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Research Article

Numerical Solutions of Time Fractional Zakharov-Kuznetsov Equation via Natural Transform Decomposition Method with Nonsingular Kernel Derivatives

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In this paper, we have studied the time-fractional Zakharov-Kuznetsov equation (TFZKE) via natural transform decomposition method (NTDM) with nonsingular kernel derivatives. The fractional derivative considered in Caputo-Fabrizio (CF) and Atangana-Baleanu derivative in Caputo sense (ABC). We employed natural transform (NT) on TFZKE followed by inverse natural transform, to obtain the solution of the equation. To validate the method, we have considered a few examples and compared with the actual results. Numerical results are in accordance with the existing results.

1. Introduction

Fractional calculus is an emerging field in various branches of engineering science. Fractional differential equations attracted researchers as they used to model a variety of diverse applications such as visco elasticity, heat conduction, biology, and dynamical systems [1–7]. Due to its importance in diverse fields, considerable methods developed to study the exact and computational solutions of fractional differential equations. Other than the modelling, divergence and convergence of the solutions are also equally important. A suitable definition is essential for a fractional generalization of a physical model. Several fractional derivative definitions developed in the last few decades. Some of the popular definitions in the literature are Riemann-Liouville (R-L), Caputo, CF, ABC, Grunwald-Letnikov, and Riesz fractional derivatives. For more details, we refer to [8, 9] and the references therein. R-L and Caputo fractional derivatives have a singular kernel. Recently, two nonsingular kernel fractional derivative definitions are developed by Atangana-Baleanu and Caputo-Fabrizio. Several methods are being investigated for the analysis of fractional differential equations for accuracy and reliable solutions. Some of the popular semi analytical and numerical methods are variational iteration method (VIM) [10], fractional differential transform method [11–14], homotopy perturbation transform method (HPTM) [15], homotopy analysis transform method [16, 17], residual power series method (RPS) [18], q-homotopy analysis transform method (q-HATM) [19–21], operational matrix method [22], tension spline method [23], parametric cubic

Table 1: Absolute errors of $NTDM_{CF}$ and $NTDM_{ABC}$ with PIA [31] and RPSM [31] of Example 1 for $\mu = 1$ and $\lambda = 0.001$.

x	ζ	t	$NTDM_{CF}$	$NTDM_{ABC}$	PIA [31]	RPSM [31]
		0.2	3.8519E-07	3.8519E-07	3.8521E-07	3.8521E-07
0.1	0.1	0.3	5.7584E-07	5.7584E-07	5.7591E-07	5.7591E-07
		0.4	7.6517E-07	7.6517E-07	7.6535E-07	7.6535E-07
		0.2	4.6474E-05	4.6474E-05	4.6634E-05	4.6639E-05
0.6	0.6	0.3	6.8157E-05	6.8157E-05	6.8606E-05	6.8631E-05
		0.4	8.8582E-05	8.8582E-05	8.9824E-05	8.9905E-05
		0.2	4.9249E-04	4.9249E-04	5.1213E-04	5.1424E-04
0.9	0.9	0.3	6.7503E-04	6.7503E-04	7.3819E-04	7.4845E-04
		0.4	8.1510E-04	8.1510E-04	9.5794E-04	9.8914E-04

spline [24], exponential B-spline method [25–27], fractional natural decomposition method (FNDM) [28], Adam Bashforth's Moulton method [29], and references therein.

In this paper, we have considered the TFZKE

$$D_t^{\mu}\mathbf{u}(x,\zeta,t) + a\left(\mathbf{u}^{\xi}\right)_x + b(\mathbf{u}^{\eta})_{xxx} + c\left(\mathbf{u}^{\delta}\right)_{x\zeta\zeta} = 0, \quad (1)$$

with initial condition

$$\mathbf{u}(x,\zeta,0) = \phi(x,\zeta). \quad (2)$$

This model illustrates the behavior of weakly nonlinear ion-acoustic waves in a plasma bearing cold ions and hot isothermal electrons in the presence of a uniform magnetic field. This problem has been solved by many techniques such as VIM [10], homotopy perturbation method [30], HPTM [15], perturbation iteration algorithm and RPS [31], and new iterative Sumudu transform method [32]. Recently, Veeresha and Prakasha [28] presented the applications of q-HATM and FNDM for solving TFZKE.

The aim of this paper is to implement NTDM to solve TFZKE. Rawashdeh and Maitama [33] introduced NTDM for a class of nonlinear partial differential equations. NTDM do not require linearization; prescribe assumptions, perturbation, or discretization; and prevent any round-off errors. Recently, NTDM employed to time-fractional Fisher's equation [34] and $(2+1)$ – dimensional time-fractional coupled Burger equations [35]. The paper is organized as follows. Basic definitions of singular and nonsingular definitions of fractional calculus and NT and its fractional derivatives are discussed briefly in Section 2. In Section 4, we presented the convergence and uniqueness of the solutions. In Section 3, we presented the NTDM for nonsingular definitions to solve TFZKE. In Section 5, few examples of TFZKE are given to validate the present methods. Section 6 presents the results and discussions. In Section 7, brief conclusion of this paper is presented.

2. Basic Definitions

There are various fractional derivative definitions that are available in the literature; for more details, we refer [8, 36–38]. In this section, we give the definitions of R-L, Caputo, CF, and ABC fractional derivatives for the benefit of the readers.

Definition 1 (see [36]). The R-L left-sided fractional integral operator of a function $f \in C_{\nu}$, $\nu \geq -1$ is given as

$$I^{\mu}f(\omega) = \frac{1}{\Gamma(\mu)}\int_0^{\omega}(\omega-\varsigma)^{\mu-1}f(\varsigma)d\varsigma, \mu > 0, \omega > 0, \quad (3)$$

and $I^0 f(\omega) = f(\omega)$.

Definition 2 (see [1]). The Caputo sense fractional derivative of $f(\omega)$is defined by

$${}_0^C D_{\omega}^{\mu} f(\omega) = I^{m-\mu}D^m f(\omega) = \frac{1}{m-\mu}\int_0^{\omega}(\omega-\varsigma)^{m-\mu-1}f^m(\varsigma)d\varsigma, \quad (4)$$

for $m-1 < \mu \leq m$, $m \in \mathbb{N}$, $\omega > 0$, $f \in C_{\nu}^m$, and $\nu \geq -1$.

Definition 3 (see [39]). The CF fractional derivative of $f(\omega)$ is given by

$${}_0^{CF} D_{\omega}^{\mu}(f(\omega)) = \frac{B(\mu)}{1-\mu}\int_0^{\omega}\exp\left(\frac{-\mu(\omega-\varsigma)}{1-\mu}\right)D(f(\varsigma))d\varsigma, \quad (5)$$

where $0 < \mu < 1$ and $B(\mu)$ is a normalization function, where $B(0) = B(1) = 1$.

Definition 4 (see [40]). The ABC fractional derivative of $f(\omega)$ is presented as

$${}_0^{ABC} D_{\omega}^{\mu} f(\omega) = \frac{B(\mu)}{1-\mu}\int_0^{\omega}E_{\mu}\left(\frac{-\mu(\omega-\varsigma)}{1-\mu}\right)D(f(\varsigma))d\varsigma, \quad (6)$$

where $0 < \mu < 1$. Normalization function is $B(\mu)$, and the Mittag-Leffler function is $E_{\mu}(z) = \sum_{l=0}^{\infty}(z^l/\Gamma(\mu l+1))$. These definitions widely used to study the fractional differential equation solutions using numerous integral transform techniques such as Sumudu transform, Shehu transform, and Laplace transform. Recently, natural transform of these definitions applied to study various differential equations; for more details, we refer [33, 41, 42]. Computational time can

Table 2: Approximate solutions of $NTDM_{CF}$ and $NTDM_{ABC}$ for $\mu = 0.67$ and $\mu = 0.75$ of Example 1 for $t = 1$ and $\lambda = 0.001$.

x	ζ	$\mu = 0.67$		$\mu = 0.75$	
		$NTDM_{CF}$	$NTDM_{ABC}$	$NTDM_{CF}$	$NTDM_{ABC}$
0.02	0.02	1.7406E-06	1.7189E-06	1.7380E-06	1.7176E-06
	0.04	4.1951E-06	4.1584E-06	4.1923E-06	4.1581E-06
	0.06	7.7192E-06	7.6674E-06	7.7161E-06	7.6679E-06
	0.08	1.2317E-05	1.2250E-05	1.2313E-05	1.2251E-05
	0.10	1.7994E-05	1.7911E-05	1.7990E-05	1.7913E-05
0.04	0.02	4.1951E-06	4.1584E-06	4.1923E-06	4.1581E-06
	0.04	7.7192E-06	7.6674E-06	7.7161E-06	7.6679E-06
	0.06	1.2317E-05	1.2250E-05	1.2313E-05	1.2251E-05
	0.08	1.7994E-05	1.7911E-05	1.7990E-05	1.7913E-05
	0.10	2.4757E-05	2.4658E-05	2.4752E-05	2.4660E-05
0.06	0.02	7.7192E-06	7.6674E-06	7.7161E-06	7.6679E-06
	0.04	1.2317E-05	1.2250E-05	1.2313E-05	1.2251E-05
	0.06	1.7994E-05	1.7911E-05	1.7990E-05	1.7913E-05
	0.08	2.4757E-05	2.4658E-05	2.4752E-05	2.4660E-05
	0.10	3.2616E-05	3.2500E-05	3.2610E-05	3.2503E-05
0.08	0.02	1.2317E-05	1.2250E-05	1.2313E-05	1.2251E-05
	0.04	1.7994E-05	1.7911E-05	1.7990E-05	1.7913E-05
	0.06	2.4757E-05	2.4658E-05	2.4752E-05	2.4660E-05
	0.08	3.2616E-05	3.2500E-05	3.2610E-05	3.2503E-05
	0.10	4.1581E-05	4.1448E-05	4.1575E-05	4.1451E-05
0.10	0.02	1.7994E-05	1.7911E-05	1.7990E-05	1.7913E-05
	0.04	2.4757E-05	2.4658E-05	2.4752E-05	2.4660E-05
	0.06	3.2616E-05	3.2500E-05	3.2610E-05	3.2503E-05
	0.08	4.1581E-05	4.1448E-05	4.1575E-05	4.1451E-05
	0.10	5.1665E-05	5.1513E-05	5.1657E-05	5.1516E-05

be reduced in this transform than other traditional methods while preserving the efficiency. When $v = 1$ and $s = 1$, the NT reduced to the Laplace transform and Sumudu integral transform, respectively.

Definition 5. The natural transform of $\mathfrak{u}(t)$ is defined by

$$\mathbf{NT}(\mathfrak{u}(t)) = \mathfrak{U}(s, v) = \int_{-\infty}^{\infty} e^{-st}\mathfrak{u}(vt)dt, s, v \in (-\infty,\infty). \quad (7)$$

For $t \in (0,\infty)$, natural transform of $\mathfrak{u}(t)$ is defined by

$$\mathbf{NT}(\mathfrak{u}(t)H(t)) = \mathbf{NT}^{+}(\mathfrak{u}(t)) = \mathfrak{U}^{+}(s, v) = \int_{0}^{\infty} e^{-st}\mathfrak{u}(vt)dt, s, v \in (0,\infty), \quad (8)$$

where $H(t)$ is the Heaviside function.

Definition 6. The inverse natural transform of $\mathfrak{U}(s, v)$ is given by

$$\mathbf{NT}^{-1}[\mathfrak{U}(s, v)] = \mathfrak{u}(t), \forall t \geq 0. \quad (9)$$

Lemma 7 (linearity property). *If natural transform of $\mathfrak{u}_1(t)$ is $\mathfrak{u}_1(s, v)$ and $\mathfrak{u}_2(t)$ is $\mathfrak{u}_2(s, v)$, then*

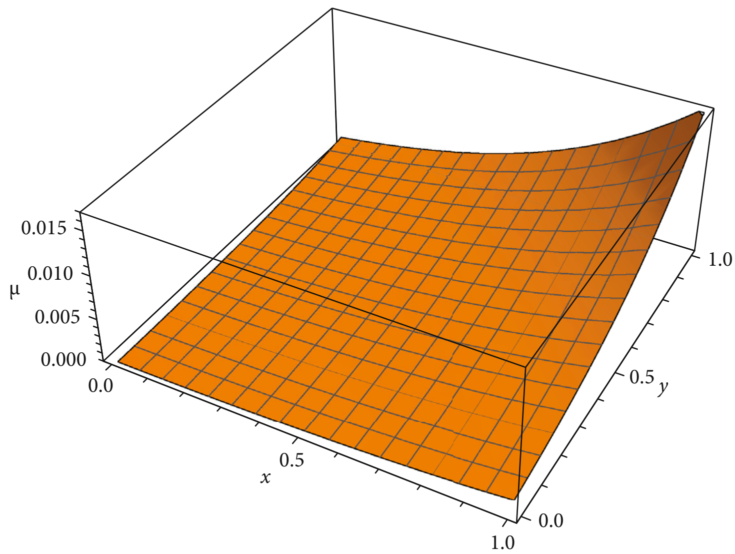
$$\begin{aligned}\mathbf{NT}[c_1\mathfrak{u}_1(t) + c_2\mathfrak{u}_2(t)] &= c_1\mathbf{NT}[\mathfrak{u}_1(t)] + c_2\mathbf{NT}[\mathfrak{u}_2(t)] \\ &= c_1\mathfrak{u}_1(s, v) + c_2\mathfrak{u}_2(s, v),\end{aligned} \quad (10)$$

where c_1 and c_2 are constants.

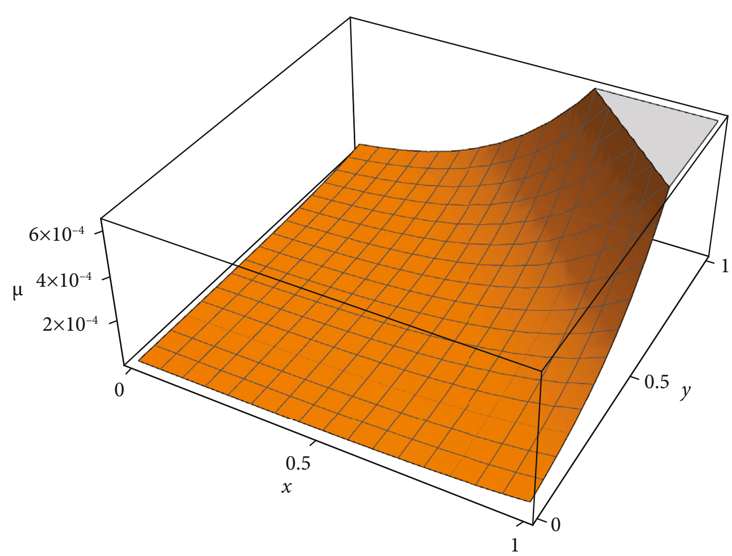
Lemma 8 (inverse linearity property). *If inverse natural transform of $\mathfrak{u}_1(s, v)$ and $\mathfrak{u}_2(s, v)$ is $\mathfrak{u}_1(t)$ and $\mathfrak{u}_2(t)$, respectively, then*

$$\begin{aligned}\mathbf{NT}^{-1}[c_1\mathfrak{u}_1(s, v) + c_2\mathfrak{u}_2(s, v)] &= c_1\mathbf{NT}^{-1}[\mathfrak{u}_1(s, v)] + c_2\mathbf{NT}^{-1}[\mathfrak{u}_2(s, v)] \\ &= c_1\mathfrak{u}_1(t) + c_2\mathfrak{u}_2(t),\end{aligned} \quad (11)$$

where c_1 and c_2 are constants.

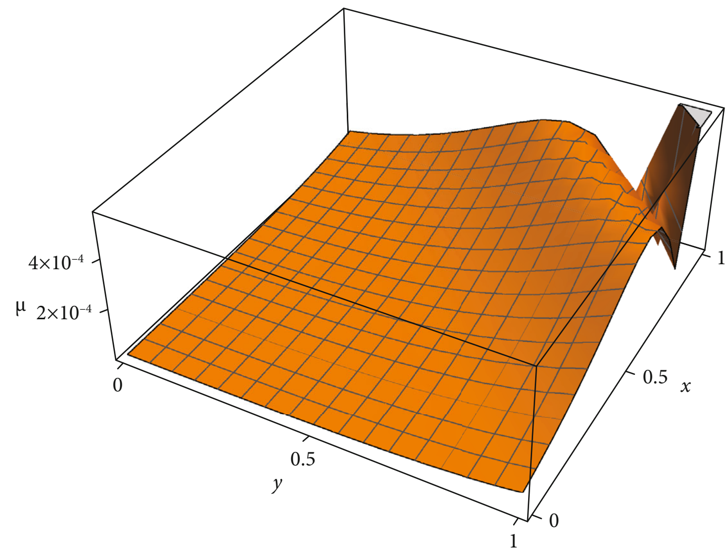


(a)



(b)

Figure 1: Continued.



(c)

Figure 1: (a) Exact solution, (b) absolute error of $\mathrm{NTDM}_{\mathrm{CF}}$, and (c) absolute error of $\mathrm{NTDM}_{\mathrm{ABC}}$ of Example 1 for $\mu = 1$, $t = 0.5$, and $\lambda = 0.001$.

Definition 9 (see [43]). Natural transform of $D_t^{\mu}\mathbf{u}(t)$ by means of Caputo sense is given as

$$\mathbf{NT}\left[{}_0^C D_t^{\mu}\mathbf{u}(t)\right] = \left(\frac{s}{\nu}\right)^{\mu}\left(\mathbf{NT}[\mathbf{u}(t)] - \left(\frac{1}{s}\right)\mathbf{u}(0)\right). \tag{12}$$

Definition 10 (see [44]). Natural transform of $D_t^{\mu}\mathbf{u}(t)$ by means of CF is defined as

$$\mathbf{NT}\left[{}_0^{\mathrm{CF}} D_t^{\mu}\mathbf{u}(t)\right] = \frac{1}{1-\mu+\mu(\nu/s)}\left(\mathbf{NT}[\mathbf{u}(t)] - \frac{1}{s}\mathbf{u}(0)\right). \tag{13}$$

With this motivation, we defined natural transform of ABC derivative as follows.

Definition 11. Natural transform of $D_t^{\mu}\mathbf{u}(t)$ by means of ABC derivative is defined as

$$\mathbf{NT}\left[{}_0^{\mathrm{ABC}} D_t^{\mu}\mathbf{u}(t)\right] = \frac{M[\mu]}{1-\mu+\mu(\nu/s)^{\mu}}\left(\mathbf{NT}[\mathbf{u}(t)] - \left(\frac{1}{s}\right)\mathbf{u}(0)\right). \tag{14}$$

3. Methodology

In this section, we present a novel approximate analytical procedure based on natural transform [42] to the following equation

$$D_t^{\mu}\mathbf{u}(\zeta,t) = \mathscr{L}(\mathbf{u}(\zeta,t)) + \mathscr{N}(\mathbf{u}(\zeta,t)) + h(\zeta,t), \tag{15}$$

with the initial condition

$$\mathbf{u}(\zeta,0) = \phi(\zeta), \tag{16}$$

where $\mathscr{N}$, $\mathscr{L}$, and $h(\zeta,t)$ are nonlinear, linear, and source terms, respectively. Now we employing NT on equation (15) by considering fractional derivative by means of three fractional definitions.

Case 1. ($\mathrm{NTDM}_{\mathrm{CF}}$). By taking natural transform of equation (15), by means of CF fractional derivative, we obtain

$$\frac{1}{p(\mu,\nu,s)}\left(\mathbf{NT}[\mathbf{u}(\zeta,t)] - \frac{\phi(\zeta)}{s}\right) = \mathbf{NT}[M(\zeta,t)], \tag{17}$$

where

$$p(\mu,\nu,s) = 1-\mu+\mu\left(\frac{\nu}{s}\right). \tag{18}$$

By taking inverse natural transform using (8), we rewrite (17) as

$$\mathbf{u}(\zeta,t) = \mathbf{NT}^{-1}\left[\frac{\phi(\zeta)}{s} + p(\mu,\nu,s)\mathbf{NT}[M(\zeta,t)]\right]. \tag{19}$$

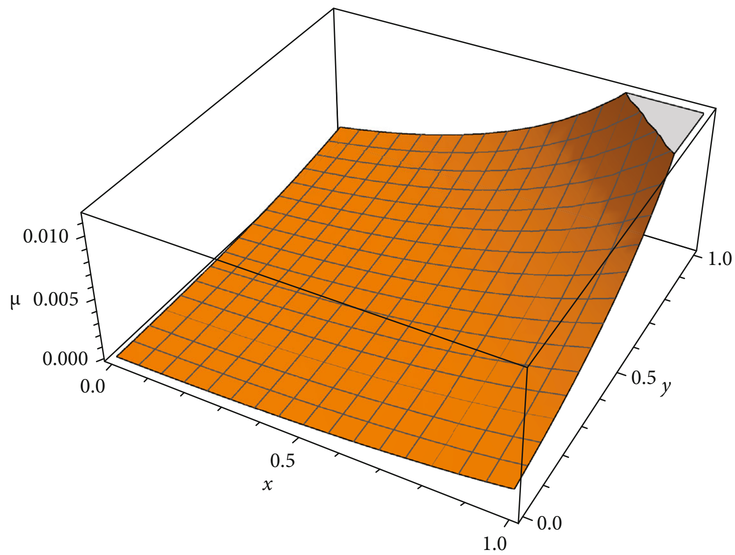
$\mathscr{N}(\mathbf{u}(\zeta,t))$ can be decomposed into

$$\mathscr{N}(\mathbf{u}(\zeta,t)) = \sum_{l=0}^{\infty} A_l, \tag{20}$$

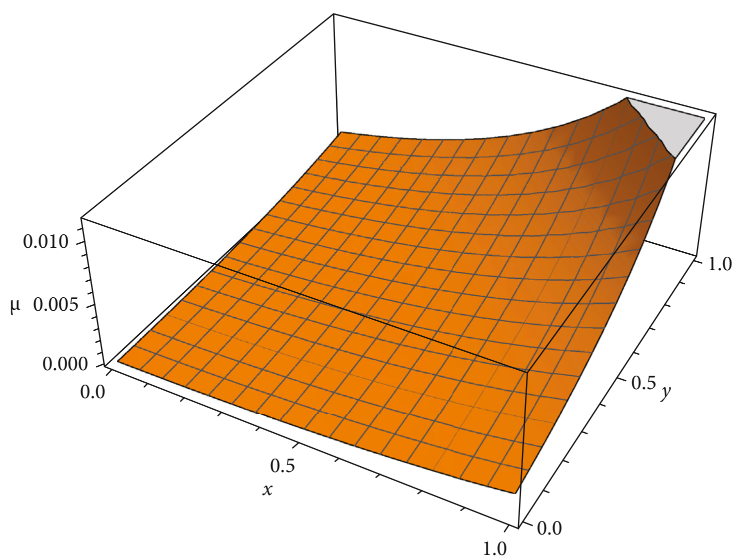
where A_l is the Adomian polynomials [45, 46]. We assume that equation (15) has the analytical expansion

$$\mathbf{u}(\zeta,t) = \sum_{l=0}^{\infty} \mathbf{u}_l(\zeta,t). \tag{21}$$

By substituting equations (20) and (21) into (19), we obtain

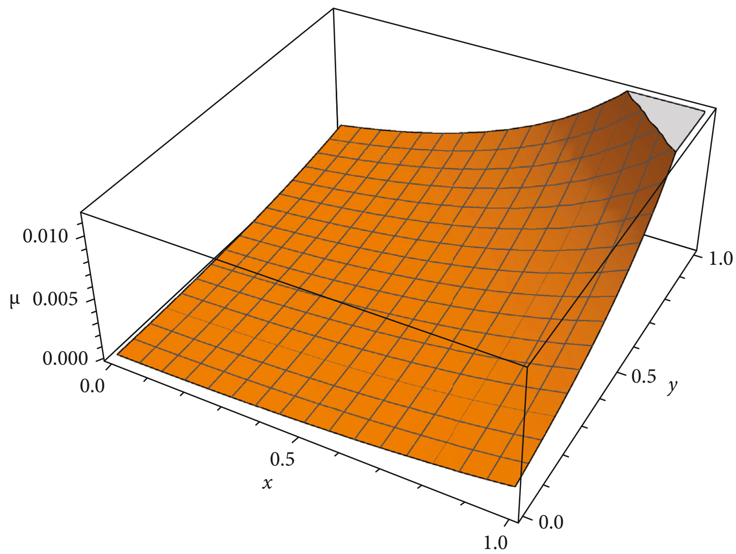


(a)

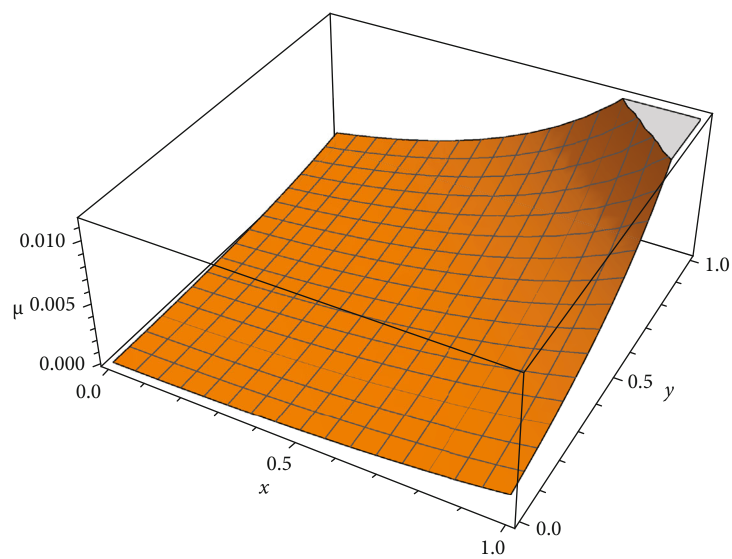


(b)

Figure 2: Continued.



(c)



(d)

FIGURE 2: (a) $\mathrm{NTDM}_{\mathrm{CF}}$ for $\mu = 0.60$, (b) $\mathrm{NTDM}_{\mathrm{CF}}$ for $\mu = 0.75$, (c) $\mathrm{NTDM}_{\mathrm{ABC}}$ for $\mu = 0.60$, and (d) $\mathrm{NTDM}_{\mathrm{ABC}}$ for $\mu = 0.75$, when $t = 0.5$ and $\lambda = 0.001$ of Example 1.

$$\sum_{l=0}^{\infty} \mathbf{u}_l(\zeta, t) = \mathbf{NT}^{-1}\left[\frac{\phi(\zeta)}{s} + p(\mu, \nu, s)\mathbf{NT}[h(\zeta, t)]\right] + \mathbf{NT}^{-1}\left[p(\mu, \nu, s)\mathbf{NT}\left[\sum_{l=0}^{\infty} \mathscr{L}(\mathbf{u}_l(\zeta, t)) + A_l\right]\right]. \tag{22}$$

From (22), we get

$$\begin{aligned} \mathbf{u}_0^{\mathrm{CF}}(\zeta, t) &= \mathbf{NT}^{-1}\left[\frac{\phi(\zeta)}{s} + p(\mu, \nu, s)\mathbf{NT}[h(\zeta, t)]\right] \mathbf{u}_1^{\mathrm{CF}}(\zeta, t) \\ &= \mathbf{NT}^{-1}[p(\mu, \nu, s)\mathbf{NT}[\mathscr{L}(\mathbf{u}_0(\zeta, t)) + A_0]] \vdots \mathbf{u}_{l+1}^{\mathrm{CF}}(\zeta, t) \\ &= \mathbf{NT}^{-1}[p(\mu, \nu, s)\mathbf{NT}[\mathscr{L}(\mathbf{u}_l(\zeta, t)) + A_l]], l = 1, 2, \cdots. \end{aligned} \tag{23}$$

By substituting (23) into (21), we get the $\mathrm{NTDM}_{\mathrm{CF}}$ solution of (15) and (16) as

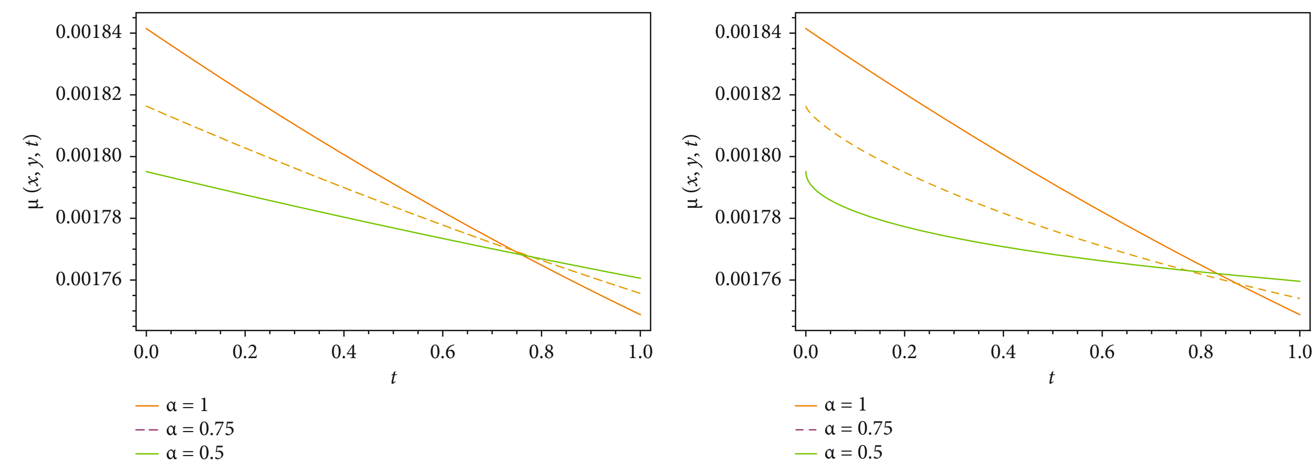
$$\mathbf{u}^{\mathrm{CF}}(\zeta, t) = \mathbf{u}_0^{\mathrm{CF}}(\zeta, t) + \mathbf{u}_1^{\mathrm{CF}}(\zeta, t) + \mathbf{u}_2^{\mathrm{CF}}(\zeta, t) + \cdots. \tag{24}$$

Case 2. ($\mathrm{NTDM}_{\mathrm{ABC}}$). By taking natural transform of equation (15), by means of ABC derivative, we acquire

$$\frac{1}{q(\mu, \nu, s)}\left(\mathbf{NT}[\mathbf{u}(\zeta, t)] - \frac{\phi(\zeta)}{s}\right) = \mathbf{NT}[M(\zeta, t)], \tag{25}$$

where

$$q(\mu, \nu, s) = \frac{1 - \mu + \mu(\nu/s)^{\mu}}{B(\mu)}. \tag{26}$$



(a) (b)

FIGURE 3: (a) $\mathrm{NTDM_{CF}}$ and (b) $\mathrm{NTDM_{ABC}}$ of Example 1, for $x = 0.5$, $\zeta = 0.5$, and $\lambda = 0.001$ for different values of μ.

By taking inverse natural transform using (8), we rewrite (25), as

$$\mathbf{u}(\zeta, t) = \mathbf{NT}^{-1}\left[\frac{\phi(\zeta)}{s} + q(\mu, \nu, s)\mathbf{NT}[M(\zeta, t)]\right]. \tag{27}$$

$\mathscr{N}(\mathbf{u}(\zeta, t))$ can be decomposed into

$$\mathscr{N}(\mathbf{u}(\zeta, t)) = \sum_{l=0}^{\infty} A_l, \tag{28}$$

where A_l is the Adomian polynomials. We assume that equation (15) has the analytical expansion

$$\mathbf{u}(\zeta, t) = \sum_{l=0}^{\infty} \mathbf{u}_l(\zeta, t). \tag{29}$$

By substituting equations (28) and (29) into (27), we obtain

$$\begin{aligned}\sum_{l=0}^{\infty} \mathbf{u}_l(\zeta, t) = {} & \mathbf{NT}^{-1}\left[\frac{\phi(\zeta)}{s} + q(\mu, \nu, s)\mathbf{NT}[h(\zeta, t)]\right] \\ & + \mathbf{NT}^{-1}\left[q(\mu, \nu, s)\mathbf{NT}\left[\left(\sum_{l=0}^{\infty} \mathscr{L}(\mathbf{u}_l(\zeta, t)) + A_l\right)\right]\right].\end{aligned} \tag{30}$$

From (30), we get

$$\begin{aligned}\mathbf{u}_0^{\mathrm{ABC}}(\zeta, t) &= \mathbf{NT}^{-1}\left[\frac{\phi(\zeta)}{s}\right] + \mathbf{NT}^{-1}[q(\mu, \nu, s)\mathbf{NT}[h(\zeta, t)]]\,\mathbf{u}_1^{\mathrm{ABC}}(\zeta, t) \\ &= \mathbf{NT}^{-1}[q(\mu, \nu, s)\mathbf{NT}[\mathscr{L}(\mathbf{u}_0(\zeta, t)) + A_0]] \,\vdots\, \mathbf{u}_{l+1}^{\mathrm{ABC}}(\zeta, t) \\ &= \mathbf{NT}^{-1}[q(\mu, \nu, s)\mathbf{NT}[\mathscr{L}(\mathbf{u}_l(\zeta, t)) + A_l]], l = 1, 2, \cdots.\end{aligned} \tag{31}$$

By substituting (31) into (29), we get the $\mathrm{NTDM_{ABC}}$ solution of (15)–(16) as

$$\mathbf{u}^{\mathrm{ABC}}(\zeta, t) = \mathbf{u}_0^{\mathrm{ABC}}(\zeta, t) + \mathbf{u}_1^{\mathrm{ABC}}(\zeta, t) + \mathbf{u}_2^{\mathrm{ABC}}(\zeta, t) + \cdots \tag{32}$$

4. Convergence Analysis

We have presented uniqueness and convergence of the $\mathrm{NTDM_{CF}}$ and $\mathrm{NTDM_{ABC}}$ in this section.

Theorem 12. *The $\mathrm{NTDM_{CF}}$ solution of (15) is unique when $0 < (\delta_1 + \delta_2)(1 - \mu + \mu t) < 1$.*

Proof. Let $F = (C[J], \|.\|)$ be the Banach space with the norm $\|\phi(t\| = \max_{t\in J}|\phi(t)|$, $\forall$ continuous functions on J. Let $G : F \longrightarrow F$ is a nonlinear mapping, where

$$\mathbf{u}_{l+1}^{C}(\zeta, t) = \mathbf{u}_0^{C} + \mathbf{NT}^{-1}[p(\mu, \nu, s)\mathbf{NT}[\mathscr{L}(\mathbf{u}_l(\zeta, t)) + \mathscr{N}(\mathbf{u}_l(\zeta, t))]], l \geq 0. \tag{33}$$

Suppose that $|\mathscr{L}(\mathbf{u}) - \mathscr{L}(\mathbf{u}^*)| < \delta_1|\mathbf{u} - \mathbf{u}^*|$ and $|\mathscr{N}(\mathbf{u}) - \mathscr{N}(\mathbf{u}^*)| < \delta_2|\mathbf{u} - \mathbf{u}^*|$, where δ_1 and δ_2 are Lipschitz constants and $\mathbf{u} \coloneqq \mathbf{u}(\zeta, t)$ and $\mathbf{u}^* \coloneqq \mathbf{u}^*(\zeta, t)$ are two different function values.

$$\begin{aligned}\|G\mathbf{u} - G\mathbf{u}^*\| &\leq \max_{t\in J}\left|\mathbf{NT}^{-1}[p(\mu, \nu, s)\mathbf{NT}[\mathscr{L}(\mathbf{u}) - \mathscr{L}(\mathbf{u}^*)]\right. \\ &\quad \left. + p(\mu, \nu, s)\mathbf{NT}[\mathscr{N}(\mathbf{u}) - \mathscr{N}(\mathbf{u}^*)]]\right| \\ &\leq \max_{t\in J}\left[\delta_1\mathbf{NT}^{-1}[p(\mu, \nu, s)\mathbf{NT}|\mathbf{u} - \mathbf{u}^*|]\right. \\ &\quad \left. + \delta_2\mathbf{NT}^{-1}[p(\mu, \nu, s)\mathbf{NT}|\mathbf{u} - \mathbf{u}^*|]\right] \\ &\leq \max_{t\in J}(\delta_1 + \delta_2)\left[\mathbf{NT}^{-1}[p(\mu, \nu, s)\mathbf{NT}|\mathbf{u} - \mathbf{u}^*|]\right] \\ &\leq (\delta_1 + \delta_2)\left[\mathbf{NT}^{-1}[p(\mu, \nu, s)\mathbf{NT}\|\mathbf{u} - \mathbf{u}^*\|]\right] \\ &= (\delta_1 + \delta_2)(1 - \mu + \mu t)\|\mathbf{u} - \mathbf{u}^*\|.\end{aligned} \tag{34}$$

G is contraction as $0 < (\delta_1 + \delta_2)(1 - \mu + \mu t) < 1$. The solution of (15) is unique from Banach fixed point theorem. □

Table 3: Absolute errors of $NTDM_{CF}$ and $NTDM_{ABC}$ with existing methods of Example 2 when $\lambda = 0.001$.

x, ζ	t	$NTDM_{CF}$	$NTDM_{ABC}$	FNDM [28]	$q -$ HATM [28]
	0.02	4.9926E-09	4.9926E-09	4.9926E-09	4.9926E-09
	0.04	9.9852E-09	9.9852E-09	9.9852E-09	9.9852E-09
0.02	0.06	1.4979E-08	1.4979E-08	1.4979E-08	1.4979E-08
	0.08	1.9970E-08	1.9970E-08	1.9970E-08	1.9970E-08
	0.10	2.4963E-08	2.4963E-08	2.4963E-08	2.4963E-08
	0.02	4.9929E-09	4.9929E-09	4.9929E-09	4.9929E-09
	0.04	9.9859E-09	9.9859E-09	9.9859E-09	9.9859E-09
0.04	0.06	1.4979E-08	1.4979E-08	1.4979E-08	1.4979E-08
	0.08	1.9972E-08	1.9972E-08	1.9972E-08	1.9972E-08
	0.10	2.4965E-08	2.4965E-08	2.4965E-08	2.4965E-08
	0.02	4.9934E-09	4.9934E-09	4.9934E-09	4.9934E-09
	0.04	9.9869E-09	9.9869E-09	9.9869E-09	9.9869E-09
0.06	0.06	1.4980E-08	1.4980E-08	1.4980E-08	1.4980E-08
	0.08	1.9974E-08	1.9974E-08	1.9974E-08	1.9974E-08
	0.10	2.4967E-08	2.4967E-08	2.4967E-08	2.4967E-08
	0.02	4.9942E-09	4.9942E-09	4.9942E-09	4.9942E-09
	0.04	9.9884E-09	9.9884E-09	9.9884E-09	9.9884E-09
0.08	0.06	1.4983E-08	1.4983E-08	1.4983E-08	1.4983E-08
	0.08	1.9977E-08	1.9977E-08	1.9977E-08	1.9977E-08
	0.10	2.4971E-08	2.4971E-08	2.4971E-08	2.4971E-08
	0.02	4.9952E-09	4.9952E-09	4.9952E-09	4.9952E-09
	0.04	9.9904E-09	9.9904E-09	9.9904E-09	9.9904E-09
0.10	0.06	1.4986E-08	1.4986E-08	1.4986E-08	1.4986E-08
	0.08	1.9981E-08	1.9981E-08	1.9981E-08	1.9981E-08
	0.10	2.4976E-08	2.4976E-08	2.4976E-08	2.4976E-08

Theorem 13. *The $NTDM_{ABC}$ solution of (15) is unique when $0 < (\delta_1 + \delta_2)(1 - \mu + \mu(t^\mu/\Gamma(\mu + 1))) < 1$.*

Proof. Let $F = (C[J], \|.\|)$ be the Banach space with the norm $\|\phi(t\| = \max_{t\in J}|\phi(t)|$, $\forall$ continuous functions on J. Let $G : F \longrightarrow F$ is a nonlinear mapping, where

$$\mathbf{u}^C_{l+1}(\zeta, t) = \mathbf{u}^C_0 + \mathbf{NT}^{-1}[q(\mu, \nu, s)\mathbf{NT}[\mathcal{L}(\mathbf{u}_l(\zeta, t)) + \mathcal{N}(\mathbf{u}_l(\zeta, t))]], l \geq 0. \tag{35}$$

Suppose that $|\mathcal{L}(\mathbf{u}) - \mathcal{L}(\mathbf{u}^*)| < \delta_1|\mathbf{u} - \mathbf{u}^*|$ and $|\mathcal{N}(\mathbf{u}) - \mathcal{N}(\mathbf{u}^*)| < \delta_2|\mathbf{u} - \mathbf{u}^*|$, where δ_1 and δ_2 are Lipschitz constants and $\mathbf{u} \coloneqq \mathbf{u}(\zeta, t)$ and $\mathbf{u}^* \coloneqq \mathbf{u}^*(\zeta, t)$ are two different function values.

$$\begin{aligned}
\|G\mathbf{u} - G\mathbf{u}^*\| &\leq \max_{t\in J}\left|\mathbf{NT}^{-1}[q(\mu, \nu, s)\mathbf{NT}[\mathcal{L}(\mathbf{u}) - \mathcal{L}(\mathbf{u}^*)]\right. \\
&\quad \left. + q(\mu, \nu, s)\mathbf{NT}[\mathcal{N}(\mathbf{u}) - \mathcal{N}(\mathbf{u}^*)]]\right| \\
&\leq \max_{t\in J}\left[\delta_1\mathbf{NT}^{-1}[q(\mu, \nu, s)\mathbf{NT}|\mathbf{u} - \mathbf{u}^*|]\right. \\
&\quad \left. + \delta_2\mathbf{NT}^{-1}[q(\mu, \nu, s)\mathbf{NT}|\mathbf{u} - \mathbf{u}^*|]\right] \\
&\leq \max_{t\in J}(\delta_1 + \delta_2)\left[\mathbf{NT}^{-1}[q(\mu, \nu, s)\mathbf{NT}|\mathbf{u} - \mathbf{u}^*|]\right] \\
&\leq (\delta_1 + \delta_2)\left[\mathbf{NT}^{-1}[q(\mu, \nu, s)\mathbf{NT}\|\mathbf{u} - \mathbf{u}^*\|]\right] \\
&= (\delta_1 + \delta_2)\left(1 - \mu + \mu\frac{t^\mu}{\Gamma(\mu + 1)}\right)\|\mathbf{u} - \mathbf{u}^*\|.
\end{aligned} \tag{36}$$

G is contraction as $0 < (\delta_1 + \delta_2)(1 - \mu + \mu(t^\mu/\Gamma(\mu + 1))) < 1$. The solution of (15) is unique from Banach fixed point theorem. □

Theorem 14. *$NTDM_{CF}$ solution of (15) is convergent.*

Proof. Let $\mathbf{u}_m = \sum_{r=0}^m \mathbf{u}_r(\zeta, t)$. To prove that $\mathbf{u}_m$ is a Cauchy sequence in F. Consider

$$\begin{aligned}
\|\mathbf{u}_m - \mathbf{u}_n\| &= \max_{t\in J}|\mathbf{u}_m - \mathbf{u}_n| = \max_{t\in J}\left|\sum_{r=n+1}^{m}\mathbf{u}_r\right|, n = 1, 2, 3, \cdots \leq \max_{t\in J}\left|\mathbf{NT}^{-1}\right. \\
&\quad \left.\cdot\left[p(\mu, \nu, s)\mathbf{NT}\left[\sum_{r=n+1}^{m}(\mathcal{L}(\mathbf{u}_{r-1}) + \mathcal{N}(\mathbf{u}_{r-1}))\right]\right]\right| \\
&= \max_{t\in J}\left|\mathbf{NT}^{-1}\left[p(\mu, \nu, s)\mathbf{NT}\left[\sum_{r=n}^{m-1}(\mathcal{L}(\mathbf{u}_r) + \mathcal{N}(\mathbf{u}_r))\right]\right]\right| \\
&\leq \max_{t\in J}\left|\mathbf{NT}^{-1}[p(\mu, \nu, s)\mathbf{NT}[(\mathcal{L}(\mathbf{u}_{m-1}) - \mathcal{L}(\mathbf{u}_{n-1})\right. \\
&\quad \left. + \mathcal{N}(\mathbf{u}_{m-1}) - \mathcal{N}(\mathbf{u}_{n-1}))]]\right| \leq \delta_1 \max_{t\in J} \\
&\quad \cdot\left|\mathbf{NT}^{-1}[p(\mu, \nu, s)\mathbf{NT}[(\mathcal{L}(\mathbf{u}_{m-1}) - \mathcal{L}(\mathbf{u}_{n-1}))]]\right| \\
&\quad + \delta_2 \max_{t\in J}\left|\mathbf{NT}^{-1}[p(\mu, \nu, s)\mathbf{NT}[(\mathcal{N}(\mathbf{u}_{m-1}) - \mathcal{N}(\mathbf{u}_{n-1}))]]\right| \\
&= (\delta_1 + \delta_2)(1 - \mu + \mu t)\|\mathbf{u}_{m-1} - \mathbf{u}_{n-1}\|.
\end{aligned} \tag{37}$$

Table 4: Approximate solutions of $NTDM_{CF}$ and $NTDM_{ABC}$ for $\mu = 0.67$ and $\mu = 0.75$ of Example 2 when $t = 1$ and $\lambda = 0.001$.

x	ζ	$\mu = 0.67$		$\mu = 0.75$	
		$NTDM_{CF}$	$NTDM_{ABC}$	$NTDM_{CF}$	$NTDM_{ABC}$
0.02	0.02	9.9997E-06	9.9997E-06	9.9997E-06	9.9997E-06
	0.04	1.5000E-05	1.5000E-05	1.5000E-05	1.5000E-05
	0.06	2.0000E-05	2.0000E-05	2.0000E-05	2.0000E-05
	0.08	2.5001E-05	2.5001E-05	2.5001E-05	2.5001E-05
	0.10	3.0002E-05	3.0002E-05	3.0002E-05	3.0002E-05
0.04	0.02	1.5000E-05	1.5000E-05	1.5000E-05	1.5000E-05
	0.04	2.0000E-05	2.0000E-05	2.0000E-05	2.0000E-05
	0.06	2.5001E-05	2.5001E-05	2.5001E-05	2.5001E-05
	0.08	3.0002E-05	3.0002E-05	3.0002E-05	3.0002E-05
	0.10	3.5003E-05	3.5003E-05	3.5003E-05	3.5003E-05
0.06	0.02	2.0000E-05	2.0000E-05	2.0000E-05	2.0000E-05
	0.04	2.5001E-05	2.5001E-05	2.5001E-05	2.5001E-05
	0.06	3.0002E-05	3.0002E-05	3.0002E-05	3.0002E-05
	0.08	3.5003E-05	3.5003E-05	3.5003E-05	3.5003E-05
	0.10	4.0004E-05	4.0004E-05	4.0004E-05	4.0004E-05
0.08	0.02	2.5001E-05	2.5001E-05	2.5001E-05	2.5001E-05
	0.04	3.0002E-05	3.0002E-05	3.0002E-05	3.0002E-05
	0.06	3.5003E-05	3.5003E-05	3.5003E-05	3.5003E-05
	0.08	4.0004E-05	4.0004E-05	4.0004E-05	4.0004E-05
	0.10	4.5006E-05	4.5006E-05	4.5006E-05	4.5006E-05
10	0.02	3.0002E-05	3.0002E-05	3.0002E-05	3.0002E-05
	0.04	3.5003E-05	3.5003E-05	3.5003E-05	3.5003E-05
	0.06	4.0004E-05	4.0004E-05	4.0004E-05	4.0004E-05
	0.08	4.5006E-05	4.5006E-05	4.5006E-05	4.5006E-05
	0.10	5.0009E-05	5.0009E-05	5.0009E-05	5.0009E-05

Let $m = n + 1$, then

$$\begin{aligned}\|\mathfrak{u}_{n+1} - \mathfrak{u}_n\| &\le \delta\|\mathfrak{u}_n - \mathfrak{u}_{n-1}\| \le \delta^2\|\mathfrak{u}_{n-1} - \mathfrak{u}_{n-2}\| \\ &\le \cdots \le \delta^n\|\mathfrak{u}_1 - \mathfrak{u}_0\|,\end{aligned} \tag{38}$$

where $\delta = (\delta_1 + \delta_2)(1 - \mu + \mu t)$. Similarly, we have

$$\begin{aligned}\|\mathfrak{u}_m - \mathfrak{u}_n\| &\le \|\mathfrak{u}_{n+1} - \mathfrak{u}_n\| + \|\mathfrak{u}_{n+2} - \mathfrak{u}_{n+1}\| + \cdots + \|\mathfrak{u}_m - \mathfrak{u}_{m-1}\|, \\ &\le \left(\delta^n + \delta^{n+1} + \cdots + \delta^{m-1}\right)\|\mathfrak{u}_1 - \mathfrak{u}_0\| \\ &\le \delta^n\left(\frac{1-\delta^{m-n}}{1-\delta}\right)\|\mathfrak{u}_1\|,\end{aligned} \tag{39}$$

As $0 < \delta < 1$, we get $1 - \delta^{m-n} < 1$. Therefore

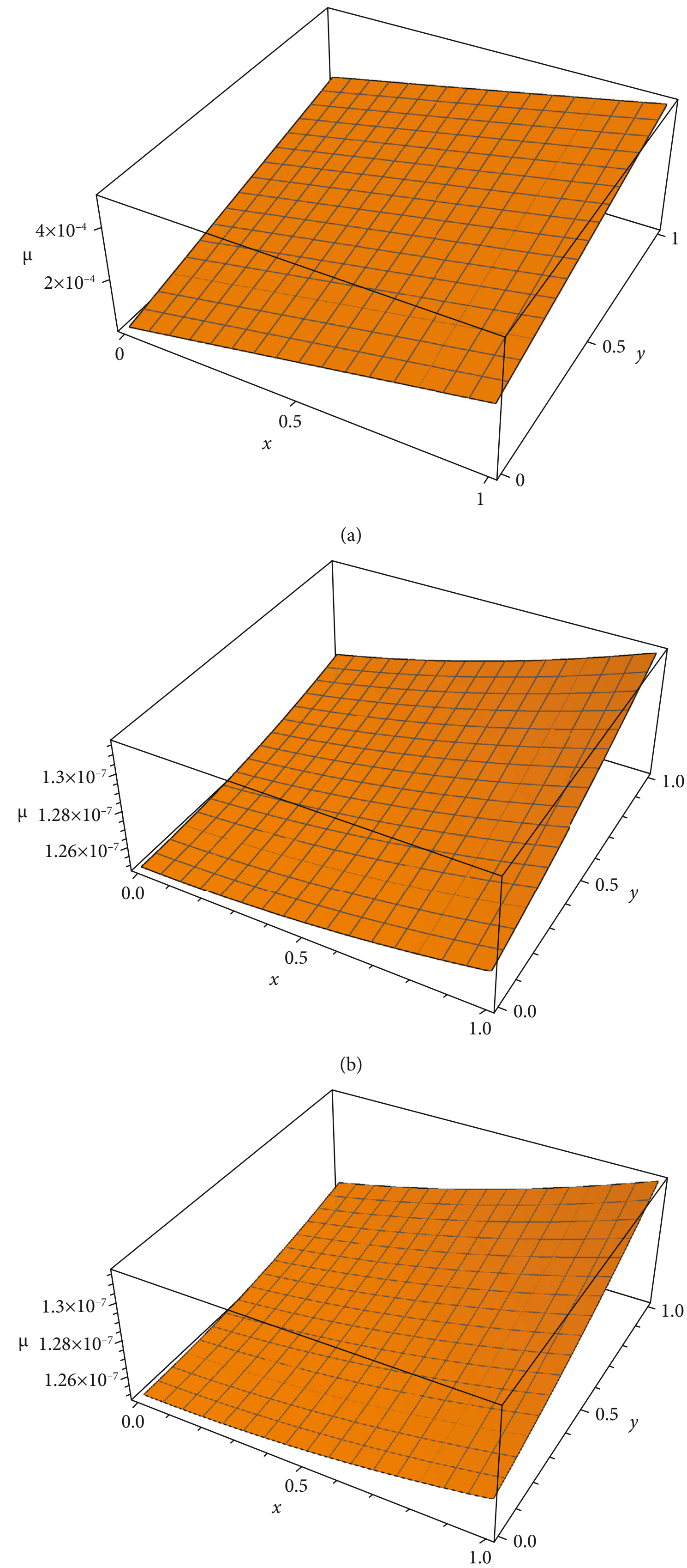
$$\|\mathfrak{u}_m - \mathfrak{u}_n\| \le \frac{\delta^n}{1-\delta}\max_{t\in J}\|\mathfrak{u}_1\|. \tag{40}$$

Since $\|\mathfrak{u}_1\| < \infty$. $\|\mathfrak{u}_m - \mathfrak{u}_n\| \longrightarrow 0$ when $n \longrightarrow \infty$. Hence, $\mathfrak{u}_m$ is a Cauchy sequence in F; therefore, the series $\mathfrak{u}_m$ is convergent. □

Theorem 15. *$NTDM_{ABC}$ solution of (15) is convergent.*

Proof. Let $\mathfrak{u}_m = \sum_{r=0}^{m}\mathfrak{u}_r(\zeta, t)$. To prove that $\mathfrak{u}_m$ is a Cauchy sequence in F. Consider

$$\begin{aligned}\|\mathfrak{u}_m - \mathfrak{u}_n\| &= \max_{t\in J}|\mathfrak{u}_m - \mathfrak{u}_n| = \max_{t\in J}\left|\sum_{r=n+1}^{m}\mathfrak{u}_r\right|, n = 1, 2, 3, \cdots. \le \max_{t\in J}\Bigg|\mathbf{NT}^{-1} \\ &\quad \cdot \left[q(\mu, \nu, s)\mathbf{NT}\left[\sum_{r=n+1}^{m}(\mathscr{L}(\mathfrak{u}_{r-1}) + \mathscr{N}(\mathfrak{u}_{r-1}))\right]\right]\Bigg| \\ &= \max_{t\in J}\left|\mathbf{NT}^{-1}\left[q(\mu, \nu, s)\mathbf{NT}\left[\sum_{r=n}^{m-1}(\mathscr{L}(\mathfrak{u}_r) + \mathscr{N}(\mathfrak{u}_r))\right]\right]\right| \\ &\le \max_{t\in J}\big|\mathbf{NT}^{-1}[q(\mu, \nu, s)\mathbf{NT}[(\mathscr{L}(\mathfrak{u}_{m-1}) - \mathscr{L}(\mathfrak{u}_{n-1}) \\ &\quad + \mathscr{N}(\mathfrak{u}_{m-1}) - \mathscr{N}(\mathfrak{u}_{n-1}))]]\big| \le \delta_1\max_{t\in J} \\ &\quad \cdot \left|\mathbf{NT}^{-1}[q(\mu, \nu, s)\mathbf{NT}[(\mathscr{L}(\mathfrak{u}_{m-1}) - \mathscr{L}(\mathfrak{u}_{n-1}))]]\right| \\ &\quad + \delta_2\max_{t\in J}\left|\mathbf{NT}^{-1}[q(\mu, \nu, s)\mathbf{NT}[(\mathscr{N}(\mathfrak{u}_{m-1}) - \mathscr{N}(\mathfrak{u}_{n-1}))]]\right| \\ &= (\delta_1 + \delta_2)\left(1 - \mu + \mu\frac{t^\mu}{\Gamma(\mu+1)}\right)\|\mathfrak{u}_{m-1} - \mathfrak{u}_{n-1}\|.\end{aligned} \tag{41}$$

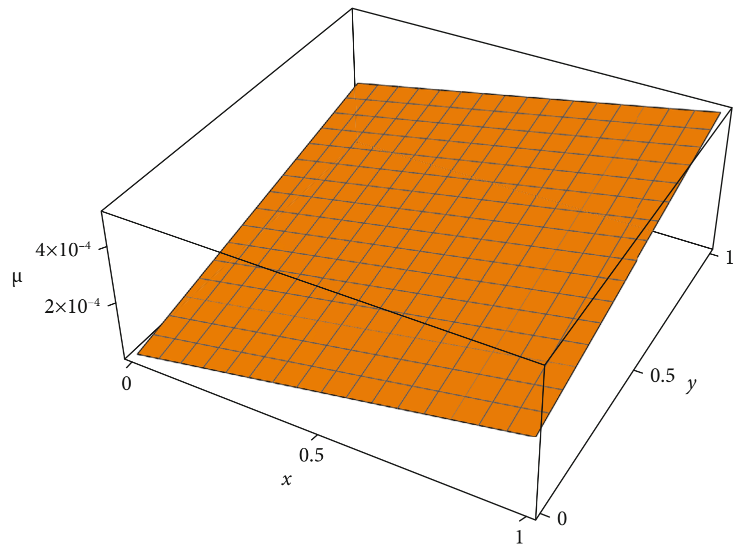


(a)

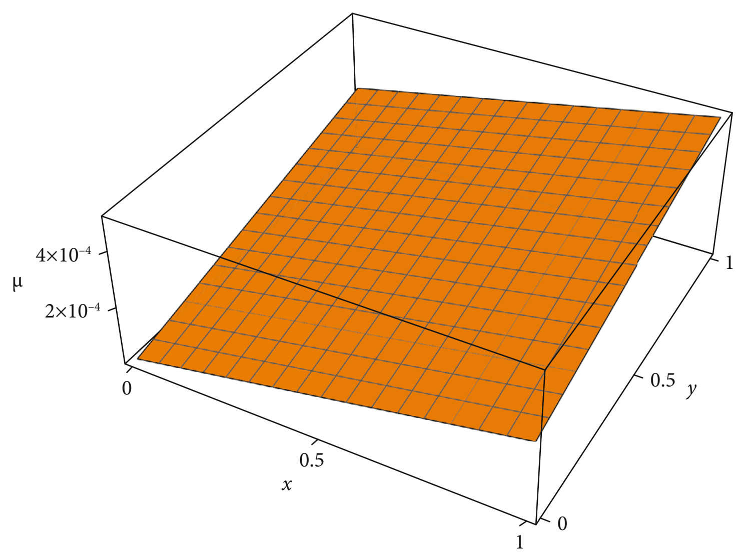
(b)

(c)

Figure 4: (a) Exact solution, (b) absolute error of $\mathrm{NTDM}_{\mathrm{CF}}$, and (c) absolute error of $\mathrm{NTDM}_{\mathrm{ABC}}$, for $\mu = 1$, $t = 0.5$, and $\lambda = 0.001$ of Example 2.

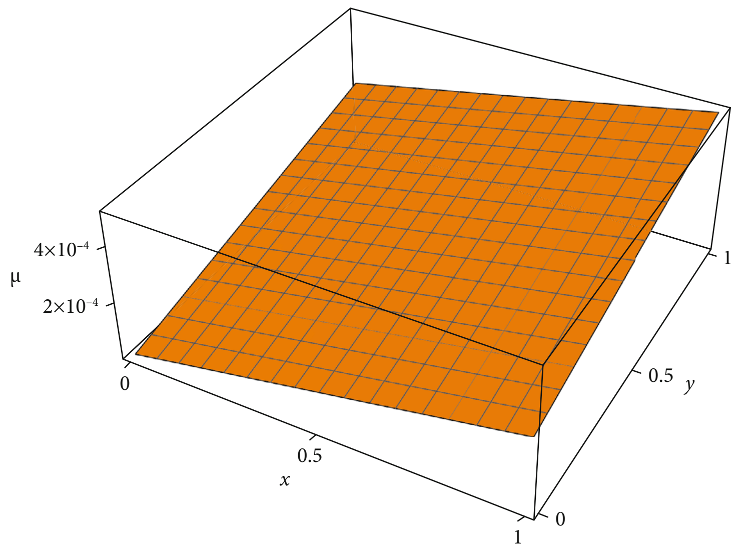


(a)

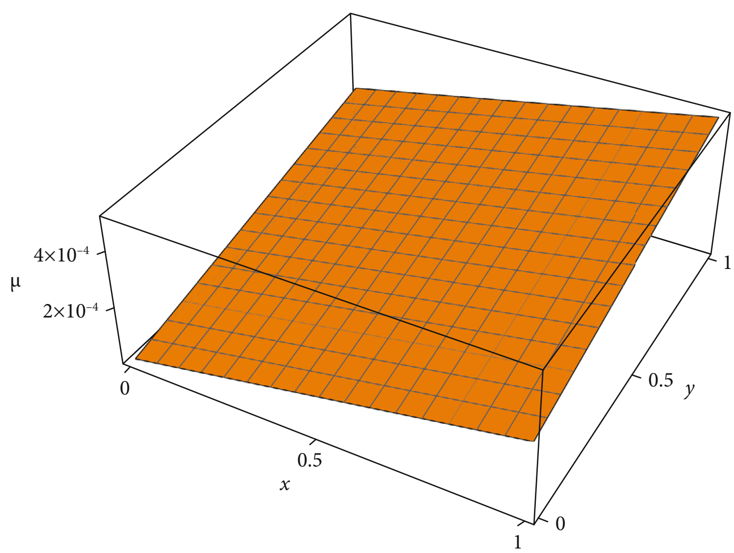


(b)

Figure 5: Continued.



(c)



(d)

Figure 5: Example 2. (a) $\mathrm{NTDM}_{\mathrm{CF}}$ for $\mu = 0.60$. (b) $\mathrm{NTDM}_{\mathrm{CF}}$ for $\mu = 0.75$. (c) $\mathrm{NTDM}_{\mathrm{ABC}}$ for $\mu = 0.60$. (d) $\mathrm{NTDM}_{\mathrm{ABC}}$ for $\mu = 0.75$ when $t = 0.5$ and $\lambda = 0.001$.

Let $m = n + 1$, then

$$\begin{aligned}\|\mathfrak{u}_{n+1} - \mathfrak{u}_n\| &\le \delta\|\mathfrak{u}_n - \mathfrak{u}_{n-1}\| \le \delta^2\|\mathfrak{u}_{n-1} - \mathfrak{u}_{n-2}\| \\ &\le \cdots \le \delta^n\|\mathfrak{u}_1 - \mathfrak{u}_0\|,\end{aligned} \tag{42}$$

where $\delta = (\delta_1 + \delta_2)(1 - \mu + \mu(t^\mu/\Gamma(\mu + 1)))$. Similarly, we have

$$\begin{aligned}\|\mathfrak{u}_m - \mathfrak{u}_n\| &\le \|\mathfrak{u}_{n+1} - \mathfrak{u}_n\| + \|\mathfrak{u}_{n+2} - \mathfrak{u}_{n+1}\| + \cdots + \|\mathfrak{u}_m - \mathfrak{u}_{m-1}\|, \\ &\le \left(\delta^n + \delta^{n+1} + \cdots + \delta^{m-1}\right)\|\mathfrak{u}_1 - \mathfrak{u}_0\| \\ &\le \delta^n\left(\frac{1 - \delta^{m-n}}{1 - \delta}\right)\|\mathfrak{u}_1\|,\end{aligned} \tag{43}$$

As $0 < \delta < 1$, we get $1 - \delta^{m-n} < 1$. Therefore

$$\|\mathfrak{u}_m - \mathfrak{u}_n\| \le \frac{\delta^n}{1 - \delta} \max_{t \in J}\|\mathfrak{u}_1\|. \tag{44}$$

Since $\|\mathfrak{u}_1\| < \infty$. $\|\mathfrak{u}_m - \mathfrak{u}_n\| \longrightarrow 0$ when $n \longrightarrow \infty$. Hence, $\mathfrak{u}_m$ is a Cauchy sequence in F; therefore, the series $\mathfrak{u}_m$ is convergent. □

5. Numerical Examples

This section includes the approximate analytical solutions for a few examples of TFZKE. We have chosen these equations as the closed form solutions are available and also well-known methods employed to study the solutions in the literature.

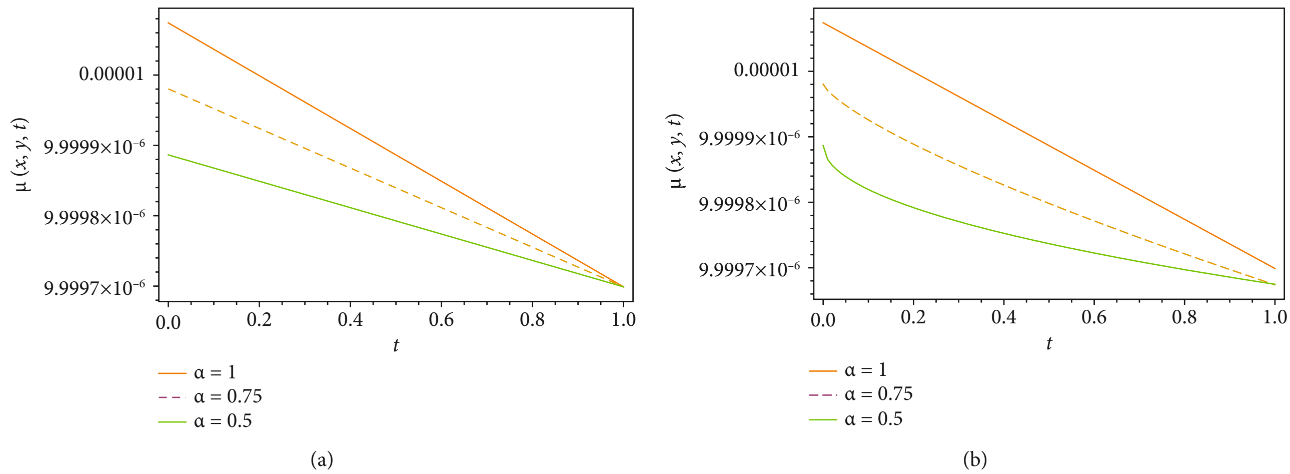


Figure 6: (a) $\mathrm{NTDM}_{\mathrm{CF}}$ and (b) $\mathrm{NTDM}_{\mathrm{ABC}}$, for $x = 0.02$, $\zeta = 0.02$, and $\lambda = 0.001$ for Example 2 with various values of μ.

Example 1. TFZKE (1) is considered with the following parameters. Let $\xi = \eta = \delta = 2$, $a = 1$, $b = c = 1/8$, and $\mathbf{u}(x, \zeta, 0) = (4/3)\lambda \sinh^2(x+\zeta)$ [47, 48]. When $\mu = 1$, exact solution [49] is $\mathbf{u}(x, \zeta, t) = (4/3)\lambda \sinh^2(x+\zeta-\lambda t)$.

$\mathrm{NTDM}_{\mathrm{CF}}$: By employing $\mathrm{NTDM}_{\mathrm{CF}}$, we get

$$\begin{aligned}
\mathbf{u}_0^{\mathrm{CF}}(x,\zeta,t) &= \frac{4}{3}\lambda \sinh^2(x+\zeta),\\
\mathbf{u}_1^{\mathrm{CF}}(x,\zeta,t) &= -\frac{8}{9}\lambda^2(\mu(t-1)+1)(5\sinh(4(x+\zeta)) - 4\sinh(2(x+\zeta))),\\
\mathbf{u}_2^{\mathrm{CF}}(x,\zeta,t) &= \frac{32}{27}\lambda^3\left(2+2\mu^2-4\mu+\left(4\mu-4\mu^2\right)t+\mu^2t^2\right)\\
&\quad \times (13\cosh(2(x+\zeta)) + 75\cosh(6(x+\zeta))\\
&\quad - 70\cosh(4(x+\zeta))), \vdots
\end{aligned} \tag{45}$$

Substituting $\mathbf{u}_0^{\mathrm{CF}}(x,\zeta,t)$, $\mathbf{u}_1^{\mathrm{CF}}(x,\zeta,t)$, in (24), we obtain the $\mathrm{NTDM}_{\mathrm{CF}}$ solution as

$$\begin{aligned}
\mathbf{u}^{\mathrm{CF}}(x,\zeta,t) \approx{}& \frac{4}{3}\lambda \sinh^2(x+\zeta) - \frac{8}{9}\lambda^2(\mu(t-1)+1)\\
&\cdot (5\sinh(4(x+\zeta)) - 4\sinh(2(x+\zeta)))\\
&+ \frac{32}{27}\lambda^3\left(2+2\mu^2-4\mu+\left(4\mu-4\mu^2\right)t+\mu^2t^2\right)\\
&\times (13\cosh(2(x+\zeta)) + 75\cosh(6(x+\zeta))\\
&- 70\cosh(4(x+\zeta))).
\end{aligned} \tag{46}$$

$\mathrm{NTDM}_{\mathrm{ABC}}$: By employing the $\mathrm{NTDM}_{\mathrm{ABC}}$, we get

$$\begin{aligned}
\mathbf{u}_0^{\mathrm{ABC}}(x,\zeta,t) &= \frac{4}{3}\lambda \sinh^2(x+\zeta),\\
\mathbf{u}_1^{\mathrm{ABC}}(x,\zeta,t) &= -\frac{8\lambda^2(-\mu\Gamma(\mu)+\Gamma(\mu)+t^{\mu})(5\sinh(4(x+\zeta)))}{9\Gamma(\mu) - 4\sinh(2(x+\zeta))},\\
\mathbf{u}_2^{\mathrm{ABC}}(x,\zeta,t) &= \frac{64\lambda^3(13\cosh(2(x+y)) - 70\cosh(4(x+y)) + 75\cosh(6(x+y)))}{27\Gamma(\mu)\Gamma(2\mu+1)}\\
&\quad \times \left(\mu\Gamma(\mu+1)t^{2\mu} + (\mu-1)\Gamma(2\mu+1)((\mu-1)\Gamma(\mu) - 2t^{\mu})\right), \vdots
\end{aligned} \tag{47}$$

Substituting $\mathbf{u}_0^{\mathrm{ABC}}(x,\zeta,t)$, $\mathbf{u}_1^{\mathrm{ABC}}(x,\zeta,t)$, in (28), we obtain the $\mathrm{NTDM}_{\mathrm{ABC}}$ solution as

$$\begin{aligned}
\mathbf{u}^{\mathrm{ABC}}(x,\zeta,t) \approx{}& \frac{4}{3}\lambda \sinh^2(x+\zeta)\\
&- \frac{8\lambda^2(-\mu\Gamma(\mu)+\Gamma(\mu)+t^{\mu})(5\sinh(4(x+\zeta)) - 4\sinh(2(x+\zeta)))}{9\Gamma(\mu)},\\
&\cdot \frac{64\lambda^3(13\cosh(2(x+y)) - 70\cosh(4(x+y)) + 75\cosh(6(x+y)))}{27\Gamma(\mu)\Gamma(2\mu+1)}\\
&\times \left(\mu\Gamma(\mu+1)t^{2\mu} + (\mu-1)\Gamma(2\mu+1)((\mu-1)\Gamma(\mu) - 2t^{\mu})\right).
\end{aligned} \tag{48}$$

Example 2. TFZKE (1) is considered with the following parameters. Let $\xi = \eta = \delta = 3$, $a = 1$, $b = c = 2$, and $\mathbf{u}(x,\zeta,0) = (3/2)\lambda \sinh((x+\zeta)/6)$ [47, 48]. When $\mu = 1$, exact solution [49] is given by $\mathbf{u}(x,\zeta,t) = (3/2)\lambda \sinh((x+\zeta-\lambda t)/6)$.

$\mathrm{NTDM}_{\mathrm{CF}}$: By employing the $\mathrm{NTDM}_{\mathrm{CF}}$, we get

$$\begin{aligned}
\mathbf{u}_0^{\mathrm{CF}}(x,\zeta,t) &= \frac{3}{2}\lambda \sinh\left(\frac{x+\zeta}{6}\right),\\
\mathbf{u}_1^{\mathrm{CF}}(x,\zeta,t) &= \frac{3}{32}\lambda^3(\mu(t-1)+1)\left(5\cosh\left(\frac{x+\zeta}{6}\right) - 9\cosh\left(\frac{x+\zeta}{2}\right)\right),\\
\mathbf{u}_2^{\mathrm{CF}}(x,\zeta,t) &= \frac{3\lambda^5}{1024}\left(-621\sinh\left(\frac{x+\zeta}{2}\right) + 70\sinh\left(\frac{x+\zeta}{6}\right)\right.\\
&\quad \left. + 765\sinh\left(\frac{5(x+\zeta)}{6}\right)\right)\\
&\quad \times \left(\mu^2((t-4)t+2) + 4\mu(t-1) + 2\right), \vdots
\end{aligned} \tag{49}$$

Substituting $\mathbf{u}_0^{\mathrm{CF}}(x,\zeta,t)$, $\mathbf{u}_1^{\mathrm{CF}}(x,\zeta,t)$, in (24), we obtain the $\mathrm{NTDM}_{\mathrm{CF}}$ solution as

$$\mathbf{u}^{\mathrm{CF}}(x,\zeta,t)\approx\frac{3}{2}\lambda\sinh\left(\frac{x+\zeta}{6}\right)+\frac{3}{32}\lambda^3(\mu(t-1)+1)\cdot\left(5\cosh\left(\frac{x+\zeta}{6}\right)-9\cosh\left(\frac{x+\zeta}{2}\right)\right)+\frac{3\lambda^5}{1024}\left(-621\sinh\left(\frac{x+\zeta}{2}\right)+70\sinh\left(\frac{x+\zeta}{6}\right)+765\sinh\left(\frac{5(x+\zeta)}{6}\right)\right)\times\left(\mu^2((t-4)t+2)+4\mu(t-1)+2\right)+. \tag{50}$$

$\mathrm{NTDM}_{\mathrm{ABC}}$: By employing the $\mathrm{NTDM}_{\mathrm{ABC}}$, we get

$$\mathbf{u}_0^{\mathrm{ABC}}(x,\zeta,t)=\frac{3}{2}\lambda\sinh\left(\frac{x+\zeta}{6}\right),$$

$$\mathbf{u}_1^{\mathrm{ABC}}(x,\zeta,t)=-\frac{3\lambda^3(-\mu\Gamma(\mu)+\Gamma(\mu)+t^\mu)\cosh((x+\zeta)/6)(9\cosh((x+\zeta)/3)-7)}{16\Gamma(\mu)},$$

$$\mathbf{u}_2^{\mathrm{ABC}}(x,\zeta,t)=\frac{3\lambda^5\left(\mu\Gamma(\mu+1)t^{2\mu}+(\mu-1)\Gamma(2\mu+1)((\mu-1)\Gamma(\mu)-2t^\mu)\right)}{512\Gamma(\mu)\Gamma(2\mu+1)}\times\left(-621\sinh\left(\frac{x+\zeta}{2}\right)+70\sinh\left(\frac{x+\zeta}{6}\right)+765\sinh\left(\frac{5(x+\zeta)}{6}\right)\right),\vdots \tag{51}$$

Substituting $\mathbf{u}_0^{\mathrm{ABC}}(x,\zeta,t)$ $\mathbf{u}_1^{\mathrm{ABC}}(x,\zeta,t)$, in (32), we obtain the $\mathrm{NTDM}_{\mathrm{ABC}}$ solution as

$$\mathbf{u}^{\mathrm{ABC}}(x,\zeta,t)\approx\frac{3}{2}\lambda\sinh\left(\frac{x+\zeta}{6}\right)-\frac{3\lambda^3(-\mu\Gamma(\mu)+\Gamma(\mu)+t^\mu)\cosh((x+\zeta)/6)(9\cosh((x+\zeta)/3)-7)}{16\Gamma(\mu)}+\frac{3\lambda^5\left(\mu\Gamma(\mu+1)t^{2\mu}+(\mu-1)\Gamma(2\mu+1)((\mu-1)\Gamma(\mu)-2t^\mu)\right)}{512\Gamma(\mu)\Gamma(2\mu+1)}\times\left(-621\sinh\left(\frac{x+\zeta}{2}\right)+70\sinh\left(\frac{x+\zeta}{6}\right)+765\sinh\left(\frac{5(x+\zeta)}{6}\right)\right)+. \tag{52}$$

6. Numerical Results and Discussion

Tables 1 and 2 demonstrates the comparison of absolute errors with the existing methods and approximate solutions for different fractional orders with different fractional derivatives, respectively, of Example 1. Absolute errors of Example 1 graphically represented in Figure 1 for fixed t when $\mu=1$. In Figure 2, we plotted approximate solutions for different values of μ for fixed t of Example 1. Figure 3 presents the comparison of $\mathrm{NTDM}_{\mathrm{CF}}$ and $\mathrm{NTDM}_{\mathrm{ABC}}$ solutions of Example 1 with exact solution for different values of fractional order μ for fixed x and ζ. In Table 3, we presented absolute errors of two fractional derivative solutions and existing results of Example 2. We have tabulated the approximate solution of Example 2 for noninteger fractional values in Table 4. For Example 2, $\mathrm{NTDM}_{\mathrm{CF}}$ and $\mathrm{NTDM}_{\mathrm{ABC}}$ absolute errors graphically represent in Figure 4 for fixed t when $\mu=1$, and in Figure 5, we plotted approximate solutions for different values of μ for specific value of t. Figure 6 presents the comparison of $\mathrm{NTDM}_{\mathrm{CF}}$ and $\mathrm{NTDM}_{\mathrm{ABC}}$ with exact solution for noninteger values of μ for fixed t and ζ of Example 2. It is observed from tables and figures that the two-term approximate solution is having good accordance with the existing results and exact solution. For noninteger values, $\mathrm{NTDM}_{\mathrm{CF}}$ and $\mathrm{NTDM}_{\mathrm{ABC}}$ are showing same behavior.

7. Conclusions

In this paper, we have studied the TFZKE through natural transformation by means of CF and ABC derivatives. We compared numerical results with the existing results. It is observed that the present method results are in accordance with existing methods. The NTDM is simple in its principles; also, NTDM is effective in solving nonlinear fractional differential equations, and promising method for a large varieties of such equations arises in mathematical physics.

Data Availability

There is no any data available.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to this work. And all the authors have read and approved the final version manuscript.

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