

## Research Article

# A Study of the Anisotropic Static Elasticity System in Thin Domain

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We study the asymptotic behavior of solutions of the anisotropic heterogeneous linearized elasticity system in thin domain of  $\mathbb{R}^3$  which has a fixed cross-section in the  $\mathbb{R}^2$  plane with Tresca friction condition. The novelty here is that stress tensor has given by the most general form of Hooke's law for anisotropic materials. We prove the convergence theorems for the transition 3D-2D when one dimension of the domain tends to zero. The necessary mathematical framework and (2D) equation model with a specific weak form of the Reynolds equation are determined. Finally, the properties of solution of the limit problem are given, in which it is confirmed that the limit problem is well defined.

## 1. Introduction

In this paper, we are interested of the asymptotic behavior of the linear elasticity system in a domain of  $\mathbb{R}^3$  with a Tresca friction condition where the boundary of this domain has a fixed cross-section in dimension 2 and a small thickness. One of the objectives of this study is to obtain two-dimensional equation that allows a reasonable description of the phenomenon occurring in the three-dimensional domain by passing the limit to 0 on the small thickness of the domain (3D). Let us mention for example [1–8] in which the authors worked on the asymptotic behavior for the linearized elasticity system with different boundary conditions. Some problems of Newtonian or non-Newtonian fluids are considered in [9–11] where the authors proved a limit problem that gives a distribution of velocity and pressure through the weak form of the Reynolds equation. In [6, 7], the authors demonstrate the transition 3D-1D in anisotropic heterogeneous linearized elasticity; so, we mention here that this phenomenon has been studied only about strong solutions, without friction law. Benseridi in [2] investigated the asymptotic analysis of a dynamical problem of linear elasticity with

Tresca's friction. The static case with a nonlinear term for linear elastic materials has been considered in [3]. See another situation in [4] where the paper concerns asymptotic derivation of frictionless contact models for elastic rods on a foundation with normal compliance. Recently, the authors in [5, 12] have proved the asymptotic behavior of a frictionless contact problem between two elastic bodies, when the vertical dimension of the two domain reaches zero. However, all these papers have been only restricted in a homogeneous and isotropic case of elastic materials.

The present work is a follow-up of [2, 3, 5] to study the heterogeneous and anisotropic situation with Tresca's friction. Here, the stress tensor with its components is given by the generalized Hooke's law (see [13]):  $\sigma^\varepsilon = A^\varepsilon e(u^\varepsilon)$ , where  $u^\varepsilon$  denotes the displacement vector,  $e(u^\varepsilon)$  is the linearized strain tensor, and  $A^\varepsilon$  is the fourth order tensor which describes the elastic properties of the material. Many materials that follow the linear elastic model, although they are well made, are not subject to the assumptions of isotropy, for example, wood, reinforced concrete, composite materials, and many biological materials, where the mechanical properties of these materials differ according to the directions of space; in that case, the elasticity

operator depends on the location of the point (see [14, 15]). Necas in [7] and Sofonea in [16] established the existence of a weak solution for the static frictional contact problem involving linearly elastic and viscoelastic materials, by using a results of convex optimization [17], and numerical approximation of this problem was studied in [18]. For the variational analysis of various contact problems, we mention excellent references in [14, 15]. Mathematically, the asymptotic analysis is more difficult since in general, the limit problem involves an equation that takes into account the anisotropy of the medium, and it is thus important to identify the elastic components of  $A^\varepsilon$  that appear in the (2D) equation model.

The paper is organized as follows; in section 2, the strong and weak formulation of the problem is given in terms of  $u^\varepsilon$  and also the related existence and uniqueness of the weak solution. In section 3, we introduce a scaling, and we find some estimates on the displacement which are independent of the parameter  $\varepsilon$ . In section 4, we state the main results concerning the existence of a weak limit  $u^*$  of  $u^\varepsilon$ , the (2D) equation model with a specific weak form of the Reynolds equation is proved, the limit form of the Tresca boundary conditions is formulated, and finally, the uniqueness of  $u^*$  is given.

## 2. Mathematical Formulation

Let  $\omega$  be an open set in  $\mathbb{R}$  with Lipschitz boundary, and we consider a smooth function  $h : \omega \rightarrow \mathbb{R}$  be a class  $C^1$  such that  $0 < h_{\min} \leq h(x) \leq h_{\max}$ , for all  $x \in \omega$ , where  $h_{\min}$  and  $h_{\max}$  are constants. We define the smooth bounded domain  $\Omega$  whose boundary has a flat part  $\omega$ ,

$$\Omega = \{(x, z) \in \mathbb{R}^3, x \in \omega, 0 < z < h(x)\}. \quad (1)$$

We denote by  $\Gamma_1$  is the upper boundary of the equation  $z = h(x)$ , and  $\Gamma_L$  is the lateral boundary.

Let  $\varepsilon > 0$  be a small parameter, and we define  $\Omega^\varepsilon$  be the change of scale  $z = x_3/\varepsilon$  and the points of  $\Omega$ ,

$$\Omega^\varepsilon = \{(x, x_3) \in \mathbb{R}^3, x \in \omega, 0 < x_3 < \varepsilon h(x)\}. \quad (2)$$

We have  $\Gamma^\varepsilon = \bar{\omega} \cup \bar{\Gamma}_1^\varepsilon \cup \bar{\Gamma}_L^\varepsilon$  which its boundary of  $\Omega^\varepsilon$  and where  $\Gamma_1^\varepsilon$  is the upper surface defined by  $x_3 = \varepsilon h(x)$ , and  $\Gamma_L^\varepsilon$  is the lateral boundary. The unit outward normal to  $\Gamma^\varepsilon$  is denoted by  $\nu$ . It follows that there is correspondence between the functions  $\phi : \Omega^\varepsilon \rightarrow \mathbb{R}^n$  and  $\hat{\phi} : \Omega \rightarrow \mathbb{R}^n$  ( $n = 1, 2, 3$ ) given by  $\hat{\phi}(x, z) = \phi(x, x_3)$ .

Let  $H^{1/2}(\Gamma)^\varepsilon$  be the space of traces of functions on  $\Gamma$  of functions from  $H^1(\Omega^\varepsilon)^\varepsilon$ , and we use the vector function  $g \in H^{1/2}(\Gamma)^\varepsilon$  such that

$$\int_{\Gamma} g \cdot \nu ds = 0. \quad (3)$$

We denote by  $\mathbb{S}_n$  the space of symmetric tensors on  $\mathbb{R}^n$  and  $|\cdot|$  the Euclidean norm on  $\mathbb{R}^n$  and  $\mathbb{S}_n$ . Here and below, the indices  $i, j, k, l$  run between 1 and 3, and the summation convention overrepeated indices is adopted.

The basic equations of frictionless contact problem for the anisotropic heterogeneous elastic body occupy the domain  $\Omega^\varepsilon$  as follows:

The equations of equilibrium are as follows:

$$\frac{\partial \sigma_{ij}^\varepsilon}{\partial x_j} + f_i^\varepsilon = 0 \text{ in } \Omega^\varepsilon, \quad (4)$$

$$\sigma_{ij}^\varepsilon(u^\varepsilon) = A_{ijkl}^\varepsilon e_{kl}(u^\varepsilon) \text{ in } \Omega^\varepsilon, \quad (5)$$

where the vector  $f^\varepsilon = (f_1^\varepsilon, f_2^\varepsilon, f_3^\varepsilon)$  represents the forces of density,  $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon)$  is the displacement field, the elements  $A_{ijkl}^\varepsilon$  denote the components of elasticity tensor  $A^\varepsilon$ , and  $e_{ij}(u^\varepsilon)$  is the rate of deformation operator,

$$e_{ij}(u^\varepsilon) = (e(u^\varepsilon))_{ij} = \frac{1}{2} \left( \frac{\partial u_i^\varepsilon}{\partial x_j} + \frac{\partial u_j^\varepsilon}{\partial x_i} \right). \quad (6)$$

On  $\Gamma_L^\varepsilon$ , the displacement is known:

$$u^\varepsilon = g^\varepsilon \text{ on } \Gamma_L^\varepsilon. \quad (7)$$

On  $\Gamma_1^\varepsilon$ , we assume that the elastic body is held fixed:

$$u^\varepsilon = g^\varepsilon = 0 \text{ on } \Gamma_1^\varepsilon. \quad (8)$$

On the surface  $\omega$ , we assume that the contact is bilateral:

$$u^\varepsilon \cdot \nu = g^\varepsilon \cdot \nu = 0 \quad (9)$$

and satisfies the Tresca boundary condition [7] with friction function  $k^\varepsilon$ ;

$$\begin{cases} |\sigma_\tau^\varepsilon| < k^\varepsilon u_\tau^\varepsilon = s, \\ |\sigma_\tau^\varepsilon| = k^\varepsilon \exists \lambda \geq 0 \text{ such that } u_\tau^\varepsilon = s - \lambda \sigma_\tau^\varepsilon, \end{cases} \quad (10)$$

where  $s = g^\varepsilon$  on  $\omega$ .  $u_\tau^\varepsilon$ ,  $\sigma_\tau^\varepsilon$ , and  $\sigma_\nu^\varepsilon$  are the tangential displacement, the tangential, and the normal stress tensor, respectively, with

$$u_{\tau_i}^\varepsilon = u_i^\varepsilon - u_j^\varepsilon \nu_j \nu_i, \quad \sigma_{\tau_i}^\varepsilon = \sigma_{ij}^\varepsilon \nu_j - (\sigma_\nu^\varepsilon) \cdot \nu_i, \quad \sigma_\nu^\varepsilon = (\sigma^\varepsilon \cdot \nu) \cdot \nu. \quad (11)$$

Consider now the following closed convex subset of  $H^1(\Omega^\varepsilon)^\varepsilon$  given by

$$K^\varepsilon = \left\{ \phi \in H^1(\Omega^\varepsilon)^\varepsilon : \phi = g^\varepsilon \text{ on } \Gamma_1^\varepsilon \cup \Gamma_L^\varepsilon, \phi \cdot \nu = 0 \text{ on } \omega \right\}. \quad (12)$$

Let us introduce the form  $a : K^\varepsilon \times K^\varepsilon \rightarrow \mathbb{R}$  and the functional  $J^\varepsilon : K^\varepsilon \rightarrow \mathbb{R}^+$  defined by

$$\begin{aligned} a(u^\varepsilon, \phi) &= \int_{\Omega^\varepsilon} A_{ijkl}^\varepsilon e_{kl}(u^\varepsilon) e_{ij}(\phi) dx dx_3, \\ J^\varepsilon(\phi) &= \int_{\omega} k^\varepsilon |\phi - s| dx. \end{aligned} \quad (13)$$

In the study of the mechanical problem (3)–(10), we assume that all components  $A_{ijkl}^\varepsilon$  belong to  $L^\infty(\Omega^\varepsilon)$  and satisfy the usual properties of symmetry and ellipticity [19], i.e.,

$$A_{ijkl}^\varepsilon = A_{jikl}^\varepsilon = A_{klij}^\varepsilon \in L^\infty(\Omega^\varepsilon), \quad (14)$$

and there exists a constant  $\mu > 0$  such that

$$A_{ijkl}^\varepsilon(y)\xi_k\xi_l \geq \mu|\xi|^2 \forall \xi \in \mathbb{S}_3, \text{ a.e. } y \in \Omega^\varepsilon. \quad (15)$$

*Remark 1.* It follows from previous properties and by Korn's inequality (see [16], pp. 79), that the bilinear form  $a$  is coercive and continuous, i.e.,

$$a(\phi, \phi) \geq \mu C_K \|\nabla \phi\|_{L^2(\Omega^\varepsilon)}^2 \forall \phi \in K^\varepsilon, \quad (16)$$

$$|a(\phi, \psi)| \leq M \|\nabla \phi\|_{L^2(\Omega^\varepsilon)} \|\nabla \psi\|_{L^2(\Omega^\varepsilon)} \forall \phi, \psi \in K^\varepsilon, \quad (17)$$

where  $M = \max_{1 \leq i, j, k, l \leq 3} \|A_{ijkl}^\varepsilon\|_{L^\infty(\Omega^\varepsilon)}$  and  $C_K$  denoting a positive constant depends on  $\Omega^\varepsilon, \Gamma_1^\varepsilon$ , and  $\Gamma_L^\varepsilon$ .

**Lemma 2.** Assuming that  $f^\varepsilon \in L^2(\Omega^\varepsilon)^3$  and  $k^\varepsilon \in L^\infty(\omega)$ , the variational formulation of problem (3)–(10) is equivalent to Find  $u^\varepsilon \in K^\varepsilon$  satisfying

$$a(u^\varepsilon, \phi - u^\varepsilon) + J^\varepsilon(\phi) - J^\varepsilon(u^\varepsilon) \geq \int_{\Omega^\varepsilon} f^\varepsilon \cdot (\phi - u^\varepsilon) dx dx_3 \forall \phi \in K^\varepsilon, \quad (18)$$

for every  $\varepsilon$  small fixed.

Moreover, if the assumptions of (14) and (15) hold, then the variational inequality (18) has a unique solution  $u^\varepsilon \in K^\varepsilon$ .

*Remark 3.* A problem of the form (18) is called an elliptic variational inequality of the second kind ([17]). The following theorem (see [19], Theorem 6) allows us to replace the variational inequality (18) by a minimization problem. Thus, we will not repeat the proof, but our goal is to study the asymptotic behavior.

### 3. Some Estimates in Fixed Domain

To be able to study the asymptotic behavior of the solutions of (18), we use the change of variable  $z = x_3/\varepsilon$ , to return to the fixed domain  $\Omega$ , and then we define the following functions in  $\Omega$ :

$$\widehat{u}_i^\varepsilon(x, z) = u_i^\varepsilon(x, x_3) \quad \text{for } i = 1, 2, 3. \quad (19)$$

For the data  $\widehat{A}_{ijkl}, \widehat{f}_i$ , and  $\widehat{k}$ , we have the following relations:

$$\widehat{A}_{ijkl}(x, z) = A_{ijkl}^\varepsilon(x, x_3), \quad \widehat{f}_i(x, z) = \varepsilon^2 f_i^\varepsilon(x, x_3) \text{ and } \widehat{k} = \varepsilon k^\varepsilon, \quad (20)$$

(for  $1 \leq i, j, k, l \leq 3$ ).

Let

$$K = \{v \in H^1(\Omega)^3 : v = \text{gon}\Gamma_L \cup \Gamma_1, v \cdot \nu = 0 \text{ on } \omega\},$$

$$V_z = \left\{ v = (v_1, v_2, v_3) \in L^2(\Omega)^3 : \frac{\partial v_i}{\partial z} \in L^2(\Omega), i = 1, 2, 3 ; v = 0 \text{ on } \Gamma_1 \right\}. \quad (21)$$

$V_z$  is a Banach space for the following norm:

$$\|v\|_{V_z} = \left[ \sum_{i=1}^3 \left( \|v_i\|_{L^2(\Omega)}^2 + \left\| \frac{\partial v_i}{\partial z} \right\|_{L^2(\Omega)}^2 \right) \right]^{1/2}. \quad (22)$$

Everywhere in the sequel, the indexes  $\alpha, \beta, \gamma$  and  $\delta$  run from 1 to 2, and summation over repeated indices is implied. Follow the same steps as in [6, 12], passing to the fixed domain  $\Omega$ , and using the symmetry of  $\sigma_{ij}^\varepsilon$  and  $A_{ijkl}^\varepsilon$ , after multiplication by  $\varepsilon$ , we have (18) that is equivalent to

Find  $\widehat{u}^\varepsilon \in K$ , such that

$$\begin{aligned} \widehat{a}(\widehat{u}^\varepsilon, \widehat{\phi} - \widehat{u}^\varepsilon) + \int_\omega \widehat{k} |\widehat{\phi} - s| dx - \int_\omega \widehat{k} \widehat{u}^\varepsilon - s dx \\ \geq \sum_{i=1}^3 \int_\Omega \widehat{f}_i (\widehat{\phi}_i - \widehat{u}_i^\varepsilon) dx dz, \forall \widehat{\phi} \in K, \end{aligned} \quad (23)$$

where

$$\begin{aligned} \widehat{a}(\widehat{u}^\varepsilon, \widehat{\phi}) = \varepsilon^2 \int_\Omega \widehat{A}_{\alpha\beta\gamma\theta} \widehat{e}_{\gamma\theta}(\widehat{u}^\varepsilon) \frac{\partial \widehat{\phi}_\alpha}{\partial x_\beta} dx dz \\ + 2\varepsilon \int_\Omega \widehat{A}_{\alpha\beta\gamma\theta} \widehat{e}_{\gamma\theta}(\widehat{u}^\varepsilon) \frac{\partial \widehat{\phi}_\alpha}{\partial z} dx dz \\ + 2\varepsilon^2 \int_\Omega \widehat{A}_{\alpha\beta\gamma 3} \widehat{e}_{\gamma 3}(\widehat{u}^\varepsilon) \frac{\partial \widehat{\phi}_\alpha}{\partial x_\beta} dx dz \\ + 4\varepsilon \int_\Omega \widehat{A}_{\alpha\beta\gamma 3} \widehat{e}_{\gamma 3}(\widehat{u}^\varepsilon) \frac{\partial \widehat{\phi}_\alpha}{\partial z} dx dz \\ + \varepsilon^2 \int_\Omega \widehat{A}_{\alpha\beta 33} \widehat{e}_{33}(\widehat{u}^\varepsilon) \frac{\partial \widehat{\phi}_\alpha}{\partial x_\beta} dx dz \\ + \varepsilon \int_\Omega \widehat{A}_{33\alpha\beta} \widehat{e}_{\alpha\beta}(\widehat{u}^\varepsilon) \frac{\partial \widehat{\phi}_3}{\partial z} dx dz \\ + 2\varepsilon \int_\Omega \widehat{A}_{\alpha 333} \widehat{e}_{33}(\widehat{u}^\varepsilon) \frac{\partial \widehat{\phi}_\alpha}{\partial z} dx dz \\ + 2\varepsilon \int_\Omega \widehat{A}_{33\alpha 3} \widehat{e}_{\alpha 3}(\widehat{u}^\varepsilon) \frac{\partial \widehat{\phi}_3}{\partial z} dx dz \\ + \varepsilon \int_\Omega \widehat{A}_{3333} \widehat{e}_{33}(\widehat{u}^\varepsilon) \frac{\partial \widehat{\phi}_3}{\partial z} dx dz, \end{aligned} \quad (24)$$

and  $\widehat{e}(\widehat{u}^\varepsilon) = (\widehat{e}_{ij}(\widehat{u}^\varepsilon))_{ij}$  is given by the relations

$$\begin{aligned}\widehat{e}_{ij}(\widehat{u}^\varepsilon) &= \frac{1}{2} \left( \frac{\partial \widehat{u}_i^\varepsilon}{\partial x_j} + \frac{\partial \widehat{u}_j^\varepsilon}{\partial x_i} \right), & i, j = 1, 2, \\ \widehat{e}_{i3}(\widehat{u}^\varepsilon) &= \widehat{e}_{3i}(\widehat{u}^\varepsilon) = \frac{1}{2} \left( \frac{1}{\varepsilon} \frac{\partial \widehat{u}_i^\varepsilon}{\partial z} + \frac{\partial \widehat{u}_3^\varepsilon}{\partial x_i} \right), & i = 1, 2, \\ \widehat{e}_{33}(\widehat{u}^\varepsilon) &= \frac{1}{\varepsilon} \frac{\partial \widehat{u}_3^\varepsilon}{\partial z},\end{aligned}\quad (25)$$

**Lemma 4.** *Under the assumptions of Lemma 2, there exists a constant  $C > 0$  independent of  $\varepsilon$ , such that*

$$\sum_{i=1}^3 \left\| \frac{\partial \widehat{u}_i^\varepsilon}{\partial z} \right\|_{L^2(\Omega)}^2 + \varepsilon^2 \sum_{i,j=1}^2 \left\| \frac{\partial \widehat{u}_i^\varepsilon}{\partial x_j} \right\|_{L^2(\Omega)}^2 + \varepsilon^2 \sum_{j=1}^2 \left\| \frac{\partial \widehat{u}_3^\varepsilon}{\partial x_j} \right\|_{L^2(\Omega)}^2 \leq C. \quad (26)$$

*Proof.* Assume that  $u^\varepsilon$  is a solution of (2.12). As  $J^\varepsilon(u^\varepsilon) \geq 0$ , then

$$\begin{aligned}a(u^\varepsilon, u^\varepsilon) &\leq a(u^\varepsilon, \phi) + J^\varepsilon(\phi) + \int_{\Omega^\varepsilon} f^\varepsilon \cdot u^\varepsilon dx dx_3 \\ &\quad - \int_{\Omega^\varepsilon} f^\varepsilon \cdot \phi dx dx_3, \quad \forall \phi \in K^\varepsilon.\end{aligned}\quad (27)$$

Using the Young's inequality

$$ab \leq \eta^2 \frac{a^2}{2} + \eta^{-2} \frac{b^2}{2} \quad (28)$$

in (17) for  $\eta = \sqrt{\mu C_K/2}$ , we find

$$|a(u^\varepsilon, \phi)| \leq \frac{\mu C_K}{4} \|\nabla u^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2 + \frac{M}{\mu C_K} \|\nabla \phi\|_{L^2(\Omega^\varepsilon)}^2. \quad (29)$$

Also, by the Cauchy-Schwarz and Poincaré's inequalities, we get

$$\left| \int_{\Omega^\varepsilon} f^\varepsilon \cdot \phi dx dx_3 \right| \leq \varepsilon h_{\max} \|f^\varepsilon\|_{L^2(\Omega^\varepsilon)} \|\nabla \phi\|_{L^2(\Omega^\varepsilon)}, \quad (30)$$

then using Young's inequality for  $\eta = \sqrt{1/\mu C_K}$  to obtain

$$\left| \int_{\Omega^\varepsilon} f^\varepsilon \cdot u^\varepsilon dx dx_3 \right| \leq \frac{(\varepsilon h_{\max})^2}{2\mu C_K} \|f^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2 + \frac{\mu C_K}{2} \|\nabla \phi\|_{L^2(\Omega^\varepsilon)}^2. \quad (31)$$

Using (16), (29), and (31) in (27), we get

$$\begin{aligned}\frac{\mu C_K}{4} \|\nabla u^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2 &\leq \frac{M}{\mu C_K} \|\nabla \phi\|_{L^2(\Omega^\varepsilon)}^2 + \frac{(\varepsilon h_{\max})^2}{\mu C_K} \|f^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2 \\ &\quad + \frac{\mu C_K}{2} \|\nabla \phi\|_{L^2(\Omega^\varepsilon)}^2 + \int_\omega k^\varepsilon |\phi - s| dx.\end{aligned}\quad (32)$$

Taking into account the  $g$  function introduced in (5) and using [20] (lemma 2 pp.24), there exists a function  $\tilde{g} \in H^1(\Omega)^3$  such that

$$\tilde{g} = g \text{ on } \Gamma_L \text{ and } \tilde{g} \cdot \nu = 0 \text{ on } \omega \cup \Gamma_1. \quad (33)$$

Thus, choosing  $\widehat{\phi} = \tilde{g}$  in (3.6), then multiplying product inequality by  $\varepsilon$ , and the fact that  $g = s$  on  $\omega$ , we obtain

$$\begin{aligned}\frac{\mu C_K}{4} \varepsilon \|\nabla u^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2 &\leq \frac{(h_{\max})^2}{\mu C_K} \|\widehat{f}\|_{L^2(\Omega)}^2 \\ &\quad + \left( \frac{M}{\mu C_K} + \frac{\mu C_K}{2} \right) \|\nabla \tilde{g}\|_{L^2(\Omega)}^2.\end{aligned}\quad (34)$$

From [6], we can see the constant Korn  $C_K$  contained in Remark 1 does not depend on  $\varepsilon$  and  $\phi$ , for  $\varepsilon \in ]0, 1]$ ; moreover, by changing the data of  $A^\varepsilon$ , remark that  $\mu$  and  $M$  are independent of  $\varepsilon$ . Therefore, passing to the fixed domain  $\Omega$ , we get

$$\begin{aligned}\varepsilon \|\nabla u^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2 &= \sum_{i=1}^3 \left\| \frac{\partial \widehat{u}_i^\varepsilon}{\partial z} \right\|_{L^2(\Omega)}^2 + \varepsilon^2 \left( \sum_{i,j=1}^2 \left\| \frac{\partial \widehat{u}_i^\varepsilon}{\partial x_j} \right\|_{L^2(\Omega)}^2 \right. \\ &\quad \left. + \sum_{j=1}^2 \left\| \frac{\partial \widehat{u}_3^\varepsilon}{\partial x_j} \right\|_{L^2(\Omega)}^2 \right) \leq C,\end{aligned}\quad (35)$$

with

$$C = \frac{4}{\mu C_K} \left[ \frac{(h_{\max})^2}{\mu C_K} \|\widehat{f}\|_{L^2(\Omega)}^2 + \left( \frac{M}{\mu C_K} + \frac{\mu C_K}{2} \right) \|\nabla \tilde{g}\|_{L^2(\Omega)}^2 \right]. \quad (36)$$

□

**Lemma 5.** *Under the assumptions of Lemma 4, there exists  $u^* = (u_1^*, u_2^*, u_3^*) \in V_z$  such that*

$$\widehat{u}^\varepsilon \rightharpoonup u^* \text{ weakly in } V_z, \quad (37)$$

$$\varepsilon \frac{\partial \widehat{u}_i^\varepsilon}{\partial x_\alpha} \rightharpoonup 0 \text{ weakly in } L^2(\Omega) \quad (i = 1, 2, 3 \text{ and } \alpha = 1, 2), \quad (38)$$

$$\varepsilon \widehat{e}_{\alpha\beta}(\widehat{u}^\varepsilon) \rightharpoonup 0, \text{ weakly in } L^2(\Omega) \quad (\alpha, \beta = 1, 2), \quad (39)$$

$$\varepsilon \widehat{e}_{\gamma 3}(\widehat{u}^\varepsilon) \rightharpoonup \frac{1}{2} \frac{\partial u_\gamma^*}{\partial z} \text{ weakly in } L^2(\Omega) \quad (\gamma = 1, 2), \quad (40)$$

$$\varepsilon \widehat{e}_{33}(\widehat{u}^\varepsilon) \rightharpoonup \frac{\partial u_3^*}{\partial z} \text{ weakly in } L^2(\Omega). \quad (41)$$

*Proof.* From (26), there exists a fixed constant  $C > 0$  such that

$$\left\| \frac{\partial \widehat{u}_i^\varepsilon}{\partial z} \right\|_{L^2(\Omega)}^2 \leq C, \quad \text{for } i = 1, 2, 3. \quad (42)$$

Using Poincaré's inequality in the domain  $\Omega$

$$\|\widehat{u}_i^\varepsilon\|_{L^2(\Omega)} \leq h_{\max} \left\| \frac{\partial \widehat{u}_i^\varepsilon}{\partial z} \right\|_{L^2(\Omega)}, \quad \text{for } i = 1, 2, 3, \quad (43)$$

we deduce that  $\widehat{u}^\varepsilon$  is bounded in  $V_z$ . From the last two estimates, there exists  $u^* = (u_1^*, u_2^*, u_3^*) \in V_z$  and satisfies (37). From (26), we can extract a subsequence such that  $\varepsilon(\partial \widehat{u}_i^\varepsilon / \partial x_\alpha) \rightharpoonup \eta$  in  $L^2(\Omega)$ ; on the other hand, from (37), we deduce (38). Also, (39)–(41) follow from (37) and (38).  $\square$

#### 4. Limit Problem and Main Result

At the limit  $\varepsilon = 0$ , we give the satisfactory equations of  $u^*$  and the properties of solution of the limit problem for the system (3)-(10).

**Theorem 6.** *With the same assumptions as Lemma 5,  $u^*$  satisfies*

$$\begin{aligned} & \left\langle A^* \frac{\partial u^*}{\partial z}, \frac{\partial}{\partial z} (\widehat{\phi} - u^*) \right\rangle_{L^2(\Omega)} + \int_\omega \widehat{k} (|\widehat{\phi} - s| - |u^* - s|) dx \\ & \geq \int_\Omega \widehat{f} \cdot (\widehat{\phi} - u^*) dx dz, \quad \forall \widehat{\phi} \in K, \end{aligned} \quad (44)$$

where the symmetric matrix  $A^*$  is given by

$$A^* = \begin{pmatrix} 4\widehat{A}_{1313} & 4\widehat{A}_{1323} & 2\widehat{A}_{1333} \\ 4\widehat{A}_{2313} & 4\widehat{A}_{2323} & 2\widehat{A}_{2333} \\ 2\widehat{A}_{3313} & 2\widehat{A}_{3323} & \widehat{A}_{3333} \end{pmatrix}. \quad (45)$$

Moreover, we have

$$\left. \begin{aligned} -\frac{\partial}{\partial z} \left\{ 4\widehat{A}_{\alpha 3\gamma 3} \frac{\partial u_\gamma^*}{\partial z} + 2\widehat{A}_{\alpha 333} \frac{\partial u_3^*}{\partial z} \right\} &= \widehat{f}_\alpha, \quad (\alpha = 1, 2) \\ -\frac{\partial}{\partial z} \left\{ 2\widehat{A}_{33\alpha 3} \frac{\partial u_\alpha^*}{\partial z} + \widehat{A}_{3333} \frac{\partial u_3^*}{\partial z} \right\} &= \widehat{f}_3 \end{aligned} \right\} \text{in } L^2(\Omega). \quad (46)$$

*Proof.* As (23) can be written,

$$\begin{aligned} & \widehat{a}(\widehat{u}^\varepsilon, \widehat{\phi}) + \int_\omega \widehat{k} |\widehat{\phi} - s| dx - \sum_{i=1}^3 \int_\Omega \widehat{f}_i (\widehat{\phi}_i - \widehat{u}_i^\varepsilon) dx dz \\ & \geq \widehat{a}(\widehat{u}^\varepsilon, \widehat{u}^\varepsilon) + \int_\omega \widehat{k} |\widehat{u}^\varepsilon - s| dx. \end{aligned} \quad (47)$$

Since the form  $\widehat{a}(\cdot, \cdot)$  is a symmetry and  $K$ -elliptic, and the fact that  $\widehat{\phi} \rightarrow \int_\omega \widehat{k} |\widehat{\phi} - s| dx$  is convex and lower semi-continuous, we deduce

$$\begin{aligned} & \liminf_{\varepsilon \rightarrow 0} \left[ \widehat{a}(\widehat{u}^\varepsilon, \widehat{u}^\varepsilon) + \int_\omega \widehat{k} |\widehat{u}^\varepsilon - s| dx \right] \\ & \geq \widehat{a}(u^*, u^*) + \int_\omega \widehat{k} |u^* - s| dx. \end{aligned} \quad (48)$$

Using Lemma 5, we let  $\varepsilon$  tend to 0 in (47), to obtain

$$\begin{aligned} & 4 \int_\Omega \widehat{A}_{\alpha 3\gamma 3} \frac{\partial u_\gamma^*}{\partial z} \frac{\partial}{\partial z} (\widehat{\phi}_\alpha - u_\alpha^*) dx dz \\ & + 2 \int_\Omega \widehat{A}_{\alpha 333} \frac{\partial u_3^*}{\partial z} \frac{\partial}{\partial z} (\widehat{\phi}_\alpha - u_\alpha^*) dx dz \\ & + 2 \int_\Omega \widehat{A}_{33\alpha 3} \frac{\partial u_\alpha^*}{\partial z} \frac{\partial}{\partial z} (\widehat{\phi}_3 - u_3^*) dx dz \\ & + \int_\Omega \widehat{A}_{3333} \frac{\partial u_3^*}{\partial z} \frac{\partial}{\partial z} (\widehat{\phi}_3 - u_3^*) dx dz \\ & + \int_\omega \widehat{k} (|\widehat{\phi} - s| - |u^* - s|) dx \\ & \geq \sum_{i=1}^3 \int_\Omega \widehat{f}_i (\widehat{\phi}_i - u_i^*) dx dz. \end{aligned} \quad (49)$$

This completes the proof of (44) if we cross (49) in the matrix form  $A^*$ . We choose in the variational inequality (49)  $\widehat{\phi}_i = u_i^* \pm \psi_i$ , where  $\psi_i \in H_0^1(\Omega)$  (for  $i = 1, 2, 3$ ), and using Green's formula, we find

$$\begin{aligned} & - \int_\Omega \frac{\partial}{\partial z} \left\{ 4\widehat{A}_{\alpha 3\gamma 3} \frac{\partial u_\gamma^*}{\partial z} + 2\widehat{A}_{\alpha 333} \frac{\partial u_3^*}{\partial z} \right\} \psi_\alpha dx dz \\ & - \int_\Omega \frac{\partial}{\partial z} \left\{ 2\widehat{A}_{33\alpha 3} \frac{\partial u_\alpha^*}{\partial z} + \widehat{A}_{3333} \frac{\partial u_3^*}{\partial z} \right\} \psi_3 dx dz \\ & = \sum_{i=1}^3 \int_\Omega \widehat{f}_i \psi_i dx dz, \end{aligned} \quad (50)$$

choosing  $\psi_3 = 0$  and  $\psi_\alpha \in H_0^1(\Omega)$ ; then,  $\psi_\alpha = 0$  and  $\psi_3 \in H_0^1(\Omega)$ , we get (59).  $\square$

**Theorem 7.** *Under the assumptions of Theorem 6 then, the solution of the limit problem (44)–(46) is unique in  $V_z$ .*

*Proof.* Suppose that there exists two solutions  $u^*$  and  $v^*$  of the variational inequality (44), and taking  $\widehat{\phi} = v^*$  in (44), then  $\widehat{\phi} = u^*$  in the inequality relating to  $v^*$ . By subtracting the two obtained inequalities, we have

$$\left\langle A^* \cdot \frac{\partial}{\partial z} (v^* - u^*), \frac{\partial}{\partial z} (v^* - u^*) \right\rangle_{L^2(\Omega)} \leq 0. \quad (51)$$

We must now check that  $A^*$  is ellipticity. So, we return to the properties of  $\widehat{A}$  mentioned in (14) and (15); in particular, we choose symmetric tensors  $\zeta$  that are given by  $\xi_{\alpha\beta} = 0$  (for  $\alpha, \beta = 1, 2$ ); otherwise, the rest of the components ( $\xi_{i3}$ ) let it be whatever. Putting  $\eta_i = \xi_{i3}$ , for  $i = 1, 2, 3$ , we will get

$$\begin{aligned} \widehat{A}_{ijkl}\xi_{kl}\xi_{ij} &= 4\widehat{A}_{\alpha 3\beta 3}\eta_{\beta}\eta_{\alpha} + 2\widehat{A}_{\alpha 333}\eta_3\eta_{\alpha} \\ &+ 2\widehat{A}_{33\alpha 3}\eta_{\alpha}\eta_3 + \widehat{A}_{3333}\eta_3\eta_3 = A_{ij}^*\eta_j\eta_i. \end{aligned} \quad (52)$$

Consequently, and as  $|\xi|^2 \geq |\eta|^2$ , there exists a positive constant  $\mu$ , and for all vectors  $\eta$  in  $\mathbb{R}^3$ , we have

$$A_{ij}^*\eta_j\eta_i \geq \mu|\eta|^2. \quad (53)$$

So,  $A^*$  is ellipticity. Thus, the relation (44) implies that

$$\mu \left\| \frac{\partial}{\partial z} (v^* - u^*) \right\|_{L^2(\Omega)}^2 \leq 0. \quad (54)$$

Using Poincaré's inequality, we obtain

$$\|v^* - u^*\|_{L^2(\Omega)}^2 \leq (h_{\max})^2 \left\| \frac{\partial}{\partial z} (v^* - u^*) \right\|_{L^2(\Omega)}^2 = 0, \quad (55)$$

and the proof of uniqueness of  $u^*$  is complete.  $\square$

**Theorem 8.** Under the assumptions of Theorem 7, the traces  $(s^*, \pi^*)$  with  $s^* = (s_i^*)_{1 \leq i \leq 3}$  and  $\pi^* = (\pi_i^*)_{1 \leq i \leq 3}$  defined by

$$\begin{aligned} s_i^*(x) &= u_i^*(x, 0), \\ \pi_{\alpha}^*(x) &= \left[ 4\widehat{A}_{\alpha 3\gamma 3} \frac{\partial u_{\gamma}^*}{\partial z} + 2\widehat{A}_{\alpha 333} \frac{\partial u_3^*}{\partial z} \right] (x, 0); \pi_3^*(x) \\ &= \left[ 2\widehat{A}_{33\gamma 3} \frac{\partial u_{\gamma}^*}{\partial z} + \widehat{A}_{3333} \frac{\partial u_3^*}{\partial z} \right] (x, 0), \end{aligned} \quad (56)$$

satisfy the following limit form of the Tresca boundary conditions:

$$\int_{\omega} \widehat{k}|\psi + s^* - s| - |s^* - s| dx - \int_{\omega} \pi^* \cdot \psi dx \geq 0 \quad \forall \psi \in L^2(\omega)^3, \quad (57)$$

$$\left. \begin{aligned} |\pi^*| < \widehat{k} &\Rightarrow s^* = s \\ |\pi^*| = \widehat{k} &\Rightarrow \exists \lambda > 0 \text{ such that } s^* = s + \lambda \pi^* \end{aligned} \right\} \text{a.e.in}\omega. \quad (58)$$

Moreover, if the coefficients  $\widehat{A}_{i3j3}$  for  $1 \leq i, j \leq 3$ , depending only on the variable  $x$ , we have the following weak form of the Reynolds equation:

$$\int_{\omega} \left( \widetilde{F} - \frac{h}{2} s^* + \int_0^h u^*(x, z) dz \right) \cdot \nabla \psi(x) dx = 0, \quad \forall \psi \in H^1(\omega), \quad (59)$$

where  $\text{Inv}A^*(x)$  denotes the inverse of  $A^*(x)$  and

$$\widetilde{F}(x) = \int_0^h \widetilde{F}(x, \rho) d\rho - \frac{h}{2} \widetilde{F}(x, h), \quad (60)$$

$$\widetilde{F}(x, \rho) = \text{Inv}A^*(x) \cdot \int_0^{\rho} \int_0^{\theta} \widehat{f}(x, y) dy d\theta. \quad (61)$$

*Proof.* We now choose in the variational inequality (49)  $\widehat{\phi}_i = u_i^* + \psi_i$ , where  $\psi_i \in H_{\Gamma_1 \cup \Gamma_L}^1(\Omega)$  for  $i = 1, 2, 3$ , and then using Green's formula, we obtain

$$\begin{aligned} & - \int_{\Omega} \frac{\partial}{\partial z} \left\{ 4\widehat{A}_{\alpha 3\gamma 3} \frac{\partial u_{\gamma}^*}{\partial z} + 2\widehat{A}_{\alpha 333} \frac{\partial u_3^*}{\partial z} \right\} \psi_{\alpha} dx dz \\ & - \int_{\omega} \left( 4\widehat{A}_{\alpha 3\gamma 3} \frac{\partial u_{\gamma}^*}{\partial z} + 2\widehat{A}_{\alpha 333} \frac{\partial u_3^*}{\partial z} \right) \psi_{\alpha}(x, 0) dx \\ & - \int_{\Omega} \frac{\partial}{\partial z} \left\{ 2\widehat{A}_{33\gamma 3} \frac{\partial u_{\gamma}^*}{\partial z} + \widehat{A}_{3333} \frac{\partial u_3^*}{\partial z} \right\} \psi_3 dx dz \\ & - \int_{\omega} \left( 2\widehat{A}_{33\gamma 3} \frac{\partial u_{\gamma}^*}{\partial z} + \widehat{A}_{3333} \frac{\partial u_3^*}{\partial z} \right) \psi_3(x, 0) dx \\ & + \int_{\omega} \widehat{k}(|\psi + s^* - s| - |u^* - s|) dx \\ & \geq \sum_{i=1}^3 \int_{\Omega} \widehat{f}_i \psi_i dx dz. \end{aligned} \quad (62)$$

On the other hand, from (46), we have

$$\int_{\omega} \widehat{k}(|\psi + s^* - s| - |s^* - s|) dx - \int_{\omega} \left( \sum_{\alpha} \pi_{\alpha}^* \psi_{\alpha} + \pi_3^* \psi_3 \right) dx \geq 0. \quad (63)$$

By density theorems, we find (57). For (58), we use the analogue of [10].

To prove (59), we use those similar steps as in [2, 5, 9–11], by integrating (46) from 0 to  $z$ , and taking into account  $\widehat{A}_{i3j3}$  depending only on  $x$ , we obtain

$$\left\{ \begin{aligned} & - \left[ 4\widehat{A}_{\alpha 3\gamma 3}(x) \frac{\partial u_{\gamma}^*}{\partial z}(x, z) + 2\widehat{A}_{\alpha 333}(x) \frac{\partial u_3^*}{\partial z}(x, z) \right] + \pi_{\alpha}^*(x) = \int_0^z \widehat{f}_{\alpha}(x, y) dy, \\ & - \left[ 2\widehat{A}_{33\gamma 3}(x) \frac{\partial u_{\gamma}^*}{\partial z}(x, z) + \widehat{A}_{3333}(x) \frac{\partial u_3^*}{\partial z}(x, z) \right] + \pi_3^*(x) = \int_0^z \widehat{f}_3(x, y) dy, \end{aligned} \right. \quad (64)$$

$$-A^*(x) \cdot \frac{\partial u^*}{\partial z} + \pi^*(x) = \int_0^z \widehat{f}(x, y) dy. \quad (65)$$

It follows from (51) that it is a invertible matrix  $A^*(x)$ , for almost every  $x \in \omega$ . Therefore,

$$-\frac{\partial u^*}{\partial z} + \text{Inv}A^*(x) \cdot \pi^*(x) = \text{Inv}A^*(x) \cdot \int_0^z \tilde{f}(x, y) dy. \quad (66)$$

By integrating between 0 and  $z$ , we obtain

$$-u^*(x, z) + s^*(x) + z \text{Inv}A^*(x) \cdot \pi^*(x) = \tilde{F}(x, z). \quad (67)$$

As  $u_i^*(x, h(x)) = 0$ , we have

$$s^*(x) + h(x) \text{Inv}A^*(x) \cdot \pi^*(x) = \int_0^{h(x)} \tilde{F}(x, y) dy. \quad (68)$$

We integrate (67) from 0 to  $h(x)$ , and we obtain

$$\begin{aligned} -\int_0^{h(x)} u^*(x, z) dz + h s^*(x) + \frac{h(x)^2}{2} \text{Inv}A^*(x) \cdot \pi^*(x) \\ = \int_0^{h(x)} \tilde{F}(x, y) dy, \end{aligned} \quad (69)$$

and by (68), we deduce that

$$-\int_0^{h(x)} u^*(x, z) dz + \frac{h(x)}{2} s^*(x) - \tilde{F}(x) = 0, \quad (70)$$

such that  $\tilde{F}$  is already defined in (61), and let us finally get the weak form (59) after multiplying (70) by  $\nabla \psi(x)$  and integrate it in  $\omega$ .  $\square$

## 5. Conclusions

We were able to find a framework to conclude that solving our original problem leads to solving a well-defined problem as in (44), (46) and (57)–(59) for the “small” parameter  $\varepsilon$ .

The key of the problem lies in the relation between the matrices  $A^\varepsilon$  and  $A^*$ . Note that they have the same properties despite the difference in dimensions, therefore it played a key role in the transition from  $u^\varepsilon$  to  $u^*$ .

Indeed, the special case

$$A_{ijkl}^\varepsilon = \mu^\varepsilon (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) + \lambda^\varepsilon \delta_{ij} \delta_{kl} \quad (71)$$

where  $\lambda^\varepsilon, \mu^\varepsilon > 0$  are the Lamé coefficients (see [13] pp. 102–103) corresponds to the homogeneous and isotropic case of elastic materials, and has been studied in [2, 3, 5]. Thus also, the Stokes flow in [11] can be recovered when  $\lambda^\varepsilon$  tends to 0.

## Data Availability

No data were used.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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