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Research Article

Common Fixed Points of Two *G*-Nonexpansive Mappings via a Faster Iteration Procedure

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In this work, we study the convergence of a new faster iteration in which two G-nonexpansive mappings are involved in the setting of uniformly convex Banach spaces with a directed graph. Moreover, by constructing a numerical example, we show the fastness of our iteration procedure over other existing iteration procedures in the literature.

1. Introduction

In 1922, Banach originated a great tool for solving the existence problems of nonlinear mappings, which is familiar as Banach's contraction principle [1]. After it, this principle has been generalized in so many different directions.

Over the last 50 years, many authors introduced and studied various iteration schemes for different classes of contractive and nonexpansive mappings. In 2008, Jachymski [2] introduced the concept of joining fixed point theory and graph theory and established the Banach contraction principle in a complete metric space endowed with a directed graph. In 2012, Aleomraninejad et al. [3] presented some iterative scheme for G-contraction and G-nonexpansive mappings in a Banach space with a graph. Tiammee et al. [4] familiarized Browder's convergence theorem for G-nonexpansive mappings in a Hilbert space with a directed graph. In 2016, Tripak [5] showed the convergence of a sequence developed by the Ishikawa iteration to some common fixed points of two G-nonexpansive mappings in a Banach space combined with a graph. In 2017, Suparatulatorn et al. [6] introduced and studied the modified S-iteration for two G-nonexpansive mappings in a uniformly convex Banach space associated with a graph. Recently, Thianwan and Yambangwai [7] introduced a new two-step iteration process involving two G-nonexpansive mappings and studied its convergence analysis in a uniformly convex Banach space endowed with a graph. There is vast literature in this direction for more details (see [8–19] and references therein).

Recently, Ullah et al. [20] introduced a new three-step iteration process known as the K-iteration process and proved its convergence for Suzuki's generalized nonexpansive mapping.

Inspired by the above work, we proposed a modified iteration process containing two G-nonexpansive mappings, by generating the sequence $\{r_n\}$ as follows:

Let $\mathscr{C}$ be a nonempty convex subset of a Banach space $\mathscr{X}$, for any random $r_0 \in \mathscr{C}$,

$$t_n = (1-\varsigma_n)r_n + \varsigma_n P_2 r_n, \tag{1}$$

$$s_n = P_2((1-\sigma_n)P_1 r_n + \sigma_n P_2 t_n), \tag{2}$$

$$r_{n+1} = P_1 s_n, \tag{3}$$

where $\{\sigma_n\}$ and $\{\varsigma_n\}$ are appropriate real sequences in $(0, 1)$ and $P_1, P_2 : \mathscr{C} \to \mathscr{C}$ are G-nonexpansive mappings. Under some certain conditions, we demonstrate the convergence analysis of (1) for approximating common fixed points of two G-nonexpansive mappings in a uniformly convex

Banach space $\mathscr{X}$ with graph. We also construct a numerical example, and by using MATLAB R2018a, we clarify that the proposed iteration procedure converges faster than modified Ishikawa iteration, modified S-iteration, and Thianwan's new iteration (see [5–7]).

2. Preliminaries

In this part, we gather some familiar concepts and applicable conclusions which will be used often.

Let $\mathscr{C}$ be a nonempty subset of a Banach space $\mathscr{X}$. Let Δ denote the diagonal of the cartesian product $\mathscr{C}\times\mathscr{C}$, i.e., $\Delta=\{(r,r): r\in\mathscr{C}\}$.

$V(G)$ denotes the set of vertices that coincides with $\mathscr{C}$ in a directed graph G, and the set $E(G)$ of its edges contains all loops, i.e., $E(G)\supseteq\Delta$. By assuming G has no parallel edge to identify the graph G with the pair $(V(G),E(G)$. G^{-1} denotes the conversion of a graph G. So we have

$$E\left(G^{-1}\right)=\{(r,s)\in\mathscr{C}\times\mathscr{C} : (s,r)\in E(G)\}. \tag{4}$$

A set B is said to be dominated by r_0 if for each $r\in B$, $(r_0,r)\in E(G)$ and dominates r_0 if for each $r\in B$, $(r,r_0)\in E(G)$.

Let $P:\mathscr{C}\to\mathscr{C}$ be a self map. An edge preserving mapping, i.e., $(r,s)\in E(G)\Rightarrow(Pr,Ps)\in E(G)$, is said to be G-nonexpansive if

$$\|Pr-Ps\|\leq\|r-s\| \quad \forall (r,s)\in E(G). \tag{5}$$

A mapping $P:\mathscr{C}\to\mathscr{C}$ is said to be G-demiclosed at 0, if for any sequence $\{r_n\}$ in $\mathscr{C}$ such that $r_n\rightharpoonup r$ for all $(r_n,r_{n+1})\in E(G)$ and $Pr_n\to 0$ then $Pr=0$.

Let us recall that a Banach space $\mathscr{X}$ is said to satisfy Opial's property if $r_n\rightharpoonup r$ and then

$$\limsup_{n\to\infty}\|r_n-r\|<\limsup_{n\to\infty}\|r_n-s\|, \quad \forall s\neq r. \tag{6}$$

Lemma 1 [21]. *Let $\mathscr{C}$ be a subset of a metric space $(\mathscr{X},d)$. A mapping $P:\mathscr{C}\to\mathscr{C}$ is semicompact if for a sequence $\{r_n\}$ in $\mathscr{C}$ with $\lim_{n\to\infty}d(r_n,Pr_n)=0$ there exists a subsequence $\{r_{n_j}\}$ of $\{r_n\}$ such that $r_{n_j}\to p\in\mathscr{C}$.*

Let $\mathscr{C}$ be a subset of a normed space $\mathscr{X}$ and let $G=(V(G),E(G))$ be a directed graph such that $V(G)=\mathscr{C}$. Then, $\mathscr{C}$ is said to have property WG (SG) if for each sequence $\{r_n\}$ in $\mathscr{C}$ converging weakly (strongly) to $r\in\mathscr{C}$ and $(r_n,r_{n+1})\in E(G)$ there is a subsequence $\{r_{n_j}\}$ of $\{r_n\}$ such that $(r_{n_j},r)\in E(G)$ for all $n\in\mathbb{N}$.

Lemma 2 [6]. *Suppose that $\mathscr{X}$ is the Banach space having Opial's condition, $\mathscr{C}$ has property WG, and let $P:\mathscr{C}\to\mathscr{C}$ be a G-nonexpansive mapping. Then, $I-P$ is G-demiclosed at 0, i.e., if $r_n\rightharpoonup r$ and $(r_n-Pr_n)\to 0$, then $r\in F(P)$, where $F(P)$ is the set of fixed points of P.*

Lemma 3 [22]. *Let $\mathscr{X}$ be uniformly convex Banach space and $\{\sigma_n\}$ a sequence in $[\delta,1-\delta]$ for some $\delta\in(0,1)$. Suppose that the sequences $\{r_n\}$ and $\{s_n\}$ in $\mathscr{X}$ are such that $\limsup_{n\to\infty}\|r_n\|\leq c$, $\limsup_{n\to\infty}\|s_n\|\leq c$, and $\limsup_{n\to\infty}\|\sigma_n r_n+(1-\sigma_n)s_n\|=c$ for some $c\geq 0$. Then, $\lim_{n\to\infty}\|r_n-s_n\|=0$.*

Lemma 4 [23]. *Let $\mathscr{X}$ be the Banach space that satisfies Opial's condition and let $\{r_n\}$ be a sequence in $\mathscr{X}$. Let $u,v\in X$ be such that $\lim_{n\to\infty}\|r_n-u\|$ and $\lim_{n\to\infty}\|r_n-v\|$ exist. If $\{r_{n_j}\}$ and $\{r_{n_k}\}$ are subsequences of $\{r_n\}$ that converges weakly to u and v, respectively, then $u=v$.*

Lemma 5 [24]. *Let $\{r_n\}$ be a bounded sequence in a reflexive Banach space $\mathscr{X}$. If for any weakly convergent subsequences $\{r_{n_j}\}$ of $\{r_n\}$, both $\{r_{n_j}\}$ and $\{r_{n_{j+1}}\}$ converge weakly to the same point in $\mathscr{X}$, then the sequence $\{r_n\}$ is weakly convergent.*

Lemma 6 [7]. *Let $\mathscr{C}$ be a nonempty closed convex subset of a uniformly convex Banach space $\mathscr{X}$ and suppose that $\mathscr{C}$ has property WG. Let P be a G-nonexpansive mapping on $\mathscr{C}$. Then, $I-P$ is G-demiclosed at 0.*

3. Main Results

We initiate this section by proving the following proposition.

Proposition 7. *Let P_1 and P_2 be two self G-nonexpansive mappings on $\mathscr{C}$ with $F=F(P_1)\cap F(P_2)$ nonempty, where $\mathscr{C}$ is a nonempty closed convex subset of a uniformly convex Banach space $\mathscr{X}$ endowed with a directed graph. Let $V(G)=\mathscr{C}$, $E(G)$ is convex and the graph G is transitive. For random $r_0\in C$, define the sequence $\{r_n\}$ by (1). Let $p_0\in F$ be such that (r_0,p_0), (p_0,r_0) are in $E(G)$. Then, (r_n,p_0), (s_n,p_0), (t_n,p_0), (p_0,r_n), (p_0,s_n), (p_0,t_n), (r_n,s_n), (r_n,t_n), and (r_n,r_{n+1}) are in $E(G)$.*

Proof. We go ahead by induction. By using the edge preserving property of P_2 and assumption $(r_0,p_0)\in E(G)$, we get $(P_2r_0,p_0)\in E(G)$. By convexity of $E(G)$, we obtain $(t_0,p_0)\in E(G)$. Again, by edge preservingness of P_2 and $(t_0,p_0)\in E(G)$, we have $(P_2t_0,p_0)\in E(G)$. By convexity of $E(G)$ and $(P_1r_0,p_0),(P_2t_0,p_0)\in E(G)$ and applying the edge preservingness of P_2 again, we have $(s_0,p_0)\in E(G)$. Again, by using the property of edge preserving of P_1 and $(s_0,p_0)\in E(G)$, we get $(r_1,p_0)\in E(G)$. Again, by applying edge preserving of P_2, $(P_2r_1,p_0)\in E(G)$, we get $(t_1,p_0)\in E(G)$ as $E(G)$ is convex. Thus, by edge preserving of P_2, $(P_2t_1,p_0)\in E(G)$. Again, by using convexity of $E(G)(P_1r_1,p_0),(P_2t_1,p_0)\in E(G)$ and edge preservingness of P_2, we get $(s_1,p_0)\in E(G)$. By edge preserving of P_1, we get $(P_1s_1,p_0)\in E(G)$, and we get $(r_2,p_0)\in E(G)$. Next, we assume that $(r_k,p_0)\in E(G)$. By edge preserving of P_2 and convexity of $E(G)$, we get $(P_2r_k,p_0)\in E(G)$ and $(t_k,p_0)\in E(G)$. By applying edge preserving of P_2 on $(t_k,p_0)\in E(G)$, we get $(P_2t_k,p_0)\in E(G)$. By using convexity of $E(G)$ and $(P_1r_k,p_0),(P_2t_k,p_0)\in E(G)$ and edge preserving property of P_2, we have $(s_k,p_0)\in E(G)$. As P_1 is edge preserving, we get $(r_{k+1}, p_0) \in E(G)$. Owing to edge preserving of P_2, we obtain $(P_2 r_{k+1}, p_0) \in E(G)$ and so $(t_{k+1}, p_0) \in E(G)$, since $E(G)$ is convex. By convexity of $E(G)$ and $(P_1 r_{k+1}, p_0), (P_2 t_{k+1}, p_0) \in E(G)$ and applying the edge preservingness of P_2 again, we have $(s_{k+1}, p_0) \in E(G)$. Therefore, $(r_n, p_0), (s_n, p_0), (t_n, p_0) \in E(G)$ for all $n \geq 1$. Using a similar argument, we can show that $(p_0, r_n), (p_0, s_n), (p_0, t_n) \in E(G)$ under the assumption that $(p_0, r_0) \in E(G)$. By using transitivity of G, we get $(r_n, s_n), (r_n, t_n), (s_n, t_n), (r_n, r_{n+1}) \in E(G)$. This completes the proof.

Lemma 8. *Let $\mathscr{X}$, $\mathscr{C}$, F, P_1, P_2, and $\{r_n\}$ be the same as in Proposition 7. Suppose that $\{\sigma_n\}$ and $\{\varsigma_n\}$ are real sequences in $(0, 1)$ and $(r_0, p_0), (p_0, r_0) \in E(G)$ for arbitrary $r_0 \in \mathscr{C}$ and $p_0 \in F$. Then,*

(i) $\lim_{n\to\infty} \|r_n - p_0\|$ *exists*

(ii) $\lim_{n\to\infty} \|P_1 r_n - r_n\| = 0 = \lim_{n\to\infty} \|P_2 r_n - r_n\|$

Proof.

(i) Let $p_0 \in F$. By Proposition 7, we have $(r_n, p_0), (s_n, p_0), (t_n, p_0) \in E(G)$. As P_1 and P_2 are G-nonexpansive mappings and using the iterative sequence $\{r_n\}$, we have

$$\|t_n - p_0\| = \|(1-\varsigma_n) r_n + \varsigma_n P_2 r_n - p_0\| \leq (1-\varsigma_n)\|r_n - p_0\| + \varsigma_n \|P_2 r_n - p_0\| \leq (1-\varsigma_n)\|r_n - p_0\| + \varsigma_n \|r_n - p_0\| = \|r_n - p_0\|, \tag{7}$$

$$\begin{aligned}\|s_n - p_0\| &= \|P_2((1-\sigma_n)P_1 r_n + \sigma_n P_2 t_n) - p_0\| \\ &\leq \|(1-\sigma_n)P_1 r_n + \sigma_n P_2 t_n - p_0\| \\ &\leq (1-\sigma_n)\|P_1 r_n - p_0\| + \sigma_n \|P_2 t_n - p_0\| \\ &\leq (1-\sigma_n)\|r_n - p_0\| + \sigma_n \|t_n - p_0\| \\ &\leq (1-\sigma_n)\|r_n - p_0\| + \sigma_n \|r_n - p_0\| \leq \|r_n - p_0\|,\end{aligned} \tag{8}$$

$$\|r_{n+1} - p_0\| = \|P_1 s_n - p_0\| \leq \|s_n - p_0\| \leq \|r_n - p_0\|. \tag{9}$$

This implies that sequence $\{\|r_n - p_0\|\}$ is decreasing and bounded below for all $p_0 \in F(P)$. Hence, $\lim_{n\to\infty} \|r_n - p_0\|$ exists.

(ii) Assume that $\lim_{n\to\infty} \|r_n - p_0\| = c$. If $c = 0$; then, by G-nonexpansiveness of P_1 and P_2, we get

$$\|r_n - P_i r_n\| \leq \|r_n - p_0\| + \|p_0 - P_i r_n\| \leq \|r_n - p_0\| + \|p_0 - r_n\|. \tag{10}$$

Therefore, the result follows.
Suppose that $c > 0$.

Taking the lim sup on both sides in the inequality (7) and (8), we obtain

$$\limsup_{n\to\infty} \|t_n - p_0\| \leq \limsup_{n\to\infty} \|r_n - p_0\| = c, \tag{11}$$

$$\limsup_{n\to\infty} \|s_n - p_0\| \leq \limsup_{n\to\infty} \|r_n - p_0\| = c. \tag{12}$$

On the other hand, using (1), we have

$$\begin{aligned}\|r_{n+1} - p_0\| &= \|P_1 s_n - p_0\| \leq \|s_n - p_0\| \\ &\leq \|P_2((1-\sigma_n)P_1 r_n + \sigma_n P_2 t_n) - p_0\| \\ &\leq \|(1-\sigma_n)P_1 r_n + \sigma_n P_2 t_n - p_0\| \\ &= \|(1-\sigma_n)(P_1 r_n - p_0) + \sigma_n (P_2 t_n - p_0)\| \\ &\leq (1-\sigma_n)\|r_n - p_0\| + \sigma_n \|t_n - p_0\| \\ &\leq \|r_n - p_0\| - \sigma_n \|r_n - p_0\| + \sigma_n \|t_n - p_0\|.\end{aligned} \tag{13}$$

This implies that

$$\frac{\|r_{n+1} - p_0\| - \|r_n - p_0\|}{\sigma_n} \leq \|t_n - p_0\| - \|r_n - p_0\|. \tag{14}$$

So

$$\begin{aligned}\|r_{n+1} - p_0\| - \|r_n - p_0\| &\leq \frac{\|r_{n+1} - p_0\| - \|r_n - p_0\|}{\sigma_n} \\ &\leq \|t_n - p_0\| - \|r_n - p_0\|.\end{aligned} \tag{15}$$

This implies

$$\|r_{n+1} - p_0\| \leq \|t_n - p_0\|. \tag{16}$$

By taking lim inf both sides, we have

$$c \leq \liminf_{n\to\infty} \|t_n - p_0\|. \tag{17}$$

From (11) and (17), we get

$$c = \lim_{n\to\infty} \|t_n - p_0\|, \tag{18}$$

$$c = \lim_{n\to\infty} \|(1-\varsigma_n) r_n + \varsigma_n P_2 r_n - p_0. \tag{19}$$

By using (11) and (19) and Lemma 3, we get

$$\lim_{n\to\infty} \|P_2 r_n - r_n\| = 0, \tag{20}$$

$$\|r_{n+1} - p_0\| \leq \|P_1 s_n - p_0\| \leq \|s_n - p_0\|. \tag{21}$$

By taking lim inf both sides, we have

$$c \leq \liminf_{n\to\infty} \|s_n - p_0\|. \tag{22}$$

By using (12) and (22), we have

$$\lim_{n\to\infty}\|s_n - p_0\| = c,$$
$$\lim_{n\to\infty}\|P_2((1-\sigma_n)P_1r_n + \sigma_nP_2t_n) - p_0\| = c. \tag{23}$$

We also have

$$\|P_2((1-\sigma_n)P_1r_n + \sigma_nP_2t_n) - p_0\| \le \|(1-\sigma_n)P_1r_n + \sigma_nP_2t_n - p_0\|. \tag{24}$$

By taking limit infimum on both sides, we get

$$c \le \liminf_{n\to\infty}\|(1-\sigma_n)P_1r_n + \sigma_nP_2t_n - p_0\|. \tag{25}$$

By using edge preserving property of P_1 and P_2, we have

$$\|P_1r_n - p_0\| \le \|r_n - p_0\|, \tag{26}$$

$$\|P_2t_n - p_0\| \le \|t_n - p_0\| \le \|r_n - p_0\|. \tag{27}$$

By taking limit sup on both sides in (26) and (27), we get

$$\limsup_{n\to\infty}\|P_1r_n - p_0\| \le c, \tag{28}$$

$$\limsup_{n\to\infty}\|P_2t_n - p_0\| \le c. \tag{29}$$

By using (28) and (29), we obtain

$$\|(1-\sigma_n)P_1r_n + \sigma_nP_2t_n - p_0\| \le (1-\sigma_n)\|P_1r_n - p_0\| + \sigma_n\|P_2t_n - p_0\| \le (1-\sigma_n)\|r_n - p_0\| + \sigma_n\|t_n - p_0\| \le \|r_n - p_0\|. \tag{30}$$

By taking limit sup on both sides in (30), we get

$$\limsup_{n\to\infty}\|(1-\sigma_n)P_1r_n + \sigma_nP_2t_n - p_0\| \le c. \tag{31}$$

By using (25) and (31), we have

$$\lim_{n\to\infty}\|(1-\sigma_n)P_1r_n + \sigma_nP_2t_n - p_0\| = c. \tag{32}$$

By (28), (29), and (32) and Lemma 3, we have

$$\lim_{n\to\infty}\|P_1r_n - P_2t_n\| = 0. \tag{33}$$

In addition,

$$\|t_n - r_n\| = \|(1-\varsigma_n)r_n + \varsigma_nP_2r_n - r_n\| = \|\varsigma_n(P_2r_n - r_n)\| \to 0 (\text{as} n \to \infty). \tag{34}$$

By using (20), (33), and (34), we have

$$\|P_1r_n - r_n\| = \|P_1r_n - P_2t_n + P_2t_n - r_n\| \le \|P_1r_n - P_2t_n\| + \|P_2t_n - r_n\| \le \|P_1r_n - P_2t_n\| + \|P_2t_n - P_2r_n\| + \|P_2r_n - r_n\| \le \|P_1r_n - P_2t_n\| + \|t_n - r_n\| + \|P_2r_n - r_n\| \to 0 (\text{as } n \to \infty). \tag{35}$$

Therefore, we conclude $\lim_{n\to\infty}\|P_1r_n - r_n\| = 0 = \lim_{n\to\infty}\|P_2r_n - r_n\|$.

The next proof is for the weak convergence of the sequence generated by (1) in a uniformly convex Banach space with directed graph satisfying Opial's condition.

Theorem 9. *Let $\mathcal{X}$, $\mathcal{C}$, F, P_1, P_2, and $\{r_n\}$ be the same as in Proposition 7 with $\mathcal{X}$ satisfying Opial's condition and $\mathcal{C}$ has property WG. Suppose that $\{\sigma_n\}$ and $\{\varsigma_n\}$ are real sequences in $(0, 1)$ and $(r_0, p_0), (p_0, r_0) \in E(G)$ for arbitrary $r_0 \in \mathcal{C}$ and $p_0 \in F$ and then $\{r_n\}$ weakly converges to a common fixed point of P_1 and P_2.*

Proof. Let $p_0 \in F$ be such that $(r_0, p_0), (p_0, r_0) \in E(G)$. From Lemm 8(i), $\lim_{n\to\infty}\|r_n - p_0\|$ exists, so $\{r_n\}$ is bounded. It follows from Lemma 8(ii) that $\lim_{n\to\infty}\|P_1r_n - r_n\| = 0 = \lim_{n\to\infty}\|P_2r_n - r_n\|$. Since $\mathcal{X}$ is uniformly convex and $\{r_n\}$ is bounded, we may assume that $r_n \rightharpoonup u$ as $n \to \infty$, without loss of generality. By Lemma 2, we have $u \in F$. Suppose that subsequences $\{r_{n_k}\}$ and $\{r_{n_j}\}$ of $\{r_n\}$ converges weakly to u and v, respectively. By Lemma 8(ii), we obtain that $\|P_ir_{n_k} - r_{n_k}\| \to 0$ and $\|P_ir_{n_j} - r_{n_j}\| \to 0$ as $k, j \to \infty$. Using Lemma 2, we have $u, v \in F$. By Lemma 8(i), $\lim_{n\to\infty}\|r_n - u\|$ and $\lim_{n\to\infty}\|r_n - v\|$ exist. It follows from Lemma 4 that $u = v$. Therefore, $\{r_n\}$ converges weakly to a common fixed point of P_1 and P_2.

Next, we prove weak convergence of the sequence $\{r_n\}$ generated by (1) without assuming Opial's condition in a uniformly convex Banach space with a directed graph.

Theorem 10. *Let $\mathcal{X}$, $\mathcal{C}$, F, P_1, P_2, and $\{r_n\}$ be the same as in Proposition 7 with C having property WG, $\{\sigma_n\}$ and $\{\varsigma_n\}$ are real sequences in $(0, 1)$, F is dominated by r_0, and F dominates r_0. Then, $\{r_n\}$ weakly converges to a common fixed point of P_1 and P_2.*

Proof. Let $p_0 \in F$ be such that $(r_0, p_0), (p_0, r_0) \in E(G)$. From Lemma 8(i), $\lim_{n\to\infty}\|r_n - p_0\|$ exists, so $\{r_n\}$ is bounded in $\mathcal{C}$. Since $\mathcal{C}$ is nonempty closed convex subset of a uniformly convex Banach space $\mathcal{X}$, it is weakly compact and hence there exists a subsequence $\{r_{n_j}\}$ of the sequence $\{r_n\}$ which converges weakly to some point $p \in \mathcal{C}$. By Lemma 8(ii), we obtain that

$$\|r_{n_j} - P_1r_{n_j}\| = 0 = \|r_{n_j} - P_2r_{n_j}\|. \tag{36}$$

By using (20), (34), and (35), we have

$$\begin{aligned}\|s_n - r_n\| &= \|P_2((1-\sigma_n)P_1 r_n + \sigma_n P_2 t_n) - r_n\| \\ &\le \|P_2((1-\sigma_n)P_1 r_n + \sigma_n P_2 t_n) - P_2 r_n\| + \|P_2 r_n - r_n\| \\ &\le \|(1-\sigma_n)P_1 r_n + \sigma_n P_2 t_n - r_n\| + \|P_2 r_n - r_n\| \\ &\le \|(1-\sigma_n)(P_1 r_n - r_n) + \sigma_n(P_2 t_n - r_n)\| + \|P_2 r_n - r_n\| \\ &\le (1-\sigma_n)\|P_1 r_n - r_n\| + \sigma_n\|P_2 t_n - r_n\| + \|P_2 r_n - r_n\| \\ &\le (1-\sigma_n)\|P_1 r_n - r_n\| + \sigma_n\|P_2 t_n - P_2 r_n\| + \|P_2 r_n - r_n\| \\ &\le (1-\sigma_n)\|P_1 r_n - r_n\| + \sigma_n\|t_n - r_n\| \\ &\quad + \|P_2 r_n - r_n\| \to 0 \ (\text{as } n \to \infty).\end{aligned} \tag{37}$$

In addition,

$$\begin{aligned}\|P_2 t_n - t_n\| &\le \|P_2 t_n - P_1 r_n\| + \|P_1 r_n - t_n\| \le \|P_2 t_n - P_1 r_n\| \\ &\quad + \|P_1 r_n - r_n\| + \|r_n - t_n\| \to 0 \ (\text{as } n \to \infty).\end{aligned} \tag{38}$$

And by using (35) and (37),

$$\begin{aligned}\|P_1 s_n - s_n\| &\le \|P_1 s_n - P_1 r_n\| + \|P_1 r_n - s_n\| \le \|s_n - r_n\| \\ &\quad + \|P_1 r_n - r_n\| + \|r_n - s_n\| \le 2\|s_n - r_n\| \\ &\quad + \|P_1 r_n - r_n\| \to 0 \ (\text{as } n \to \infty).\end{aligned} \tag{39}$$

Using Lemma 6, $I - P_1$ and $I - P_2$ are G-demiclosed at 0 so that $p \in F$. To complete the proof, it suffices to show that $\{r_n\}$ converges weakly to p. To this end, we need to show that $\{r_n\}$ satisfies the hypothesis of Lemma 5. Let $\{r_{n_j}\}$ be a subsequence of $\{r_n\}$ which converges weakly to some $q \in \mathscr{C}$. By similar argument as above $q \in F$. Now, for each $j \ge 1$, using (1), we have

$$r_{n_{j+1}} = P_1 s_{n_j}. \tag{40}$$

By using (34), we get

$$P_1 r_{n_j} = \left(P_1 r_{n_j} - r_{n_j}\right) + r_{n_j} \rightharpoonup q. \tag{41}$$

By using (20), we have

$$P_2 r_{n_j} = \left(P_2 r_{n_j} - r_{n_j}\right) + r_{n_j} \rightharpoonup q. \tag{42}$$

We have

$$t_{n_j} = \left(1 - \varsigma_{n_j}\right) r_{n_j} + \varsigma_{n_j} P_2 r_{n_j} \rightharpoonup q. \tag{43}$$

It follows from (38)

$$P_2 t_{n_j} = \left(P_2 t_{n_j} - t_{n_j}\right) + t_{n_j} \rightharpoonup q. \tag{44}$$

By using (41) and (44), we get

$$s_{n_j} = P_2\left(\left(1 - \sigma_{n_j}\right) P_1 r_{n_j} + \sigma_{n_j} P_2 t_{n_j}\right) \rightharpoonup q. \tag{45}$$

It follows from (39) and (45), we get

$$\begin{aligned}P_1 s_{n_j} &= \left(P_1 s_{n_j} - s_{n_j}\right) + s_{n_j} \rightharpoonup q, \\ r_{n_{j+1}} &= P_1 s_{n_j} \rightharpoonup q.\end{aligned} \tag{46}$$

Therefore, the sequence $\{r_n\}$ satisfies the hypothesis of Lemma 5 which in turn implies that $\{r_n\}$ weakly converges to q so that $p = q$.

Next, we recall condition (B) for strong convergence.

Let $\mathscr{C}$ be a nonempty closed convex subset of a uniformly convex Banach space $\mathscr{X}$. The mappings P_1 and P_2 on $\mathscr{C}$ are said to satisfy condition (B) [21] if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(z) > 0$ for all $z > 0$ such that for all $r \in \mathscr{C}$,

$$\max\{\|r - P_1 r\|, \|r - P_2 r\|\} \ge f(d(r, F)), \tag{47}$$

where $d(r, F) = \inf\{\|r - q\| : q \in F\}$.

Theorem 11. *Let $\mathscr{X}$, $\mathscr{C}$, F, P_1, P_2, and $\{r_n\}$ be the same as in Proposition 7. Suppose that $\{\sigma_n\}$ and $\{\varsigma_n\}$ are real sequences in $(0, 1)$, $P_i (i = 1, 2)$ satisfy condition (B), F is dominated by r_0 , and F dominates r_0. Then, the sequence $\{r_n\}$ converges strongly to a common fixed point of P_1 and P_2.*

Proof. From Lemma 8(i), $\lim_{n\to\infty} \|r_n - q\|$ exists and so $\lim_{n\to\infty} d(r_n, F)$ exists for any $q \in F$. Also, from Lemma 8(ii), $\lim_{n\to\infty} \|r_n - P_1 r_n\| = 0 = \lim_{n\to\infty} \|r_n - P_2 r_n\|$. Owing to condition (B),

$$f(d(r, F)) \le \max\{\|r - P_1 r\|, \|r - P_2 r\|\}. \tag{48}$$

We have $\lim_{n\to\infty} f(d(r_n, F)) = 0$. As $f : [0,\infty) \to [0,\infty)$ is a nondecreasing function satisfying $f(0) = 0$, $f(z) > 0$ for all $z \in [0,\infty)$, we obtain that $\lim_{n\to\infty} d(r_n, F) = 0$.

Hence, we can find a subsequence $\{r_{n_j}\}$ of $\{r_n\}$ and a sequence $\{u_j\} \subset F$ such that $\|r_{n_j} - u_j\| \le (1/2^j)$. Put $n_{j+1} = n_j + h$ for some $h \ge 1$. Then,

$$\begin{aligned}&\|r_{n_{j+1}} - u_j\| \le \|r_{n_j + h - 1} - u_j\| \le \|r_{n_j} - u_j\| \le \frac{1}{2^j}, \\ &\|u_{j+1} - u_j\| \le \|u_{j+1} - r_{n_{j+1}}\| + \|r_{n_{j+1}} - u_j\| \le \frac{1}{2^{j+1}} + \frac{1}{2^j} \\ &\qquad < \frac{1}{2^{j-1}} \to 0 \ (\text{as } j \to \infty).\end{aligned} \tag{49}$$

So $\{u_j\}$ is a Cauchy sequence. We assume that $u_j \to q_0 \in C$ as $j \to \infty$. Since F is closed, we get $q_0 \in F$. So we have

Table 1: Convergence of iterative schemes.

Iteration no.	Modified Ishikawa iteration	Modified S-iteration	Thianwan's new iteration	Proposed iteration
1.0	0.5000000000000	0.5000000000000	0.5000000000000	0.5000000000000
3.0	0.5839729465079	0.9045303242056	0.9160290275994	0.9953507671826
5.0	0.7228781259509	0.9831124491843	0.9875389515921	0.9999589583885
7.0	0.8483210253860	0.9970590402716	0.9982054854975	0.9999996090513
9.0	0.9299336356828	0.9995059080444	0.9997519971246	0.9999999961503
11.0	0.9720289950766	0.9999208072730	0.9999673170979	0.9999999999616
13.0	0.9901611017681	0.9999879343875	0.9999958969892	0.9999999999996
15.0	0.9969024367323	0.9999982522932	0.9999995085433	1.00000000000000

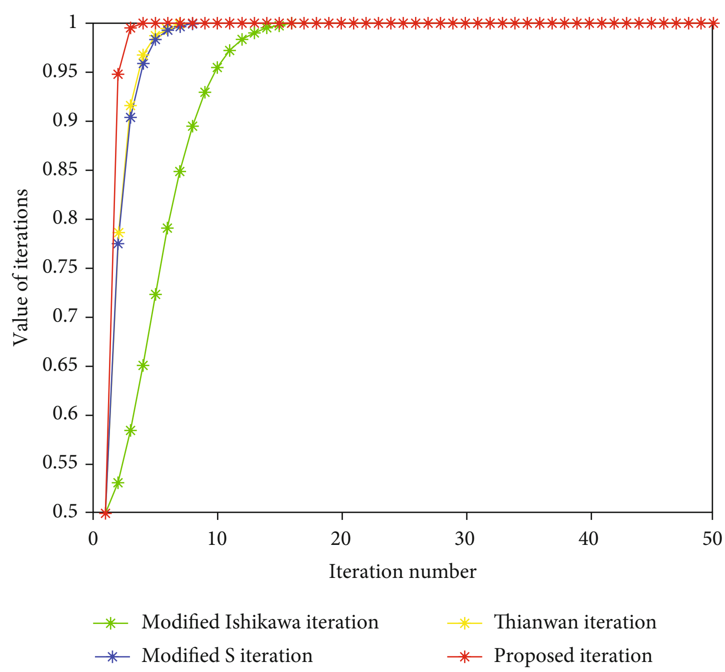

Figure 1: Numerical experiment of Example 1 using the modified Ishikawa iteration, modified S-iteration, Thianwan new iteration, and proposed iteration.

$r_{n_j} \to q_0$ as $j \to \infty$. Since $\lim_{n\to\infty} \|r_n - q_0\|$ exists, we get $r_n \to q_0$.

We prove another strong convergence theorem as follows.

Theorem 12. *Let $\mathscr{X}$, $\mathscr{C}$, F, P_1, P_2, and $\{r_n\}$ be the same as in Proposition 7 with C having property SG, $\{\sigma_n\}$ and $\{\varsigma_n\}$ are real sequences in $(0, 1)$, F is dominated by r_0, and F dominates r_0. If one of $P_i (i = 1, 2)$ is semicompact, then $\{r_n\}$ converges strongly to a common fixed point of P_1 and P_2.*

Proof. It follows from Lemma 8 that $\{r_n\}$ is bounded and $\lim_{n\to\infty} \|P_1 r_n - r_n\| = 0 = \lim_{n\to\infty} \|P_2 r_n - r_n\|$. Since one of P_1 and P_2 is semicompact, then there exist subsequences $\{r_{n_j}\}$ of $\{r_n\}$ such that $r_{n_j} \to q \in \mathscr{C}$ as $j \to \infty$. Since $\mathscr{C}$ has property SG and transitivity of graph G, we obtain $(r_{n_j}, q) \in E(G)$. Notice that, for each $i \in 0, 1$, $\lim_{j\to\infty} \|r_{n_j} - P_i r_{n_j}\| = 0$.

Then,

$$\begin{aligned} \|q - P_i q\| &\le \|q - r_{n_j}\| + \|r_{n_j} - P_i r_{n_j}\| + \|P_i r_{n_j} - P_i q\| \\ &\le \|q - r_{n_j}\| + \|r_{n_j} - \mathrm{P}_i r_{n_j}\| + \|r_{n_j} - q\| \to 0 \,(\text{as } j \to \infty). \end{aligned} \tag{50}$$

Hence, $q \in F$. Thus, $\lim_{n\to\infty} d(r_n, F)$ exists by Theorem 11. We note that $d(r_{n_j}, F) \le d(r_{n_j}, q) \to 0$ as $j \to \infty$. Hence, $\lim_{n\to\infty} d(r_n, F) = 0$. It follows, as in the proof of Theorem 11, that $\{r_n\}$ converges strongly to a common fixed point of P_1 and P_2.

4. Numerical Examples

This section contains a numerical example which supports our main theorem. It is worth mentioning here that this example is motivated by [25].

Example 1. Let $\mathscr{X} = \mathbb{R}$ and $\mathscr{C} = [0, 2]$. Let $G = (V(G), E(G))$ be a directed graph defined by $V(G) = \mathscr{C}$ and $(r, s) \in E(G)$ if and only if $0.50 \leq r \neq s \leq 1.70$ or $r = s \in \mathscr{C}$. Define mappings $P_1, P_2 : \mathscr{C} \to \mathscr{C}$

$$\begin{aligned} P_1 r &= r^{1/3}, \\ P_2 r &= \frac{40}{62} \arcsin (r - 1) + 1, \end{aligned} \tag{51}$$

for any $r \in \mathscr{C}$.

It is easy to show that P_1, P_2 are G-nonexpansive mappings but P_1, P_2 are not nonexpansive mappings because

$$\begin{aligned} |P_1 r - P_1 s| &> |r - s|, \\ |P_2 u - P_2 v| &> |u - v|, \end{aligned} \tag{52}$$

when $r = 0.5$, $s = .03$, $u = 1.9$, and $v = 1.5$. Choose $\sigma_n = n/(n + 5)$, $\varsigma_n = n/(n + 5)$, for all $n \in N$ and initial points $r_0 = s_0 = t_0 = 0.5$. In Table 1 and Figure 1, we have shown the convergence rate of the modified Ishikawa iteration, modified S-iteration, Thianwan iteration, and proposed iteration (1).

Figure 1 shows the convergence of the modified Ishikawa iteration, modified S-iteration, Thianwan new iteration, and proposed iteration (1) to the common fixed point of P_1 and P_2 which is 1 in this numerical experiment, and it is clear that the proposed iteration process converges faster than others.

5. Conclusion

The purpose of this paper was to study the convergence of a new faster iteration in which two G-nonexpansive mappings were involved in the setting of uniformly convex Banach spaces with a directed graph. Also, we constructed a numerical example to show the fastness of our iteration procedure over other existing iteration procedures in the literature.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally and significantly in writing this paper. All authors have read and agreed to the published version of the manuscript.

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References

[1] S. Banach, "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales," *Fundamenta Mathematicae*, vol. 3, pp. 133–181, 1922.

[2] J. Jachymski, "The contraction principle for mappings on a metric space with a graph," *Proceedings of American Mathematical Society*, vol. 136, no. 4, pp. 1359–1373, 2008.

[3] S. M. A. Aleomraninejad, S. Rezapour, and N. Shahzad, "Some fixed point results on a metric space with a graph," *Topology and its Applications*, vol. 159, no. 3, pp. 659–663, 2012.

[4] J. Tiammee, A. Kaewkhao, and S. Suantai, "On Browder's convergence theorem and Halpern iteration process for G-nonexpansive mappings in Hilbert spaces endowed with graphs," *Fixed Point Theory and Applications*, vol. 2015, no. 1, Article ID 187, 2015.

[5] O. Tripak, "Common fixed points of G-nonexpansive mappings on Banach spaces with a graph," *Fixed Point Theory and Applications*, vol. 2016, no. 1, Article ID 83, 2016.

[6] R. Suparatulatorn, W. Cholamjiak, and S. Suantai, "A modified S-iteration process for G-nonexpansive mappings in Banach spaces with graphs," *Numerical Algorithms*, vol. 77, no. 2, pp. 479–490, 2018.

[7] T. Thianwan and D. Yambangwai, "Convergence analysis for a new two-step iteration process for G-nonexpansive mappings with directed graphs," *Journal of Fixed Point Theory and Applications*, vol. 21, no. 2, 2019.

[8] M. R. Alfuraidan, "Fixed points of monotone nonexpansive mappings with a graph," *Fixed Point Theory and Applications*, vol. 2015, no. 1, Article ID 49, 2015.

[9] M. R. Alfuraidan and M. A. Khamsi, "Fixed points of monotone nonexpansive mappings on a hyperbolic metric space with a graph," *Fixed Point Theory and Applications*, vol. 2015, no. 1, Article ID 44, 2015.

[10] F. Ali, J. Ali, and I. Uddin, "A novel approach for the solution of BVPs via Green's function and fixed point iterative method," *Journal of Applied Mathematics and Computing*, 2020.

[11] J. Ali and F. Ali, "Approximation of common fixed points and the solution of image recovery problem," *Results in Mathematics*, vol. 74, no. 4, p. 130, 2019.

[12] M. F. Bota, L. Guran, and A. Petrusel, "New fixed point theorems on b-metric spaces with applications to coupled fixed point theory," *Journal of Fixed Point Theory and Applications*, vol. 22, no. 3, 2020.

[13] L. Guran and M. F. Bota, "Existence of the solutions of nonlinear fractional differential equations using the fixed point technique in extended b-metric spaces," *Symmetry*, vol. 13, no. 2, p. 158, 2021.

[14] H. Işık and C. Park, "Existence of a common solution to systems of integral equations via fixed point results," *Open Mathematics*, vol. 18, no. 1, pp. 249–261, 2020.

[15] H. Işık, V. Parvaneh, and M. R. Haddadi, "Strong and pure fixed point properties of mappings on normed spaces," *Mathematical Sciences*, 2021.

[16] S. Khatoon, I. Uddin, and D. Baleanu, "Approximation of fixed point and its application to fractional differential equation," *Journal of Applied Mathematics and Computing*, 2020.

[17] S. Khatoon and I. Uddin, "Convergence analysis of modified Abbas iteration process for two G-nonexpansive mappings," *Rendiconti del Circolo Matematico di Palermo Series 2*, vol. 70, no. 1, pp. 31–44, 2021.

[18] S. Khatoon, I. Uddin, N. Pakkaranang, and N. Wairojjana, "Common fixed pointd of modified Picard-S iteration process involving two G-nonexpansive mappings in CAT (0) space with directed graph," *Thai Journal of Mathematics*, vol. 18, no. 1, pp. 1–13, 2020.

[19] L. Liu, "Shrinking projection method for solving zero point and fixed point problems in Banach spaces," *Journal of Nonlinear and Variational Analysis*, vol. 4, no. 3, pp. 439–454, 2020.

[20] K. Ullah, K. Iqbal, and M. Arshad, "Some convergence results using K iteration process in $CAT(0)$ spaces," *Fixed Point Theory and Applications*, vol. 2018, no. 1, Article ID 11, 2018.

[21] S. Shahzad and R. al-Dubiban, "Approximating common fixed points of nonexpansive mappings in Banach spaces," *Georgian Mathematical Journal*, vol. 13, no. 3, pp. 529–537, 2006.

[22] J. Schu, "Weak and strong convergence to fixed points of asymptotically nonexpansive mappings," *Bulletin of the Australian Mathematical Society*, vol. 43, no. 1, pp. 153–159, 1991.

[23] S. Suantai, "Weak and strong convergence criteria of Noor iterations for asymptotically nonexpansive mappings," *Journal of Mathematical Analysis and Applications*, vol. 311, no. 2, pp. 506–517, 2005.

[24] M. G. Sangago, "Convergence of iterative schemes for nonexpansive mappings," *Asian-European Journal of Mathematics*, vol. 4, no. 4, pp. 671–682, 2011.

[25] N. van Dung and N. Trung Hieu, "Convergence of a new three-step iteration process to common fixed points of three G-nonexpansive mappings in Banach spaces with directed graphs," *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, vol. 114, no. 3, 2020.