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Research Article

A New Application of the Sumudu Transform for the Falling Body Problem

Esra Karatas Akgül[1], **Ali Akgül**[1], **and Rubayyi T. Alqahtani**[2]

[1]*Siirt University, Art and Science Faculty, Department of Mathematics, TR-56100 Siirt, Turkey*
[2]*Department of Mathematics and Statistics, College of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh 11432, Saudi Arabia*

Correspondence should be addressed to Ali Akgül; aliakgul00727@gmail.com

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In this study, we investigate the falling body problem with three different fractional derivatives. We acquire the solutions of the model by the Sumudu transform. We show the accuracy of the Sumudu transform by some theoretic results and implementations.

1. Introduction

The Sumudu transform is an important integral transformation. This transformation has been begun with Watugala [1] who has presented a new integral transform to solve differential equations and control engineering problems. Weerakoon [2, 3] has made a big contribution with the application of the Sumudu transform to partial differential equations and complex inversion formula for the Sumudu transform. Then, this transformation has attracted great attention, and lots of work have been done related to this transformation by the authors. Some of them are as follows: Belgacem et al. [4] have searched the analytical investigations of the Sumudu transform. Fundamental properties of the Sumudu transform have been studied by Belgacem and Karaballi [5]. Atangana and Akgül [6] have obtained the transfer function and Bode diagram by the Sumudu transform.

Fractional differential equations have taken much interest recently. Arqub and Maayah [7, 8] have studied a fitted fractional reproducing kernel algorithm for the numerical solutions of ABC-fractional Volterra integro-differential equations and solution of the fractional epidemic model by the homotopy analysis method. Jangid et al. [9] have investigated some fractional calculus findings associated with the incomplete functions. Some new fractional-calculus connections between Mittag-Leffler functions have been studied by Srivastava et al. [10]. Singh et al. [11] have investigated the fractional epidemiological model for computer viruses. Ghanbari and Atangana [12, 13] have presented an efficient numerical approach for fractional diffusion partial differential equations and a new application of fractional Atangana-Baleanu derivatives. Abdeljawad et al. [14, 15] have investigated an efficient sustainable algorithm for numerical solutions of systems of fractional-order differential equations by the Haar wavelet collocation method and more general fractional integration by part formulae and applications. For more details, see [16–25].

In this study, we examine the falling body problem depending on Newton's second law that represents that the acceleration of a particle relied on the mass of the particle and the net force action on the particle. Take into consideration an object of mass m falling through the air from a height h with velocity $v(0)$ in a gravitational field. If we use Newton's second law, we acquire

$$\frac{mdv}{dt} + mkv = -mg, \tag{1}$$

where k is the positive constant rate and g expresses the gravitational constant. The solution of this equation is presented as [26]

$$v(t) = -\frac{g}{k} + \exp(-kt)\left(v(0) + \frac{g}{k}\right), \quad (2)$$

and by integrating for $z(0) = h$, we get

$$z(t) = h - \frac{gt}{k} + \frac{1}{k}(1 - \exp(-kt))\left(v(0) + \frac{g}{k}\right). \quad (3)$$

Considering all the information presented above, we organize this study as follows. In Section 2, some fundamental definitions and lemmas about nonlocal fractional calculus are given. In Section 3, the fractional falling body problem is investigated by means of Caputo, Caputo-Fabrizio sense of Caputo, and ABC. Also, some outstanding consequences are clarified in Section 4.

2. Preliminaries

After giving some introduction, we want to present some significant definition and lemma properties of fractional calculus for setting up a mathematically sound theory that will serve the purpose of the current study.

Definition 1. Over

$$B = \left\{v(t) | \exists N, \eta_1, \eta_2 > 0, |v(t)| < N \exp\left(|t|/\eta_j, \text{if } t \in (-1)^j \times [0,\infty)\right\}, \quad (4)$$

the Sumudu transform is identified by [17]

$$V(u) = S[v(t)] = \int_0^\infty f(ut)\exp(-t)dt, \quad u \in (-\eta_1, \eta_2). \quad (5)$$

Definition 2. We define the classical Mittag-Leffler function $E_\alpha(z)$ as [27]

$$E_\alpha(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m + 1)} (z \in \mathbb{C}, \text{Re}(\alpha) > 0), \quad (6)$$

and the Mittag-Leffler kernel with two parameters is presented by [27]

$$E_{\alpha,\beta}(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m + \beta)} (z, \beta \in \mathbb{C}, \text{Re}(\alpha) > 0). \quad (7)$$

Definition 3. The generalized Mittag-Leffler function is defined by [28]

$$E^\rho_{\alpha,\beta}(z) = \sum_{m=0}^{\infty} \frac{z^m (\rho)_m}{\Gamma(\alpha m + \beta) m!} (z, \beta, \rho, \alpha \in \mathbb{C}, \text{Re}(\alpha) > 0), \quad (8)$$

where $(\rho)_m = \rho(\rho + 1) \cdots (\rho + m - 1)$ is the Pochhammer symbol introduced by Prabhakar. As seen clearly, $(1)_m = m$! and $E^1_{\alpha,\beta}(z) = E_{\alpha,\beta}(z)$.

Definition 4. Let $\zeta, \psi : [0,\infty) \longrightarrow \mathfrak{R}$; then, the convolution of ζ, ψ is given as [27]

$$(\zeta * \psi) = \int_0^t \zeta(t - u)\psi(u)du, \quad (9)$$

and assume that $\zeta, \psi : [0,\infty) \longrightarrow \mathfrak{R}$; then, we have

$$S\{(\zeta * \psi)(t)\} = uS\{\zeta(t)\}S\{\psi(t)\}. \quad (10)$$

Definition 5. Let $v : [0,\infty] \longrightarrow \mathbb{R}$ be a smooth function. Then, the Caputo fractional derivative is defined as follows [18]:

$${}_0^C D^\alpha v(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - z)^{n-\alpha-1} v^{(n)}(z)dz, \quad (11)$$

where $\alpha \in \mathbb{C}$, $\text{Re}(\alpha) > 0$ and $n = [\text{Re}(\alpha)] + 1$.

Lemma 6. *The Sumudu transform of the Caputo fractional derivative is presented by [6]*

$$S\left[{}_0^C D_t^\alpha v(t)\right] = \frac{V[u] - v(0)}{u^\alpha}, \quad (12)$$

where $V[u] = S[v(t)]$.

Definition 7. We identify the Caputo-Fabrizio fractional derivative as [29]

$${}_0^{\text{CFC}} D^\alpha v(t) = \frac{M(\alpha)}{1 - \alpha} \int_0^t v'(z)\exp(-\lambda(t - z))dz. \quad (13)$$

Lemma 8. *The Sumudu transform of the CFC fractional derivative is acquired by [6]*

$$S\left[{}_0^{CFC} D_t^\alpha v(t)\right] = \frac{M(\alpha)}{(1 - \alpha)} \frac{V[u]}{(1 + (\alpha/(1 - \alpha))u)} - \frac{M(\alpha)}{(1 - \alpha)} \frac{v(0)}{(1 + (\alpha/(1 - \alpha))u)}. \quad (14)$$

Definition 9. We describe the Atangana-Baleanu fractional derivative by [27]

$${}_0^{\text{ABC}} D^\alpha v(t) = \frac{\text{AB}(\alpha)}{1 - \alpha} \int_0^t v'(z)E_\alpha(-\lambda(t - z)^\alpha)dz. \quad (15)$$

Lemma 10. *The Sumudu transform of the ABC fractional derivative is acquired by [6]*

$$S\left[{}_0^{ABC} D_t^\alpha v(t)\right] = \frac{AB(\alpha)}{(1 - \alpha)} \frac{V[u]}{(1 + (\alpha/(1 - \alpha))u^\alpha)} - \frac{AB(\alpha)}{(1 - \alpha)} \frac{v(0)}{(1 + (\alpha/(1 - \alpha))u^\alpha)}. \quad (16)$$

Definition 11. The generalized fractional integral is given by [28]

$${}_0 I^{\alpha,\beta} v(t) = \frac{1}{\Gamma(\alpha)\beta^{\alpha-1}} \int_0^t \left(t^\beta - z^\beta\right)^{\alpha-1} v(z)z^{\beta-1}dz. \quad (17)$$

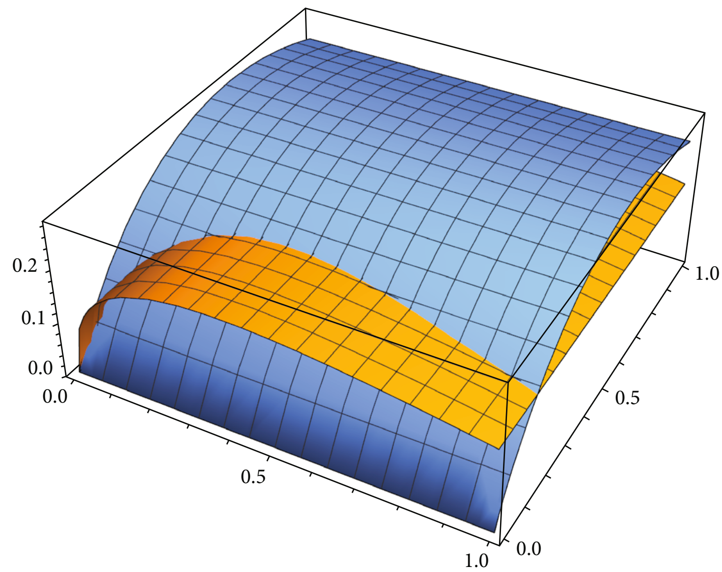

Figure 1: Simulations of the solution for $\alpha = 0.5$ and $\alpha = 0.9$.

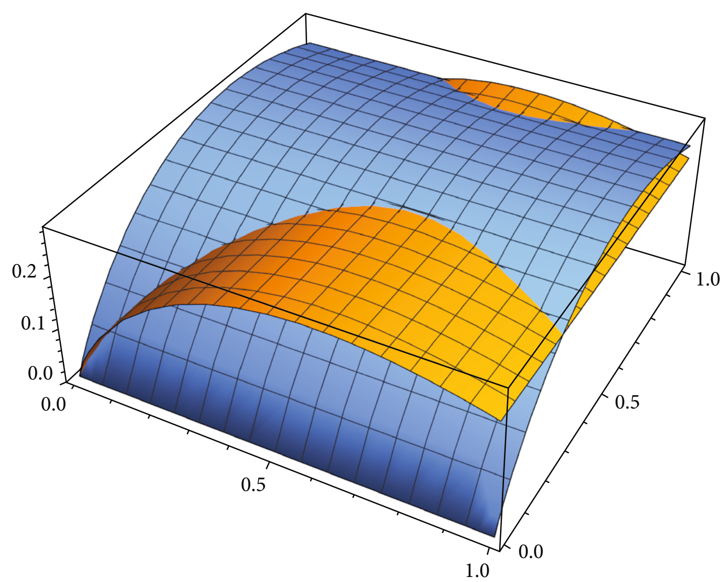

Figure 2: Simulations of the solution for $\alpha = 0.8$ and $\alpha = 0.9$.

Definition 12. The generalized fractional derivatives in the Caputo sense are defined, respectively, by [28]

$$
\begin{aligned}
{}_0^C D^{\alpha,\beta} v(t) &= {}_0 I^{n-\alpha,\beta} \left(t^{1-\beta} \frac{d}{dt} \right)^n v(t) \\
&= \frac{1}{\Gamma(n-\alpha)\beta^{n-\alpha-1}} \int_0^t \left(t^\beta - z^\beta \right)^{n-\alpha-1} \left(t^{1-\beta} \frac{d}{dt} \right)^n v(z) z^{\beta-1} dz.
\end{aligned} \tag{18}
$$

Lemma 13. *We have [6]*

$$
S[E_\alpha(-\lambda t^\alpha)] = \frac{1}{1+\lambda u^\alpha}, \tag{19}
$$

$$
S[1 - E_\alpha(-\lambda t^\alpha)] = \frac{\lambda u^\alpha}{1+\lambda u^\alpha}. \tag{20}
$$

3. Main Results

The aim of this section is to obtain the solutions for the fractional falling body problem by means of some nonlocal fractional derivative operators such as Caputo, Caputo-Fabrizio, and ABC.

3.1. The Falling Body Problem in the Sense of Caputo. The falling body problem in the sense of Caputo depended on Newton's second law which is given as follows:

$$
{}_0^C D^\alpha v(t) + k\sigma^{1-\alpha} v(t) = -g\sigma^{1-\alpha}, \tag{21}
$$

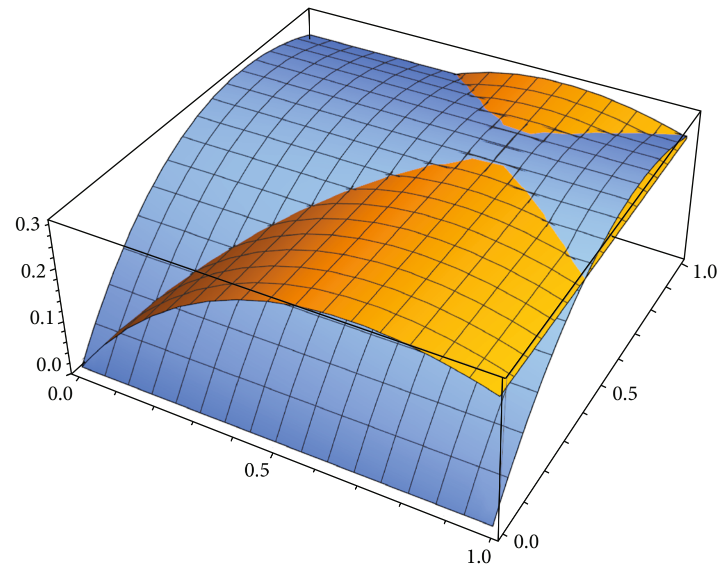

FIGURE 3: Simulations of the solution for $\alpha = 0.98$ and $\alpha = 0.99$.

where the initial velocity $v(0) = v_0$, g is the gravitational constant, and m and k are the positive constant rate which indicates the mass of the body. We have

$$S\{{}_0^C D^\alpha v(t)\} + k\sigma^{1-\alpha} S\{v(t)\} = S\{-g\sigma^{1-\alpha}\}. \quad (22)$$

Using the relation equation (9), we can write

$$\frac{S\{v(t)\} - v(0)}{u^\alpha} + k\sigma^{1-\alpha} S\{v(t)\} = -g\sigma^{1-\alpha}, \quad (23)$$

$$S\{v(t)\} = \frac{v(0)}{1 + k\sigma^{1-\alpha}u^\alpha} - \frac{g\sigma^{1-\alpha}u^\alpha}{1 + k\sigma^{1-\alpha}u^\alpha}. \quad (24)$$

If we apply the inverse Sumudu transform, we will obtain

$$v(t) = v(0)E_\alpha\left(-k\sigma^{1-\alpha}t^\alpha\right) - \frac{g}{k}\left[1 - E_\alpha\left(-k\sigma^{1-\alpha}t^\alpha\right)\right]. \quad (25)$$

Because of $\alpha = \sigma k$, $0 < \sigma \le 1/k$, the velocity $v(t)$ can be put down as follows:

$$v(t) = v(0)E_\alpha\left(-k^\alpha\alpha^{1-\alpha}t^\alpha\right) - \frac{g}{k}\left[1 - E_\alpha\left(-k^\alpha\alpha^{1-\alpha}t^\alpha\right)\right]. \quad (26)$$

Note that we put the condition $v_0 = -g/k$ in order to satisfy the initial condition $v(0) = v_0$. By benefiting from the velocity (33), vertical distance $z(t)$ can be obtained in the following way:

$$ {}_0^C D^\alpha z(t) = v(0)\sigma^{1-\alpha}E_\alpha\left(-k\sigma^{1-\alpha}t^\alpha\right) - \frac{g\sigma^{1-\alpha}}{k}\left[1 - E_\alpha\left(-k\sigma^{1-\alpha}t^\alpha\right)\right]. \quad (27)$$

Applying the Sumudu transformation to the above equation, we have

$$\begin{aligned} S\{{}_0^C D^\alpha z(t)\} &= v(0)\sigma^{1-\alpha} S\{E_\alpha\left(-k\sigma^{1-\alpha}t^\alpha\right)\} - \frac{g\sigma^{1-\alpha}}{k} S\left[1 - E_\alpha\left(-k\sigma^{1-\alpha}t^\alpha\right)\right], \\ \frac{S\{z(t)\} - z(0)}{u^\alpha} &= v(0)\sigma^{1-\alpha}\frac{1}{1 + k\sigma^{1-\alpha}u^\alpha} - \frac{g\sigma^{1-\alpha}}{k} + \frac{g\sigma^{1-\alpha}}{k}\frac{1}{1 + k\sigma^{1-\alpha}u^\alpha}, \\ S\{z(t)\} &= z(0) + v(0)\sigma^{1-\alpha}\frac{u^\alpha}{1 + k\sigma^{1-\alpha}u^\alpha} - \frac{g\sigma^{1-\alpha}u^\alpha}{k} + \frac{g\sigma^{1-\alpha}}{k}\frac{u^\alpha}{1 + k\sigma^{1-\alpha}u^\alpha}. \end{aligned} \quad (28)$$

Using the inverse Sumudu transformation for the last equation and taking the $z(0) = h$, we acquire the vertical distance $z(t)$ as

$$z(t) = h + \frac{v(0)}{k}\left[1 - E_\alpha\left(-k\sigma^{1-\alpha}t^\alpha\right)\right] - \frac{g\sigma^{1-\alpha}t^\alpha}{k\Gamma(\alpha+1)} + \frac{g}{k^2}\left[1 - E_\alpha\left(-k\sigma^{1-\alpha}t^\alpha\right)\right]. \quad (29)$$

Because of $\alpha = \sigma k$, $0 < \sigma \le 1/k$, the vertical distance $z(t)$ can be put down as

$$\begin{aligned} z(t) = h &+ \frac{v(0)}{k}\left[1 - E_\alpha\left(-k^\alpha\alpha^{1-\alpha}t^\alpha\right)\right] - \frac{g\alpha^{1-\alpha}t^\alpha}{k^{(2-\alpha)}\Gamma(\alpha+1)} \\ &+ \frac{g}{k^2}\left[1 - E_\alpha\left(-k^\alpha\alpha^{1-\alpha}t^\alpha\right)\right]. \end{aligned} \quad (30)$$

We demonstrate the simulations of the above solution for different values of α as shown in Figures 1–3. Similar simulations can be shown easily for other solutions.

3.2. The Falling Body Problem in the Sense of Caputo-Fabrizio. The falling body problem in the sense of Caputo-Fabrizio depended on Newton's second law which is given as follows:

$$ {}_0^{CFC}D^\alpha v(t) + k\sigma^{1-\alpha} v(t) = -g\sigma^{1-\alpha}. \quad (31)$$

where the initial velocity $v(0)=v_0$, g is the gravitational constant, and m and k are the positive constant rate which indicates the mass of the body. We have

$$S\left\{{}_0^{\mathrm{CFC}}D^{\alpha}v(t)\right\}+k\sigma^{1-\alpha}S\{v(t)\}=S\left\{-g\sigma^{1-\alpha}\right\}. \quad (32)$$

If we use the relation equation(19), we can write

$$\frac{M(\alpha)}{(1-\alpha)}\frac{S\{v(t)\}}{(1+(\alpha/1-\alpha)u)}-\frac{M(\alpha)}{(1-\alpha)}\frac{v(0)}{(1+(\alpha/(1-\alpha))u)}+k\sigma^{1-\alpha}S\{v(t)\}=-g\sigma^{1-\alpha}, \quad (33)$$

$$S\{v(t)\}=\frac{M(\alpha)v(0)/(1-\alpha)(1+(\alpha/1-\alpha)u)}{(M(\alpha)/(1-\alpha)(1+(\alpha/(1-\alpha))u))+k\sigma^{1-\alpha}}-\frac{g\sigma^{1-\alpha}}{(M(\alpha)/(1-\alpha)(1+(\alpha/(1-\alpha))u))+k\sigma^{1-\alpha}}. \quad (34)$$

Applying the inverse Sumudu transform yields

$$v(t)=S^{-1}\left\{\frac{M(\alpha)v(0)/(1-\alpha)(1+(\alpha/1-\alpha)u)}{(M(\alpha)/(1-\alpha)(1+(\alpha/1-\alpha)u))+k\sigma^{1-\alpha}}\right\}-S^{-1}\left\{\frac{g\sigma^{1-\alpha}}{(M(\alpha)/(1-\alpha)(1+(\alpha/1-\alpha)u))+k\sigma^{1-\alpha}}\right\}.$$

$$v(t)=\frac{M(\alpha)v(0)}{M(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\exp\left(\frac{-\alpha k\sigma^{1-\alpha}t}{M(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\right)-\frac{g\left[M(\alpha)+k\sigma^{1-\alpha}(1-\alpha)-M(\alpha)\exp\left(-\alpha k\sigma^{1-\alpha}t/\left(M(\alpha)+k\sigma^{1-\alpha}(1-\alpha)\right)\right)\right]}{k(M(\alpha)+k\sigma^{1-\alpha}(1-\alpha))}. \quad (35)$$

Because of $\alpha=\sigma k$, $0<\sigma\le 1/k$, the velocity $v(t)$ can be put down as follows:

$$v(t)=\frac{M(\alpha)v(0)}{M(\alpha)+k^{\alpha}\alpha^{1-\alpha}(1-\alpha)}\exp\left(\frac{-k^{\alpha}\alpha^{2-\alpha}t}{M(\alpha)+k^{\alpha}\alpha^{1-\alpha}(1-\alpha)}\right)-\frac{g\left[M(\alpha)+k^{\alpha}\alpha^{1-\alpha}(1-\alpha)-M(\alpha)\exp\left(-k^{\alpha}\alpha^{2-\alpha}t/\left(M(\alpha)+k^{\alpha}\alpha^{1-\alpha}(1-\alpha)\right)\right)\right]}{k(M(\alpha)+k^{\alpha}\alpha^{1-\alpha}(1-\alpha))}. \quad (36)$$

Note that we put the condition $v_0=-g/k$ in order to satisfy the initial condition $v(0)=v_0$. By benefiting from the velocity (33), vertical distance $z(t)$ can be obtained in the following way:

$${}_0^{\mathrm{CFC}}D^{\alpha}z(t)=\frac{M(\alpha)\sigma^{1-\alpha}v(0)}{M(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\exp\left(\frac{-\alpha k\sigma^{1-\alpha}t}{M(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\right)-\frac{g\sigma^{1-\alpha}\left[M(\alpha)+k\sigma^{1-\alpha}(1-\alpha)-M(\alpha)\exp\left(-\alpha k\sigma^{1-\alpha}t/\left(M(\alpha)+k\sigma^{1-\alpha}(1-\alpha)\right)\right)\right]}{k(M(\alpha)+k\sigma^{1-\alpha}(1-\alpha))}. \quad (37)$$

If we implement the Sumudu transformation to the last equation, we will obtain

$$S\left\{{}_0^{\mathrm{CFC}}D^{\alpha}z(t)\right\}=\frac{M(\alpha)\sigma^{1-\alpha}v(0)}{M(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}S\left\{\exp\left(\frac{-\alpha k\sigma^{1-\alpha}t}{M(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\right)\right\}-S\left\{\frac{g\sigma^{1-\alpha}}{k}\right\}+\frac{g\sigma^{1-\alpha}M(\alpha)}{k(M(\alpha)+k\sigma^{1-\alpha}(1-\alpha))}S\cdot\left\{\exp\left(\frac{-\alpha k\sigma^{1-\alpha}t}{M(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\right)\right\},$$

$$\frac{M(\alpha)}{(1-\alpha)}\frac{S\{z(t)\}}{(1+(\alpha/(1-\alpha))u)}-\frac{M(\alpha)}{(1-\alpha)}\frac{z(0)}{(1+(\alpha/(1-\alpha))u)}=\frac{M(\alpha)\sigma^{1-\alpha}v(0)}{M(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\frac{1}{1+(\alpha k\sigma^{1-\alpha}u/(M(\alpha)+k\sigma^{1-\alpha}(1-\alpha)))}-\frac{g\sigma^{1-\alpha}}{k}+\frac{g\sigma^{1-\alpha}M(\alpha)}{k(M(\alpha)+k\sigma^{1-\alpha}(1-\alpha))}\frac{1}{1+(\alpha k\sigma^{1-\alpha}u/(M(\alpha)+k\sigma^{1-\alpha}(1-\alpha)))},$$

$$S\{z(t)\}=z(0)+\sigma^{1-\alpha}v(0)\frac{(1-\alpha+\alpha u)}{M(\alpha)+k\sigma^{1-\alpha}(1-\alpha+\alpha u)}\Bigg\}-\frac{g\sigma^{(1-\alpha)}(1-\alpha)}{M(\alpha)k}-\frac{g\sigma^{(1-\alpha)}\alpha u}{M(\alpha)k}+\frac{g\sigma^{1-\alpha}}{k}\frac{(1-\alpha+\alpha u)}{M(\alpha)+k\sigma^{1-\alpha}(1-\alpha+\alpha u)}\Bigg\}, \quad (38)$$

If we utilize the inverse Sumudu transformation for equation (13) and take the $z(0)=h$, we acquire the vertical distance $z(t)$ as

$$z(t)=h+\frac{v(0)}{k}\frac{\left[M(\alpha)+k\sigma^{1-\alpha}(1-\alpha)-M(\alpha)\exp\left(-\alpha k\sigma^{1-\alpha}t/\left(M(\alpha)+k\sigma^{1-\alpha}(1-\alpha)\right)\right)\right]}{M(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}-\frac{g\sigma^{1-\alpha}}{M(\alpha)k}-\frac{g\sigma^{1-\alpha}\alpha t}{M(\alpha)k}+\frac{g}{k^2}\frac{\left[M(\alpha)+k\sigma^{1-\alpha}(1-\alpha)-M(\alpha)\exp\left(-\alpha k\sigma^{1-\alpha}t/\left(M(\alpha)+k\sigma^{1-\alpha}(1-\alpha)\right)\right)\right]}{M(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}. \quad (39)$$

Because of $\alpha=\sigma k$, $0<\sigma\le 1/k$, the vertical distance $z(t)$ can be put down as

$$z(t)=h+\frac{v(0)}{k}\frac{\left[M(\alpha)+k^{\alpha}\alpha^{1-\alpha}(1-\alpha)-M(\alpha)\exp\left(-k^{\alpha}\alpha^{2-\alpha}t/\left(M(\alpha)+k^{\alpha}\alpha^{1-\alpha}(1-\alpha)\right)\right)\right]}{M(\alpha)+k^{\alpha}\alpha^{1-\alpha}(1-\alpha)}-\frac{g\alpha^{1-\alpha}}{M(\alpha)k}-\frac{g\alpha^{2-\alpha}t}{M(\alpha)k}+\frac{g}{k^2}\frac{\left[M(\alpha)+k^{\alpha}\alpha^{1-\alpha}(1-\alpha)-M(\alpha)\exp\left(-k^{\alpha}\alpha^{2-\alpha}t/\left(M(\alpha)+k^{\alpha}\alpha^{1-\alpha}(1-\alpha)\right)\right)\right]}{M(\alpha)+k^{\alpha}\alpha^{1-\alpha}(1-\alpha)}. \quad (40)$$

3.3. The Falling Body Problem in the Sense of ABC.

The falling body problem in the sense of ABC depended on Newton's second law which is given as follows:

$${}_0^{\mathrm{ABC}}D^{\alpha}v(t)+k\sigma^{1-\alpha}v(t)=-g\sigma^{1-\alpha}, \quad (41)$$

where the initial velocity $v(0)=v_0$, g is the gravitational constant, and m and k are the positive constant rate which indicates the mass of the body. We have

$$S\left\{{}_0^{\mathrm{ABC}}D^{\alpha}v(t)\right\}+k\sigma^{1-\alpha}S\{v(t)\}=S\left\{-g\sigma^{1-\alpha}\right\}. \quad (42)$$

Using the relation equation (15), we can write

$$\frac{AB(\alpha)}{(1-\alpha)}\frac{S\{v(t)\}}{(1+(\alpha/(1-\alpha))u^{\alpha})}-\frac{AB(\alpha)}{(1-\alpha)}\frac{v(0)}{(1+(\alpha/(1-\alpha))u^{\alpha})}+k\sigma^{1-\alpha}S\{v(t)\}=-g\sigma^{1-\alpha},$$
$$S\{v(t)\}=\frac{AB(\alpha)v(0)-g\sigma^{1-\alpha}(1-\alpha)(1+(\alpha/(1-\alpha))u^{\alpha})}{AB(\alpha)+k\sigma^{1-\alpha}(1-\alpha)(1+(\alpha/(1-\alpha))u^{\alpha})}. \tag{43}$$

If we apply the inverse Sumudu transform, we will reach

$$v(t)=S^{-1}\left\{\frac{AB(\alpha)v(0)}{AB(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\frac{1}{1+(\alpha k\sigma^{1-\alpha}u^{\alpha}/(AB(\alpha)+k\sigma^{1-\alpha}(1-\alpha)))}\right\}$$
$$-S^{-1}\left\{\frac{g\sigma^{1-\alpha}(1-\alpha)(1+(\alpha/(1-\alpha))u^{\alpha})}{AB(\alpha)+k\sigma^{1-\alpha}(1-\alpha)(1+(\alpha/(1-\alpha))u^{\alpha})}\right\}, \tag{44}$$

$$v(t)=\frac{AB(\alpha)v(0)}{AB(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}E_{\alpha}\left(-\frac{\alpha k\sigma^{1-\alpha}t^{\alpha}}{AB(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\right)$$
$$-\frac{g\sigma^{1-\alpha}(1-\alpha)}{AB(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}E_{\alpha}\left(-\frac{\alpha k\sigma^{1-\alpha}t^{\alpha}}{AB(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\right)$$
$$-\frac{g}{k}\left[1-E_{\alpha}\left(-\frac{\alpha k\sigma^{1-\alpha}t^{\alpha}}{AB(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\right)\right]. \tag{45}$$

Because $\alpha=\sigma k$, $0<\sigma\le 1/k$, the velocity $v(t)$ can be put down as follows:

$$v(t)=\frac{AB(\alpha)v(0)}{AB(\alpha)+k^{\alpha}\alpha^{1-\alpha}(1-\alpha)}E_{\alpha}\left(-\frac{\alpha^{2-\alpha}(kt)^{\alpha}}{AB(\alpha)+k^{\alpha}\alpha^{1-\alpha}(1-\alpha)}\right)$$
$$-\frac{gk^{\alpha-1}\alpha^{1-\alpha}(1-\alpha)}{AB(\alpha)+k^{\alpha}\alpha^{1-\alpha}(1-\alpha)}E_{\alpha}\left(-\frac{\alpha^{2-\alpha}(kt)^{\alpha}}{AB(\alpha)+k^{\alpha}\alpha^{1-\alpha}(1-\alpha)}\right)$$
$$-\frac{g}{k}\left[1-E_{\alpha}\left(-\frac{\alpha^{2-\alpha}(kt)^{\alpha}}{AB(\alpha)+k^{\alpha}\alpha^{1-\alpha}(1-\alpha)}\right)\right]. \tag{46}$$

Note that we put the condition $v_0=-g/k$ in order to satisfy the initial condition $v(0)=v_0$. By benefiting from the velocity (33), vertical distance $z(t)$ can be obtained in the following way:

$${}_{0}^{ABC}D^{\alpha}z(t)=\frac{AB(\alpha)\sigma^{1-\alpha}v(0)}{AB(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}E_{\alpha}\left(-\frac{\alpha k\sigma^{1-\alpha}t^{\alpha}}{AB(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\right)$$
$$-\frac{g\sigma^{2}1-\alpha(1-\alpha)}{AB(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}E_{\alpha}\left(-\frac{\alpha k\sigma^{1-\alpha}t^{\alpha}}{AB(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\right)$$
$$-\frac{g\sigma^{1-\alpha}}{k}\left[1-E_{\alpha}\left(-\frac{\alpha k\sigma^{1-\alpha}t^{\alpha}}{AB(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\right)\right]. \tag{47}$$

If we apply the Sumudu transformation to the above equation, we have

$$S\left\{{}_{0}^{ABC}D^{\alpha}z(t)\right\}=\frac{AB(\alpha)\sigma^{1-\alpha}v(0)}{AB(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}S\left\{E_{\alpha}\left(-\frac{\alpha k\sigma^{1-\alpha}t^{\alpha}}{AB(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\right)\right\}$$
$$-\frac{g\sigma^{2}1-\alpha(1-\alpha)}{AB(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}S\left\{E_{\alpha}\left(-\frac{\alpha k\sigma^{1-\alpha}t^{\alpha}}{AB(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\right)\right\}$$
$$-S\left\{\frac{g\sigma^{1-\alpha}}{k}\right\}+\frac{g\sigma^{1-\alpha}}{k}S\left\{E_{\alpha}\left(-\frac{\alpha k\sigma^{1-\alpha}t^{\alpha}}{AB(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\right)\right\},$$

$$\frac{AB(\alpha)}{(1-\alpha)}\frac{S\{z(t)\}}{(1+(\alpha/(1-\alpha))u^{\alpha})}-\frac{AB(\alpha)}{(1-\alpha)}\frac{z(0)}{(1+(\alpha/(1-\alpha))u^{\alpha})}$$
$$=\frac{AB(\alpha)\sigma^{1-\alpha}v(0)}{AB(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\frac{1}{1+(\alpha k\sigma^{1-\alpha}u^{\alpha}/(AB(\alpha)+k\sigma^{1-\alpha}(1-\alpha)))}\Bigg\}$$
$$-\frac{g\sigma^{2}1-\alpha(1-\alpha)}{AB(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\frac{1}{1+(\alpha k\sigma^{1-\alpha}u^{\alpha}/(AB(\alpha)+k\sigma^{1-\alpha}(1-\alpha)))}\Bigg\}$$
$$-\frac{g\sigma^{1-\alpha}}{k}+\frac{g\sigma^{1-\alpha}}{k}\frac{1}{1+(\alpha k\sigma^{1-\alpha}u^{\alpha}/(AB(\alpha)+k\sigma^{1-\alpha}(1-\alpha)))}\Bigg\},$$

$$S\{z(t)\}=z(0)+\frac{\sigma^{1-\alpha}(1-\alpha)v(0)}{AB(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\frac{1}{1+(\alpha k\sigma^{1-\alpha}u^{\alpha}/(AB(\alpha)+k\sigma^{1-\alpha}(1-\alpha)))}\Bigg\}$$
$$+\frac{\sigma^{1-\alpha}v(0)\alpha}{AB(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\frac{u^{\alpha}}{1+(\alpha k\sigma^{1-\alpha}u^{\alpha}/(AB(\alpha)+k\sigma^{1-\alpha}(1-\alpha)))}\Bigg\}$$
$$-\frac{g\sigma^{2(1-\alpha)}(1-\alpha)^{2}}{AB(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\frac{1}{1+(\alpha k\sigma^{1-\alpha}u^{\alpha}/(AB(\alpha)+k\sigma^{1-\alpha}(1-\alpha)))}\Bigg\}$$
$$-\frac{g\sigma^{2(1-\alpha)}\alpha(1-\alpha)}{AB(\alpha)(AB(\alpha)+k\sigma^{1-\alpha}(1-\alpha))}\frac{u^{\alpha}}{1+(\alpha k\sigma^{1-\alpha}u^{\alpha}/(AB(\alpha)+k\sigma^{1-\alpha}(1-\alpha)))}\Bigg\}$$
$$-\frac{g\sigma^{1-\alpha}\alpha u^{\alpha}}{AB(\alpha)k}-\frac{g\sigma^{1-\alpha}(1-\alpha)}{AB(\alpha)k}+\frac{g\sigma^{1-\alpha}\alpha}{AB(\alpha)k}\frac{u^{\alpha}}{1+(\alpha k\sigma^{1-\alpha}u^{\alpha}/(AB(\alpha)+k\sigma^{1-\alpha}(1-\alpha)))}\Bigg\}$$
$$+\frac{g\sigma^{1-\alpha}(1-\alpha)}{AB(\alpha)k}\frac{1}{1+(\alpha k\sigma^{1-\alpha}u^{\alpha}/(AB(\alpha)+k\sigma^{1-\alpha}(1-\alpha)))}\Bigg\}. \tag{48}$$

Using the inverse Sumudu transformation for the last equation and taking the $z(0)=h$, we acquire the vertical distance $z(t)$ as

$$z(t)=h+\frac{\sigma^{1-\alpha}(1-\alpha)v(0)}{AB(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}E_{\alpha}\left(-\frac{\alpha k\sigma^{1-\alpha}t^{\alpha}}{AB(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\right)$$
$$+\frac{v(0)}{k}\left[1-E_{\alpha}\left(-\frac{\alpha k\sigma^{1-\alpha}t^{\alpha}}{AB(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\right)\right]$$
$$-\frac{g\sigma^{2(1-\alpha)}(1-\alpha)^{2}}{AB(\alpha)(AB(\alpha)+k\sigma^{1-\alpha}(1-\alpha))}E_{\alpha}\left(-\frac{\alpha k\sigma^{1-\alpha}t^{\alpha}}{AB(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\right)$$
$$-\frac{g\sigma^{(1-\alpha)}(1-\alpha)}{AB(\alpha)k}E_{\alpha}\left(-\frac{\alpha k\sigma^{1-\alpha}t^{\alpha}}{AB(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\right)$$
$$-\frac{g\sigma^{(1-\alpha)}}{AB(\alpha)k}\left[1-\alpha+\frac{\alpha t^{\alpha}}{\Gamma(1+\alpha)}\right]$$
$$+\frac{g\sigma^{(1-\alpha)}(1-\alpha)}{AB(\alpha)k}E_{\alpha}\left(-\frac{\alpha k\sigma^{1-\alpha}t^{\alpha}}{AB(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\right)$$
$$+\frac{gAB(\alpha)+gk\sigma^{(1-\alpha)}(1-\alpha)}{AB(\alpha)k^{2}}\left[1-E_{\alpha}\left(-\frac{\alpha k\sigma^{1-\alpha}t^{\alpha}}{AB(\alpha)+k\sigma^{1-\alpha}(1-\alpha)}\right)\right], \tag{49}$$

where $v_0=g\sigma^{1-\alpha}/AB(\alpha)$. Because of $\alpha=\sigma k$, $0<\sigma\le 1/k$, the vertical distance $z(t)$ can be put down as

$$
\begin{aligned}
z(t) = h &+ \frac{\alpha^{1-\alpha}k^{\alpha-1}(1-\alpha)\nu(0)}{\mathrm{AB}(\alpha)+k^{\alpha}\alpha^{1-\alpha}(1-\alpha)}E_{\alpha}\left(-\frac{\alpha^{2-\alpha}(kt)^{\alpha}}{\mathrm{AB}(\alpha)+k^{\alpha}\alpha^{1-\alpha}(1-\alpha)}\right) \\
&+ \frac{\nu(0)}{k}\left[1-E_{\alpha}\left(-\frac{\alpha^{2-\alpha}(kt)^{\alpha}}{\mathrm{AB}(\alpha)+k^{\alpha}\alpha^{1-\alpha}(1-\alpha)}\right)\right] \\
&- \frac{g\alpha^{2(1-\alpha)}k^{2(\alpha-1)}(1-\alpha)^2}{\mathrm{AB}(\alpha)(\mathrm{AB}(\alpha)+k^{\alpha}\alpha^{1-\alpha}(1-\alpha))}E_{\alpha}\left(-\frac{\alpha^{2-\alpha}(kt)^{\alpha}}{\mathrm{AB}(\alpha)+k^{\alpha}\alpha^{1-\alpha}(1-\alpha)}\right) \\
&- \frac{g\alpha^{(1-\alpha)}k^{(\alpha-1)}(1-\alpha)}{\mathrm{AB}(\alpha)k}E_{\alpha}\left(-\frac{\alpha^{2-\alpha}(kt)^{\alpha}}{\mathrm{AB}(\alpha)+k^{\alpha}\alpha^{1-\alpha}(1-\alpha)}\right) \\
&- \frac{g\alpha^{(1-\alpha)}k^{\alpha}}{\mathrm{AB}(\alpha)k^2}\left[1-\alpha+\frac{\alpha t^{\alpha}}{\Gamma(1+\alpha)}\right] \\
&+ \frac{g\sigma^{(1-\alpha)k^{\alpha}}(1-\alpha)}{\mathrm{AB}(\alpha)k}E_{\alpha}\left(-\frac{\alpha^{2-\alpha}(kt)^{\alpha}}{\mathrm{AB}(\alpha)+k^{\alpha}\alpha^{1-\alpha}(1-\alpha)}\right) \\
&+ \frac{gAB(\alpha)+gk^{\alpha}\alpha^{(1-\alpha)}(1-\alpha)}{\mathrm{AB}(\alpha)k^2}\left[1-E_{\alpha}\left(-\frac{\alpha^{2-\alpha}(kt)^{\alpha}}{\mathrm{AB}(\alpha)+k^{\alpha}\alpha^{1-\alpha}(1-\alpha)}\right)\right].
\end{aligned} \tag{50}
$$

4. Conclusions

We searched the falling body problem in detail by the Sumudu transform in this study. We obtained the exact solutions of this problem with Caputo, Caputo-Fabrizio, and Atangana-Baleanu derivatives. We demonstrated the effect of the Sumudu transform by these results.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they do not have any conflict of interest.

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