Research Article

A Generalized Strong Convergence Algorithm in the Presence of Errors for Variational Inequality Problems in Hilbert Spaces

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Received 3 April 2021; Revised 8 May 2021; Accepted 9 June 2021; Published 21 June 2021

Academic Editor: Ismat Beg

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In this paper, we study the strong convergence of an algorithm to solve the variational inequality problem which extends a recent paper (Thong et al., Numerical Algorithms. 78, 1045-1060 (2018)). We reduce and refine some of their algorithm conditions and we prove the convergence of the algorithm in the presence of some computational errors. Then, using the MATLAB software, the result will be illustrated with some numerical examples. Also, we compare our algorithm with some other well-known algorithms.

1. Introduction

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$, and C be a nonempty, closed, and convex subset of H . The variational inequality (VI) is to find a point $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0, \forall y \in C, \quad (1)$$

where A is a mapping of C into H . The solution set of (1) is denoted by $VI(A, C)$. Variational inequalities arise in the study of network equilibriums, optimization problems, saddle point problem, Nash equilibrium problems in non-cooperative games etc.; see, for example, [1–12] and the references therein.

A new algorithm was proposed by Korpelevich [13] for solving the problem (VI) in the Euclidean space which is known as the extragradient method. Let x_1 be an arbitrary element in H and consider

$$\begin{cases} y_n = P_C(x_n - \lambda Ax_n), \\ x_{n+1} = P_C(x_n - \lambda Ay_n). \end{cases} \quad (2)$$

where λ is a number in $(0, 1)$, P_C is the Euclidean least distance projection of H onto C , and $A : C \rightarrow H$ is a monotone operator. The next algorithm (3) was introduced by Tseng [14] and applying the modified forward-backward (F-B) method is a good alternative to the extragradient method (TEGM):

$$\begin{cases} y_n = P_C(x_n - \lambda Ax_n), \\ x_{n+1} = P_X(y_n - \lambda(Ay_n - Ax_n)), \end{cases} \quad (3)$$

where $X = C$ and $X = H$ if A is Lipschitz continuous. The following algorithm (4) was proposed by Shehu and Iyiola [15] which a viscosity type subgradient extragradient method (VSEGM):

$$\begin{cases} y_n = P_C(x_n - \lambda_n Ax_n), \\ T_n = \{z \in H : \langle x_n - \lambda_n Ax_n - y_n, z - y_n \rangle \leq 0\}, \\ z_n = P_{T_n}(x_n - \lambda_n Ay_n), \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)z_n, \end{cases} \quad (4)$$

where the operator A is monotone and Lipschitz continuous, f is a strict contraction mapping, $l \in (0, 1)$, $\mu \in (0, 1)$, and $\lambda_n = l^{m_n}$ where m_n is the smallest nonnegative integer m such that

$$\lambda \|Ax_n - Ay_n\| \leq \mu \|r^{m_n}(x_n)\|, \quad (5)$$

where $r^{m_n}(x) = x - P_C(x - l^{m_n}A(x))$ for all $x \in C$. Recently, the sequence produced by the following algorithm was introduced by Thong and Hieu [3] based on Tseng's method (THEGM):

$$\begin{cases} y_n = P_C(x_n - \lambda_n Ax_n), \\ z_n = y_n - \lambda_n (Ay_n - Ax_n), \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) z_n, \end{cases} \quad (6)$$

where the operator A is monotone and Lipschitz continuous, $\gamma > 0$, $l \in (0, 1)$, $\mu \in (0, 1)$, and λ_n is chosen to be the largest $\lambda \in \{\gamma, \gamma l, \gamma l^2, \dots\}$ satisfying

$$\lambda \|Ax_n - Ay_n\| \leq \mu \|x_n - y_n\|. \quad (7)$$

In this paper, substituting a sequence $\{\beta_n\} \subset (0, 1)$ of coefficients instead of the sequence $\{1 - \alpha_n\}$ in the algorithm (6), we extend algorithm (6). Moreover, condition (7) will be removed just by a slight change in the coefficients $\{\lambda_n\}$. Also, a sequence of computational errors in our algorithm is considered. The strong convergence of the proposed algorithm to a point of the variational inequality $VI(C, A)$ will be proved under the presence of computational errors. Finally, some examples will be presented which will examine the convergence of the proposed algorithm in different situations.

2. Preliminaries

In this section, some basic concepts are presented.

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$ and suppose that C is a nonempty closed convex subset of H and $A : C \rightarrow H$ is an operator. The operator A is said to be

(i) Monotone if

$$\langle Ax - Ay, x - y \rangle \geq 0, \forall x, y \in C, \quad (8)$$

(ii) L -Lipchitz continuous if there exist $L > 0$ such that

$$\|Ax - Ay\| \leq L \|x - y\|, \forall x, y \in C. \quad (9)$$

For the main results of this paper, we need the following useful lemmas.

Lemma 1. Let H be a real Hilbert space. Then, we have the following well-known results:

$$\begin{aligned} \|x + y\|^2 &= \|x\|^2 + 2\langle x, y \rangle + \|y\|^2, \forall x, y \in H. \\ \|x + y\|^2 &\leq \|x\|^2 + 2\langle y, x + y \rangle, \forall x, y \in H. \end{aligned} \quad (10)$$

Lemma 2 (Xu, see [16]). Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:

$$a_{n+1} \leq (1 - \eta_n)a_n + \eta_n\sigma_n + \gamma_n, \quad n \geq 1, \quad (11)$$

where

- (a) $\{\eta_n\} \subset [0, 1]$, $\sum_{n=1}^{\infty} \eta_n = \infty$
- (b) $\limsup \sigma_n \leq 0$
- (c) $\gamma_n \geq 0$ ($n \geq 1$), $\sum_{n=1}^{\infty} \gamma_n < \infty$

Then, $a_n \rightarrow 0$ as $n \rightarrow \infty$.

Lemma 3 (see [17]). Let C be a closed and convex subset in a real Hilbert space H . Then, $z = P_C x$ if and only if $\langle x - z, y - z \rangle \leq 0 \forall y \in C$.

Lemma 4 (see [18]). Let $\{a_n\}$ be a sequence of nonnegative real numbers such that there exists a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ such that $a_{n_j} < a_{n_j+1}$ for all $j \in \mathbb{N}$. Then, there exists a nondecreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $\lim_{k \rightarrow \infty} m_k = \infty$ and the following properties are satisfied by all (sufficiently large) number $k \in \mathbb{N}$:

$$a_{m_k} \leq a_{m_k+1} \text{ and } a_k \leq a_{m_k+1}. \quad (12)$$

In fact, m_k is the largest number n in the set $\{1, 2, \dots, k\}$ such that $a_n < a_{n+1}$.

Lemma 5 (see [3]). Let $\{x_n\}$ be a sequence generated by algorithm (3). Then,

$$\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2 - (1 - \mu^2) \|x_n - y_n\| \forall p \in VI(C, A). \quad (13)$$

3. Main Results

In this section, we prove a strong convergence theorem for finding a common element of the set of solutions of an equilibrium problem and a fixed point problem.

Theorem 6. Let C be a nonempty closed convex subset of a real Hilbert space H , A be a monotone, and L -Lipschitz continuous mapping on C and $\lambda \in (0, 1)$ such that $\lambda L < 1$. Suppose that $f : H \rightarrow H$ is a contraction mapping with a constant $\rho \in [0, 1)$. Let $\{e_n\} \subseteq H$ be a sequence of computational errors, $x_0 \in H$ be arbitrary and $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be the sequences generated by

$$\begin{cases} y_n = P_C(x_n - \lambda Ax_n), \\ z_n = y_n - \lambda(Ay_n - Ax_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n z_n + e_n, \end{cases} \quad (14)$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1]$ such that $\alpha_n + \beta_n \leq 1$ for each $n \geq 1$. Also, assume the following conditions:

$$\begin{aligned} \sum_{n=1}^{\infty} \alpha_n \beta_n &= \infty, \\ \sum_{n=1}^{\infty} (1 - \alpha_n - \beta_n) &< \infty, \\ \lim_{n \rightarrow \infty} \frac{(1 - \alpha_n - \beta_n)}{\alpha_n} &= \lim_{n \rightarrow \infty} \alpha_n = 0, \\ \sum_{n=1}^{\infty} \|e_n\| &< \infty. \end{aligned} \quad (15)$$

Then,

- (i) $VI(C, A) \neq \emptyset$ if and only if $\{x_n\}$ is bounded and $\liminf_{n \rightarrow \infty} \|x_n - y_n\| = 0$

Suppose $VI(C, A) \neq \emptyset$. Then,

- (ii) If $\lim_{n \rightarrow \infty} \|e_n\|/\alpha_n = 0$, then $\{x_n\}$ converges strongly to $q = P_{VI(C, A)} \circ f(q)$, where $P_{VI(C, A)} \circ f : H \rightarrow VI(C, A)$ is the mapping defined by $P_{VI(C, A)} \circ f(x) = P_{VI(C, A)}(f(x))$ for each $x \in H$

Proof. (i) Assume that $\{x_n\}$ is a bounded sequence and $\liminf_{n \rightarrow \infty} \|x_n - y_n\| = 0$. Then, there exists a subsequence $\{n_i\} \subset \mathbb{N}$ such that $\|x_{n_i} - y_{n_i}\| \rightarrow 0$ when $i \rightarrow \infty$. Note $\{x_{n_i}\}$ is a bounded sequence. Hence, there exists a subsequence $\{x_{n_{i_k}}\}$ of $\{x_{n_i}\}$ such that $\{x_{n_{i_k}}\}$ converges weakly to some $x \in C$. Now, noting that since $y_{n_{i_k}} = P_C(x_{n_{i_k}} - \lambda Ax_{n_{i_k}})$, for all $z \in C$, we have

$$\begin{aligned} 0 &\leq \langle x_{n_{i_k}} - \lambda Ax_{n_{i_k}} - y_{n_{i_k}}, z - y_{n_{i_k}} \rangle \\ &= \langle x_{n_{i_k}} - y_{n_{i_k}}, z - y_{n_{i_k}} \rangle - \lambda \langle Ax_{n_{i_k}}, z - y_{n_{i_k}} \rangle \\ &= \langle x_{n_{i_k}} - y_{n_{i_k}}, z - y_{n_{i_k}} \rangle - \lambda \langle Ax_{n_{i_k}}, z - x_{n_{i_k}} \rangle - \lambda \langle Ax_{n_{i_k}}, x_{n_{i_k}} - y_{n_{i_k}} \rangle \\ &= \langle x_{n_{i_k}} - y_{n_{i_k}}, z - y_{n_{i_k}} \rangle - \lambda \langle Ax_{n_{i_k}} - Az, z - x_{n_{i_k}} \rangle \\ &\quad - \lambda \langle Az, z - x_{n_{i_k}} \rangle - \lambda \langle Ax_{n_{i_k}}, x_{n_{i_k}} - y_{n_{i_k}} \rangle \\ &\geq \langle x_{n_{i_k}} - y_{n_{i_k}}, z - y_{n_{i_k}} \rangle - \lambda \langle Az, z - x_{n_{i_k}} \rangle \\ &\quad - \lambda \langle Ax_{n_{i_k}}, x_{n_{i_k}} - y_{n_{i_k}} \rangle. \end{aligned} \quad (16)$$

Now, we have

$$-\lambda \langle Az, z - x_{n_{i_k}} \rangle \leq \langle x_{n_{i_k}} - y_{n_{i_k}}, y_{n_{i_k}} - z \rangle + \lambda \langle Ax_{n_{i_k}}, x_{n_{i_k}} - y_{n_{i_k}} \rangle. \quad (17)$$

Therefore,

$$-\lambda \langle Az, z - x_{n_{i_k}} \rangle \leq \|y_{n_{i_k}} - x_{n_{i_k}}\| \|y_{n_{i_k}} - z\| + \lambda \|Ax_{n_{i_k}}\| \|x_{n_{i_k}} - y_{n_{i_k}}\|. \quad (18)$$

From $\lim_{k \rightarrow \infty} \|x_{n_{i_k}} - y_{n_{i_k}}\| = 0$, we have $-\lambda \langle Az, z - x \rangle \leq 0$ for all $z \in C$. Now, let $y \in C$ and $0 < t < 1$, and from the convexity C we have $y_t = [ty + (1-t)x] \in C$. Therefore,

$$0 \leq \langle Ay_t, y_t - x \rangle = \langle Ay_t, ty - tx \rangle = t \langle Ay_t, y - x \rangle. \quad (19)$$

Since $0 < t < 1$ then $\langle Ay_t, y - x \rangle \geq 0$ for all $y \in C$. Because the mapping A and multiplication are continuous, if $t \rightarrow 0$, then we have $\langle Ax, y - x \rangle \geq 0$ for all $y \in C$, i.e; $x \in VI(C, A)$.

For the converse fix $p \in VI(C, A)$, using Lemma 5, we have

$$\|z_n - p\|^2 \leq \|x_n - p\|^2 - (1 - (\lambda L)^2) \|x_n - y_n\|^2. \quad (20)$$

Therefore,

$$\|z_n - p\| \leq \|x_n - p\|. \quad (21)$$

Using the above inequality, we have

$$\begin{aligned} 1 \|x_{n+1} - p\| &= \|\alpha_n f(x_n) + \beta_n z_n + e_n - p\| \\ &= \|\alpha_n (f(x_n) - p) + \beta_n (z_n - p) - (1 - \alpha_n - \beta_n)p + e_n\| \\ &\leq \alpha_n \|f(x_n) - p\| + \beta_n \|z_n - p\| + (1 - \alpha_n - \beta_n) \|p\| + \|e_n\| \\ &\leq \alpha_n \|f(x_n) - f(p)\| + \alpha_n \|f(p) - p\| \\ &\quad + \beta_n \|x_n - p\| + (1 - \alpha_n - \beta_n) \|p\| + \|e_n\| \\ &\leq \alpha_n \rho \|x_n - p\| + \alpha_n \|f(p) - p\| + \beta_n \|x_n - p\| \\ &\quad + (1 - \alpha_n - \beta_n) \|p\| + \|e_n\| \\ &= (\alpha_n \rho + \beta_n) \|x_n - p\| + \alpha_n (1 - \rho) \frac{\|f(p) - p\|}{1 - \rho} \\ &\quad + (1 - \alpha_n - \beta_n) \|p\| + \|e_n\| \\ &\leq \max \left\{ \|x_n - p\|, \frac{\|f(p) - p\|}{1 - \rho}, \|p\| \right\} + \|e_n\|, \end{aligned} \quad (22)$$

so the sequence $\{x_n\}$ is bounded.

(ii) Let $p \in VI(C, A)$. From part (i), we have that $\{x_n\}$ is bounded. Then $\{f(x_n)\}$, $\{y_n\}$ and $\{z_n\}$ are bounded. Now, using Lemma 5, we have

$$\|z_n - p\|^2 \leq \|x_n - p\|^2 - (1 - (\lambda L)^2) \|x_n - y_n\|^2 \forall p \in VI(C, A), \quad (23)$$

(note that our sequence $\{z_n\}$ in the algorithm (14) replaces $\{x_{n+1}\}$ in Lemma 5). $\square$

Since $\alpha_n + \beta_n + (1 - \alpha_n - \beta_n) = 1$, by the convexity of $\|\cdot\|^2$ and the relation $\|x + y\|^2 = \|x\|^2 + 2\langle x, y \rangle + \|y\|^2$, we conclude that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n f(x_n) + \beta_n z_n + e_n - p\|^2 \\
&= \|\alpha_n(f(x_n) + e_n - p) + \beta_n(z_n + e_n - p) + (1 - \alpha_n - \beta_n)(e_n - p)\|^2 \\
&\leq \alpha_n \|f(x_n) + e_n - p\|^2 + \beta_n \|z_n + e_n - p\|^2 + (1 - \alpha_n - \beta_n) \|e_n - p\|^2 \\
&\leq \alpha_n \|f(x_n) - p\|^2 + (1 - \alpha_n) \|z_n - p\|^2 + (1 - \alpha_n - \beta_n) \|p\|^2 \\
&\quad + (\alpha_n + \beta_n + (1 - \alpha_n - \beta_n)) \|e_n\|^2 \\
&\quad + 2\langle \alpha_n f(x_n) - \alpha_n p + \beta_n z_n - \beta_n p - (1 - \alpha_n - \beta_n)p, e_n \rangle \\
&\leq \alpha_n \|f(x_n) - p\|^2 + (1 - \alpha_n) \|x_n - p\|^2 \\
&\quad - (1 - \alpha_n)(1 - (\lambda L)^2) \|x_n - y_n\|^2 + (1 - \alpha_n - \beta_n) \|p\|^2 \\
&\quad + \|e_n\|^2 + 2\langle \alpha_n f(x_n) + \beta_n z_n - p, e_n \rangle \text{ (by (3.3))} \\
&\leq \alpha_n \|f(x_n) - p\|^2 + \|x_n - p\|^2 - (1 - \alpha_n)(1 - (\lambda L)^2) \|x_n - y_n\|^2 \\
&\quad + (1 - \alpha_n - \beta_n) \|p\|^2 + \|e_n\|^2 + 2\|\alpha_n f(x_n) + \beta_n z_n - p\| \|e_n\|. \tag{24}
\end{aligned}$$

Therefore,

$$\begin{aligned}
(1 - \alpha_n)(1 - (\lambda L)^2) \|x_n - y_n\|^2 &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\quad + \alpha_n \|f(x_n) - p\|^2 + (1 - \alpha_n - \beta_n) \|p\|^2 + \|e_n\|^2 \\
&\quad + 2\|\alpha_n f(x_n) + \beta_n z_n - p\| \|e_n\|. \tag{25}
\end{aligned}$$

Also, we have,

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|\alpha_n f(x_n) + \beta_n z_n + e_n - x_n\| \\
&= \|\alpha_n(f(x_n) - x_n) + \beta_n(z_n - x_n) - (1 - \alpha_n - \beta_n)x_n + e_n\| \\
&\leq \alpha_n \|f(x_n) - x_n\| + \beta_n \|z_n - x_n\| + (1 - \alpha_n - \beta_n) \|x_n\| + \|e_n\| \\
&= \alpha_n \|f(x_n) - x_n\| + \beta_n \|y_n - \lambda(Ay_n - Ax_n) - x_n\| \\
&\quad + (1 - \alpha_n - \beta_n) \|x_n\| + \|e_n\| \leq \alpha_n \|f(x_n) - x_n\| \\
&\quad + \beta_n \|y_n - x_n\| + \beta_n \lambda L \|y_n - x_n\| \\
&\quad + (1 - \alpha_n - \beta_n) \|x_n\| + \|e_n\|. \tag{26}
\end{aligned}$$

Note that $P_{VI(C,A)} \circ f$ is a contraction mapping. Then, by the Banach contraction principle, there exists a unique element $q \in H$ such that $q = P_{VI(C,A)} \circ f(q)$. Now, we claim that $\{x_n\}$ converges strongly to $q = P_{VI(C,A)} \circ f(q)$. It is enough to consider two cases:

Case 1. Suppose there exists some $n_0 \in \mathbb{N}$ such that $\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2$ for all $n \geq n_0$. Then, $\lim_{n \rightarrow \infty} \|x_n - q\|$ exists. From (25) and our assumptions, $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$. From the conditions (b), (c), and (d), $\lim_{n \rightarrow \infty} (1 - \alpha_n - \beta_n) = \lim_{n \rightarrow \infty} \alpha_n = \lim_{n \rightarrow \infty} \|e_n\| = 0$. Then, from (26) and the boundedness of the sequences $\{\|x_n\|\}$, $\{\|f(x_n) - x_n\|\}$ and $\{\beta_n\}$, we obtain that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \tag{27}$$

Now, we have

$$\|x_{n+p} - x_n\| \leq \|x_{n+p} - x_{n+p-1}\| + \|x_{n+p-1} - x_{n+p-2}\| + \cdots + \|x_{n+1} - x_n\|, \tag{28}$$

for all $p \in \mathbb{N}$. Note that from (27), it is concluded that $\lim_{n \rightarrow \infty} \|$

$x_{n+p-i} - x_{n+p-(i+1)}\| = 0$, for all $0 \leq i \leq p-1$ and $p \in \mathbb{N}$. Then, from (28), $\lim_{n \rightarrow \infty} \|x_{n+p} - x_n\| = 0$, for all $p \in \mathbb{N}$, i.e., $\{x_n\}$ is a Cauchy sequence in the Hilbert space H ; therefore, $\{x_n\}$ is convergent. Now we show that $\{x_n\}$ converges strongly to q . Note that

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \|\alpha_n f(x_n) + \beta_n z_n + e_n - q\|^2 \\
&= \|\alpha_n(f(x_n) + e_n - q) + \beta_n(z_n + e_n - q) \\
&\quad + (1 - \alpha_n - \beta_n)(e_n - q)\|^2 \\
&\leq \beta_n^2 \|z_n + e_n - q\|^2 + 2\langle \alpha_n(f(x_n) + e_n - q) \\
&\quad + (1 - \alpha_n - \beta_n)(e_n - q), x_{n+1} - q \rangle \text{ (by Lemma 2.1)} \\
&= \beta_n^2 \|z_n + e_n - q\|^2 + 2\alpha_n \langle f(x_n) + e_n - q, x_{n+1} - q \rangle \\
&\quad + 2(1 - \alpha_n - \beta_n) \langle e_n - q, x_{n+1} - q \rangle \\
&= \beta_n^2 \|z_n + e_n - q\|^2 + 2\alpha_n \langle f(x_n) - q, x_{n+1} - q \rangle \\
&\quad + 2\alpha_n \langle e_n, x_{n+1} - q \rangle + 2(1 - \alpha_n - \beta_n) \langle e_n - q, x_{n+1} - q \rangle \\
&\leq \beta_n^2 \|z_n + e_n - q\|^2 + 2\alpha_n \rho \|x_n - q\| \|x_{n+1} - q\| \\
&\quad + 2\alpha_n \langle f(q) - q, x_{n+1} - q \rangle + 2\alpha_n \langle e_n, x_{n+1} - q \rangle \\
&\quad + 2(1 - \alpha_n - \beta_n) \langle e_n - q, x_{n+1} - q \rangle \leq \beta_n^2 \|z_n - q\|^2 \\
&\quad + 2\beta_n^2 \langle e_n, z_n + e_n - q \rangle + 2\alpha_n \rho \|x_n - q\|^2 \text{ (by Lemma 2.1)} \\
&\quad + 2\alpha_n \langle f(q) - q, x_{n+1} - q \rangle + 2\alpha_n \langle e_n, x_{n+1} - q \rangle \\
&\quad + 2(1 - \alpha_n - \beta_n) \langle e_n - q, x_{n+1} - q \rangle \\
&\leq (1 - \alpha_n)^2 \|x_n - q\|^2 + 2\alpha_n \rho \|x_n - q\|^2 \\
&\quad + 2\alpha_n \langle f(q) - q, x_{n+1} - q \rangle + 2\beta_n^2 \langle e_n, z_n + e_n - q \rangle \\
&\quad + 2\alpha_n \langle e_n, x_{n+1} - q \rangle + 2(1 - \alpha_n - \beta_n) \langle e_n - q, x_{n+1} - q \rangle \\
&= (1 - 2\alpha_n(1 - \rho)) \|x_n - q\|^2 + 2\alpha_n(1 - \rho) \\
&\quad \cdot \left[\frac{\alpha_n \|x_n - q\|^2}{2(1 - \rho)} + \frac{\langle f(q) - q, x_{n+1} - q \rangle}{1 - \rho} \right] \\
&\quad + 2\beta_n^2 \langle e_n, z_n + e_n - q \rangle + 2\alpha_n \langle e_n, x_{n+1} - q \rangle \\
&\quad + 2(1 - \alpha_n - \beta_n) \langle e_n - q, x_{n+1} - q \rangle \\
&\leq (1 - 2\alpha_n(1 - \rho)) \|x_n - q\|^2 + 2\alpha_n(1 - \rho) \\
&\quad \cdot \left[\frac{\alpha_n \|x_n - q\|^2}{2(1 - \rho)} + \frac{\langle f(q) - q, x_{n+1} - q \rangle}{1 - \rho} \right] \\
&\quad + 2\beta_n^2 \|e_n\| \|z_n + e_n - q\| + 2\alpha_n \|e_n\| \|x_{n+1} - q\| \\
&\quad + 2(1 - \alpha_n - \beta_n) \|e_n - q\| \|x_{n+1} - q\|. \tag{29}
\end{aligned}$$

Since $\{x_n\}$ is strongly convergent, so $x_n \rightarrow z$ for some $z \in C$, then we conclude that $x_n \rightarrow z$. Also, as in the proof of part (i), we conclude $z \in VI(C, A)$. Therefore, from Lemma 3

$$\limsup_{n \rightarrow \infty} \langle f(q) - q, x_{n+1} - q \rangle = \langle f(q) - q, z - q \rangle \leq 0. \tag{30}$$

Next, we consider the sequences $\{\eta_n\}$, $\{a_n\}$, $\{\sigma_n\}$ and $\{\gamma_n\}$ in (3.8) as follows:

$$\eta_n := 2\alpha_n(1 - \rho),$$

$$a_n := \|x_n - q\|^2,$$

$$\sigma_n := \frac{\alpha_n \|x_n - q\|^2}{2(1 - \rho)} + \frac{\langle f(q) - q, x_{n+1} - q \rangle}{1 - \rho},$$

$$\begin{aligned}
\gamma_n &:= 2\beta_n^2 \|e_n\| \|z_n + e_n - q\| + 2\alpha_n \|e_n\| \|x_{n+1} - q\| \\
&\quad + 2(1 - \alpha_n - \beta_n) \|e_n - q\| \|x_{n+1} - q\|. \tag{31}
\end{aligned}$$

FIGURE 1: Example 1. Convergence behavior $\{f_n\}$ in Example 1.

FIGURE 2: The number of iteration for convergence to zero is, respectively, 15, 27, 25, and 27 in Example 2.

From the above, we have

$$a_{n+1} \leq (1 - \eta_n)a_n + \eta_n\sigma_n + \gamma_n. \quad (32)$$

From (c), there exists an integer $n_1 \in \mathbb{N}$ such that $\{\eta_n\} \subset [0, 1]$, for each $n \geq n_1$. Without loss of generality, we may assume that $\{\eta_n\} \subset [0, 1]$, for each $n \geq 1$. From condition (a), note $\infty = \sum_{n=1}^{\infty} \alpha_n \beta_n \leq \sum_{n=1}^{\infty} \alpha_n$, so $\sum_{n=1}^{\infty} \alpha_n = \infty$, and hence, $\sum_{n=1}^{\infty} \eta_n = 2(1 - \rho) \sum_{n=1}^{\infty} \alpha_n = \infty$. Therefore, condition (a) of

Lemma 2 holds. From (c), the boundedness of $\|x_n - q\|^2$ and (3.9), we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \sigma_n &= \limsup_{n \rightarrow \infty} \left(\frac{\alpha_n \|x_n - q\|^2}{2(1 - \rho)} + \frac{\langle f(q) - q, x_{n+1} - q \rangle}{1 - \rho} \right) \\ &\leq \limsup_{n \rightarrow \infty} \frac{\alpha_n \|x_n - q\|^2}{2(1 - \rho)} + \limsup_{n \rightarrow \infty} \frac{\langle f(q) - q, x_{n+1} - q \rangle}{1 - \rho} \\ &\leq 0. \end{aligned} \quad (33)$$

Therefore, condition (b) of Lemma 2 holds. Next, note $\gamma_n \geq 0$ and $\|z_n + e_n - q\|$, $\|x_{n+1} - q\|$, and $\|e_n - q\|$ are bounded. Hence, there exist some positive constants M_1 , M_2 , and M_3 such that, $\|z_n + e_n - q\| \leq M_1$, $\|x_{n+1} - q\| \leq M_2$ and $\|e_n - q\| \leq M_3$, for each $n \geq 1$. Then, from (b) and (d), we have

$$\begin{aligned} \Sigma_{n=1}^{\infty} \gamma_n &= \Sigma_{n=1}^{\infty} [2\beta_n^2 \|e_n\| \|z_n + e_n - q\| + 2\alpha_n \|e_n\| \|x_{n+1} - q\| \\ &\quad + 2(1 - \alpha_n - \beta_n) \|e_n - q\| \|x_{n+1} - q\|] \\ &\leq \Sigma_{n=1}^{\infty} [2\beta_n^2 \|e_n\| M_1 + 2\alpha_n \|e_n\| M_2 + 2(1 - \alpha_n - \beta_n) M_3 M_2] \\ &\leq 2M_1 \Sigma_{n=1}^{\infty} \|e_n\| + 2M_2 \Sigma_{n=0}^{\infty} \|e_n\| + 2M_3 M_2 \Sigma_{n=1}^{\infty} (1 - \alpha_n - \beta_n) \\ &< \infty. \end{aligned} \quad (34)$$

Therefore, condition (c) of Lemma 2 holds. Then, from Lemma 2, it follows that $a_n \rightarrow 0$ as $n \rightarrow \infty$, i.e., $\lim_{n \rightarrow \infty} \|x_n - q\|^2 = 0$.

Case 2. Suppose there exists a subsequence $\{\|x_{n_j} - q\|^2\}$ of $\{\|x_n - q\|^2\}$ such that $\|x_{n_j} - q\|^2 < \|x_{n_{j+1}} - q\|^2$ for all $j \in \mathbb{N}$. Now from Lemma 4, there exists a nondecreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $\lim_{k \rightarrow \infty} m_k = \infty$ and the following inequalities hold for all $k \in \mathbb{N}$:

$$\|x_{m_k} - q\|^2 \leq \|x_{m_k+1} - q\|^2 \text{ and } \|x_k - q\|^2 \leq \|x_{m_k+1} - q\|^2. \quad (35)$$

Now, from (25), we have

$$\begin{aligned} &(1 - \alpha_{m_k})(1 - (\lambda L)^2) \|x_{m_k} - y_{m_k}\|^2 \\ &\leq \|x_{m_k} - p\|^2 - \|x_{m_k+1} - p\|^2 + \alpha_{m_k} \|f(x_{m_k}) - p\|^2 \\ &\quad + (1 - \alpha_{m_k} - \beta_{m_k}) \|p\|^2 + \|e_{m_k}\|^2 \\ &\quad + 2\|\alpha_{m_k} f(x_{m_k}) + \beta_{m_k} z_{m_k} - p\| \|e_{m_k}\|. \end{aligned} \quad (36)$$

Hence, $\lim_{k \rightarrow \infty} \|x_{m_k} - y_{m_k}\| = 0$, therefore from (26) (adjusted) $\lim_{k \rightarrow \infty} \|x_{m_k+1} - x_{m_k}\| = 0$.

From (29) (adjusted), we conclude that

$$\begin{aligned} \|x_{m_k+1} - q\|^2 &\leq (1 - 2\alpha_{m_k}(1 - \rho)) \|x_{m_k} - q\|^2 + 2\alpha_{m_k}(1 - \rho) \\ &\quad \cdot \left[\frac{\alpha_{m_k} \|x_{m_k} - q\|^2}{2(1 - \rho)} + \frac{\langle f(q) - q, x_{m_k+1} - q \rangle}{1 - \rho} \right] \\ &\quad + 2\beta_{m_k}^2 \|e_{m_k}\| \|z_{m_k} + e_{m_k} - q\| + 2\alpha_{m_k} \|e_{m_k}\| \|x_{m_k+1} - q\| \\ &\quad + 2(1 - \alpha_{m_k} - \beta_{m_k}) \|e_{m_k} - q\| \|x_{m_k+1} - q\| \\ &\leq (1 - 2\alpha_{m_k}(1 - \rho)) \|x_{m_k+1} - q\|^2 + 2\alpha_{m_k}(1 - \rho) \\ &\quad \cdot \left[\frac{\alpha_{m_k} \|x_{m_k} - q\|^2}{2(1 - \rho)} + \frac{\langle f(q) - q, x_{m_k+1} - q \rangle}{1 - \rho} \right] \\ &\quad + 2\beta_{m_k}^2 \|e_{m_k}\| \|z_{m_k} + e_{m_k} - q\| \\ &\quad + 2\alpha_{m_k} \|e_{m_k}\| \|x_{m_k+1} - q\| \\ &\quad + 2(1 - \alpha_{m_k} - \beta_{m_k}) \|e_{m_k} - q\| \|x_{m_k+1} - q\|. \end{aligned} \quad (37)$$

TABLE 1: Numerical results of convergence for $x_1 = 2$ in Example 3.

n	x_n	y_n	z_n
1	2	1	1.5
2	0.5	0.25	0.375
3	0.4429	0.2214	0.3321
93	0.0005	0.0002	0.0003
94	0.0005	0.0002	0.0003
95	0.0004	0.0002	0.0003
185	0.0001	0.0001	0.0001
186	0.0001	0.0001	0.0001
187	0.0001	0.0001	0.0001
277	0.0001	0.0000	0.0000
278	0.0001	0.0000	0.0000
279	0.0000	0.0000	0.0000

Therefore, we have

$$\begin{aligned} \|x_k - q\|^2 &\leq \|x_{m_k+1} - q\|^2 \leq \frac{\alpha_{m_k} \|x_{m_k} - q\|^2}{2(1 - \rho)} \\ &\quad + \frac{\langle f(q) - q, x_{m_k+1} - q \rangle}{1 - \rho} + \frac{\|e_{m_k}\|}{\alpha_{m_k}} M_1 \\ &\quad + \frac{(1 - \alpha_{m_k} - \beta_{m_k})}{\alpha_{m_k}} M_2, \end{aligned} \quad (38)$$

where

$$\begin{aligned} M_1 &= \frac{\beta_{m_k}^2}{1 - \rho} (\|z_{m_k} + e_{m_k} - q\| + 2\alpha_{m_k} \|x_{m_k+1} - q\|), \\ M_2 &= \frac{1}{1 - \rho} \|e_{m_k} - q\| \|x_{m_k+1} - q\|. \end{aligned} \quad (39)$$

From our assumptions $\lim_{k \rightarrow \infty} \alpha_{m_k} = 0$, $\lim_{k \rightarrow \infty} (1 - \alpha_{m_k} - \beta_{m_k}) / \alpha_{m_k} = 0$, $\lim_{k \rightarrow \infty} \|e_{m_k}\| / \alpha_{m_k} = 0$, and (30) (adjusted), we conclude $\limsup_{k \rightarrow \infty} \|x_k - q\|^2 = 0$. Hence, $x_k \rightarrow q$ which completes the proof of part (ii).

Open problem 1. Can we remove the condition $\liminf_{n \rightarrow \infty} \|x_n - y_n\| = 0$ in (i) in Theorem 6?

4. Numerical Example

In this section, the algorithm (14) is illustrated with some examples.

Example 1. Put $\alpha_n = 1/n + 1$, $\beta_n = 1 - 1/n + 1$, $e_n = 0$, $H = L_2[0, 1]$, $A \equiv I$, $F(x) \equiv 1$, $\lambda = 1/2$, $C = B(0, 1) = \{f \in L_2[0, 1]: \|f\| \leq 1\}$.

FIGURE 3: Convergence behavior $\{x_n\}$ in Example 3.

Then, from the algorithm (14), we have the following sequences:

$$\begin{aligned} g_n &= P_C(f_n - \lambda A f_n) = P_C\left(\frac{f_n}{2}\right), \\ k_n &= g_n - \lambda(Ag_n - A f_n) = \frac{1}{2}(f_n + g_n), \end{aligned} \quad (40)$$

$$f_{n+1} = \alpha_n F(f_n) + \beta_n k_n + e_n = \frac{1}{n+1} + \frac{n}{n+1} k_n,$$

where P_C is as follows

$$P_{B(z,\rho)}(x) = \begin{cases} x & \|x - z\| \leq \rho, \\ z + \frac{\rho}{\|x - z\|}(x - z) & \|x - z\| > \rho, \end{cases} \quad (41)$$

where $\rho > 0$ (see [19]). We have,

$$\begin{aligned} \text{VI}(C, A) &= \{f \in C : \langle Af, g - f \rangle \geq 0, \forall g \in C\} \\ &= \{f \in C : \langle f, g - f \rangle \geq 0, \forall g \in C\}. \end{aligned} \quad (42)$$

Obviously, $0 \in \text{VI}(C, A)$; hence, $\text{VI}(C, A) \neq \emptyset$.

Now, with $f_1 = 2$ and using the MATLAB software, we see that $\{f_n\}$ converges to 0 (Figure 1).

In the following example, using the MATLAB software, we compare some similar algorithms and their convergence speed and behavior. In particular, the TEGM algorithm (3), VSEGM algorithm (4), THEGM algorithm (6), and the algorithm (14) are compared. We see that the algorithm (14) has a higher convergence speed than the other algorithms (Figure 2).

Example 2. Let $\alpha_n = 1/\sqrt{n}$, $\beta_n = 1 - 1/\sqrt{n} - 1/n^2$, $\lambda = 1/2$, $e_n = 0$, $H = \mathbb{R}$, $A \equiv I$, $f(x) = x/2$, and $C = [0, \infty)$, $x_1 = 2$ (Figure 2).

Now, we examine the convergence of the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ in Theorem 6 in the following example.

Example 3. Put $\alpha_n = 1/\sqrt{n}$, $\beta_n = 1 - 1/\sqrt{n} - 1/n^2$, $\lambda = 1/2$, $e_n = 1/n^2$, $H = \mathbb{R}$, $A \equiv I$, $f \equiv 1$, and $C = [0, \infty)$, $x_1 = 2$.

Then, we have

$$y_n = P_C(x_n - \lambda A x_n) = P_C\left(\frac{1}{2}x_n\right),$$

$$z_n = y_n - \lambda(Ay_n - A x_n) = \frac{1}{2}(y_n + x_n),$$

$$x_{n+1} = \alpha_n f(x_n) + \beta_n z_n + e_n = \frac{1}{\sqrt{n}} + \left(1 - \frac{1}{\sqrt{n}} - \frac{1}{n^2}\right)z_n + \frac{1}{n^2}. \quad (43)$$

Hence,

$$\begin{aligned} \text{VI}(C, A) &= \{x \in C : \langle Ax, y - x \rangle \geq 0, \forall y \in C\} \\ &= \{x \in [0, \infty) : \langle x, y - x \rangle \geq 0, \forall y \in C\} = \{0\}. \end{aligned} \quad (44)$$

Therefore, $q = P_{\text{VI}(A,C)} \circ f(q) = P_{\{0\}} \circ (q/2) = 0$, and now, by Theorem 6, the sequence $\{x_n\}$ converges strongly to 0.

In the following example, we examine the case that $\text{VI}(C, A) = \emptyset$, and the sequence generated by the algorithm (14) is divergent (Table 1 and Figure 3).

Example 4. Put $\alpha_n = 1/n$, $\beta_n = 1 - 1/n$, $\lambda = 1/2$, $e_n = 0$, $H = \mathbb{R}$, $A \equiv -1$, $f \equiv 1$, and $C = [0, \infty)$, $x_1 = 2$.

FIGURE 4: Convergence behavior $\{x_n\}$ in Example 4.

Then, we have

$$\begin{aligned} y_n &= P_C(x_n - \lambda A x_n) = P_C\left(x_n + \frac{1}{2}\right), \\ z_n &= y_n - \lambda(A y_n - A x_n) = y_n, \\ x_{n+1} &= \alpha_n f(x_n) + \beta_n z_n + e_n = \frac{1}{n} + \left(1 - \frac{1}{n}\right) P_C\left(x_n + \frac{1}{2}\right). \end{aligned} \quad (45)$$

Hence,

$$\begin{aligned} \text{VI}(C, A) &= \{x \in C : \langle Ax, y - x \rangle \geq 0, \forall y \in C\} \\ &= \{x \in [0, \infty) : \langle -1, y - x \rangle \geq 0, \forall y \in C\} \\ &= \emptyset. \end{aligned} \quad (46)$$

Now, we will prove by induction that, for all $n \in \mathbb{N}$

$$y_n = P_C\left(x_n + \frac{1}{2}\right) = x_n + \frac{1}{2}. \quad (47)$$

When $n = 1$, then we have $x_1 = 2 \geq -1/2$.

Induction step: suppose for $n = k$ the inequality $x_k \geq -1/2$ holds.

Then,

$$\begin{aligned} x_{k+1} &= \frac{1}{k} + \left(1 - \frac{1}{k}\right) P_C\left(x_k + \frac{1}{2}\right) \\ &= \frac{1}{k} + \left(1 - \frac{1}{k}\right) \left(x_k + \frac{1}{2}\right) \\ &= \frac{1}{k} + \left(1 - \frac{1}{k}\right) x_k + \left(1 - \frac{1}{k}\right) \frac{1}{2} \\ &\geq \frac{1}{k} + \left(1 - \frac{1}{k}\right) \left(-\frac{1}{2}\right) + \left(1 - \frac{1}{k}\right) \frac{1}{2} \\ &\geq -\frac{1}{2}. \end{aligned} \quad (48)$$

Hence, $y_n = x_n + 1/2$. Consequently $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 1/2 \neq 0$.

Next, we show that $\{x_n\}$ is an unbounded sequence. Note $x_3 = 5/4 > 3/4$.

Induction step: suppose for $n = k$ the inequality $x_k > k/4$ holds.

Then,

$$\begin{aligned} x_{k+1} &= \frac{1}{k} + \left(1 - \frac{1}{k}\right) \left(x_k + \frac{1}{2}\right) > \frac{1}{k} + \left(1 - \frac{1}{k}\right) \left(\frac{k}{4} + \frac{1}{2}\right) \\ &= \frac{1}{k} + \frac{k}{4} - \frac{1}{4} + \frac{1}{2} - \frac{1}{2k} = \frac{1}{2k} + \frac{k}{4} + \frac{1}{4} > \frac{k+1}{4}, \end{aligned} \quad (49)$$

thus $\{x_n\}$ is an unbounded sequence (Figure 4).

5. Conclusions

In this paper, we proposed a Tseng-type viscosity algorithm based on the viscosity method, which is an extension of the Thong et al.'s algorithm [3]. We showed that the sequence generated by the proposed algorithm strongly converges to an element of $VI(C, A)$.

The following are the results in this paper:

- (i) We extended the results of Thong et al.'s method [3] and provided necessary and sufficient conditions for the $VI(C, A)$ to be nonempty
- (ii) In our generated sequence $\{x_n\}$, in the algorithm (14), we put a sequence of computational errors $\{e_n\}$ and proved the convergence of the sequence in the presence of computational errors
- (iii) We provided some numerical examples to compare our algorithm with the algorithms TEGM, VSEGM, and THEGM

Data Availability

No data were used to support the study.

Conflicts of Interest

This work does not have any conflicts of interest.

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