1))^{\nu-1}(\psi\circ\sigma)(\beta_1)d\beta_1\right| \\
&\quad \cdot\left|\left(s'\circ\sigma\right)(l)\right|\sigma'(l)dl
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{\Gamma(\nu)}\left(\int_{\sigma^{-1}(d_1)}^{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)}\left|\int_{\sigma^{-1}(d_1)}^{l}\sigma'(\beta_1)(\sigma(\nu)\right.\right.\\
&\quad\left.\left.-d_1)^{\nu-1}(\psi\circ\sigma)(\beta_1)d\beta_1\right|^p\sigma'(l)dl\right)^{1/p}\\
&\quad\times\left(\int_{\sigma^{-1}(d_1)}^{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)}\left|\left(s'^{\circ}\sigma\right)(l)\right|^q\sigma'^{(l)}dl\right)^{1/q}\\
&\quad+\frac{1}{\Gamma(\nu)}\left(\int_{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)}^{\sigma^{-1(d_2)}}\left|\int_{l}^{\sigma^{-1}(d_2)}\sigma'(\beta_1)\right.\right.\\
&\quad\left.\left.\cdot(d_2-\sigma(\beta_1))^{\nu-1}(\psi\circ\sigma)(\beta_1)d\beta_1\right|^p\sigma'(l)dl\right)^{1/p}\\
&\quad\times\left(\int_{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)}^{\sigma^{-1(d_2)}}\left|\left(s'^{\circ}\sigma\right)(l)\right|^q\sigma'(l)dl\right)^{1/q}\\
&\le \frac{\|\psi\|_{[d_1,(d_1+d_2/2)],\infty}}{\Gamma(\nu)}\\
&\quad\cdot\left(\int_{\sigma^{-1}(d_1)}^{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)}\left|\int_{\sigma^{-1}(d_1)}^{l}\sigma'(\beta_1)(\sigma(\beta_1)-d_1)^{\nu-1}d\beta_1\right|^p\sigma'(l)dl\right)^{1/p}\\
&\quad\times\left(\int_{\sigma^{-1}(d_1)}^{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)}\left|\left(s'\circ\sigma\right)(l)\right|^q\sigma'(l)dl\right)^{1/q}\\
&\quad+\frac{\|\psi\|_{[(d_1+d_2/2),d_2],\infty}}{\Gamma(\nu)}\\
&\quad\cdot\left(\int_{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)}^{\sigma^{-1}(d_2)}\left|\int_{l}^{\sigma^{-1}(d_2)}\sigma'(\beta_1)(d_2-\sigma(\beta_1))^{\nu-1}d\beta_1\right|^p\sigma'(l)dl\right)^{1/p}\\
&\quad\times\left(\int_{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)}^{\sigma^{-1}(d_2)}\left|\left(s'\circ\sigma\right)(l)\right|^q\sigma'(l)dl\right)^{1/q}\\
&\le \frac{\|\psi\|_{[d_1,(d_1+d_2/2)],\infty}}{\Gamma(\nu)}\\
&\quad\cdot\left(\int_{\sigma^{-1}(d_1)}^{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)}\left|\int_{\sigma^{-1}(d_1)}^{l}\sigma'(\beta_1)(\sigma(\nu)-d_1)^{\nu-1}d\beta_1\right|^p\sigma'(l)dl\right)^{1/p}\\
&\quad\times\left[\int_{\sigma^{-1}(d_1)}^{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)}\left(\frac{d_2-\sigma(l)}{d_2-d_1}\left|s'd_1\right|^q+\frac{\sigma(l)-d_2}{d_2-d_1}\right.\right.\\
&\quad\left.\left.\cdot\left|s'd_2\right|^q-a(d_2-\sigma(l))(\sigma(l)-d_2)\right)\sigma'(l)dl\right]\Big)^{1/q}\\
&\quad+\frac{\|\psi\|_{[(d_1+d_2/2),d_2],\infty}}{\Gamma(\nu)}\\
&\quad\cdot\left(\int_{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)}^{\sigma^{-1}(d_2)}\left|\int_{l}^{\sigma^{-1}(d_2)}\sigma'(\beta_1)(d_2-\sigma(\beta_1))^{\nu-1}d\beta_1\right|^p\sigma'(l)dl\right)^{1/p}\\
&\quad\times\left[\int_{\sigma^{-1}\left(\frac{d_1+d_2}{2}\right)}^{\sigma^{-1}(d_2)}\left(\frac{d_2-\sigma(l)}{d_2-d_1}\left|s'(d_1)\right|^q\right.\right.\\
&\quad\left.\left.+\frac{\sigma(l)-d_2}{d_2-d_1}\left|s'(d_2)\right|^q-a(d_2-\sigma(l))(\sigma(l)-d_2)\right)\sigma'(l)dl\right]^{1/q}\\
&= \frac{(d_2-d_1)^{\nu+1}}{2^{\nu+1+(2/q)}(p\nu+2)^{1/p}\Gamma(\nu+1)}\\
&\quad\cdot\left\{\|\psi\|_{\left[d_1,\frac{d_1+d_1}{2}\right],\infty}\left[3\left|s'(d_1)\right|^q+\left|s'(d_2)\right|^q-\frac{2a}{\nu}(d_2-d_1)\right]\right\}\\
&\quad+\|\psi\|_{\left[\frac{d_1+d_2}{2},d_2\right],\infty}\left[\left|s'(d_1)\right|^q+3\left|s'(d_2)\right|^q-\frac{2a(d_2-d_1)^2}{\nu}\right]^{1/q}\Bigg\}\\
&\le \frac{(d_2-d_1)^{\nu+1}\|\psi\|_{[d_1,d_2],\infty}}{2^{\nu+1+(2/q)}(p\nu+1)^{1/p}\Gamma(\nu+1)}\\
&\quad\cdot\left\{\left[3\left|s'(d_1)\right|^q+\left|s'(d_2)\right|^q-2a(d_2-d_1)^2\right]^{1/q}\right.\\
&\quad\left.+\left[(\nu+1)\left|s'(d_1)\right|^q+(\nu+3)\left|s'(d_2)\right|^q-2a(d_2-d_1)^2\right]^{1/q}\right\}
\end{aligned}
\tag{45}
$$

Hence, the proof is completed. □

Remark 13. From Theorem 12, we can obtain some special cases as follows:

(1) Theorem 4 of [32] is obtained if we take $a=0$, $\omega=\psi$, and $\rho=\sigma$ in Theorem 12

(2) If $a=0$, $\sigma(\beta_1)=x$, inequality (43) becomes

$$
\begin{aligned}
&\left|s\left(\frac{d_1+d_2}{2}\right)\left[{}_{(d1+d22)^+}I^{\nu}\psi(d_2)+I^{\nu}_{\left(\frac{d_1+d_2}{2}\right)_-}\psi(d_1)\right]\right.\\
&\quad\left.-\left[\psi(d_2)\left({}_{(d1+d22)}I^{\nu}_{\psi}s\right)(d_1)+\psi(d_2)\left({}_{w}I^{\nu}_{\left(\frac{d_1+d_2}{2}\right)_-}s\right)(d_2)\right]\right|\\
&\le \frac{(d_2-d_1)^{\nu+1}}{2^{\nu+1+(2/q)}(p\nu+1)^{1/p}\Gamma(\nu+1)}\\
&\quad\cdot\left\{\|\psi\|_{\left[d_1,\frac{d_1+d_2}{2}\right],\infty}\left[3\left|s'(d_1)\right|^q+\left|s'(d_2)\right|^q\right]^{1/q}\right.\\
&\quad\left.+\|\psi\|_{\left[\frac{d_1+d_2}{2},d_2\right],\infty}\left[\left|s'(d_1)\right|^q+3\left|s'(d_2)\right|^q-\right]^{1/q}\right\}\\
&\le \frac{(d_2-d_1)^{\nu+1}\|\psi\|_{[d_1,d_2],\infty}}{2^{\nu+1+(1/q)}(p\nu+2)^{1/q}\Gamma(\nu+2)}\\
&\quad\cdot\left\{\left[3\left|s'(d_1)\right|^q+\left|s'(d_2)\right|^q\right]^{1/q}+\left[\left|s'(d_1)\right|^q+3\left|s'(d_2)\right|^q\right\}.
\end{aligned}
\tag{46}
$$

(3) If $a=0$, $\sigma(\beta_1)=\rho(x)=x$, and $\psi(\beta_1)=\omega(x)=1$, then inequality (43) becomes

$$
\begin{aligned}
&\left|\frac{2^{\nu-1}\Gamma(\nu+1)}{(d_2-d_1)^{\nu}}\left[{}_{(d1+d22)^+}I^{\nu}\psi(d_2)+I^{\nu}_{\left(\frac{d_1+d_2}{2}\right)_-}\psi(d_1)\right]-s\left(\frac{d_1+d_2}{2}\right)\right|\\
&\le \frac{(d_2-d_1)^{\nu+1}\|\psi\|_{[d_1,d_2]},\infty}{2^{\nu+1+(2/q)}(p\nu+1)^{1/p}\Gamma(\nu+1)}\\
&\quad\cdot\left\{\left[3\left|s'(d_1)\right|^q+\left|s'(d_2)\right|^q\right]^{1/q}+\left|s'(d_1)\right|^q+\left[3\left|s'(d_2)\right|^q\right]^{\frac{1}{q}}\right\},
\end{aligned}
\tag{47}
$$

which is already obtained in ((39) [Theorem 6])

(4) If $a=0$, $\sigma(\beta_1)=\rho(x)=x$, $\psi(\beta_1)=\omega(x)=1$, and $\nu=1$, then inequality (43) becomes

$$
\begin{aligned}
&\left|\frac{1}{d_2-d_1}\int_{d_1}^{d_2}s(\beta_1)d\beta_1-s\left(\frac{d_1+d_2}{2}\right)\right|\\
&\le \frac{c_2-c_1}{16}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\\
&\quad\cdot\left\{\left[3\left|s'(d_1)\right|^q+\left|s'(d_2)\right|^q\right]^{1/q}+\left|s'(d_1)\right|^q+\left[3\left|s'(d_2)\right|^q\right]^{\frac{1}{q}}\right\},
\end{aligned}
\tag{48}
$$

which is already obtained.

4. Conclusions

In this paper, we established the midpoint type inequalities for the strong convex function by using positive weighted symmetry kernels. As an application, our established inequalities can be applied to the special means of real numbers. Our results can be used to estimate error for the midpoint formula. It is interesting to establish midpoint type inequalities for the strong convex function in the setting of a different version of fractional integral operators.

Data Availability

All data required for this research is included within this paper.

Conflicts of Interest

The authors declare that they do not have any competing interests.

Authors' Contributions

Hengxiao Qi designed the problem, Waqas Nazeer proved the main results, Sami Ullah Zakir wrote the first version of the paper, and Kamsing Nonlaopon analyzed the results, wrote the final version of the paper, and arranged funding for this paper.

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