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Research Article

Toeplitz Operators with Lagrangian Invariant Symbols Acting on the Poly-Fock Space of $\mathbb{C}^n$

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We introduce the so-called extended Lagrangian symbols, and we prove that the C^*-algebra generated by Toeplitz operators with these kind of symbols acting on the homogeneously poly-Fock space of the complex space $\mathbb{C}^n$ is isomorphic and isometric to the C^*-algebra of matrix-valued functions on a certain compactification of $\mathbb{R}^n$ obtained by adding a sphere at the infinity; moreover, the matrix values at the infinity points are equal to some scalar multiples of the identity matrix.

1. Introduction

Let $m \in \mathbb{N}$, the one-dimensional m poly-Fock space $F_m^2(\mathbb{C}) \subset L_2(\mathbb{C}, d\mu)$ consists of all m-analytic functions φ which satisfy

$$\left(\frac{\partial}{\partial \bar{z}}\right)^m \varphi = \frac{1}{2^m}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)^m \varphi = 0, \qquad (1)$$

where $d\mu = \pi^{-1}e^{-z\cdot\bar{z}}dxdy$ is the Gaussian measure in $\mathbb{C}$ and $dxdy$ is the Euclidian measure in $\mathbb{R}^2 = \mathbb{C}$. Further, the one-dimensional true poly-Fock space of order m is given by

$$F_{(m)}^2(\mathbb{C}) = F_m^2(\mathbb{C}) \ominus F_{m-1}^2(\mathbb{C}). \qquad (2)$$

In the case of several variables, for $n \in \mathbb{N}$, the n-dimensional Gaussian measure in $\mathbb{C}^n$ is given by $d\mu_n(z) = \pi^{-n}e^{-|z|^2}dxdy$, where $dxdy$ is the Euclidian measure in $\mathbb{R}^{2n}$. We have that the space $L_2(\mathbb{C}^n, d\mu_n)$ is the tensorial product of n components

$$L_2(\mathbb{C}^n, d\mu_n) = L_2(\mathbb{C}, d\mu) \otimes \cdots \otimes L_2(\mathbb{C}, d\mu), \qquad (3)$$

and the Fock space $F^2(\mathbb{C}^n)$ is

$$F^2(\mathbb{C}^n) = F^2(\mathbb{C}) \otimes \cdots \otimes F^2(\mathbb{C}). \qquad (4)$$

Given a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}_+^n$, the poly-Fock space $F_\alpha^2(\mathbb{C}^n)$ of order α is given by

$$F_\alpha^2(\mathbb{C}^n) = \bigotimes_{j=1}^{n} F_{\alpha_j}^2(\mathbb{C}). \qquad (5)$$

Similarly, the true poly-Fock space $F_{(\alpha)}^2(\mathbb{C}^n)$ is

$$F_{(\alpha)}^2(\mathbb{C}^n) = \bigotimes_{j=1}^{n} F_{(\alpha_j)}^2(\mathbb{C}). \qquad (6)$$

In [1], Vasilevski introduced the poly-Fock spaces over $\mathbb{C}^n$ and he obtained the following decomposition formula:

$$L_2(\mathbb{C}^n, d\mu_n) = \bigoplus_{|\alpha|=n}^{\infty} F_{(\alpha)}^2(\mathbb{C}^n). \qquad (7)$$

Moreover, he showed that the true poly-Fock space $F_{(\alpha)}^2(\mathbb{C}^n)$ is isomorphic and isometric to $L_2(\mathbb{R}^n, dx) \otimes \tilde{H}_{\alpha-1}$, where $\tilde{H}_{\alpha-1}$ is the one-dimensional space generated by the function

$$\tilde{h}_{\alpha-1}(y) = \prod_{j=1}^{n} h_{\alpha_j-1}\left(y_j\right), \tag{8}$$

and each $h_{\alpha_j-1}(y_j)$ is a Hermite's function in $\mathbb{R}$.

Another treatment of the poly-Fock spaces can be found in [2], where the author characterized all lattice sampling and interpolation sequences in the poly-Fock spaces. He introduced the polyanalytic Bargmann transform from vector-valued Hilbert spaces to poly-Fock spaces, and he showed the duality between sampling and interpolation in polyanalytic spaces and multiple interpolation and sampling in analytic spaces.

The Toeplitz operators acting on the Fock space have been investigated by several authors. For example, in [3], the authors studied Toeplitz operators acting on the one-dimensional Fock space and on true poly-Fock space whose symbols are bounded radial functions that have a finite limit at the infinity. They considered an orthonormal basis of normalized complex Hermite polynomials to prove that the radial operators are diagonal. In [4], the authors studied Toeplitz operators acting on the one-dimensional poly-Fock space with horizontal symbols such that the limit values at $x = \infty$ and $x = -\infty$ exist. They proved that the C^*-algebra generated with this class of symbols is isomorphic to the C^*-algebra of functions on $\bar{\mathbb{R}}$ with values on the $m \times m$ matrices, whose limit value at $x = \infty$ and $x = -\infty$ are equal to some scalar multiples of the identity matrix. In [5], the authors introduced the Toeplitz operators with $\mathscr{L}$-invariant symbols over the Fock space $F^2(\mathbb{C}^n)$ for a Lagrangian plane $\mathscr{L}$, and they proved that the corresponding C^*-algebra generated is isometric to the C^*-algebra generated by Toeplitz operators with horizontal symbols.

On the other hand, the spaces of homogeneously polyanalytic functions have been studied recently. For example, in [6], the authors computed the reproducing kernel of the Bergman space of homogeneously polyanalytic functions on the unit ball in $\mathbb{C}^n$ and on the Siegel domain.

The main result of this paper is the following: the C^*-algebra generated by Toeplitz operators with extended Lagrangian symbols acting on the homogeneously poly-Fock space over $\mathbb{C}^n$ is isomorphic and isometric to the C^*-algebra of matrix-valued functions on a certain compactification of $\mathbb{R}^n$ with the sphere at the infinity; moreover, the values at the infinity points are scalar multiplies of the identity matrix.

This paper is organized as follows. In Section 2, we define the so-called homogeneously poly-Fock space and study some of its properties. In Section 3, we prove that every Toeplitz operator with a horizontal symbol acting on the poly-Fock (or homogeneously poly-Fock) space is unitary equivalent to a multiplication operator by a matrix-valued function. In Section 4, we introduce the concept of extended horizontal symbol and we describe the C^*-algebra generated by Toeplitz operators with this kind of symbols acting on both the poly-Fock space and the homogeneously poly-Fock space. Finally, in Section 5, we define the extended Lagrangian symbols and we prove that the C^*-algebra generate by Toeplitz operators with these symbols acting on the homogeneously poly-Fock space is isomorphic to the C^*-algebra generated by Toeplitz operators with horizontal symbols acting on the same space.

2. Poly-Fock Spaces over $\mathbb{C}^n$

In this section, we define the homogeneously poly-Fock space and we review some facts about the classic poly-Fock spaces.

For $z = (z_1, \cdots, z_n) \in \mathbb{C}^n$, we denote $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, y_n) \in \mathbb{R}^n$, where $z_j = x_j + iy_j$. The Fock space $F^2(\mathbb{C}^n)$ is given by

$$F^2(\mathbb{C}^n) = F^2(\mathbb{C}) \otimes \cdots \otimes F^2(\mathbb{C}). \tag{9}$$

We have that $F^2(\mathbb{C}^n)$ is a Hilbert space with the usual inner product of functions:

$$\langle f, g \rangle = \frac{1}{\pi^n} \int_{\mathbb{C}^n} f(z) \bar{g}(z) e^{-|z|^2} dxdy. \tag{10}$$

Moreover, the Fock space $F^2(\mathbb{C}^n)$ is a reproducing kernel Hilbert Space, whose reproducing kernel is given by

$$K(w, z) = K_w(z) = e^{z \cdot \bar{w}}. \tag{11}$$

For the multi-index $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}_+^n$, we recall the operations $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$, and $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$, with $z \in \mathbb{C}^n$. Also, for $\alpha, \beta \in \mathbb{Z}_+^n$, we have $\alpha \pm \beta = (\alpha_1 \pm \beta_1, \cdots, \alpha_n \pm \beta_n)$. We consider the set $\mathbb{Z}_+^n$ with the lexicographic order. Finally, for $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}_+^n$, we write $\mathbf{P}(\alpha) = \alpha_1 \cdots \alpha_n$. For the multi-index $1_n = (1, \cdots, 1)$, sometimes, we write 1 instead of 1_n as long as no there is no confusion.

Let $\alpha \in \mathbb{Z}_+^n$ be a multi-index and consider the poly-Fock space $F_\alpha^2(\mathbb{C}^n)$ defined in (5), since every one-dimensional poly-Fock space $F_{\alpha_j}^2(\mathbb{C})$ is a direct sum of true poly-Fock spaces whose order is less than or equal to α_j, see [1], p. 5-6, we have

$$F_\alpha^2(\mathbb{C}^n) = \left[\bigoplus_{\lambda_1=1}^{\alpha_1} F_{(\lambda_1)}^2(\mathbb{C})\right] \otimes \cdots \otimes \left[\bigoplus_{\lambda_n=1}^{\alpha_n} F_{(\lambda_n)}^2(\mathbb{C})\right] = \bigoplus_{\substack{\lambda_i \le \alpha_i \\ \lambda \in \mathbb{Z}_+^n}} F_{(\lambda)}^2(\mathbb{C}^n). \tag{12}$$

Note that the number of components in (12) is equal to $\mathbf{P}(\alpha) = \alpha_1 \cdots \alpha_n$.

Now, let $k \in \mathbb{N}$ be a natural number such that $k \ge n$.

Definition 1. The homogeneously poly-Fock space of order k over $\mathbb{C}^n$ is given by

$$F_{(k)}^2(\mathbb{C}^n) = \bigoplus_{|\alpha|=k} F_{(\alpha)}^2(\mathbb{C}^n). \tag{13}$$

The number of multi-indices whose absolute value is exactly k is equal to $\mathbf{s}_{(k)} = \begin{pmatrix} k-1 \\ n-1 \end{pmatrix}$.

Definition 2. The poly-Fock space of order k in $\mathbb{C}^n$ is given by

$$F_k^2(\mathbb{C}^n) = \bigoplus_{|\alpha|=n}^{k} F_{(\alpha)}^2(\mathbb{C}^n). \quad (14)$$

The number of multi-indices whose absolute value is less than or equal to k is equal to $\mathbf{s}_k = \begin{pmatrix} k \\ k-n \end{pmatrix}$.

Remark 3 (see [6], Proposition 2.7). The authors introduced the concept of homogeneously polyanalytic function; this concept is very important in the development of this paper. Also, they proved that homogeneously polyanalytic spaces are invariant under linear change of variables.

In [1], Vasilevski applied the "creation" and "annihilation" operators in the Fock spaces and he proved the following results:

(1) All true poly-Fock spaces are isomorphic one to each other

(2) The explicit expression of the functions $\psi(z)$ in the true poly-Fock space $F_{(\alpha)}^2(\mathbb{C}^n)$ is given by

$$\psi(z) = \sum_{\substack{\lambda\in\mathbb{Z}_+^n \\ \lambda_i\leq\alpha_i}} (-1)^{|\lambda|} \frac{\sqrt{(\alpha-1)!}}{\lambda!(\alpha-1-\lambda)!} \bar{z}^{\alpha-1-\lambda} \partial^\lambda \varphi(z), \quad (15)$$

where $\varphi(z) \in F^2(\mathbb{C}^n)$ and $\partial^\lambda \varphi = \partial^{|\lambda|}\varphi / \partial z_1^{\lambda_1} \cdots \partial z_n^{\lambda_n}$.

(3) The reproducing kernel of the true poly-Fock space $F_{(\alpha)}^2(\mathbb{C}^n)$ can be obtained applying the "creation" operator to the reproducing kernel of the Fock space $F^2(\mathbb{C}^n)$

Remark 4. Using the creation operator defined in [1], we have that the homogeneously poly-Fock space $F_{(k)}^2(\mathbb{C}^n)$ and the poly-Fock space $F_k^2(\mathbb{C}^n)$ are reproducing kernel Hilbert spaces.

On the other hand, for $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}_+^n$ and $y = (y_1, \cdots, y_n) \in \mathbb{R}^n$, consider the function introduced in (8):

$$\tilde{h}_{\alpha-1}(y) = \prod_{j=1}^{n} h_{\alpha_j-1}\left(y_j\right), \quad (16)$$

where $\alpha - 1 = (\alpha_1 - 1, \cdots, \alpha_n - 1)$ and each $h_{\alpha_j-1}(y_j)$ is Hermite's function in $\mathbb{R}$. The set $\{\tilde{h}_{\alpha-1}(y)\}$, with $|\alpha| = n, \cdots, \infty$, forms an ortonormal base in the space $L_2(\mathbb{R}^n, dy)$. We denote $\tilde{H}_{\alpha-1} \subset L_2(\mathbb{R}^n, dy)$ to the one-dimensional space generated by the function $\tilde{h}_{\alpha-1}(y)$, whose orthogonal projection is given by

$$\left(\tilde{P}_{(\alpha-1)}\psi\right)(y) = \tilde{h}_{\alpha-1}(y) \int_{\mathbb{R}^n} \psi(\eta) \tilde{h}_{\alpha-1}(\eta) d\eta. \quad (17)$$

Now, we consider the operators

$$\begin{aligned} U_1 &: L_2(\mathbb{C}^n, d\mu_n) \longrightarrow L_2\left(\mathbb{R}^{2n}, dxdy\right), (U_1\varphi)(x, y) \\ &= \pi^{-n/2} e^{-((x+iy)(x-iy))/2} \varphi(x+iy). \end{aligned}$$

$$U_2 = I \otimes F : L_2(\mathbb{R}^n, dx) \otimes L_2(\mathbb{R}^n, dy) \longrightarrow L_2(\mathbb{R}^n, dx) \otimes L_2(\mathbb{R}^n, dy), \quad (18)$$

with $(F\psi)(y) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\eta y} \psi(\eta) d\eta$.

$$U_3 : L_2\left(\mathbb{R}^{2n}\right) \longrightarrow L_2\left(\mathbb{R}^{2n}\right), (U_3 f)(x, y) = f\left(\frac{1}{\sqrt{2}}(x+y), \frac{1}{\sqrt{2}}(x-y)\right), \quad (19)$$

for $z = (z_1, \cdots, z_n) \in \mathbb{C}^n$, $x = (x_1, \cdots, x_n)$, and $y = (y_1, \cdots, y_n) \in \mathbb{R}^n$. See [1], the operator $U = U_3 U_2 U_1$ from the space $L_2(\mathbb{C}^n, d\mu_n)$ onto $L_2(\mathbb{R}^{2n}, dxdy) = L_2(\mathbb{R}^n, dx) \otimes L_2(\mathbb{R}^n, dy)$ satisfies that the image of the true poly-Fock space $F_{(\alpha)}^2(\mathbb{C}^n)$ is $L_2(\mathbb{R}^n, dx) \otimes \tilde{H}_{\alpha-1}$ and the orthogonal projection $B_{(\alpha)}$ of $L_2(\mathbb{C}^n, d\mu_n)$ onto $F_{(\alpha)}^2(\mathbb{C}^n)$ is unitary equivalent to the operator

$$U B_{(\alpha)} U^{-1} = I \otimes \tilde{P}_{(\alpha-1)}. \quad (20)$$

Analogously, for $\alpha \in \mathbb{Z}_+^n$ and $k \in \mathbb{N}$, we define the spaces

$$\tilde{H}_{\alpha-1}^{\oplus} = \bigoplus_{\substack{\lambda_i\leq\alpha_i \\ \lambda\in\mathbb{Z}_+^n}} \tilde{H}_{\lambda-1}, \tilde{H}_{(k)}^{\oplus} = \bigoplus_{\substack{|\alpha|=k \\ \alpha\in\mathbb{Z}_+^n}} \tilde{H}_{\alpha-1} \text{and} \tilde{H}_k^{\oplus} = \bigoplus_{\substack{|\alpha|=n \\ \alpha\in\mathbb{Z}_+^n}}^{k} \tilde{H}_{\alpha-1}, \quad (21)$$

with respective orthogonal projections

$$\begin{aligned} \left(\tilde{P}_{\alpha-1}\psi\right)(y) &= \sum_{\substack{\lambda_i\leq\alpha_i \\ \lambda\in\mathbb{Z}_+^n}} \tilde{h}_{\lambda-1}(y) \int_{\mathbb{R}^n} \psi(\eta) \tilde{h}_{\lambda-1}(\eta) d\eta, \\ \left(\tilde{P}_{(k)}\psi\right)(y) &= \sum_{\substack{|\alpha|=k \\ \alpha\in\mathbb{Z}_+^n}} \tilde{h}_{\alpha-1}(y) \int_{\mathbb{R}^n} \psi(\eta) \tilde{h}_{\alpha-1}(\eta) d\eta, \\ \left(\tilde{P}_k\psi\right)(y) &= \sum_{\substack{|\alpha|=n \\ \alpha\in\mathbb{Z}_+^n}}^{k} \tilde{h}_{\alpha-1}(y) \int_{\mathbb{R}^n} \psi(\eta) \tilde{h}_{\alpha-1}(\eta) d\eta. \end{aligned} \quad (22)$$

If we denote by B_α, $B_{(k)}$, and B_k the orthogonal projections of $L_2(\mathbb{C}^n, d\mu_n)$ onto $F_\alpha^2(\mathbb{C}^n)$, $F_{(k)}^2(\mathbb{C}^n)$ and $F_k^2(\mathbb{C}^n)$, respectively; thus,

$$\begin{aligned} UB_\alpha U^{-1} &= I \otimes \tilde{P}_{\alpha-1}, \\ UB_{(k)} U^{-1} &= I \otimes \tilde{P}_{(k)}, \\ UB_k U^{-1} &= I \otimes \tilde{P}_k. \end{aligned} \tag{23}$$

Consider the isometric inmersion $\tilde{R}_{0,(\alpha)} : L_2(\mathbb{R}^n, dx) \longrightarrow L_2(\mathbb{R}^{2n}, dxdy)$, defined by $(\tilde{R}_{0,(\alpha)}f)(x,y) = f(x)\tilde{h}_{\alpha-1}(y)$. Whose adjoint operator $\tilde{R}^*_{0,(\alpha)} : L_2(\mathbb{R}^{2n}, dxdy) \longrightarrow L_2(\mathbb{R}^n, dx)$ is given by $(\tilde{R}^*_{0,(\alpha)}\varphi)(x) = \int_{\mathbb{R}^n}\varphi(x,\eta)\tilde{h}_{\alpha-1}(\eta)d\eta$. These operators satisfy the following relations:

$$\begin{aligned} &\tilde{R}^*_{0,(\alpha)}\tilde{R}_{0,(\alpha)} = I : L_2(\mathbb{R}^n, dx) \longrightarrow L_2(\mathbb{R}^n, dx), \\ &\tilde{R}_{0,(\alpha)}\tilde{R}^*_{0,(\alpha)} = I \otimes \tilde{P}_{(\alpha-1)} : L_2\left(\mathbb{R}^{2n}, dxdy\right) \longrightarrow L_2\left(\mathbb{R}^{2n}, dxdy\right). \end{aligned} \tag{24}$$

Now, we define the operator $\tilde{R}_{(\alpha)} = \tilde{R}^*_{0,(\alpha)}U$ from $L_2(\mathbb{C}^n, d\mu_n)$ onto $L_2(\mathbb{R}^n, dx)$, and the adjoint operator $\tilde{R}^*_{(\alpha)} = U^*\tilde{R}_{0,(\alpha)}$, which satisfy

$$\tilde{R}^*_{(\alpha)}\tilde{R}_{(\alpha)} = B_{(\alpha)} : L_2(\mathbb{C}^n, d\mu_n) \longrightarrow F^2_{(\alpha)}(\mathbb{C}^n), \tag{25}$$

$$\tilde{R}_{(\alpha)}\tilde{R}^*_{(\alpha)} = I : L_2(\mathbb{R}^n, dx) \longrightarrow L_2(\mathbb{R}^n, dx). \tag{26}$$

For the multi-index $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}^n_+$, remember that $\mathbf{P}(\alpha) = \alpha_1 \cdots \alpha_n$, so we introduce the isometric inmersion $\tilde{R}_{0,\alpha} : (L_2(\mathbb{R}^n, dx))^{\mathbf{P}(\alpha)} \longrightarrow L_2(\mathbb{R}^n, dx) \otimes L_2(\mathbb{R}^n, dy)$, defined by

$$\left(\tilde{R}_{0,\alpha}f\right)(x,y) = \sum_{\substack{\lambda_i \le \alpha_i \\ \lambda \in \mathbb{Z}^n_+}} f_\lambda(x)\tilde{h}_{\lambda-1}(y) = f(x)[N_\alpha(y)]^T, \tag{27}$$

where the $\mathbf{P}(\alpha)$-tuples $f = (f_\lambda)_{\lambda_i \le \alpha_i}$, and $N_\alpha(y) = (\tilde{h}_{\lambda-1}(y))_{\lambda_i \le \alpha_i}$ are taken over all multi-indices $\lambda = (\lambda_1, \cdots, \lambda_n)$ whose entries are less than or equal to each of the corresponding entries of α. The adjoint operator $\tilde{R}^*_{0,\alpha} : L_2(\mathbb{R}^{2n}, dxdy) \longrightarrow (L_2(\mathbb{R}^n), dx)^{\mathbf{P}(\alpha)}$ is given by

$$\begin{aligned} \left(\tilde{R}^*_{0,\alpha}\varphi\right)(x) &= \left(\int_{\mathbb{R}^n}\varphi(x,y)\tilde{h}_{\lambda-1}(y)dy\right)_{\substack{\lambda_i \le \alpha_i \\ \lambda \in \mathbb{Z}^n_+}} \\ &= \int_{\mathbb{R}^n}\varphi(x,y)N_\alpha(y)dy. \end{aligned} \tag{28}$$

We have

$$\begin{aligned} &\tilde{R}^*_{0,\alpha}\tilde{R}_{0,\alpha} = I : (L_2(\mathbb{R}^n, dx))^{\mathbf{P}(\alpha)} \longrightarrow (L_2(\mathbb{R}^n, dx))^{\mathbf{P}(\alpha)}, \\ &\tilde{R}_{0,\alpha}\tilde{R}^*_{0,\alpha} = I \otimes \tilde{P}_{\alpha-1} : L_2\left(\mathbb{R}^{2n}, dxdy\right) \longrightarrow L_2(\mathbb{R}^n, dx) \otimes \tilde{H}^{\oplus}_{\alpha-1}. \end{aligned} \tag{29}$$

The operator $\tilde{R}_\alpha = \tilde{R}^*_{0,\alpha}U$ from $L_2(\mathbb{C}^n, d\mu_n)$ onto $(L_2(\mathbb{R}^n, dx))^{\mathbf{P}(\alpha)}$ and its adjoint $\tilde{R}^*_\alpha = U^*\tilde{R}_{0,\alpha}$ satisfy

$$\tilde{R}^*_\alpha\tilde{R}_\alpha = B_\alpha : L_2(\mathbb{C}^n, d\mu_n) \longrightarrow F^2_\alpha(\mathbb{C}^n), \tag{30}$$

$$\tilde{R}_\alpha\tilde{R}^*_\alpha = I : (L_2(\mathbb{R}^n, dx))^{\mathbf{P}(\alpha)} \longrightarrow (L_2(\mathbb{R}^n, dx))^{\mathbf{P}(\alpha)}. \tag{31}$$

Similarly to (27), we define the operators

$$\begin{aligned} &\tilde{R}_{0,(k)} : (L_2(\mathbb{R}^n, dx))^{\mathbf{s}_{(k)}} \longrightarrow L_2\left(\mathbb{R}^{2n}, dxdy\right), \left(\tilde{R}_{0,(k)}f\right)(x,y) = f(x)\left[N_{(k)}(y)\right]^T, \\ &\tilde{R}_{0,k} : (L_2(\mathbb{R}^n, dx))^{\mathbf{s}_k} \longrightarrow L_2\left(\mathbb{R}^{2n}, dxdy\right), \left(\tilde{R}_{0,k}f\right)(x,y) = f(x)[N_k(y)]^T. \end{aligned} \tag{32}$$

The arrangement $N_{(k)}(y)$ is formed by the functions $\tilde{h}_{\alpha-1}(y)$ over all multi-indices α such that $|\alpha| = k$. Analogously, the functions in $N_k(y)$ are indexed with the multi-indices whose absolute value are less or equal to k. Remember that $\mathbf{s}_{(k)} = \begin{pmatrix} k-1 \\ n-1 \end{pmatrix}$ and $\mathbf{s}_k = \begin{pmatrix} k \\ k-n \end{pmatrix}$. The adjoint operators $\tilde{R}^*_{0,(k)}$ and $\tilde{R}^*_{0,k}$ are defined similarly to (28). Finally, the operators $\tilde{R}_{(k)} = \tilde{R}^*_{0,(k)}U$, $\tilde{R}_k = \tilde{R}^*_{0,k}U$, and its adjoint operators satisfy

$$\begin{aligned} \tilde{R}^*_{(k)}\tilde{R}_{(k)} &= B_{(k)} : L_2(\mathbb{C}^n, d\mu_n) \longrightarrow F^2_{(k)}(\mathbb{C}^n), \\ \tilde{R}_{(k)}\tilde{R}^*_{(k)} &= I : (L_2(\mathbb{R}^n, dx))^{\mathbf{s}_{(k)}} \longrightarrow (L_2(\mathbb{R}^n, dx))^{\mathbf{s}_{(k)}}, \\ \tilde{R}^*_k\tilde{R}_k &= B_k : L_2(\mathbb{C}^n, d\mu_n) \longrightarrow F^2_k(\mathbb{C}^n), \\ \tilde{R}_k\tilde{R}^*_k &= I : (L_2(\mathbb{R}^n, dx))^{\mathbf{s}_k} \longrightarrow (L_2(\mathbb{R}^n, dx))^{\mathbf{s}_k}. \end{aligned} \tag{33}$$

3. Toeplitz Operators with Horizontal Symbols

In this section, we define Toeplitz operators with certain class of symbols acting on the poly-Fock, true poly-Fock, and homogeneously poly-Fock spaces over $\mathbb{C}^n$. And, we prove that this operators are unitary equivalent to certain multiplication operators. Let $a(z) = a(x_1, \cdots, x_n)$ be a function in $L_\infty(\mathbb{R}^n, dx)$ depending only on $x = (\mathrm{Re}\, z_1, \cdots, \mathrm{Re}\, z_n)$, we call to this kind of functions horizontal symbols. Henceforth, $\alpha \in \mathbb{Z}^n_+$ denote a fixed multi-index and $k \in \mathbb{N}$ a fixed natural number.

Definition 5. Let $a(x)$ be a horizontal symbol. The Toeplitz operator with symbol $a(x)$, acting on the true poly-Fock space (or poly-Fock space) of order α is defined as

$$T_{(\alpha),a} : \varphi \in F^2_{(\alpha)}(\mathbb{C}^n) \mapsto B_{(\alpha)}(a\varphi) \in F^2_{(\alpha)}(\mathbb{C}^n), \tag{34}$$

$$T_{\alpha,a} : \varphi \in F^2_{\alpha}(\mathbb{C}^n) \mapsto B_{\alpha}(a\varphi) \in F^2_{\alpha}(\mathbb{C}^n). \tag{35}$$

Similarly, the Toeplitz operator with symbol $a(x)$, acting on the homogeneously poly-Fock space (or poly-Fock space) of order k is defined as

$$T_{(k),a} : \varphi \in F^2_{(k)}(\mathbb{C}^n) \mapsto B_{(k)}(a\varphi) \in F^2_{(k)}(\mathbb{C}^n), \tag{36}$$

$$T_{k,a} : \varphi \in F^2_{k}(\mathbb{C}^n) \mapsto B_{k}(a\varphi) \in F^2_{k}(\mathbb{C}^n). \tag{37}$$

The following theorem characterizes the Toeplitz operators with horizontal symbols acting on the true poly-Fock space $F^2_{(\alpha)}(\mathbb{C}^n)$.

Theorem 6. *Let $a(x) \in L_\infty(\mathbb{R}^n, dx)$ be a horizontal symbol, then the Toeplitz operator $T_{(\alpha),a}$ acting on $F^2_{(\alpha)}(\mathbb{C}^n)$ is unitary equivalent to the multiplication operator $\gamma^{(\alpha),a} I = \tilde{R}_{(\alpha)} T_{(\alpha),a} \tilde{R}^*_{(\alpha)}$ acting on $L_2(\mathbb{R}^n, dx)$ where the function $\gamma^{(\alpha),a}$ is given by*

$$\gamma^{(\alpha),a}(x) = \int_{\mathbb{R}^n} a\left(\frac{x+y}{\sqrt{2}}\right)\left(\tilde{h}_{\alpha-1}(y)\right)\left(\tilde{h}_{\alpha-1}(y)\right)^2 dy, \tag{38}$$

and $\tilde{h}_{\alpha-1}(y)$ is defined in (8).

Proof. Remember that $\tilde{R}_{(\alpha)} = \tilde{R}^*_{0,(\alpha)} U$ and using (25) and (26), we obtain

$$\begin{aligned}
\tilde{R}_{(\alpha)} T_{(\alpha),a} \tilde{R}^*_{(\alpha)} &= \tilde{R}_{(\alpha)} B_{(\alpha)} a B_{(\alpha)} \tilde{R}^*_{(\alpha)} = \tilde{R}_{(\alpha)} \tilde{R}^*_{(\alpha)} \tilde{R}_{(\alpha)} a \tilde{R}^*_{(\alpha)} \tilde{R}_{(\alpha)} \tilde{R}^*_{(\alpha)} \\
&= \tilde{R}_{(\alpha)} a \tilde{R}^*_{(\alpha)} = \tilde{R}^*_{0,(\alpha)} U_3 U_2 U_1 a(x) U_1^{-1} U_2^{-1} U_3^{-1} \tilde{R}_{0,(\alpha)} \\
&= \tilde{R}^*_{0,(\alpha)} U_3 a(x) U_3^{-1} \tilde{R}_{0,(\alpha)} = \tilde{R}^*_{0,(\alpha)} a\left(\frac{x+y}{\sqrt{2}}\right) \tilde{R}_{0,(\alpha)}.
\end{aligned} \tag{39}$$

Explicitly for a function $f \in L_2(\mathbb{R}^n)$, we have

$$\begin{aligned}
\left(\tilde{R}^*_{0,(\alpha)} a\left(\frac{x+y}{\sqrt{2}}\right) \tilde{R}_{0,(\alpha)} f\right)(x) &= \int_{\mathbb{R}^n} a\left(\frac{x+y}{\sqrt{2}}\right) f(x) \left(\tilde{h}_{\alpha-1}(y)\right)^2 \\
&\quad \cdot dy = \gamma^{(\alpha),a}(x) \cdot f(x).
\end{aligned} \tag{40}$$

We call to the function $\gamma^{(\alpha),a}(x)$ the α^{th} spectral function for the Toeplitz operator with horizontal symbol a acting on the true poly-Fock space $F^2_{(\alpha)}(\mathbb{C}^n)$.

Naturally, we can extend the above result to the case of the Toeplitz operator with horizontal symbols acting on the poly-Fock space $F^2_{\alpha}(\mathbb{C}^n)$. □

Theorem 7. *Let $a(x) \in L_\infty(\mathbb{R}^n, dx)$ be a horizontal symbol; thus, the Toeplitz operator $T_{\alpha,a}$ acting on $F^2_{\alpha}(\mathbb{C}^n)$ is unitary equivalent to the multiplication operator $\gamma^{\alpha,a}(x) I = \tilde{R}_{\alpha} T_{\alpha,a} \tilde{R}^*_{\alpha}$, acting on $(L_2(\mathbb{R}^n, dx))^{\boldsymbol{P}(\alpha)}$, where the matrix $\gamma^{\alpha,a}$ is given by*

$$\gamma^{\alpha,a}(x) = \int_{\mathbb{R}^n} a\left(\frac{x+y}{\sqrt{2}}\right) [N_\alpha(y)]^T N_\alpha(y) dy. \tag{41}$$

That is, each component function is equal to

$$[\gamma^{\alpha,a}]_{\lambda\mu}(x) = \int_{\mathbb{R}^n} a\left(\frac{x+y}{\sqrt{2}}\right) \tilde{h}_{\lambda-1}(y) \tilde{h}_{\mu-1}(y) dy, \tag{42}$$

with $\lambda, \mu \in \mathbb{Z}^n_+$ such that $\lambda_i, \mu_i \le \alpha_i$ and $N_\alpha(y)$ is defined in (27).

Proof. Since $\tilde{R}_\alpha = \tilde{R}^*_{0,\alpha} U$ and using (30) and (31), we obtain

$$\begin{aligned}
\tilde{R}_\alpha T_{\alpha,a} \tilde{R}^*_\alpha &= \tilde{R}_\alpha B_\alpha a B_\alpha \tilde{R}^*_\alpha = \tilde{R}_\alpha \tilde{R}^*_\alpha \tilde{R}_\alpha a \tilde{R}^*_\alpha \tilde{R}_\alpha \tilde{R}^*_\alpha = \tilde{R}_\alpha a \tilde{R}^*_\alpha \\
&= \tilde{R}^*_{0,\alpha} U_3 U_2 U_1 a(x) U_1^{-1} U_2^{-1} U_3^{-1} \tilde{R}_{0,\alpha} \\
&= \tilde{R}^*_{0,\alpha} U_3 a(x) U_3^{-1} \tilde{R}_{0,\alpha} = \tilde{R}^*_{0,\alpha} a\left(\frac{x+y}{\sqrt{2}}\right) \tilde{R}_{0,\alpha}.
\end{aligned} \tag{43}$$

Calculating for a function $f \in (L_2(\mathbb{R}^n, dx))^{\mathbf{P}(\alpha)}$

$$\begin{aligned}
&\left[\tilde{R}^*_{0,\alpha} a\left(\frac{x+y}{\sqrt{2}}\right) \tilde{R}_{0,\alpha} f\right](x) = \tilde{R}^*_{0,\alpha}\left(a\left(\frac{x+y}{\sqrt{2}}\right) f(x) [N_\alpha(y)]^T\right) \\
&= \int_{\mathbb{R}^n} \left[a\left(\frac{x+y}{\sqrt{2}}\right) f(x) [N_\alpha(y)]^T\right] N_\alpha(y) \\
&\quad \cdot dy = \int_{\mathbb{R}^n} a\left(\frac{x+y}{\sqrt{2}}\right) [N_\alpha(y)]^T N_\alpha(y) f(x) dy = \gamma^{\alpha,a}(x) f(x).
\end{aligned} \tag{44}$$

We have the next two theorems, whose proofs are analogous to the above one. □

Theorem 8. *For a horizontal symbol $a(x) \in L_\infty(\mathbb{R}^n, dx)$, the Toeplitz operator $T_{(k),a}$ acting on the homogeneously poly-Fock space $F^2_{(k)}(\mathbb{C}^n)$ is unitary equivalent to the multiplication operator $\gamma^{(k),a}(x) I = \tilde{R}_{(k)} T_{(k),a} \tilde{R}^*_{(k)}$, acting on $(L_2(\mathbb{R}^n, dx))^{\boldsymbol{s}_{(k)}}$, where*

$$\gamma^{(k),a}(x) = \int_{\mathbb{R}^n} a\left(\frac{x+y}{\sqrt{2}}\right) \left[N_{(k)}(y)\right]^T N_{(k)}(y) dy. \tag{45}$$

Theorem 9. *For a horizontal symbol $a(x) \in L_\infty(\mathbb{R}^n, dx)$, the Toeplitz operator $T_{k,a}$ acting on the poly-Fock space $F^2_k(\mathbb{C}^n)$ is unitary equivalent to the multiplication operator $\gamma^{k,a}(x) I = \tilde{R}_k T_{k,a} \tilde{R}^*_k$, acting on $(L_2(\mathbb{R}^n, dx))^{\boldsymbol{s}_k}$, where*

$$\gamma^{k,a}(x) = \int_{\mathbb{R}^n} a\left(\frac{x+y}{\sqrt{2}}\right) [N_k(y)]^T N_k(y) dy. \tag{46}$$

We call to the matrices $\gamma^{\alpha,a}(x)$, $\gamma^{(k),a}(x)$, and $\gamma^{k,a}(x)$ the spectral matrices correspondent to the Toeplitz operator with horizontal symbol $a(x)$, acting on the poly-Fock space

$F^2_\alpha(\mathbb{C}^n)$, on the homogeneously poly-Fock space $F^2_{(k)}(\mathbb{C}^n)$, and on the poly-Fock space $F^2_k(\mathbb{C}^n)$, respectively.

Remark 10. The components of the spectral matrices $\gamma^{\alpha,a}(x)$, $\gamma^{(k),a}(x)$, and $\gamma^{k,a}(x)$ can be expressed as a convolution of functions $\bar{a} * (\tilde{h}_\lambda \tilde{h}_\mu)$, where $\bar{a}(x) = a(x/\sqrt{2}) \in L_\infty(\mathbb{R}^n)$. Since $\tilde{h}_\lambda \tilde{h}_\mu \in L_1(\mathbb{R}^n)$, from [7], p. 283, we guaranteed that they belong in the set of uniformily continuous functions $C_b(\mathbb{R}^n)$.

4. The C^*-Algebras Generated by Toeplitz Operators with Extended Horizontal Symbols

In this section, we introduce the concept of extended horizontal symbol and we describe the C^*-algebras generated by Toeplitz operators with these symbols acting on the poly-Fock spaces and on the homogeneously poly-Fock spaces. Following the terminology and the notation introduced in [8], Section 3, we have the following.

Definition 11. Let $a(x) \in L_\infty(\mathbb{R}^n)$ be a horizontal symbol. We say that $a(x)$ is an extended horizontal symbol if there exists a function $a_\infty(x) \in C(\mathbb{S}^{n-1})$ such that

$$\lim_{R\longrightarrow\infty} \sup_{\|x\|>R} \left| a(x) - a_\infty\left(\frac{x}{\|x\|}\right)\right| = 0. \tag{47}$$

We denote by $\mathrm{HS}(\mathbb{R}^n)$ the set of extended horizontal symbols. We note that $\mathrm{HS}(\mathbb{R}^n)$ equipped with the supremum norm is a C^*-subalgebra of $L_\infty(\mathbb{R}^n)$.

The compact of maximal ideals of the C^*-algebra HS $(\mathbb{R}^n)$ coincides with the compactification of $\mathbb{R}^n$, denoted by $\widetilde{\mathbb{R}}^n = \mathbb{R}^n \cup \mathbb{S}^{n-1}_\infty$, obtained by adding an "infinitely far" n-sphere $\mathbb{S}^{n-1}_\infty$. This compact space is isomorphic to $\bar{\mathbb{D}}^n$. We can identify the elements $s_\infty \in \mathbb{S}^{n-1}_\infty$ with the points $s \in \mathbb{S}^{n-1}$ as follows. For every extended horizontal symbol $a \in H S(\mathbb{R}^n)$, we have

$$\lim_{t\longrightarrow\infty} a(ts) = a(s_\infty) = a_\infty(s). \tag{48}$$

We identify the extended horizontal symbols $a(x)$ with its extensions to the complex space $\mathbb{C}^n$, where $x = \operatorname{Re}\, z$.

The following lemma shows that the different spectral matrices $\gamma^{\alpha,a}(x)$, $\gamma^{(k),a}(x)$, and $\gamma^{k,a}(x)$, corresponding to Toeplitz operators with extended horizontal symbol $a(x)$, posses a limit value to infinity in any direction. We write $\gamma^{\square,a}(x)$ to refer to any of this spectral matrices.

Lemma 12. *Let $a(x)$ be an extended horizontal symbol and let $x_0 \in \mathbb{S}^{n-1}$. Then the spectral matrix $\gamma^{\square,a}(x)$ satisfies*

$$\lim_{t\longrightarrow\infty} \gamma^{\square,a}(x)(tx_0) = a_\infty(x_0) I. \tag{49}$$

Proof. We apply the dominated convergence theorem. Let $\lambda, \mu \in \mathbb{Z}^n_+$ be two multi-indices corresponding to some entry of the spectral matrix. For each $m \in \mathbb{N}$, we consider the function $F_m : \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by

$$F_m(y) = a\left(\frac{mx_0 + y}{\sqrt{2}}\right) \tilde{h}_{\lambda-1}(y) \tilde{h}_{\mu-1}(y). \tag{50}$$

Since $\tilde{h}_{\lambda-1}(y), \tilde{h}_{\mu-1}(y) \in L_2(\mathbb{R}^n)$, and $a(x) \in L_\infty(\mathbb{R}^n)$ we have $F_m(y) \in L_1(\mathbb{R}^n)$. Note that the integrable function $\|a\|_\infty \|\tilde{h}_{\lambda-1}(y)\tilde{h}_{\mu-1}(y)\|$ limits to $F_m(y)$ for all $m \in \mathbb{N}$. Since the function a_∞ is continuous, we have

$$\lim_{m\longrightarrow\infty} a_\infty\left(\frac{x_0 + (1/m)y}{\|x_0 + (1/m)y\|}\right) = a_\infty(x_0), \tag{51}$$

and using (48)

$$\begin{aligned} &\lim_{m\longrightarrow\infty}\left[\lim_{t\longrightarrow\infty} a\left(\frac{t(x_0 + (1/m)y)}{\|x_0 + (1/m)y\|}\right)\right] \\ &= \lim_{m\longrightarrow\infty}\left[\lim_{t\longrightarrow\infty} a\left(\frac{t(x_0 + (1/m)y)}{\sqrt{2}}\right)\right] = a_\infty(x_0). \end{aligned} \tag{52}$$

We can take this limit along the line $t = m$; thus,

$$\lim_{m\longrightarrow\infty} a\left(\frac{mx_0 + y}{\sqrt{2}}\right) = a_\infty(x_0), \tag{53}$$

and $F_m(y) \longrightarrow a_\infty(x)\tilde{h}_{\lambda-1}(y)\tilde{h}_{\mu-1}(y)$ when $m \longrightarrow \infty$. Therefore,

$$\begin{aligned} &\lim_{m\longrightarrow\infty} \int_{\mathbb{R}^n} a\left(\frac{mx_0 + y}{\sqrt{2}}\right) \tilde{h}_{\lambda-1}(y)\tilde{h}_{\mu-1}(y)dy \\ &= a_\infty(x) \int_{\mathbb{R}^n} \tilde{h}_{\lambda-1}(y)\tilde{h}_{\mu-1}(y)dy = a_\infty(x)\delta_{\lambda\mu}. \end{aligned} \tag{54}$$

Let $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}^n_+$ be a fixed multi-index and $k \in \mathbb{N}$ be a fixed natural number. □

Definition 13. We introduce the following C^* algebras, which are very useful to our study

(i) Denote $\mathcal{G}^H_{(\alpha)} = \{\gamma^{(\alpha),a} : a(x) \in HS(\mathbb{R}^n)\}$ to the set of all horizontal spectral functions

(ii) Denote $\mathcal{G}^H_\alpha = \{\gamma^{\alpha,a} : a(x) \in HS(\mathbb{R}^n)\}$ to the set of all horizontal spectral matrices of order α

(iii) Denote $\mathcal{G}^H_{(k)} = \{\gamma^{(k),a} : a(x) \in HS(\mathbb{R}^n)\}$ to the set of all horizontal spectral matrices of order exactly k

(iv) Denote $\mathcal{G}^H_k = \{\gamma^{k,a} : a(x) \in HS(\mathbb{R}^n)\}$ to the set of all horizontal spectral matrices of order at most k

(v) Denote by $\mathcal{T}^{(\alpha)}_\infty$ the C^*-algebra generated by the set of Toeplitz operators $T_{(\alpha),a}$ acting on the true poly-Fock space $F^2_{(\alpha)}(\mathbb{C}^n)$, with $a(x) \in HS(\mathbb{R}^n)$

(vi) Denote by $\mathcal{T}_{\infty}^{\alpha}$ the C^*-algebra generated by the set of Toeplitz operators $T_{\alpha,a}$ acting in the poly-Fock space $F_{\alpha}^{2}(\mathbb{C}^n)$, with $a(x) \in HS(\mathbb{R}^n)$.

(vii) Denote by $\mathcal{T}_{\infty}^{(k)}$ the C^*-algebra generated by the set of Toeplitz operators $T_{(k),a}$ acting on the homogeneously poly-Fock space $F_{(k)}^{2}(\mathbb{C}^n)$, with $a(x) \in HS(\mathbb{R}^n)$

(viii) Denote by $\mathcal{T}_{\infty}^{k}$ the C^*-algebra generated by the set of Toeplitz operators $T_{k,a}$ acting in the poly-Fock space $F_{k}^{2}(\mathbb{C}^n)$, with $a(x) \in HS(\mathbb{R}^n)$

We have the following results.

Corollary 14. *The C^*-algebra $\mathcal{T}_{\infty}^{(\alpha)}$ is isometrically isomorphic to the C^*-algebra $\mathfrak{G}_{(\alpha)}^{H}$ generated by $\mathcal{G}_{(\alpha)}^{H}$.*

Corollary 15. *The C^* -algebra $\mathcal{T}_{\infty}^{\alpha}$ is isometrically isomorphic to the C^* -algebra $\mathfrak{G}_{\alpha}^{H}$ generated by $\mathcal{G}_{\alpha}^{H}$.*

Corollary 16. *The C^* -algebra $\mathcal{T}_{\infty}^{(k)}$ is isometrically isomorphic to the C^* -algebra $\mathfrak{G}_{(k)}^{H}$ generated by $\mathcal{G}_{(k)}^{H}$.*

Corollary 17. *The C^* -algebra $\mathcal{T}_{\infty}^{k}$ is isometrically isomorphic to the C^* -algebra $\mathfrak{G}_{k}^{H}$ generated by $\mathcal{G}_{k}^{H}$.*

Now, we describe the C^*-algebras $\mathfrak{G}_{\alpha}^{H}$, $\mathfrak{G}_{(k)}^{H}$, and $\mathfrak{G}_{k}^{H}$, generated by the different spectral matrices. First, we start with $\mathfrak{G}_{\alpha}^{H}$. Consider the C^*-algebra defined by $\mathfrak{C}_{\alpha} = M_{\mathbf{P}(\alpha)}(\mathbb{C}) \otimes C(\mathbb{R}^n \cup \mathbb{S}_{\infty}^{n-1})$, which consists of the algebra of all $\mathbf{P}(\alpha) \times \mathbf{P}(\alpha)$ matrices with entries in $C(\mathbb{R}^n \cup \mathbb{S}_{\infty}^{n-1})$, where $\mathbf{P}(\alpha) = \alpha_1 \cdots \alpha_n$. Now, we introduce the C^*-subalgebra $\mathfrak{D}_{\alpha}$ given by

$$\mathfrak{D}_{\alpha} = \left\{ M \in \mathfrak{C}_{\alpha} : \lim_{t \longrightarrow \infty} M(tx) = M(x_{\infty}) = \mathbb{C}I, \text{forall} x \in \mathbb{S}^{n-1} \right\}. \tag{55}$$

We note that $\mathfrak{G}_{\alpha}^{H}$ is a C^*-subalgebra of $\mathfrak{D}_{\alpha}$. In fact, we prove that $\mathfrak{G}_{\alpha}^{H} = \mathfrak{D}_{\alpha}$. For this, we use a Stone-Weirstrass theorem. We need to show that $\mathfrak{G}_{\alpha}^{H}$ separates the pure states of $\mathfrak{D}_{\alpha}$.

Since $\mathfrak{D}_{\alpha}$ is a C^*-bundle, the set of all its pure states is completely determined by the pure states on the fibers:

$$\mathfrak{D}_{\alpha}(x) = \begin{cases} M_{\mathbf{P}(\alpha)}(\mathbb{C}), & \text{if } x \in \mathbb{R}^n, \\ \mathbb{C}, & \text{if } x = x_{\infty} \in \mathbb{S}_{\infty}^{n-1}. \end{cases} \tag{56}$$

So each pure state of $\mathfrak{D}_{\alpha}$ has the form

$$f(M) = f_x(M(x)), M \in \mathfrak{D}_{\alpha}, \tag{57}$$

where $x \in \mathbb{R}^n \cup \mathbb{S}_{\infty}^{n-1}$ and f_x is a pure state of $\mathfrak{D}_{\alpha}(x)$. Every pure state in the matrix algebra $M_{\mathbf{P}(\alpha)}(\mathbb{C})$ is given by a functional f_{ν} defined as

$$f_{\nu}(Q) = \langle Q\nu, \nu \rangle, Q \in M_{\mathbf{P}(\alpha)}(\mathbb{C}), \tag{58}$$

with $\nu \in \mathbb{S}^{\mathbf{P}(\alpha)} = \{ z \in \mathbb{C}^{\mathbf{P}(\alpha)} : |z| = 1 \}$. Moreover, if $\nu, w \in \mathbb{S}^{\mathbf{P}(\alpha)}$ such that $f_{\nu} = f_{w}$; thus, $\nu = tw$ where $t \in \mathbb{C}$ and $|t| = 1$, see [9] for more details.

In consequence, the set of all pure states of $\mathfrak{D}_{\alpha}$ consists of all functional of the form

$$f_{x,\nu}(M) = \langle M(x)\nu, \nu \rangle, M \in \mathfrak{D}_{\alpha}, \tag{59}$$

with $x \in \mathbb{R}^n \cup \mathbb{S}_{\infty}^{n-1}$ and $\nu \in \mathbb{S}^{\mathbf{P}(\alpha)}$.

In the cases of the C^*-algebras $\mathfrak{G}_{(k)}^{H}$ and $\mathfrak{G}_{k}^{H}$, we consider the C^*-algebras $\mathfrak{C}_{(k)} = M_{\mathbf{s}_{(k)}}(\mathbb{C}) \otimes C(\mathbb{R}^n \cup \mathbb{S}_{\infty}^{n-1})$ and $\mathfrak{C}_{k} = M_{\mathbf{s}_{k}}(\mathbb{C}) \otimes C(\mathbb{R}^n \cup \mathbb{S}_{\infty}^{n-1})$. And their corresponding C^*-subalgebras $\mathfrak{D}_{(k)}$ and $\mathfrak{D}_{k}$. For this two C^*-subalgebras, the pure states are determined in a similar way to (59). Remember that $\mathbf{s}_{(k)} = \begin{pmatrix} k-1 \\ n-1 \end{pmatrix}$ and $\mathbf{s}_{k} = \begin{pmatrix} k \\ k-n \end{pmatrix}$.

Now to fixing ideas, we return to the previous case $\mathfrak{G}_{\alpha}^{H}$; the other ones are totally analogous, and we analyzed them at the end of this section.

For each element $x_{\infty} \in \mathbb{S}_{\infty}^{n-1}$, we have only one pure state for any $\nu, w \in \mathbb{S}^{\mathbf{P}(\alpha)}$, that is, $f_{x_{\infty},\nu} = f_{x_{\infty},w}$. To separate the pure states corresponding to two different elements $x_{0,\infty}$ and $x_{1,\infty}$ in $\mathbb{S}_{\infty}^{n-1}$, using the identification given by (48), we note that the corresponding elements $x_0, x_1 \in \mathbb{S}^{n-1}$ differ at least one coordinate. Suppose that the j^{th} coordinate of this vectors is different, that is, $x_{0,j} \neq x_{1,j}$. Thus, we consider the horizontal extended symbol $C^j : \mathbb{R}^n \longrightarrow \mathbb{C}$ defined by

$$C^j(x) = \begin{cases} 0, & \text{if } x = 0 \in \mathbb{R}^n, \\ \dfrac{x_j}{\|x\|}, & \text{in otherwise.} \end{cases} \tag{60}$$

For $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ with $\|x\| = 1$, we have $C_{\infty}^{j}(x) = \lim_{t \longrightarrow \infty} C^j(tx) = x_j$. Clearly, C_{∞}^{j} is a continuous function. Now, consider the spectral matrix γ^{α,C^j}, for every x_{∞}, we have

$$f_{x_{\infty},\nu}\left(\gamma^{\alpha,C^j}\right) = \left\langle \gamma^{\alpha,C^j}(x_{\infty})\nu, \nu \right\rangle = \left\langle C_{\infty}^{j}(x) I \nu, \nu \right\rangle = x_j. \tag{61}$$

Hence, for $x_{0,\infty} \neq x_{1,\infty}$, we have

$$f_{x_{0,\infty},\nu}\left(\gamma^{\alpha,C^j}(x)\right) \neq f_{x_{1,\infty},\nu}\left(\gamma^{\alpha,C^j}(x)\right). \tag{62}$$

Thus, the spectral matrix $\gamma^{\alpha,C^j}(x)$ separates the corresponding pure states.

In the case when we have the pure states corresponding to the points $x_0 \in \mathbb{R}^n$ and $x_{1,\infty} \in \mathbb{S}_{\infty}^{n-1}$, we consider the set $[0,p]=[0,p_1]\times\cdots\times[0,p_n]$ with $p=(p_1,\cdots,p_n)\in(\mathbb{R}^+)^n$, and the function $c(x)=\chi_{[0,p]}(x)$. We have $c_\infty(x)\equiv 0$. We write $\gamma^{\alpha,p}(x)$ instead of $\gamma^{\alpha,c}(x)$. Notice that

$$\begin{aligned}\gamma^{\alpha,p}(x)&=\int_{-x_n}^{\sqrt{2}p_n-x_n}\cdots\int_{-x_1}^{\sqrt{2}p_1-x_1}N_\alpha(y)[N_\alpha(y)]^T dy_1\cdots dy_n\\&=\int_{[-x,\sqrt{2}p-x]}N_\alpha(y)[N_\alpha(y)]^T dy.\end{aligned}\tag{63}$$

For $x_0\in\mathbb{R}^n$, we have

$$\begin{aligned}f_{x_0,v}(\gamma^{\alpha,p})&=\langle\gamma^{\alpha,p}x_0v,v\rangle=\left\langle\int_{[-x,\sqrt{2}p-x]}N_\alpha(y)[N_\alpha(y)]^T dyv,v\right\rangle\\&=\int_{[-x,\sqrt{2}p-x]}\left\langle N_\alpha(y)[N_\alpha(y)]^T v,v\right\rangle\\&\quad\cdot dy=\int_{[-x,\sqrt{2}p-x]}\langle v,N_\alpha(y)\rangle\langle N_\alpha(y),v\rangle\\&\quad\cdot dy=\int_{[-x,\sqrt{2}p-x]}|\langle v,N_\alpha(y)\rangle|^2dy.\end{aligned}\tag{64}$$

Note that $f_{x_0,v}(\gamma^{\alpha,p})>0$ because $|<v,N_\alpha(y)>|^2\geq 0$, except in a set of measure zero. On the other hand, if $x_1\in\mathbb{S}^{n-1}$ is the corresponding element of $x_{1,\infty}\in\mathbb{S}_\infty^{n-1}$, we have

$$f_{x_{1,\infty},v}(\gamma^{\alpha,p})=\left\langle\gamma^{\alpha,p}(x_{1,\infty})v,v\right\rangle=\langle c_\infty(x_1)Iv,v\rangle=0.\tag{65}$$

Therefore, the spectral matrix $\gamma^{\alpha,p}$ separates the pure states of the points x_0 and $x_{1,\infty}$.

To separate the pure states corresponding to two points $x_0,x_1\in\mathbb{R}^n$, we consider again the extended horizontal symbol $c(x)=\chi_{[0,p]}(x)$. From (64), we define the function $h_v(y)=|<v,N_\alpha(y)>|^2$. We can express this function as $h_v(y)=q_v(y)e^{-(y_1^2+\cdots+y_n^2)}$, with

$$q_v(y)=\left|\sum_{\substack{\lambda\in\mathbb{Z}_+^n\\ \lambda_i\leq\alpha_i}}v_\lambda\left(2^\lambda\lambda!\pi^{\frac{n}{2}}\right)^{-1/2}\tilde{H}_{\lambda-1}(y)\right|^2,\tag{66}$$

where $\tilde{H}_{\lambda-1}(y)$ is the product of the n one-dimensional Hermite polynomials $H_{\lambda_1-1}(y_1),\cdots,H_{\lambda_n-1}(y_n)$, so $q_v(y)$ is a nonnegative-valued polynomial of degree at most $2|\alpha|-2n$.

The following lemma provides us a tool to prove that the C^*-algebra $\mathfrak{G}_\alpha^H$ separates the pure states of $\mathfrak{D}_\alpha$ of the form $f_{x_0,v},f_{x_1,w}$, where $x_0\neq x_1$ and $v,w\in\mathbb{S}^{\mathbf{P}(\alpha)}$.

Lemma 18. *We assume that $v,w\in\mathbb{S}^{P(\alpha)}$,$x_0,x_1\in\mathbb{R}^n$ and $\gamma^{\alpha,p}\in\mathfrak{G}_\alpha^H$ with $p\in(\mathbb{R}^+)^n$. If $f_{x_0,v}(\gamma^{\alpha,p}(x))=f_{x_0,w}(\gamma^{\alpha,p}(x))$ for all vector p, then $x_0=x_1$. Moreover, $|\langle v,N_\alpha(y)\rangle|^2=|\langle w,N_\alpha(y)\rangle|^2$ for all $y\in\mathbb{R}^n$.*

Proof. The hypothesis $f_{x_0,v}(\gamma^{\alpha,p}(x))=f_{x_1,w}(\gamma^{\alpha,p}(x))$ is equivalent to the following equation:

$$\int_{[-x_0,\sqrt{2}p-x_0]}q_v(y)e^{-(y_1^2+\cdots+y_n^2)}dy=\int_{[-x_1,\sqrt{2}p-x_1]}q_w(y)e^{-(y_1^2+\cdots+y_n^2)}dy.\tag{67}$$

Taking the partial derivative $\partial^n/\partial p=\partial^n/\partial p_n\cdots\partial p_1$, we obtain

$$\begin{aligned}&q_v\left(\sqrt{2}p-x_0\right)e^{-\left[(\sqrt{2}p_1-x_{0,1})^2+\cdots+\left(\sqrt{2}p_n-x_{0,n}\right)^2\right]}\\&=q_w\left(\sqrt{2}p-x_1\right)e^{-\left[(\sqrt{2}p_1-x_{1,1})^2+\cdots+\left(\sqrt{2}p_n-x_{1,n}\right)^2\right]},\end{aligned}\tag{68}$$

thus,

$$\begin{aligned}q_v\left(\sqrt{2}p-x_0\right)&=q_w\left(\sqrt{2}p-x_1\right)e^{\left[\sum_{j=1}^n\left(\sqrt{2}p_j-x_{0,j}\right)^2-\left(\sqrt{2}p_j-x_{1,j}\right)^2\right]}\\&=q_w\left(\sqrt{2}p-x_1\right)e^{\left[\sum_{j=1}^n2\sqrt{2}p_j\left(x_{1,j}-x_{0,j}\right)+\left(x_{0,j}\right)^2-\left(x_{1,j}\right)^2\right]}.\end{aligned}\tag{69}$$

Since q_v and q_w are polynomials, this fact implies that the exponential part in the above equation is constant for all $p\in(\mathbb{R}^+)^n$. Hence, $x_{0,j}=x_{1,j}$ for all j; therefore, $x_0=x_1$. Using this fact, it is clear that $q_v(\sqrt{2}p-x_0)=q_w(\sqrt{2}p-x_1)$, for all p, that is, $|\langle v,N_\alpha(y)\rangle|^2=|\langle w,N_\alpha(y)\rangle|^2$ for all $y\in\mathbb{R}^n$.

As consequence of the above lemma, if $x_0\neq x_1$ and $v,w\in\mathbb{S}^{\mathbf{P}(\alpha)}$, then there exists $p_0\in(\mathbb{R}^+)^n$ such that $f_{x_0,v}(\gamma^{\alpha,p_0}(x))\neq f_{x_1,w}(\gamma^{\alpha,p_0}(x))$; hence, the spectral matrix γ^{α,p_0} separates the pure states $f_{x_0,v}$ and $f_{x_1,w}$.

To complete the proof of the fact that the C^*-algebra $\mathfrak{G}_k^H$ separates all the pure states of $\mathfrak{D}$, only missing step is to separate the pure states of the forms $f_{x,v}$ and $f_{x,w}$, where $v,w\in\mathbb{S}^{\mathbf{P}(\alpha)}$ and $x\in\mathbb{R}^n$. For this, we need to deduce some useful facts before. □

For the multi-index $\alpha=(\alpha_1,\cdots,\alpha_n)$, we consider $N_\alpha(y)=(\tilde{h}_{\lambda-1}(y))_{\lambda_i\leq\alpha_i}$. From (8), we have $\tilde{h}_{\lambda-1}(y)=h_{\lambda_1-1}(y_1)\cdots h_{\lambda_n-1}(y_n)$, where $h_{\lambda_i-1}(y_i)$ is a one-dimensional Hermite function. We can consider that all correspondent one-dimensional Hermite's polynomials $H_{\lambda_i-1}(y_i)$, whose degrees are equal to λ_i-1, are monic polynomials. Hence, for every multi-index λ such that $\lambda_i\leq\alpha_i$, we can write

$$\tilde{h}_{\lambda-1}(y)=2^{|\lambda-1|/2}\left((\lambda-1)!\sqrt{\pi^n}\right)^{-1/2}e^{-\|y\|^2/2}H_{\lambda_1-1}(y_1)\cdots H_{\lambda_n-1}(y_n).\tag{70}$$

Now, for $y \in \mathbb{R} - \{0\}$, we construct the vector $\overleftarrow{y} \in \mathbb{R}^n$ with the form

$$\overleftarrow{y} = (y^{\alpha_2 \cdots \alpha_n}, y^{\alpha_3 \cdots \alpha_n}, \cdots, y^{\alpha_{n-1} \cdot \alpha_n}, y^{\alpha_n}, y). \tag{71}$$

Evaluating this vector in Hermite's polynomials corresponding to the multi-index $\lambda = (\lambda_1, \cdots, \lambda_n)$

$$H_{\lambda_1 - 1}(y^{\alpha_2 \cdots \alpha_n}) \cdot H_{\lambda_2 - 1}(y^{\alpha_3 \cdots \alpha_n}) \cdots H_{\lambda_{n-1} - 1}(y^{\alpha_n}) \cdot H_{\lambda_n - 1}(y), \tag{72}$$

we obtain a polynomial dependents on only one variable, whose degree we can calculate with the equation:

$$p_\lambda = \left[\sum_{j=1}^{n-1} (\lambda_j - 1)\alpha_{j+1} \cdots \alpha_n\right] + (\lambda_n - 1). \tag{73}$$

Consequently, evaluating $\overleftarrow{y}$ in $\tilde{h}_{\lambda-1}$, we can write

$$\tilde{h}_{\lambda-1}\left(\overleftarrow{y}\right) = 2^{|\lambda-1|/2}\left((\lambda-1)!\sqrt{\pi^n}\right)^{-1/2} e^{-\frac{\|\overleftarrow{y}\|^2}{2}}\left(y^{p_\lambda} + a_{\lambda,1} y^{p_\lambda - 1} + \cdots + a_{\lambda,p_\lambda} y + 1\right). \tag{74}$$

From (73), we notice that for two different multi-indices λ and μ, the corresponding degrees satisfy $p_\lambda \neq p_\mu$. Moreover, for $1_n = (1, \cdots, 1)$, we have $p_{1_n} = 0$. And for $\alpha = (\alpha_1, \cdots, \alpha_n)$,

$$p_\alpha = \left[\sum_{j=1}^{n-1} (\alpha_j - 1)\alpha_{j+1} \cdots \alpha_n\right] + (\alpha_n - 1) = \alpha_1 \cdots \alpha_n - 1 = \mathbf{P}(\alpha) - 1. \tag{75}$$

Therefore, the multi-indices $\lambda \in \mathbb{Z}_+^n$ such that $\lambda_i \leq \alpha_i$ for every i generate different polynomials of degrees between 0 and $\mathbf{P}(\alpha) - 1$. Now, for each of these multi-indices λ, we consider vectors $\overleftarrow{y_\lambda}$ defined by (71), and we define the matrix N whose dimension is $\mathbf{P}(\alpha) \times \mathbf{P}(\alpha)$ and the λ^{th} row is equal to $N_\alpha(\overleftarrow{y_\lambda})$. Since the components of $N_\alpha(y)$ are sorted ascending by the lexicographic order, we claim that the matrix N has the form:

$$N = C_\alpha \cdot D \cdot \begin{pmatrix} 1 & y_{1_n} + a_0 & y_{1_n}^2 + b_1 y_{1_n} + b_0 & \cdots & y_{1_n}^{\mathbf{P}(\alpha)-1} + \cdots \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & y_\alpha + a_0 & y_\alpha^2 + b_1 y_\alpha + b_0 & \cdots & y_\alpha^{\mathbf{P}(\alpha)-1} + \cdots \end{pmatrix}, \tag{76}$$

where

$$C_\alpha = \prod_{\substack{\lambda \in \mathbb{Z}_+^n \\ \lambda_i \leq \alpha_i}} 2^{|\lambda-1|/2}\left((\lambda-1)!\sqrt{\pi^n}\right)^{-1/2}, D = \operatorname{diag}\left\{e^{-\|\overleftarrow{y}_\mu\|^2/2}\right\}_{\substack{\mu \in \mathbb{Z}_+^n \\ \mu_i \leq \alpha_i}}. \tag{77}$$

We can calculate the determinant of N using multilinearity and the VandermondeÂ's formula; we obtain

$$\det N = \prod_{\substack{\lambda \in \mathbb{Z}_+^n \\ \lambda_i \leq \alpha_i}} 2^{|\lambda-1|/2}\left((\lambda-1)!\sqrt{\pi^n}\right)^{-1/2} e^{-\sum_\mu \|y_\mu\|^2/2} \cdot \prod_{\substack{\gamma,\delta \in \mathbb{Z}_+^n \\ 1_n \leq \gamma < \delta \leq \alpha}} \left(y_\delta - y_\gamma\right) \neq 0. \tag{78}$$

Example 1. Consider $n = 3$ and the multi-index $\alpha = (2, 4, 1)$. We have $\boldsymbol{P}(\alpha) = 8$ and the multi-indices, arranged with the lexicographic order, whose coordinates are less or equal to the corresponding coordinates of α are

$$\begin{aligned} &1_n = \lambda_1 = (1,1,1), \quad \lambda_2 = (1,2,1), \quad \lambda_3 = (1,3,1), \quad \lambda_4 = (1,4,1), \\ &\lambda_5 = (2,1,1), \quad \lambda_6 = (2,2,1), \quad \lambda_7 = (2,3,1), \quad \lambda_8 = (2,4,1) = \alpha. \end{aligned} \tag{79}$$

In this case, the vector $\overleftarrow{y} = (y^4, y, y)$. From (73), we can calculate the different degrees, for example for $\lambda_7 = (2, 3, 1)$,

$$p_{\lambda_7} = 1 \cdot 4 + 2 \cdot 1 + 1 - 1 = 6. \tag{80}$$

And for the rest of multi-indices, we have

$$\begin{aligned} &p_{\lambda_1} = 0, \quad p_{\lambda_2} = 1, \quad p_{\lambda_3} = 2, \quad p_{\lambda_4} = 3, \\ &p_{\lambda_5} = 4, \quad p_{\lambda_6} = 5, \quad p_{\lambda_7} = 6, \quad p_{\lambda_8} = 7. \end{aligned} \tag{81}$$

Now, we can continue with the separation of the pure states.

Lemma 19. *Given $v, w \in \mathbb{S}^{P(\alpha)}$ and $x \in \mathbb{R}^n$ being fixed, consider the spectral matrices $\gamma^{\alpha,p}, \gamma^{\alpha,r}$. If $f_{x,v}(\gamma^{\alpha,p}) = f_{x,w}(\gamma^{\alpha,r})$, for all $p, r \in (\mathbb{R}^+)^n$ then $v = tw$, with $t \in \mathbb{C}$ and $|t| = 1$.*

Proof. From Lemma 18, we have $|<v, N_\alpha(y)>|^2 = |<w, N_\alpha(y)>|^2$ for all $y \in \mathbb{R}^n$; thus, there exists a function $\theta : \mathbb{R}^n \longrightarrow \mathbb{C}$ such that

$$\langle v, N_\alpha(y)\rangle = e^{i\theta(y)}\langle w, N_\alpha(y)\rangle. \tag{82}$$

For $u \in \mathbb{S}^{\mathbf{P}(\alpha)}$, we define the function $H_u : (\mathbb{R}^+)^{2n} \longrightarrow \mathbb{C}$ given by

$$H_u(p, r) = f_{x,u}(\gamma^{\alpha,p}\gamma^{\alpha,r}) = \int_{-x}^{\sqrt{2}p - x} \bar{u}^T N_\alpha(y)[N_\alpha(y)]^T \cdot dy \int_{-x}^{\sqrt{2}r - x} N_\alpha(y)[N_\alpha(y)]^T u dy. \tag{83}$$

Without loss of generality, we can assume that $x = (0, \cdots, 0) \in \mathbb{R}^n$, just to simplify the calculations. Taking the derivative of order $2n$ of the function H_u, with respect to p and r, we obtain

$$\frac{\partial^{2n} H_u}{\partial p \partial r}\left(\frac{p}{\sqrt{2}}, \frac{r}{\sqrt{2}}\right) = \bar{u}^T N_\alpha(p)[N_\alpha(p)]^T N_\alpha(r)[N_\alpha(r)]^T u = \langle N_\alpha(p), u\rangle \langle N_\alpha(r), N_\alpha(p)\rangle \langle u, N_\alpha(r)\rangle. \quad (84)$$

The hypothesis $H_v = H_w$ implies that $\partial^{2n}H_v/\partial p\partial r = \partial^{2n}H_w/\partial p\partial r$. Using (82) and (84),

$$\langle N_\alpha(p), w\rangle \langle N_\alpha(r), N_\alpha(p)\rangle \langle w, N_\alpha(r)\rangle = e^{i(\theta(r)-\theta(p))}\langle N_\alpha(p), w\rangle \times \times \langle N_\alpha(r), N_\alpha(p)\rangle \langle w, N_\alpha(r)\rangle, \quad (85)$$

for all $p, r \in (\mathbb{R}^+)^n$. Clearly, there exists $r_0 \in (\mathbb{R}^+)^n$ such that $\langle w, N_\alpha(r_0)\rangle \neq 0$. Thus,

$$\left(1 - e^{i(\theta(r_0)-\theta(p))}\right)\langle N_\alpha(p), w\rangle \langle N_\alpha(r_0), N_\alpha(p)\rangle = 0. \quad (86)$$

Notice that $\langle N_\alpha(p), w\rangle \langle N_\alpha(r_0), N_\alpha(p)\rangle$ is a nonzero polynomial with respect to p; thus, the above equation implies that the function θ is constant. From (82), we obtain that $\langle v - e^{i\theta_0}w, N_\alpha(y)\rangle = 0$, for all $y \in \mathbb{R}^n$. Using (78), we have $v - e^{i\theta_0}w = 0$ and $v = e^{i\theta_0}w$. □

Finally, we consider the case of the C^*-algebras $\mathfrak{G}^H_{(k)}$ and $\mathfrak{G}^H_k$. Since the proof of the Lemma 19 is independent of the dimension $\mathbf{P}(\alpha)$ and the nature of the multi-indices λ and μ, we can obtain the following analogous results.

Lemma 20. *For $v, w \in \mathbb{S}^{\mathbf{s}_{(k)}}$, $x_0, x_1 \in \mathbb{R}^n$ and $\gamma^{(k),p} \in \mathfrak{G}^H_{(k)}$ with $p \in (\mathbb{R}^+)^n$. If $f_{x_0,v}(\gamma^{(k),p}(x)) = f_{x_0,w}(\gamma^{(k),p}(x))$ for all vector p, then $x_0 = x_1$. Also, $|\langle v, N_{(k)}(y)\rangle|^2 = |\langle w, N_{(k)}(y)\rangle|^2$ for all $y \in \mathbb{R}^n$.*

Lemma 21. *For $v, w \in \mathbb{S}^{\mathbf{s}_k}$, $x_0, x_1 \in \mathbb{R}^n$ and $\gamma^{k,p} \in \mathfrak{G}^H_k$ with $p \in (\mathbb{R}^+)^n$. If $f_{x_0,v}(\gamma^{k,p}(x)) = f_{x_0,w}(\gamma^{k,p}(x))$ for all vector p, then $x_0 = x_1$. Also, $|\langle v, N_k(y)\rangle|^2 = |\langle w, N_k(y)\rangle|^2$ for all $y \in \mathbb{R}^n$.*

On the other hand, we consider the C^*-algebra $\mathfrak{G}^H_{(k)}$ and the multi-index

$$k_n = (k - (n-1), \cdots, k - (n-1)). \quad (87)$$

If $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multi-index such that $|\alpha| = k$ then all its components satisfy $\alpha_i \leq k - (n-1)$. So, if we construct the $\mathbf{P}(k_n) \times \mathbf{P}(k_n)$ invertible matrix N as in (76), corresponding to the multi-index k_n, and apply it to the vector

$$\left(0, \cdots, v - e^{i\theta_0}w, \cdots, 0\right), \quad (88)$$

where the coordinates of $v - e^{i\theta_0}w \in \mathbb{S}^{\mathbf{s}_{(k)}}$ occupy the same positions that the multi-indices whose absolute value is equal to k in the matrix N. Thus, we can prove the following.

Lemma 22. *Given $v, w \in \mathbb{S}^{\mathbf{s}_{(k)}}$ and $x \in \mathbb{R}^n$ being fixed, consider the spectral matrices $\gamma^{(k),p}, \gamma^{(k),r}$. If $f_{x,v}(\gamma^{(k),p}) = f_{x,w}(\gamma^{(k),r})$, for all $p, r \in (\mathbb{R}^+)^n$ then $v = tw$, with $t \in \mathbb{C}$ and $|t| = 1$.*

Analogously, for the case of the C^*-algebra $\mathfrak{G}^H_k$ and the set of multi-indices with absolute value is less than or equal to k, we have the following.

Lemma 23. *Given $v, w \in \mathbb{S}^{\mathbf{s}_k}$ and $x \in \mathbb{R}^n$ being fixed, consider the spectral matrices $\gamma^{k,p}, \gamma^{k,r}$. If $f_{x,v}(\gamma^{k,p}) = f_{x,w}(\gamma^{k,r})$, for all $p, r \in (\mathbb{R}^+)^n$ then $v = tw$, with $t \in \mathbb{C}$ and $|t| = 1$.*

For the noncommutative Stone-Weierstrass conjecture, let $\mathscr{B}$ be a C^*-subalgebra of a C^*-algebra $\mathscr{A}$, and suppose that $\mathscr{B}$ separates all the pure states of $\mathscr{A}$ (and 0 if $\mathscr{A}$ is nonunital). Then, $\mathscr{A} = \mathscr{B}$.

In [10], Kaplansky proved this conjecture for a C^*-algebra type I. In consequence, we prove that the algebra $\mathfrak{G}^H_\alpha$ is equal to $\mathfrak{D}_\alpha$. From Corollary 15, we have that the algebra of Toeplitz operators $\mathscr{T}^\alpha_\infty$ is isometric and isomorphic to the algebra $\mathfrak{D}_\alpha$. Analogously, applying the Corollary 16 and the Corollary 17, we have that $\mathscr{T}^{(k)}_\infty$ and $\mathscr{T}^k_\infty$ are isometric and isomorphic to $\mathfrak{D}_{(k)}$ and $\mathfrak{D}_k$, respectively. In summary, we have the following results.

Theorem 24. *The C^*-algebra $\mathscr{T}^\alpha_\infty$ is isomorphic and isometric to the C^*-algebra $\mathfrak{D}_\alpha$. The isomorphism is given by*

$$\mathscr{T}^\alpha_\infty : T_{\alpha,a} \mapsto (Sym\, T_{\alpha,a})(x) = \gamma^{\alpha,a}(x), \quad (89)$$

where $\gamma^{\alpha,a}(x)$ is given in (41).

Theorem 25. *The C^*-algebra $\mathscr{T}^{(k)}_\infty$ is isomorphic and isometric to the C^*-algebra $\mathfrak{D}_{(k)}$. The isomorphism is given by*

$$\mathscr{T}^{(k)}_\infty : T_{(k),a} \mapsto \left(Sym\, T_{(k),a}\right)(x) = \gamma^{(k),a}(x), \quad (90)$$

where $\gamma^{(k),a}(x)$ is given in (45).

Theorem 26. *The C^*-algebra $\mathscr{T}^k_\infty$ is isomorphic and isometric to the C^*-algebra $\mathfrak{D}_k$. The isomorphism is given by*

$$\mathscr{T}^k_\infty : T_{k,a} \mapsto (Sym\, T_{k,a})(x) = \gamma^{k,a}(x), \quad (91)$$

where $\gamma^{k,a}(x)$ is given in (46).

Corollary 27. *The C^*-algebra $\mathscr{T}^{(\alpha)}_\infty$ is isomorphic and isometric to the commutative C^*-algebra $C[\mathbb{R}^n \cup \mathbb{S}^{n-1}_\infty]$. The isomorphism is given by*

$$\mathscr{T}^{(\alpha)}_\infty : T_{(\alpha),a} \mapsto \left(Sym\, T_{(\alpha),a}\right)(x) = \gamma^{(\alpha),a}(x), \quad (92)$$

where $\gamma^{(\alpha),a}(x)$ is given in (38).

5. Toeplitz Operators with $\mathscr{L}$-Invariant Symbols

In this section, we introduce the extended Lagrangian symbols, and we prove that the C^*-algebra generated by Toeplitz operators with this kind of symbols acting on the homogeneously poly-Fock space is isomorphic and isometric to the C^*-algebra generated by Toeplitz operators with extended horizontal symbols acting on this same space.

We consider the standard symplectic form ω_0 of $\mathbb{C}^n = \mathbb{R}^{2n}$ given by $\omega_0(z, w) = Jz \cdot w$, forall$z, w$, where

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}. \tag{93}$$

Recall that a n-dimensional subspace $\mathscr{L} \subset \mathbb{R}^{2n}$ is called a Lagrangian plane if for every $z, w \in \mathscr{L}$ it satisfy $\omega_0(z, w) = 0$. Clearly, $i\mathbb{R}^n = \{0\} \times \mathbb{R}^n$ is a Lagrangian plane. We denote by $\mathrm{Lag}(2n, \mathbb{R})$ the set of all Lagrangian planes in $\mathbb{R}^{2n}$. If we consider the transitive group action of $U(2n, \mathbb{R})$ onto $\mathrm{Lag}(2n, \mathbb{R})$ defined by

$$(X, \mathscr{L}) \mapsto X\mathscr{L}, \tag{94}$$

we have that for every Lagrangian plane $\mathscr{L}$ there is an unitary matrix X such that $X\mathscr{L} = i\mathbb{R}^n$. For more details, see [11], Proposition 43. Since the unitary group $U(2n, \mathbb{R})$ is isomorphic to $U(n, \mathbb{C})$, each Lagrangian plane $\mathscr{L}$ can be identified with a subspace of $\mathbb{C}^n$; abusing the notation, we denote this subspace with $\mathscr{L}$ too.

Let $\mathscr{L}$ be a Lagrangian plane, we say that a function $\varphi \in L_\infty(\mathbb{C}^n)$ is $\mathscr{L}$-invariant or Lagrangian invariant if for every $h \in \mathscr{L}$ it satisfies

$$\varphi(z - h) = \varphi(z), \text{for almost all } z \in \mathbb{C}^n, \tag{95}$$

so we can consider it like a function depending only on the elements of $\mathscr{L}^{\mathbb{C}}$.

In [5], Esmeral and Vasilevski introduced the concept of $\mathscr{L}$-invariant functions and they provided the following criterion for a function to be so.

Lemma 28. *Consider a Lagrangian plane $\mathscr{L}$ and $X \in U(n, \mathbb{C})$ such that $X\mathscr{L} = i\mathbb{R}^n$. Then, a function $\varphi \in L_\infty(\mathbb{C}^n)$ is $\mathscr{L}$-invariant if and only if there exists $a \in L_\infty(\mathbb{R}^n)$ such that*

$$\varphi(X^* z) = a(\mathrm{Re}\, z_1, \cdots, \mathrm{Re}\, z_n), \textit{for almost all } z \in \mathbb{C}^n. \tag{96}$$

Moreover, they established the following result.

Proposition 29. *The C^*-algebra generated by Toeplitz operators with horizontal symbols acting on the Fock space $F^2(\mathbb{C}^n)$ is unitary equivalent to the C^*-algebra generated by Toeplitz operators with $\mathscr{L}$-invariant symbols.*

For this, they introduced the operator $V_X : L_2(\mathbb{C}^n, d\mu_n) \longrightarrow L_2(\mathbb{C}^n, d\mu_n)$ defined by

$$(V_X f)(z) = f(X^* z), \textit{with } X \in U(2n, \mathbb{R}) \textit{ such that } X\mathscr{L} = i\mathbb{R}^n. \tag{97}$$

Since $X^* = X^{-1}$, this operator is unitary and $V_X^* = V_{X^{-1}}$. It too satisfies

$$V_X K_z = K_{Xz}, \tag{98}$$

where K_z is the reproducing kernel of $F^2(\mathbb{C}^n)$ in the point z.

In the case of the poly-Fock space and the true poly-Fock space, the above result could fail, because for some multi-index α and some unitary matrix X; the spaces $F^2_\alpha(\mathbb{C}^n)$ and $F^2_{(\alpha)}(\mathbb{C}^n)$ might not be invariant under the operator V_X.

Example 1. Consider $n = 2$, $\alpha = (2, 1)$. Using (15), we have

$$\psi(z) = \bar{z_1}(z_1 + z_2) \in F^2_{(\alpha)}(\mathbb{C}^n), \tag{99}$$

however, if $X^* = (a_{ij})$ with $i, j = 1, 2$ is an unitary matrix, it is clear that

$$\psi(X^* z) = \bar{w}_1(w_1 + w_2) \notin F^2_{(\alpha)}(\mathbb{C}^n), \tag{100}$$

where $w_1 = a_{11}z_1 + a_{12}z_2$, $w_2 = a_{21}z_1 + a_{22}z_2$.

This is the main motivation for which we consider the homogeneously poly-Fock space $F^2_{(k)}(\mathbb{C}^n)$ and the poly-Fock space $F^2_k(\mathbb{C}^n)$ for $k \in \mathbb{N}$.

Note that for $\mathscr{L} \in \mathrm{Lag}(2n, \mathbb{R})$, $X \in U(2n, \mathbb{R})$ such that $X\mathscr{L} = i\mathbb{R}^n$, and V_X defined by (97), using the explicit form of the elements in the true poly-Fock space given by (15), from [6], Proposition 2.7, we have that the homogeneously poly-Fock space $F^2_{(k)}(\mathbb{C}^n)$ and the poly-Fock space $F^2_k(\mathbb{C}^n)$ are invariant under V_X.

Now, we define the extended Lagrangian symbols; these kind of symbols is related with the extended horizontal symbols as follows.

Definition 30. For $\mathscr{L} \in \mathrm{Lag}(2n, \mathbb{R})$ and $X \in U(2n, \mathbb{R})$ such that $X\mathscr{L} = i\mathbb{R}^n$, equivalently $X\mathscr{L}^{\mathbb{C}} = \mathbb{R}^n$ and $X^*\mathbb{R}^n = \mathscr{L}^{\mathbb{C}}$, consider the following diagram.

$$\begin{tikzcd}
& R^n \arrow[dr, "a"] \arrow[dd, leftrightarrow, "X"] & \\
C^n \arrow[ur, "\pi"] \arrow[dr, "\pi"'] & & C \\
& L^c \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

We say that the $\mathscr{L}$-invariant symbol φ is an extended Lagrangian symbol or an extended $\mathscr{L}$-invariant symbol if its pullback by X^* is an extended horizontal symbol. In other words, if the function $a(\mathrm{Re}\, z) = \varphi(X^* z)$ given in (96), it is an extended horizontal symbol.

According to the above diagram, for $z \in \mathscr{L}^{\mathbb{C}}$, there exists $w \in \mathbb{C}^n$ such that $z = X^*$ Re w and $|z| = \|\mathrm{Re}\ w\|$. If φ is an extended Lagrangian symbol, using (96) and (48), we have

$$\begin{aligned}\lim_{t\longrightarrow\infty} \varphi(tz) &= \lim_{t\longrightarrow\infty} \varphi(t X^* \mathrm{Re}\ w) = \lim_{t\longrightarrow\infty} a(t\ \mathrm{Re}\ w) \\ &= a_\infty\left(\frac{\mathrm{Re}\ w}{\|\mathrm{Re}\ w\|}\right) = a_\infty\left(X\left(\frac{z}{|z|}\right)\right).\end{aligned} \tag{101}$$

If we define the continuous function $\varphi_\infty : \mathbb{S}^{2n-1} \cap \mathscr{L}^{\mathbb{C}} \longrightarrow \mathbb{C}$ given by

$$\varphi_\infty(z) = a_\infty(Xz), \tag{102}$$

we have for $z \in \mathscr{L}^{\mathbb{C}}$

$$\lim_{t\longrightarrow\infty} \varphi(tz) = \varphi_\infty\left(\frac{z}{|z|}\right). \tag{103}$$

This function φ_∞ is invariant under translations by Lagrangians elements whose norm is equal to 1. Let $h \in \mathscr{L}$ such that $|h| = 1$, so $Xh \in i\mathbb{R}^n$ and $\|Xh\| = 1$; thus, for $z \in \mathbb{S}^{2n-1} \cap \mathscr{L}^{\mathbb{C}}$, we have

$$\varphi_\infty(z) = a_\infty(Xz) = a_\infty(Xz + Xh) = a_\infty(X(z+h)) = \varphi_\infty(z+h). \tag{104}$$

Lemma 31. *Consider the unitary operator V_X for $X \in U(2n, \mathbb{R})$. If $K^{(k)}(z, w)$ denotes the reproducing kernel of the true homogeneously poly-Fock space $F^2_{(k)}(\mathbb{C}^n)$ then $V_X K^{(k)}_v = K^{(k)}_{Xv}$.*

Proof. Let $\varphi(z) \in F^2_{(k)}(\mathbb{C}^n)$. Using the reproducing property, we can express

$$\varphi(z) = \frac{1}{\pi^n}\int_{\mathbb{C}^n} \varphi(w) K^{(k)}(\bar{z}, w) e^{-|w|^2} dw. \tag{105}$$

Apply V_X

$$(V_X\varphi)(z) = \frac{1}{\pi^n}\int_{\mathbb{C}^n} \varphi(X^* w) K^{(k)}(\bar{z}, w) e^{-|w|^2} dw, \tag{106}$$

since X^* is unitary and taking $v = X^* w$, we have

$$(V_X\varphi)(z) = \frac{1}{\pi^n}\int_{\mathbb{C}^n} \varphi(v) K^{(k)}(\bar{z}, Xv) e^{-|v|^2} dv. \tag{107}$$

On the other hand,

$$\varphi(X^* z) = \frac{1}{\pi^n}\int_{\mathbb{C}^n} \varphi(v) K^{(k)}(\overline{X^* z}, v) e^{-|v|^2} dv, \tag{108}$$

for the uniqueness of the reproducing kernel, we have $K^{(k)}(v, X^* z) = K^{(k)}(Xv, z)$; therefore $V_X K^{(k)}_v = K^{(k)}_{Xv}$. □

Corollary 32. *The reproducing kernel $K^k(z, w)$ of the poly-Fock space $F^2_k(\mathbb{C}^n)$ satisfies $V_X K^k_v = K^k_{Xv}$.*

Now, consider $a \in L_\infty(\mathbb{C}^n)$ and $X \in U(2n, \mathbb{R})$. Using the Lemma 31, we obtain that the Toeplitz operator T_a acting on the true homogeneously poly-Fock space $F^2_{(k)}(\mathbb{C}^n)$ is unitary equivalent to

$$\begin{aligned}\left(V_X^{-1} T_a V_X \varphi\right)(z) &= \left\langle V_X^{-1} T_a V_X \varphi, K^{(k)}_z\right\rangle = \left\langle T_a V_X \varphi, V_X K^{(k)}_z\right\rangle \\ &= \left\langle a V_X \varphi, K^{(k)}_{Xz}\right\rangle = \left\langle a_X \varphi, K^{(k)}_z\right\rangle = \left(T_{a_X}\varphi\right)(z),\end{aligned} \tag{109}$$

where $a_X(z) = a(Xz)$. Analogously, by Corollary 32, the Toeplitz operator T_a acting in the poly-Fock space $F^2_k(\mathbb{C}^n)$ is unitary equivalent to T_{a_X}.

Using the above results, we obtain the following generalizations of Proposition 29.

Theorem 33. *The C^* -algebra $\mathscr{T}^{(k)}_\infty$ generated by Toeplitz operators with extended horizontal symbols acting on the true homogeneously poly-Fock space $F^2_{(k)}(\mathbb{C}^n)$ is unitary equivalent to the C^* -algebra $\mathscr{T}^{(k)}_{\mathscr{L},\infty}$ generated by Toeplitz operators with extended $\mathscr{L}$ -invariant symbols.*

Theorem 34. *The C^*-algebra $\mathscr{T}^k_\infty$ generated by Toeplitz operators with extended horizontal symbols acting in the poly-Fock space $F^2_k(\mathbb{C}^n)$ is unitary equivalent to the C^* -algebra $\mathscr{T}^k_{\mathscr{L},\infty}$ generated by Toeplitz operators with extended $\mathscr{L}$ -invariant symbols.*

Finally, using Theorems (25) and (26), we have the following.

Corollary 35. *The C^*-algebra $\mathscr{T}^{(k)}_{\mathscr{L},\infty}$ generated by Toeplitz operators with extended $\mathscr{L}$-invariant symbols acting on the homogeneously poly-Fock space $F^2_{(k)}(\mathbb{C}^n)$ is isomorphic and isometric to the C^*-algebra $\mathfrak{D}_{(k)}$.*

Corollary 36. *The C^*-algebra $\mathscr{T}^k_{\mathscr{L},\infty}$ generated by Toeplitz operators with extended $\mathscr{L}$-invariant symbols acting on the poly-Fock space $F^2_k(\mathbb{C}^n)$ is isomorphic and isometric to the C^*-algebra $\mathfrak{D}_k$.*

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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