Research Article

Fast High-Order Difference Scheme for the Modified Anomalous Subdiffusion Equation Based on Fast Discrete Sine Transform

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1. Introduction

In this paper, we focus on the numerical method for the time-fractional modified subdiffusion equation [1]:

$$\partial_t^\alpha u(x, t) = \left( \kappa_1 D_{0+}^{1-\alpha} + \kappa_2 D_{0+}^{1-\beta} \right) \Delta u + f(x, t), \quad (x, t) \in \Omega \times (0, T],$$

(1)

with the initial condition $u(x, 0) = v(x)$ and the homogeneous Dirichlet boundary condition. Here, $x = (x_1, x_2)$, $\Omega$ is the rectangle domain, $T > 0$, and $\Delta$ is the Laplacian defined by $\Delta = \partial^2 / \partial x_1^2 + \partial^2 / \partial x_2^2$. The parameters $\kappa_1$ and $\kappa_2$ are some fixed positive constants, $f$ and $v$ are two given functions, $0 < \alpha, \beta < 1$, and $D_{0+}^{1-\gamma}$ is the Riemann-Liouville derivative of order $\gamma$ given by:

$$D_{0+}^{1-\gamma} u(t) = \frac{1}{\Gamma(\gamma)} \frac{\partial^\gamma}{\partial t^\gamma} \int_0^t (t-s)^{\gamma-1} u(s) \, ds, \quad n - 1 < \gamma < n, \quad n \in \mathbb{N},$$

(2)

where $\Gamma(\cdot)$ is the Gamma function.

Anomalous diffusion is ubiquitous in nature and it can be characterized by the method of mean square particle displacement at the microscopic level. When the mean square displacement (MSD) is linear with time, the particle is precisely in classical Brownian motion. If the MSD grows either sublinearly or superlinearly with time, then this phenomenon is regarded as the subdiffusion and superdiffusion, respectively. Numerous experimental studies have shown that the anomalous diffusion may adequately describe a finite series of Fox special functions, which is of complex form that makes it difficult to apply to practical numerical simulations. So, one needs to resort to the numerical methods for efficiently solving equation (1). Many efficient numerical methods for solving fractional models have emerged in recent years, see the book [5] and the two review papers [2, 6]. For equation (1), some
Numerical schemes have been developed. Ding and Li applied two kinds of high-order compact finite difference methods to construct efficient numerical schemes. The stability and convergence analysis are proved by the Fourier method [7]. In [8], the authors developed the compact difference scheme based on the second-order compact approximation formula of the first-order derivative. The two papers mentioned above both focus on the one-dimensional case.

For the two-dimensional case, Chen and Li employed the modified L1 method and compact difference method and proposed a compact alternating direction implicit scheme. By utilizing the energy method, they proved that their scheme is stable with an accuracy of $O(h^2 \min (\alpha, \beta) + h^4)$ in the new defined norm which is equivalent with $H^1$-norm, under the assumption that the solution is sufficiently smooth [1]. Such assumption may be too restrictive to limit the scope of application of their scheme. To address this issue, Chen proposed two robust fully discrete finite element methods by convolution quadrature in time. He proved that the schemes are convergent under data regularity without relying on the assumption of the solution regularity. In addition, he also proposed a Crank-Nicolson finite element scheme to numerically compare and verify the robustness of the convolution-based schemes in solving nonsmooth solution problems, but no detailed theoretical analysis of the scheme was given [9]. It seems that the numerical methods for equation (1) have not been sufficiently studied. This motivates us to design efficient numerical schemes for (1), especially for high-dimensional problems where the solutions are not sufficiently smooth.

As the further work on the high-dimensional equation (1), we focus on designing numerical schemes that are computationally efficient and can handle the nonsmooth solution case. In [10], Li et al. implemented the fourth-order compact difference operator by a fast discrete sine transform (DST) via the FFT algorithm, which greatly reduces the computational cost and storage requirement. Notice that the DST technology can avoid solving matrix inversion directly and has been successfully applied in the discretization of classical models, such as Poisson equation [11] and general order semilinear evolution equations [12], just to name a few. On the other hand, the weak singularity of the fractional model has gradually attracted the attention of scholars in the fractional community, and some kinds of methods have been proposed to resolve this issue, such as nonuniform meshes [13–16] and convolution quadrature [9, 17, 18]. The method of adding correction terms is also an efficient way of dealing with nonsmooth solutions problems. However, this method is generally not very stable as the starting weights need to be obtained through a linear system which involves the ill-conditioned exponential Vandermonde matrix. To resolve this issue, Zeng et al. theoretically and numerically shown that the accuracy of numerical solution can be efficiently improved with only a few correction terms [19]. Since then, a variety of numerical schemes based on the addition of correction terms have emerged for fractional problems with nonsmooth solutions, see [3, 20, 21]. To the best of our knowledge, it seems that the method of adding correction terms with DST for solving equation (1) has not been considered in the existing literatures yet.

The contributions of this paper are as follows. First, we apply the modified L1 method to discrete the Riemann-Liouville derivative and compact difference operator with DST to discrete the Laplacian and then naturally obtain the fast Crank-Nicolson compact difference scheme for the two-dimensional problem (1), see (7). Second, the stability and error estimate are rigorously proved by the energy method, see Theorems 2 and 3. Specially, we improve the convergence results in [1] and guarantee computational efficiency but without sacrificing the accuracy of the scheme. Note that the small term added during the construction of the ADI scheme in [1] destroys the accuracy of their original scheme. Third, by using the method of adding correction terms, we successfully improve the accuracy of the proposed scheme in solving the nonsmooth problem with no impact on the stability of the original scheme, see (9). Finally, we provide concrete numerical tests to show the effectiveness of the scheme in solving the high-dimensional problem with nonsmooth solution, see Table 1 and Figures 1–3.

### Table 1: The $L^2$-norm errors in time for nonsmooth case in Example 1 with $h = 1/64$.

<table>
<thead>
<tr>
<th>$(\alpha, \beta)$</th>
<th>$n_T$</th>
<th>$L^2$ error $m = 0$</th>
<th>Rate</th>
<th>$L^2$ error $m = 1$</th>
<th>Rate</th>
<th>$L^2$ error $m = 3$</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$m = 0$</td>
<td></td>
<td>$m = 1$</td>
<td></td>
<td>$m = 3$</td>
<td></td>
</tr>
<tr>
<td>(0.3, 0.8)</td>
<td>80</td>
<td>3.58E-03</td>
<td>—</td>
<td>1.71E-03</td>
<td>—</td>
<td>3.53E-04</td>
<td>—</td>
</tr>
<tr>
<td></td>
<td>160</td>
<td>2.24E-03</td>
<td>0.67</td>
<td>1.03E-03</td>
<td>0.73</td>
<td>2.07E-04</td>
<td>0.77</td>
</tr>
<tr>
<td></td>
<td>320</td>
<td>1.42E-03</td>
<td>0.66</td>
<td>6.28E-04</td>
<td>0.72</td>
<td>1.22E-04</td>
<td>0.76</td>
</tr>
<tr>
<td></td>
<td>640</td>
<td>9.06E-04</td>
<td>0.65</td>
<td>3.83E-04</td>
<td>0.71</td>
<td>7.31E-05</td>
<td>0.74</td>
</tr>
<tr>
<td>(0.5, 0.6)</td>
<td>80</td>
<td>1.45E-03</td>
<td>—</td>
<td>7.86E-04</td>
<td>—</td>
<td>2.93E-04</td>
<td>—</td>
</tr>
<tr>
<td></td>
<td>160</td>
<td>8.52E-04</td>
<td>0.76</td>
<td>4.11E-04</td>
<td>0.94</td>
<td>1.54E-04</td>
<td>0.93</td>
</tr>
<tr>
<td></td>
<td>320</td>
<td>5.24E-04</td>
<td>0.70</td>
<td>2.15E-04</td>
<td>0.93</td>
<td>8.05E-05</td>
<td>0.93</td>
</tr>
<tr>
<td></td>
<td>640</td>
<td>3.36E-04</td>
<td>0.64</td>
<td>1.13E-04</td>
<td>0.93</td>
<td>4.21E-05</td>
<td>0.94</td>
</tr>
<tr>
<td>(0.7, 0.4)</td>
<td>80</td>
<td>2.04E-03</td>
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<td>1.05E-03</td>
<td>—</td>
<td>3.14E-04</td>
<td>—</td>
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<tr>
<td></td>
<td>160</td>
<td>1.22E-03</td>
<td>0.73</td>
<td>5.86E-04</td>
<td>0.85</td>
<td>1.72E-04</td>
<td>0.87</td>
</tr>
<tr>
<td></td>
<td>320</td>
<td>7.54E-04</td>
<td>0.70</td>
<td>3.28E-04</td>
<td>0.84</td>
<td>9.43E-05</td>
<td>0.87</td>
</tr>
<tr>
<td></td>
<td>640</td>
<td>4.76E-04</td>
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<td>1.84E-04</td>
<td>0.83</td>
<td>5.20E-05</td>
<td>0.86</td>
</tr>
</tbody>
</table>
The rest of the paper is organized as follows. In Section 2, we derive the fast Crank-Nicolson compact difference scheme by the modified L1 method and the compact difference operator with DST. In Section 3, we prove that the proposed numerical scheme is stable with an accuracy of \( O(\tau^{\min(1+\alpha,1+\beta)} + h^4) \) under the assumption that the solution is sufficiently smooth. To weaken the assumption and make the scheme more robust in solving nonsmooth solution problems, we present the improved version in Section 4 with the method of adding correction terms. Numerical examples are given in Section 5 to confirm the effectiveness of the proposed scheme. Finally, we present the conclusions of this paper in Section 6.

Throughout this paper, we shall let the symbol \( c \) (with or without subscript) be a positive constant which may vary at different locations but is always independent of the temporal and spatial stepsizes.

2. The Compact Difference Scheme with Fast Solver

In this section, we derive the fast compact difference scheme for (1). We first introduce the temporal discretization. The time stepsize \( \tau \) is given by \( \tau = T/n_T \) with the positive integer \( n_T \). The grid point is denoted by \( t_n = n\tau \) for \( n \geq 0 \). Let \( t_{n+1/2} = \ldots \)
$(t_n + t_{n+1})/2$. For $g(t) \in C^2[0, T]$, the modified L1 method for the approximation of Riemann-Liouville derivative \( R^\gamma_{\Omega} \) of $g(t)$ with $\gamma \in (0, 1)$ at $t = t_{n+1/2}$ is described as:

$$
R^\gamma_{\Omega} g(t_{n+1/2}) = \tilde{D}^\gamma g(t_{n+1/2}) + R_{n+1/2}^{\gamma},
$$

where $[R_{n+1/2}^{\gamma}] \leq C \tau^{1-\gamma} \max_{{t \in [0, T]}|g''(t)|}$ ([22], Lemma 1).

The operator $\tilde{D}^\gamma$ in (2) is given by

$$
\tilde{D}^\gamma g(t_{n+1/2}) = b^{(y)}_0 g(t_{n+1/2}) - \frac{1}{2} \sum_{k=1}^{d} (b^{(y)}_{n-k} - b^{(y)}_{n-k+1}) g(t_{k-1/2}) - \left(b^{(y)}_{n-k} - B^{(y)}_{n-k}\right) g(t_0),
$$

where

$$
\begin{align*}
\left\{ 
\begin{array}{ll}
b^{(y)}_k &= \frac{\tau^{-\gamma}}{\Gamma(2-\gamma)} \left((k+1)^{-\gamma} - k^{-\gamma}\right), \\
B^{(y)}_k &= \frac{\tau^{-\gamma}}{\Gamma(2-\gamma)} \left((k+\frac{1}{2})^{-\gamma} - k^{-\gamma}\right), \\
A^{(y)}_n &= B^{(y)}_n - \left(1 - \frac{1}{\gamma}\right) \frac{\tau^{-\gamma}}{\Gamma(2-\gamma)} \left(n + \frac{1}{2}\right)^{-\gamma}.
\end{array}
\right.
\end{align*}
$$

So, applying the difference discretization $\delta_t u(x, t_{n+1/2}) = \delta_t u^{n+1/2} + O(\tau^2)$ with $\delta_t u^{n+1/2} = (u^{n+1} - u^{n})/\tau$, we derive that

$$
\delta_t u^{n+1/2} = \left(\kappa_1 \tilde{D}^{1-a} + \kappa_2 \tilde{D}^{1-b}\right) \Delta u^{n+1/2} + f^{n+1/2} + R_{n+1/2}^{\delta_t},
$$

where the local truncation error $R_{n+1/2}^{\delta_t} = O(\tau^{\min\{1+a, 1+b\}})$ and $u^{n+1/2} = (u^{n+1} + u^n)/2$.

Next, we consider the spatial discretization for (4). In order to make our discussion more general, we follow the notations presented in [10] and always set the symbol $d = 2$ unless otherwise noted. Denote the domain $\Omega = [x_1, x_1^2] \times [x_2, x_2] \times \cdots \times [x_d, x_d^2]$. Let $M_k (1 \leq k \leq d)$ be a positive integer. The spatial stepsize is then denoted as $h_k = (x_k^2 - x_k)/M_k$ and $x_{k, h_k} = x_k + j_k h_k$ for $j_k = 0, 1, \ldots, M_k$. The full discrete grids in space are denoted by $\Omega_h = \{x_h = (x_{1,h_1}, x_{2,h_2}, \ldots, x_{d,h_d}) | 0 \leq j_k \leq M_k, 1 \leq k \leq d\}$. We further denote that $\Omega_{h_k} = \Omega_h \cap \Omega$ and the boundary $\partial \Omega = \partial \Omega_h \cap \partial \Omega$. The space of grid function is denoted as $V_h = \{v \mid v = (v_h)_{x_h} \text{ and } v_h = 0 \text{ for } x_h \notin \Omega_h\}$. For the grid function $v_h = v(x_h)$ with the index vector $h = (i_1, i_2, \ldots, i_d)$ at $k$th position, we denote the compact difference operator as $\Delta_h v_h = \delta^2_h(\mathcal{F} v_h)$, with the difference operator $\mathcal{F} v_h := (I + h_1^2 + 12h_2^2) v_h$. Here, $I$ is the identity operator, $\delta^2_h v_h = (\delta h v_h \delta^{-1}_h)(1/2)/h_k$ and $\delta^4_h v_{h+1/2} = (v_h + v_{h+1} - v_{h-1})/h_k$. So, the fourth-order spatial approximation of $\Delta v(x_h)$ for $x_h \in \Omega_h$ is given by $\Delta_h v_h := \sum \Delta h v_h$.

Combining the compact difference approximation in space with (4), we have

$$
\delta_t u(x_h)^{n+1/2} = \left(\kappa_1 \tilde{D}^{1-a} + \kappa_2 \tilde{D}^{1-b}\right) \Delta_h u(x_h)^{n+1/2} + f^{n+1/2} + R_{x_h}^{\delta_t},
$$

where the local truncation error is given by $R_{x_h}^{\delta_t} = O(\tau^{\min\{1+a, 1+b\}} + h^{4})$. Here, $h^4 = h_1^4 + h_2^4 + \cdots + h_d^4$. Omitting the small term $R_{x_h}^{\delta_t}$, we obtain the following Crank-Nicolson compact difference scheme for (1): find $u_h^n$ of $u(x_h, t_n)$ for $n \geq 1$, such that

$$
\delta_t u_h^{n+1/2} = \left(\kappa_1 \tilde{D}^{1-a} + \kappa_2 \tilde{D}^{1-b}\right) \Delta_h u_h^{n+1/2} + f^{n+1/2},
$$

where $u_h^0 = v(x_h)$ and $u(x_h)|_{x_h \in \partial \Omega_h} = 0$. 

**Figure 3:** Comparison of CPU execution time between original and fast schemes with fixed $\alpha = 0.7$ and $\beta = 0.4$.
If we solve the discretized system (6) directly, the computational cost will be \(O((M_1M_2 \cdots M_d)^3)\) on each time level due to the calculation of matrix inversion. Next, we employ the fast discrete sine transform based on FFT to reduce the computational cost to \(O((M_1M_2 \cdots M_d) \log (M_1M_2 \cdots M_d))\) [11], which greatly improves the computational performance. Since the discrete sine transform of the grid function \(v_h\) at the \(k\)th position is provided by
\[
v_h^{(k)} = \sum_{j=1}^{M_k} v_{jk} \sin (j \pi/M_k),
\]
where \(v_{jk} = \cos (j \pi/M_k)\) and \(1 \leq j_k \leq M_k - 1\). One can refer to [11] or [10] for the derivation. Denote the index set \(v = \{j_1, j_2, \cdots, j_d\} | 1 \leq j_k \leq M_k - 1, 1 \leq k \leq d\}. Therefore, the scheme (6) is equivalent to
\[
\delta_h \tilde{u}_h^{n+1/2} = \left( \kappa_1 D_{-\tau} + \kappa_2 D_{-\tau}^{-\beta} \right) \left( \sum_{i=1}^d \lambda(i,M_i) \right) \tilde{u}_h^{n+1/2} + f \lambda^{n+1/2}.
\]

The computational procedure is described as follows:

(a) For \(n \geq 0\), we first computed \(\tilde{u}_h^n\) and \(f \lambda^{n+1/2}\) by means of DST

(b) And then we solve equation (10) from which the numerical solution \(u_h^n\) is obtained from \(\tilde{u}_h^n\) by the inverse of DST.

3. Stability and Error Estimates

In this part, we demonstrate the stability and error estimates for the compact difference scheme (6).

We first introduce some useful notations. For any grid function \(v \in V_h\), the discrete \(L^2\)-norm is given by \(\|v\|_h^2 = \sqrt{(v, v)_h}\) with the discrete inner product \((u, v)_h = (\prod_{k=1}^d h_k) \sum_{k \in \Omega_h} u_k v_k\). The discrete \(H^1\) seminorm and \(H^1\) norm are denoted as \(\|v\|_1^2 = \sqrt{\|v\|_h^2} + \sqrt{\sum_{k=1}^d \|\Delta_h v_k\|^2}\) and \(\|v\|_1 = \sqrt{\|v\|_h^2} + |v|_1\). Here, \(V_h = (\Delta_1, \Delta_2, \cdots, \Delta_d)\). One can readily have the equivalence of \(\|v\|_1\) and \(\|v\|_1\) for any \(v \in V_h\) in view of the embedding theorem.

We shall first need the following lemma.

Lemma 1. The operator \(\tilde{D}_h^n\) given by (3) satisfies the inequality:
\[
-2(\tilde{D}_h^n v_{n+1/2}, v_{n+1/2})_h \leq \sum_{k=1}^d b_{n,k}^0 \|v_{n+1/2}\|^2 - \sum_{k=1}^d b_{n+1,k}^0 \|v_{n+1/2}\|^2 + A_n \|v\|^2,
\]
where \(v^n \in V_h, n \geq 0\).

Proof. The proof of the lemma can be obtained in view of Lemma 4.2 in [22] or Lemma 4.4 in [1], thus, the details are omitted here.

We are ready to present the stability of the scheme (6).

Theorem 2. The Crank-Nicolson compact difference scheme (6) is stable in the sense that
\[
\|u_h^{n+1}\|^2 - \|u_h^n\|^2 \leq c \left( \|u_h^n\|^2 + \left( c^1 \tau + c^2 \tau \right) \|v_h^n\|^2 + \tau \sum_{k=1}^n \|f_{k-1/2}\|^2 \right).
\]

Proof. By taking the discrete inner product on both sides of (6) with \(2\tau u_h^{n+1/2}\), we get
\[
2\tau \left( \tilde{D}_h^n u_h^{n+1/2}, u_h^{n+1/2} \right)_h = 2\tau \left( \kappa_1 D_{-\tau} + \kappa_2 D_{-\tau}^{-\beta} \right) \left( \sum_{i=1}^d \lambda(i,M_i) \right) \tilde{u}_h^{n+1/2} + 2\tau \left( f^{n+1/2}, u_h^{n+1/2} \right)_h.
\]

Notice that the difference operator \(\tilde{D}_h\) is bounded in discrete inner product ([10], Theorem 2):
\[
\frac{3}{2} \left( \tilde{D}_h u_h^{n+1/2}, u_h^{n+1/2} \right)_h < \left( \tilde{D}_h u_h^{n+1/2}, u_h^{n+1/2} \right)_h < \left( \tilde{D}_h u_h^{n+1/2}, u_h^{n+1/2} \right)_h,
\]
with the notation \(\tilde{D}_h u_h^n = \sum_{k=1}^d \delta^k u_h^n\). By the identity \(\tilde{D}_h u_h^{n+1/2}, u_h^{n+1/2} = -\left( \Delta_h u_h^{n+1/2}, \Delta_h u_h^{n+1/2} \right)_h\), the Lemma 1 yields
\[
\|u_h^{n+1}\|^2 - \|u_h^n\|^2 \leq c \left( \sum_{k=1}^n b_{n,k}^0 \|v_h^{k-1/2}\|^2 - \sum_{k=1}^n b_{n+1,k}^0 \|v_h^{k-1/2}\|^2 + A_n \|v_h^n\|^2 \right) + 2\tau \left( f^{n+1/2}, u_h^{n+1/2} \right)_h,
\]
where \(b_k = b_k^{(1-n)} + b_k^{(1-\beta)}\), \(B_k = B_k^{(1-n)} + B_k^{(1-\beta)}\), and \(A_k = A_k^{(1-n)} + A_k^{(1-\beta)}\). With \(G^n = \|u_h^n\|^2 + \tau \sum_{k=1}^n b_{n,k} \|v_h^{k-1/2}\|^2\), we write the above inequality as:
\[
G^{n+1} \leq G^n + \tau A_n \|v_h^n\|^2 + 2\tau \left( f^{n+1/2}, u_h^{n+1/2} \right)_h.
\]

We sum up \(n\) from 1 to \(m\) and replace \(m\) with \(n\) to get
\[
G^{n+1} \leq G^n + \tau \sum_{k=1}^m A_k \|v_h^n\|^2 + 2\tau \sum_{k=1}^n \left( f^{k+1/2}, u_h^{k+1/2} \right)_h.
\]

By the Cauchy-Schwarz inequality and the inequality \(\|v_h^n\|_1 \geq c_1 \|v_h^n\|_1\) with the equivalence of \(\|v_h^n\|_1\) and \(\|v_h^n\|_1\), we obtain the estimate:
\[ 2\tau \sum_{k=1}^{n} \left( f_k^{1/2} + u_h^{1/2} \right)_h \leq \tau \sum_{k=1}^{n+1} b_{n+1-k} \| \nabla u_h^{1/2} \|^2 + \tau \sum_{k=2}^{n+1} b_{n+1-k} \| f^{1/2} \|^2, \] 

(18)

from which we derive that

\[ \| u_h^{n+1} \|^2 \leq G^1 + \tau \sum_{k=1}^{n+1} A_k \| \nabla u_h^0 \|^2 + \tau \sum_{k=2}^{n+1} \frac{1}{b_{n+1-k}} \| f^{1/2} \|^2. \] 

(19)

Next, we consider the case \( n = 0 \) for the scheme (6) for the estimate of \( G^1 \). By a similar procedure, we take the discrete inner product for (6) with \( 2\tau n_h^{1/2} \) when \( n = 0 \), we have

\[ 2\tau (\delta_t, u_h^{1/2}, u_h^{1/2})_h = 2\tau \left( (k_1 D_t^{1-a} + k_2 D_t^{1-b}) \Delta_t u_h^{1/2}, u_h^{1/2} \right)_h + 2\tau \left( f^{1/2}, u_h^{1/2} \right)_h, \] 

from which we have

\[ \| u_h^0 \|^2 + 2\tau B_0 \| \nabla u_h^{1/2} \|^2 = \| u_h^0 \|^2 + 2\tau A_0 \| (\nabla u_h^0, \nabla u_h^{1/2})_h + \tau \left( f^{1/2}, u_h^{1/2} \right)_h, \] 

(20)

where \( B_0 = k_1 B_0^{1+a} + k_2 B_0^{1-b} \) and \( A_0 = k_1 A_0^{1+a} + k_2 A_0^{1-b} \). Utilizing the Cauchy-Schwarz inequality again, we arrive at the estimate for the last two terms on the right-hand of the above inequality:

\[ 2\tau \| (\nabla u_h^0, \nabla u_h^{1/2})_h + 2\tau \left( f^{1/2}, u_h^{1/2} \right)_h \leq 2\tau \left( 1/4 + \frac{1}{4} \right) \| \nabla u_h^0 \|^2 + 2\tau \left( f^{1/2}, u_h^{1/2} \right)_h \] 

(22)

By letting the constants \( \epsilon_1 = B_0/(4A_0) \) and \( \epsilon_2 = B_0/4 \) and the equivalence of the \( \| v_h \|_1 \) and \( \| v_h \|_1 \), we further get

\[ 2\tau A_0 \| (\nabla u_h^0, \nabla u_h^{1/2})_h + 2\tau \left( f^{1/2}, u_h^{1/2} \right)_h \leq 2\tau A_0^2 \| \nabla u_h^0 \|^2 + B_0 \| \nabla u_h^{1/2} \|^2 + 2\tau \left( f^{1/2}, u_h^{1/2} \right)_h \] 

(23)

which implies that the \( G^1 \) has the following estimate:

\[ G^1 = \| u_h^0 \|^2 + 2\tau \| \nabla u_h^{1/2} \|^2 \leq \| u_h^0 \|^2 + 2\tau B_0 \| \nabla u_h^{1/2} \|^2, \] 

(24)

Therefore, the inequality (8) yields

\[ \| u_h^{n+1} \|^2 \leq \| u_h^0 \|^2 + 2\tau \frac{A_0^2}{B_0} \| \nabla u_h^0 \|^2 + 2\tau \frac{A_0}{B_0} \| f^{1/2} \|^2 \] 

(25)

By the mean value theorem, one can readily check that the coefficients appearing in the above inequality are all bounded, that is, we formally have \( A_0^2/B_0 = c_1 \tau^{1-\alpha} + c_2 \tau^{1-b-1} \), \( 1/B_0 = c_3 \tau^{1-\alpha} + c_4 \tau^{1-b} \), \( 1/B_0 = c_5 \tau^{1-\alpha} + c_6 \tau^{1-b} \), and \( \tau \sum_{k=1}^{n} A_k \leq c_7 \tau^{1-\alpha} + c_8 \tau^{1-b} \), thus, the proof is completed.

By means of the error equation and the stability conclusion, we have the following convergence result.

**Theorem 3.** Suppose that \( u \in C^2(0, T; C^0(\Omega)) \), then we have the discrete \( L^2 \)-norm error estimate: For \( n \geq 1 \),

\[ \| u(t_n) - u_h^0 \| \leq c \left( \tau^{min (1+a,1+b)} + h^4 \right). \] 

(26)

**Proof.** The error equation can be obtained by subtracting (6) from (5), that is, by letting the error \( e_h^n = u(x_h,t_n) - u_h^n \) for \( x_h \in \Omega_h \), we have

\[ \delta_t e_h^{n+1/2} = \left( k_1 D_t^{1-a} + k_2 D_t^{1-b} \right) \nabla u_h^{n+1/2} + B_0 e_h^{n+1/2}. \] 

(27)

It follows from Theorem 2 that

\[ \| e_h^n \|^2 \leq c \left( \| e_h^0 \|^2 + \left( c_1 \tau^a + c_2 \tau^b \right) \| \nabla u_h^0 \|^2 + \tau \sum_{k=1}^{n} \| \nabla u_h^{1/2} \|^2 \right) \] 

(28)

which leads to the desired convergence result.

### 4. Numerical Implementation for Nonsmooth Problems

In general, the solution of equation (1) may not have the regularity required in Theorem 3. If the nonsmooth solution problems are directly solved by the fast Crank-Nicolson compact difference scheme (7), unsatisfactory accuracy may be obtained. In this part, we apply the method of adding suitable correction terms when dealing with such nonsmooth issue.

Following the idea presented in [3], we, respectively, take the numerical approximations of \( \tau D_t u_h^{n+1/2} g(t) \) and the first-order time derivative \( dg(t)/dt \) at \( t = t_{n+1/2} \) as follows:
\[ m D_t^\nu g(t) \bigg|_{t=t_{n+1/2}} \approx D_t^\nu g(t_{n+1/2}) \text{red} + \sum_{k=1}^{m} w_{nk}(g(t_k) - g(0)), \]
\[ \frac{dg(t)}{dt} \bigg|_{t=t_{n+1/2}} \approx \delta g(t_{n+1/2}) \text{red} + \sum_{k=1}^{m} w_{nk}(g(t_k) - g(0)). \]

\[ (29) \]

Here, \( w_{nk}^{(y)} \) and \( w_{nk}^{(1)} \) are the starting weights which are chosen such that the above schemes are exact for some power functions \( g(t) = t^\gamma \) with \( 0 < \zeta_j < \zeta_{j+1} \) and \( 0 \leq j \leq m \), that is, they can be determined by the two linear systems:

\[ \sum_{k=1}^{m} w_{nk}^{(y)} \zeta_j = \frac{\Gamma(1 + \zeta_j)}{\Gamma(1 + \zeta_j - \gamma)} \zeta_j^{\gamma-\nu+\gamma} - D_t^{\nu+1/2} \zeta_j, \]
\[ \sum_{k=1}^{m} w_{nk}^{(1)} \zeta_j = \zeta_j^{\nu+1/2} - \frac{\zeta_j^{\nu-1} - \zeta_j^{\nu}}{\tau}, \]

\[ (30) \]

respectively. So, we have the following fast Crank-Nicolson compact difference scheme with correction terms: for \( n \geq 0, \)

\[ \delta \tilde{u}_n^{\nu+1/2} + \sum_{k=1}^{m} \left( \left( \kappa_1 D_t^\alpha + \kappa_2 D_t^\beta \right) \left( \sum_{j=1}^{d} \lambda_{j} L_{j,n} \right) \right) \tilde{u}_n^{\nu+1/2} \]
\[ + \sum_{k=1}^{m} \left( w_{nk}(l) + w_{nk}(l+1) \right) \left( \sum_{j=1}^{d} \lambda_{j} L_{j,n} \right) \tilde{u}_n^{\nu+1/2} + D_t^{3/2} \]
\[ + f A^{n+1/2}. \]

\[ (31) \]

The execution procedure of the above scheme is similar to that of (7). We can observe that the scheme (9) is stable and effective in solving nonsmooth problems, which will be verified by numerical examples in the next section. We remark that the method of adding correction terms is based on the assumption that the problem solution can be divided into two terms: low regularity and high regularity terms (with respect to time). Such assumption is valid for equation (1) in view of the solution formulation discussed in [4]. By using the starting weights in the correction terms, one can improve the accuracy of the proposed scheme for dealing with the nonsmooth problem solution. For further details about the parameters \( m \) and \( \zeta_j \), one may refer to [19].

5. Numerical Examples

In this part, we present two numerical examples to verify the accuracy and effectiveness of the scheme (9). The \( L^2 \)-norm error at \( t = t_n \) is obtained by \( e(n, h) = \| u(x_0, t_n) - u^h_0 \| \), and the convergence orders in time and in space are calculated by \( \log \| e(n, h) \| / \log (2n, h) \) and \( \log (e(n, h)) / \log (e(n, h/2)) \), respectively. For simplicity, we set the parameters \( \kappa_1 \) and \( \kappa_2 \) in (1) to be one and restrict the computational domain to be \( \Omega = (0, 1)^2 \). We remark that the numerical tests in this paper are implemented by MATLAB software (R2020a) on an Apple OS platform with a quad-core 2.3 GHz processor and 8 GB of memory.

Example 1. (Accuracy). Consider the following problem with zero Dirichlet boundary conditions:

\[ \partial_t u(x, y, t) = \left( \partial_{x}D_t^{\alpha} + \partial_{y}D_t^{\beta} \right) \Delta u(x, y, t) + f(x, y, t), \]
\[ u(x, y, 0) = \frac{1}{2} \sin (\tau x) \sin (\tau y), \]
\[ (32) \]

where

\[ f(x, y, t) = \sin (\tau x) \sin (\tau y) \left( \frac{\partial_t^{\nu-1} + 2\pi^2 c}{\Gamma(\alpha)} + \frac{\tau^{\beta-1}}{\Gamma(\beta)} \right). \]
\[ (33) \]

The exact solution is \( u = \sin (\tau x) \sin (\tau y)(c \tau^t) \) with the two given nonnegative parameters \( c \) and \( \gamma \).

We verify the accuracy of the proposed scheme (9) using two cases: the smooth and nonsmooth solutions. We first let \( c = 1 \) and \( \gamma = 2.1 \). The numerical results are obtained at \( T = 1 \) by fast Crank-Nicolson compact difference scheme (9) with no correction terms and demonstrated in Tables 2 and 3. One can observe that accuracy of the scheme is \( O(\tau^min(1-\alpha,1+\beta)) + h^4 \) for different fractional orders \( \alpha \) and \( \beta \), which is in agreement with the theoretical analysis.

Next, for the nonsmooth case, we let \( c = 0 \) and \( \gamma = 0.4 \). One can see that the first-order partial derivative of \( u \) with respect to \( t \) is \( \partial_t u(x, y, t) = \tau^{\nu-1} \sin (\tau x) \sin (\tau y) \), which is unbounded at \( t = 0 \) when \( \gamma = 0.4 \). By using the fast Crank-Nicolson compact difference scheme with correction terms (9), we compute the \( L^2 \)-norm errors at \( T = 0.5 \) and present the results in Tables 1 and 4. We can see from Table 1 that when \( m = 0 \), that is, no correction term is added to the scheme, the accuracy of the numerical solution suffers from the low regularity of the analytic solution. In contrast, when \( m \) is greater than 0, the accuracy of the numerical solution seems to be improved to some extents. Similar phenomenon is also observed in Table 4. This suggests that adding a small number of correction terms does improve the accuracy of the numerical solution in nonsmooth problems. Thus, the fast Crank-Nicolson compact difference scheme with correction terms (9) is valid for solving non-smooth solution problems.

Example 2. (Computational efficiency). In this example, we investigate the computational efficiency of the fast Crank-Nicolson compact difference scheme (7). So, we consider the comparison between results from the schemes with fast solver and the direct solver, that is, fast scheme (7) and original scheme (6). We separately solve the smooth solution case in Example 1 with the two numerical schemes and report the numerical results obtained in Figures 1–3. For the given fractional orders \( \alpha \) and \( \beta \), by fixing the time stepsize \( \tau = 1/4 \) and
level, and such operation would be extremely ine-
direct solver requires to solve matrix inversion on each time
stepsize is getting smaller. This is due to the fact that the
that using the DST technique, especially when the spatial
the direct solver in numerical scheme is more expensive than
The comparison shows that the execution time spent using
the proposed scheme (7) has more potential than the direct
solver (6) in high-dimensional problems.

6. Conclusions

In this paper, we propose the efficient compact difference
scheme for solving the modified anomalous subdiffusion
equation based on the modified L1 method in time and
compact difference operator in space. By combining the
DST technology, we improve the effectiveness of the scheme
for the two-dimensional problem. The stability and error
estimate of the scheme are provided rigorously. We also
improve the accuracy of the scheme for the nonsmooth solu-
tion problems with the method of adding correction terms.
Numerical examples illustrate the effectiveness and accuracy
of the proposed scheme.

The results of this paper can be readily generalized to
tree-dimensional problems. In addition, for inhomoge-
ous boundary conditions, one can convert them into
homogeneous boundary condition problems by variable sub-
stitution. For other types of boundary condition problems,
such as Neumann, Robin, or other combinations of boundary
conditions, we do not discuss them in this paper. In [23], the
authors introduced the augmented matched interface and
boundary (AMIB) method to e-
derent types of boundary
problems with complex boundary conditions, and this is
the possible one of the future research directions.

Data Availability

The data of numerical simulation used to support the find-
ings of this study are included within the article.
Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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