Research Article

New Estimates of Solution to Coupled System of Damped Wave Equations with Logarithmic External Forces

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Received 18 March 2021; Revised 30 March 2021; Accepted 2 April 2021; Published 10 April 2021

Academic Editor: Liliana Guran

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In the paper, we consider new stability results of solution to class of coupled damped wave equations with logarithmic sources in \( \mathbb{R}^n \). We prove a new scenario of stability estimates by introducing a suitable Lyapunov functional combined with some estimates.

1. Introduction

In the present paper, we consider an initial boundary value problem with damping terms and logarithmic sources, for \( x \in \mathbb{R}^n, t > 0 \)

\[
\begin{aligned}
th_{x} v_1 + bv_2 &= \phi(x)\Delta v_1 - \int_0^t \omega_1(t-p)\nu_1(p)dp + kv_1 \ln |v_1|, \\
th_{x} v_2 + bv_1 &= \phi(x)\Delta v_2 - \int_0^t \omega_2(t-p)\nu_2(p)dp + kv_2 \ln |v_2|, \\
v_1(x,0) &= v_{10}(x), v_2(x,0) = v_{20}(x), \\
v_1(x,t) &= v_{11}(x), v_2(x,t) = v_{21}(x),
\end{aligned}
\]

(1)

where \( b > 0, n \geq 3, \) and \( k \) is a small positive real number. The density function \( \rho(x) > 0, \) for all \( x \in \mathbb{R}^n, \) where \( (\phi(x))^{-1} = 1/\phi(x) \equiv \rho(x), \) under homogeneous Dirichlet boundary conditions.

A related initial boundary value problem was considered by Han in [1]:

\[
\begin{aligned}
\partial_t u + \Delta u + u + |u|^2 u &= u \ln |u|^2, & x \in \Omega, t \in [0,T), \\
u(x,0) &= u_0(x)u_1(x,0) = u_1(x), & x \in \Omega, \\
u(x,t) &= 0, & x \in \partial \Omega, t \in [0,T),
\end{aligned}
\]

(2)

and the global existence of weak solutions was proved, for all \((u_0, u_1) \in H^1_0 \times L^2\) in \( \mathbb{R}^3 \). The weak and strong damping terms in logarithmic wave equation

\[
\begin{aligned}
u_{tt} + \mu u_t - \Delta u - \omega \Delta u_t &= u \ln |u|, & x \in \Omega, t \in (0,\infty), \\
u(x,0) &= u_0(x)u_1(x,0) = u_1(x), & x \in \Omega, \\
u(x,t) &= 0, & x \in \partial \Omega, t \in (0,\infty)
\end{aligned}
\]

(3)

were introduced by Lian and Xu [2]. The global existence, asymptotic behavior, and blowup at three different initial energy levels (subcritical energy \( E(0) < d \), critical initial energy \( E(0) = d \), and the arbitrary high initial energy \( E(0) > 0(\omega = 0) \) ) were proved. In [3], Al-Gharabli established explicit and general energy decay results for the problem

\[
\begin{aligned}
u_{tt} + \Delta^2 u + u - \int_0^t g(t-s)\Delta^2 u ds &= ku \ln |u|, & x \in \Omega, t \in (0,\infty), \\
u(x,0) &= u_0(x)u_1(x,0) = u_1(x), & x \in \Omega, \\
u(x,t) &= \frac{\partial u}{\partial n} = 0, & x \in \partial \Omega, t \in (0,\infty).
\end{aligned}
\]

(4)
When the density \( \phi(x) \neq 1 \), Papadopoulos and Stavrakakis [4] considered the following semilinear hyperbolic initial value problem:

\[
 u_{tt} + \phi(x) \Delta u + \delta u_t + \lambda f(u) = \eta(x), \quad (x, t) \in \mathbb{R}^n \times \mathbb{R}^+. \tag{5}
\]

The authors proved local existence of solutions and established the existence of a global attractor in the energy space \( \mathcal{D}^{1,2}(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \), where \( (\phi(x))^{-1} = g(x) \). Miyasita and Zennir [5] proved the global existence of the following viscoelastic wave equation:

\[
\begin{aligned}
& u_{tt} + \omega \Delta u_t - \int_0^t g(t-s) \Delta u(s) \, ds = u|u|^{p-1}u, \quad x \in \mathbb{R}^n, \quad t > 0, \\
& u(x, 0) = u_0(x), \\
& u_t(x, 0) = u_1(x), \\
& x \in \mathbb{R}^n.
\end{aligned}
\tag{6}
\]

The novelty of our work lies primarily in the use of a new condition between the weights of damping the external forces, where we outline the effects of the damping term with less conditions on the viscoelastic terms. We also propose logarithmic nonlinearities in sources and used classical arguments to estimate them. These nonlinearities make the problem very interesting in the application point of view. In order to compensate for the lack of classical Poincaré’s inequality in \( \mathbb{R}^n \), we use the weighted function to use generalized Poincaré’s one. The main contribution of this paper is introduced in Theorem 8, where we obtain decay estimates with positive initial energy under a general assumption on the kernel. The rest of the paper is outline as follows. In Section 2, we give some preliminaries and our main results. In Section 3, we will prove the general decay of energy to the problem.

**2. Preliminaries and Main Results**

We state some assumptions and definitions that will be useful in this paper. With respect to the relaxation functions \( \omega_1, \omega_2 \), we assume for \( i = 1, 2 \).

(H1) \( \omega_1, \omega_2 \in C(\mathbb{R}^+, \mathbb{R}^+) \) satisfy for any \( t \geq 0 \),

\[
\omega_i(0) > 0, \quad \int_0^\infty \omega_i(p) \, dp = l_{i0} < \infty, \quad 1 - \int_0^\infty \omega_i(p) \, dp = l_i > 0
\tag{7}
\]

(H2) There exist nonincreasing differentiable functions \( \zeta_1, \zeta_2 : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) that satisfy

\[
\zeta_i(t) > 0, \quad \zeta_i'(t) \leq -\zeta_i(t) \omega_i(t) \quad \text{for} \quad t \geq 0
\tag{8}
\]

(H3) The function \( \rho : \mathbb{R}^n \rightarrow \mathbb{R}^*_+ \), \( \rho(x) \in C^\gamma(\mathbb{R}^n) \) with \( \gamma \in (0, 1) \) and \( \rho \in L^r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \), where \( s = 2n/(2n - qn + 2q) \)

**Definition 1** (see [4]). We define the function spaces of our problem and their norms as follows:

\[
\mathcal{H} = \left\{ v \in L^{2n/(n-2)}(\mathbb{R}^n) | \nabla v \in \left( L^2(\mathbb{R}^n) \right)^n \right\}.
\tag{9}
\]

Let the function spaces \( \mathcal{H} \) as the closure of \( C_0^\infty(\mathbb{R}^n) \) with respect to the norm \( ||v||_\mathcal{H} = (v, v)^{1/2} \) for the inner product:

\[
(v, w)_{\mathcal{H}} = \int_{\mathbb{R}^n} \nabla v \cdot \nabla w \, dx,
\tag{10}
\]

and \( L^s_\rho(\mathbb{R}^n) \) be defined with the norm \( ||v||_{L^s_\rho} = (v, v)^{1/2} \) for

\[
(v, w)_{L^s_\rho} = \int_{\mathbb{R}^n} \rho \, v w \, dx.
\tag{11}
\]

For general \( q \in [1, +\infty) \), \( L^s_\rho(\mathbb{R}^n) \) is the weighted \( L^s \) space under a weighted norm

\[
||v||_{L^s_\rho} = \left( \int_{\mathbb{R}^n} \rho |v|^q \, dx \right)^{1/q}.
\tag{12}
\]

To distinguish the usual \( L^s \) space from the weighted one, we denote the standard \( L^s \) norm by

\[
||v||_s = \left( \int_{\mathbb{R}^n} |v|^q \, dx \right)^{1/q}.
\tag{13}
\]

We denote an eigenpair \( \{ (\lambda_j, w_j) \}_{j \in \mathbb{N}} \subset \mathbb{R} \times \mathcal{H} \) by

\[
-\phi(x) \Delta w_j = \lambda_j w_j, \quad x \in \mathbb{R}^n
\tag{14}
\]

for any \( j \in \mathbb{N} \). Then, according to [4],

\[
0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots \uparrow +\infty
\tag{15}
\]

holds and \( \{ w_j \} \) is a complete orthonormal system in \( \mathcal{H} \).

Now, we introduce Sobolev embedding and generalized Poincaré’s inequalities.

**Lemma 2.** Let \( \rho \) satisfy (H3). Then, there are positive constants \( C_\xi > 0 \) and \( C_\rho > 0 \) that depend only on \( n \) and \( \rho \) such
for \( v \in \mathcal{H} \).

**Lemma 3** (see Lemma 2.2 in [6]). Let \( p \) satisfy (H3). Then, we have

\[
\|v\|_{L^q_x} \leq C_q \|v\|_{\mathcal{H}}, \quad C_q = C_q(p) \|ho\|_{L^q_t}
\]

for \( v \in \mathcal{H} \), where \( s = 2n/(2n - qn + 2q) \) for \( 1 \leq q \leq 2n/(n - 2) \).

The energy functional associated to problem (1) is given by

\[
\mathcal{E}(t) = \frac{1}{2} \sum_{i=1}^{2} \|\partial_i v_i(t)\|_{L^2_x}^2 + \frac{2}{3} \sum_{i=1}^{2} \left( 1 - \int_0^t \omega_i(p) dp \right) \|\nabla v_i(t)\|^2
\]

\[
+ b \|v_1(t)v_2(t)\|^2 + \frac{2}{3} \sum_{i=1}^{2} \left( \partial_i \delta \nabla v_i(t) \right)
\]

\[
- \frac{k}{2} \sum_{i=1}^{2} \int_{\mathbb{R}^n} \rho(x) v_i^2 \ln |v_i| dx + \frac{k}{4} \sum_{i=1}^{2} \|v_i\|_{L^2_t}^2
\]

\[
\geq \frac{1}{2} \sum_{i=1}^{2} \|\partial_i v_i(t)\|_{L^2_x}^2 + \frac{2}{3} \sum_{i=1}^{2} \left( 1 - \int_0^t \omega_i(p) dp \right) \|\nabla v_i(t)\|^2
\]

\[
+ \frac{2}{3} \sum_{i=1}^{2} \left( \partial_i \delta \nabla v_i(t) \right) - \frac{k}{2} \sum_{i=1}^{2} \int_{\mathbb{R}^n} \rho(x) v_i^2 \ln |v_i| dx
\]

\[
+ \frac{k}{4} \sum_{i=1}^{2} \|v_i\|_{L^2_t}^2,
\]

where

\[
(\partial \circ v) = \int_0^t \omega(t - p) \|v(t) - v(p)\|^2_{L^2_x} dp.
\]

With direct differentiation of (18), using (1), we obtain

\[
\partial_t \mathcal{E}(t) = - \frac{1}{2} \sum_{i=1}^{2} \left( \omega_i(t) \|v_i\|_{\mathcal{H}}^2 - (\partial_i \delta \nabla v_i) \right) \leq 0,
\]

which let our system dissipative.

**Lemma 5** (see [8]) (logarithmic Gronwall inequality). Let \( c > 0, \gamma \in L^1(0, T ; \mathbb{R}^n) \), and assume that the function \( \omega : [0, T ] \longrightarrow [1, \infty) \) satisfies

\[
\omega(t) \leq c \left( 1 + \int_0^t \gamma(p) \omega(p) \ln \omega(p) dp \right), \quad 0 \leq t \leq T,
\]

then

\[
\omega(t) \leq c \exp \left( c \int_0^t \gamma(p) dp \right), \quad 0 \leq t \leq T.
\]

We define the following functionals

\[
J(t) = \frac{1}{2} \sum_{i=1}^{2} \left( 1 - \int_0^t \omega_i(p) dp \right) \|\nabla v_i(t)\|^2
\]

\[
+ \frac{1}{2} \sum_{i=1}^{2} (\partial_i \delta \nabla v_i)(t) - \frac{k}{2} \sum_{i=1}^{2} \int_{\mathbb{R}^n} \rho(x) v_i^2 \ln |v_i| dx
\]

\[
+ \frac{k}{4} \sum_{i=1}^{2} \|v_i\|_{L^2_t}^2,
\]

\[
I(t) = \frac{1}{2} \sum_{i=1}^{2} \left( 1 - \int_0^t \omega_i(p) dp \right) \|\nabla v_i(t)\|^2
\]

\[
+ \frac{1}{2} \sum_{i=1}^{2} (\partial_i \delta \nabla v_i)(t) - \frac{k}{2} \sum_{i=1}^{2} \int_{\mathbb{R}^n} \rho(x) v_i^2 \ln |v_i| dx.
\]

Then, we introduce

\[
W = \{(v_1, v_2); v_1, v_2 \in \mathcal{H} : I(t) > 0, J(t) < d \} \cup \{0\}, \quad \sum_{i=1}^{2} \|v_i\|_{L^2_t}^2 < 4d \text{ for all } t \in [0, T).
\]

**Lemma 6.** Let \( (v_{10}, v_{11}), (v_{20}, v_{21}) \in \mathcal{H} \times L^2_p(\mathbb{R}^n) \) such that \( 0 \leq \mathcal{E}(0) < d \) and \( I(t_0) > 0 \). Then, we have

\[
(v_1, v_2) \in W,
\]

**Theorem 7** (see [5]). Let \( (v_{10}, v_{11}), (v_{20}, v_{21}) \in \mathcal{H} \times L^2_p(\mathbb{R}^n) \).

Under the assumptions (H1)–(H3). Then, problem (1) has a global weak solution \( u \) in the space

\[
(v_1, v_2) \in C([0, \infty) ; \mathcal{H}) \cap C^1([0, \infty) ; L^2_p(\mathbb{R}^n))^2.
\]
Then, the main result in this paper is the general decay of energy to problem (1) which is given in the following theorem.

**Theorem 8.** Assume the assumptions (H1)–(H3) hold and 0 < \( \mathcal{E}(0) < d \). Let \( (v_1, v_2) \) be the weak solution of problem (1) with the initial data \( (v_{10}, v_{20}), (v_{20}, v_{21}) \in \mathcal{H}(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \). Then, there exist constant \( \beta > 0 \) such that the energy \( \mathcal{E}(t) \) defined by (18) satisfies for all \( t > 0 \),

\[
\mathcal{E}(t) \leq \beta \left( 1 + \int_0^t \eta \left( p \right) \right)^{-1/\varepsilon_0}, \quad \varepsilon_0 \in (0, 1). \tag{28}
\]

**3. Asymptotic Behavior for \( \mathcal{E}(0) < d \)**

The following technical lemmas are useful to prove the general decay of energy to problem (1).

**Lemma 9.** Under the assumptions in Theorem 8, then the functional \( \Phi(t) \) defined by

\[
\Phi(t) = \int_{\mathbb{R}^n} \rho(x) \left( \partial_t v_1 + v_2(t) \partial_t v_2 \right) dx \tag{29}
\]

satisfies for any \( t \geq 0 \),

\[
\Phi'(t) \leq \sum_{i=1}^{2} \left( \| \partial_t v_i(t) \|^2_{L^2} - \frac{1}{2} \sum_{i=1}^{2} l_i \| v_i(t) \|^2 \right)
+ \sum_{i=1}^{2} \frac{1 - l_i}{4\varepsilon} \left( \partial_{\varepsilon} \mathcal{E}(v_i) \right) (t)
+ k \sum_{i=1}^{2} \int_{\mathbb{R}^n} \rho(x) v_i^2 \ln |v_i| dx. \tag{30}
\]

**Proof.** We differentiate \( \Phi(t) \), using (1), we can get

\[
\Phi'(t) = \sum_{i=1}^{2} \| \partial_t v_i(t) \|^2_{L^2} - \sum_{i=1}^{2} \| v_i(t) \|^2
+ \sum_{i=1}^{2} \int_{\mathbb{R}^n} \mathcal{E}(v_i(t)) \cdot \int_0^t \partial_t (t - p) \mathcal{E}(v_i(p)) dp dx
- 2b \int_{\mathbb{R}^n} \rho(x) v_1 v_2 dx + k \sum_{i=1}^{2} \int_{\mathbb{R}^n} \rho(x) v_i^2 \ln |v_i| dx. \tag{31}
\]

Exploit Young and Poincaré’s inequalities to estimate

\[
2b \int_{\mathbb{R}^n} \rho(x) v_1 v_2 dx \leq \varepsilon_* \| v_1 \|^2_{L^2} + \frac{c}{4\varepsilon} \| v_2 \|^2_{L^2}. \tag{33}
\]

Inserting (32)–(33) into (31) yields for any \( \varepsilon > 0 \),

\[
\Phi'(t) \leq \sum_{i=1}^{2} \| \partial_t v_i(t) \|^2_{L^2} - \sum_{i=1}^{2} (1 - \varepsilon - \varepsilon_* \| v_i(t) \|^2)
+ \sum_{i=1}^{2} \frac{1 - l_i}{4\varepsilon} \left( \partial_{\varepsilon} \mathcal{E}(v_i) \right) (t)
+ k \sum_{i=1}^{2} \int_{\mathbb{R}^n} \rho(x) v_i^2 \ln |v_i| dx. \tag{34}
\]

Taking \( \varepsilon > 0 \) small enough in (34) such that

\[
l_i - \varepsilon - \varepsilon_* > \frac{1}{2}. \tag{35}
\]

The proof is hence complete.

**Lemma 10.** Under the assumptions in Theorem 8, then the functional \( \psi(t) \) defined by

\[
\psi(t) = -\int_{\mathbb{R}^n} \rho(x) \partial_t v_1(t) \int_0^t \partial_t v_1(t - p) v_1(p) dp dx
- \int_{\mathbb{R}^n} \rho(x) \partial_t v_2(t) \int_0^t \partial_t v_2(t - p) v_2(p) dp dx \tag{36}
\]

...
satisfies for any \( \delta > 0 \),

\[
\psi'(t) \leq \sum_{i=1}^{2} \delta[(1-t)^2 + 1 + bc_i] \| \nabla v_i(t) \|^2
\]

\[
- \sum_{i=1}^{2} \left[ \left( \int_{0}^{t} \omega_i(s) \, ds \right) - 2\delta \right] \| \partial_i v_i(t) \|^2
\]

\[
+ C \sum_{i=1}^{2} \left( \int_{0}^{t} \omega_i(s) \, ds \right) (\partial_i^\circ \nabla v_i)(t)
\]

\[
+ c_{\epsilon_i} \sum_{i=1}^{2} (\partial_i^\circ \nabla v_i)^{1(1+\epsilon_i)}.
\]

**Proof.** Taking the derivative of \( \psi(t) \) and using (1), we conclude that

\[
\psi'(t) = \sum_{i=1}^{2} \int_{\mathbb{R}^*} \nabla v_i(t) \int_{0}^{t} \omega_i(t-p) (\nabla v_i(t) - \nabla v_i(p)) \, dp \, dx
\]

\[
- \int_{\mathbb{R}^*} \left( \int_{0}^{t} \omega_i(t-p) \nabla v_i(p) \, dp \right) dx
\]

\[
+ \left( \int_{0}^{t} \omega_i(t-p) (\nabla v_i(t) - \nabla v_i(p)) \, dp \right) dx
\]

\[
- \int_{\mathbb{R}^*} \left( \int_{0}^{t} \omega_i(t-p) \nabla v_i(p) \, dp \right) dx
\]

\[
+ b \int_{\mathbb{R}^*} \rho(x) v_i \int_{0}^{t} \omega_i(t-p) (v_i(t) - v_i(p)) \, dp \, dx
\]

\[
+ b \int_{\mathbb{R}^*} \rho(x) v_i \int_{0}^{t} \omega_i(t-p) (v_i(t) - v_i(p)) \, dp \, dx
\]

\[
- k \sum_{i=1}^{2} \int_{\mathbb{R}^*} \rho(x) v_i \ln |v_i| \int_{0}^{t} \omega_i(t-p) (v_i(t) - v_i(p)) \, dp \, dx
\]

\[
- k \sum_{i=1}^{2} \int_{\mathbb{R}^*} \rho(x) |v_i| \omega_i(t-p) \, dp \| \partial_i v_i \|_L^2
\]

\[
- \sum_{i=1}^{2} \int_{\mathbb{R}^*} \rho(x) \partial_i v_i \int_{0}^{t} \omega_i(t-p) (v_i(t) - v_i(p)) \, dp \, dx.
\]

The second and third terms can be treated as

\[
\int_{\mathbb{R}^*} \left( \int_{0}^{t} \omega_i(t-p) \nabla v_i(p) \, dp \right) \left( \int_{0}^{t} \omega_i(t-p) (\nabla v_i(t) - \nabla v_i(p)) \, dp \right) dx
\]

\[
\leq \delta(1-t)^2 \| \nabla v_i \|^2 + \left( 1 + \frac{1}{4\delta} \right) \left( \int_{0}^{t} \omega_i(p) \, dp \right) (\partial_i^\circ \nabla v_i)(t).
\]

The fourth and fifth terms will be estimated by

\[
\int_{\mathbb{R}^*} \rho(x) v_i \int_{0}^{t} \omega_i(t-p) (v_i(t) - v_i(p)) \, dp \, dx
\]

\[
\leq \delta c_i \| \nabla v_i \|^2 + \frac{c_i}{4\delta} \left( \int_{0}^{t} \omega_i(p) \, dp \right) (\partial_i^\circ \nabla v_i)(t),
\]

\[
\int_{\mathbb{R}^*} \rho(x) v_i \int_{0}^{t} \omega_i(t-p) (v_i(t) - v_i(p)) \, dp \, dx
\]

\[
\leq \delta^2 \| \nabla v_i \|^2 + \frac{c_i}{4\delta} \left( \int_{0}^{t} \omega_i(p) \, dp \right) (\partial_i^\circ \nabla v_i)(t),
\]

respectively.

For the last term, we have

\[
\int_{\mathbb{R}^*} \rho(x) v_i \int_{0}^{t} \omega_i(t-p) (v_i(t) - v_i(p)) \, dp \, dx
\]

\[
\leq \delta \| \partial_i v_i \|_L^2 + \frac{c_i}{4\delta} \left( \int_{0}^{t} \omega_i(p) \, dp \right) (\partial_i^\circ \nabla v_i)(t).
\]

Let \( \epsilon_0 \in (0,1) \) and \( g(s) = s^\epsilon \cdot |\ln s| \). Notice that \( g \) is continuous on \( (0, \infty) \), its limit at \( 0 \) is \( 0 \), and its limit at \( \infty \) is \( \infty \). Then, \( g \) has a maximum \( m_{\epsilon_0} \) on \( (0, \infty) \), so the following inequality holds

\[
s |\ln s| \leq s^\epsilon + m_{\epsilon_0} s^{1-\epsilon}, \quad \text{for all } s > 0.
\]

Using the Cauchy-Schwartz’s inequality and applying (43), yields, for any \( \delta > 0 \),

\[
k \int_{\mathbb{R}^*} \rho(x) v_i \ln |v_i| \int_{0}^{t} \omega_i(t-p) (v_i(t) - v_i(p)) \, dp \, dx
\]

\[
\leq k \int_{\mathbb{R}^*} \rho(x) (v_i^2 + m_{\epsilon_0} |v_i|^{1-\epsilon})
\]

\[
+ \left( \int_{0}^{t} \omega_i(t-p) (v_i(t) - v_i(p)) \, dp \right)
\]

\[
\leq k \int_{\mathbb{R}^*} \rho(x) v_i^2 + \left( \int_{0}^{t} \omega_i(t-p) (v_i(t) - v_i(p)) \, dp \right)
\]

\[
\leq \delta \| v_i \|^2 + \frac{1}{4\delta} (\partial_i^\circ \nabla v_i)(t) + c_{\epsilon_0} (\partial_i^\circ \nabla v_i)^{1/1+\epsilon_0}.
\]
Combining (39)–(44) with (39) gives us (37) with
\[
C = \frac{bc_* + 2}{4\delta} + 2\delta. 
\] (45)

Therefore, the proof is complete.

Now, we define a Lyapunov functional \( \mathcal{L}(t) \) by
\[
\mathcal{L}(t) = M \mathcal{E}(t) + \varepsilon_1 \Phi(t) + \varepsilon_2 \psi(t),
\] (46)
where \( M, \varepsilon_1, \) and \( \varepsilon_2 \) are positive constants which will be taken later.

It is easy to see that \( \mathcal{L}(t) \) and \( \mathcal{E}(t) \) are equivalent in the sense that there exist two positive constants \( \beta_1 \) and \( \beta_2 \) such that
\[
\beta_1 \mathcal{E}(t) \leq \mathcal{L}(t) \leq \beta_2 \mathcal{E}(t). 
\] (47)

Remark 11 (see [3]). Since \( \zeta_i \) is nonincreasing, we have
\[
\zeta_i(t)(\varrho_i \varphi \nabla v_i)\overset{1/(1+\varepsilon_0)}{=} C\left(-\mathcal{E}'(t)\right)^{1/(1+\varepsilon_0)}. 
\] (48)

Proof of Theorem 8. For any fixed \( t_0 > 0 \), we have for any \( t \geq t_0 \),
\[
\int_0^t \varphi_i(p) dp \geq \int_0^{t_0} \varphi_i(p) dp = \varphi_{t_0}. 
\] (49)

It follows from (37), (30), and (20) that
\[
\mathcal{L}'(t) = M\mathcal{E}'(t) + \varepsilon_1 \Phi'(t) + \psi'(t)
\leq -\sum_{i=1}^2 (\varphi_{t_0} - 2\varepsilon - \varepsilon_1) \| \partial_i v_i(t) \|^2_{L^2} \\
- \sum_{i=1}^2 \left[ \frac{l}{2}\varepsilon_1 - \delta (1 - l_1)^2 + 1 + bc_* \right] \| \nabla v_i(t) \|^2_{L^2} \\
+ \sum_{i=1}^2 \left[ \sum_{j=1}^2 \| C_i \varepsilon_1 + C_{i_1} (\varrho_j \varphi \nabla v_i)(t) - \frac{M}{2} \sum_{i=1}^2 \varphi_i(t) \| v_i(t) \|^2 \\
+ \epsilon_1 k \sum_{i=1}^2 \int_{\mathbb{R}^2} \rho(x)^2 \ln |v_i| dx + \epsilon_1 c_\varepsilon \sum_{i=1}^2 (\varrho_i \varphi \nabla v_i)\overset{1/(1+\varepsilon_0)}{=} C_3 \sum_{i=1}^2 (\varphi_i \varphi \nabla v_i)(t). 
\] (50)

Using the logarithmic Sobolev inequality, we have
\[
\mathcal{L}'(t) \leq -\sum_{i=1}^2 (\varphi_{t_0} - 2\varepsilon - \varepsilon_1) \| \partial_i v_i(t) \|^2_{L^2} \\
+ C_3 \sum_{i=1}^2 \left( (\varrho_i \varphi \nabla v_i)(t) - \frac{l}{2}\varepsilon_1 - \delta (1 - l_1)^2 + 1 + bc_* - \varepsilon_1 k \sum_{i=1}^2 \| v_i(t) \|^2_{L^2} \\
+ \epsilon_1 k (1 + \ln \varphi) \sum_{i=1}^2 \| v_i(t) \|^2_{L^2} + \epsilon_1 c_\varepsilon \sum_{i=1}^2 (\varrho_i \varphi \nabla v_i)\overset{1/(1+\varepsilon_0)}{=} C_3 \sum_{i=1}^2 (\varphi_i \varphi \nabla v_i)(t). 
\] (51)

Recalling (18) and \( \mathcal{E}(t) \leq \mathcal{E}(0) < d \), we get
\[
\ln \| v_i \|^2_{L^2} < \ln \left( \frac{4}{k} \mathcal{E}(t) \right) < \ln \left( \frac{4}{k} \mathcal{E}(0) \right) < \ln \left( \frac{4}{k} d \right). 
\] (52)

Now, we take \( \varepsilon_1 \) small enough so that
\[
(\varphi_{t_0} - 2\varepsilon - \varepsilon_1) > 0. 
\] (53)

For any fixed \( \varepsilon_1 > 0 \), we pick \( \delta > 0 \) so small that
\[
\frac{l}{2}\varepsilon_1 - \delta (1 - l_1)^2 + 1 > \frac{l}{4}\varepsilon_1. 
\] (54)

On the other hand, we choose \( M > 0 \) large enough so that
\[
C_3 = \frac{M}{2} - \frac{C_3(0)}{4\delta} > 0. 
\] (55)

We can conclude that there exist two positive constant \( m \) and \( C' \) such that
\[
\mathcal{L}'(t) \leq -m \mathcal{E}(t) + C' \sum_{i=1}^2 (\varrho_i \varphi \nabla v_i)(t) + \varepsilon_1 c_\varepsilon \sum_{i=1}^2 (\varrho_i \varphi \nabla v_i)\overset{1/(1+\varepsilon_0)}{=} C_3 \sum_{i=1}^2 (\varphi_i \varphi \nabla v_i)(t). 
\] (56)

Multiplying (56) by \( \zeta(t) = \min \{ \zeta_1, \zeta_2 \} \) by (H2) and use the fact that
\[
(\varrho_i \varphi \nabla v_i)(t) \leq c(\varrho_i \varphi \nabla v_i)\overset{1/(1+\varepsilon_0)}{=} C_3 \sum_{i=1}^2 (\varphi_i \varphi \nabla v_i)(t), 
\] (57)
and (48), we get
\[
\zeta(t) \mathcal{L}'(t) \leq -m \zeta(t) \mathcal{E}(t) + c \left( \mathcal{E}'(t) \right)^{1/(1+\varepsilon_0)}. 
\] (58)
Multiply (58) by $\zeta^{n}(t) \mathcal{E}_{s}(t)$ and recall that $\zeta'(t) \leq 0$ to obtain

$$
\zeta^{n+1}(t) \mathcal{E}_{s}(t) \mathcal{L}'(t) \leq -m \zeta^{n+1}(t) \mathcal{E}_{s+1}(t) + c(\mathcal{E}) \zeta^{n+1}(t) \left( -\mathcal{E}'(t) \right)^{1/(1+\zeta)}.
$$

(59)

Using Young’s inequality, for any $\delta > 0$,

$$
\zeta^{n+1}(t) \mathcal{E}_{s}(t) \mathcal{L}'(t) \leq -m \zeta^{n+1}(t) \mathcal{E}_{s+1}(t) + c(\delta \zeta^{n+1}(t) \mathcal{E}_{s+1}(t) - c \mathcal{E}'(t)) \\
\leq -(m - \delta c) \zeta^{n+1}(t) \mathcal{E}_{s+1}(t) - c \mathcal{E}'(t),
$$

(60)

which implies

$$
\left( \zeta^{n+1} \mathcal{E}_{s} \mathcal{L} + c \mathcal{E} \right)(t) \leq -(m - \delta c) \zeta^{n+1}(t) \mathcal{E}_{s+1}(t).
$$

(61)

It is clear that to get

$$
\mathcal{L}_{1}(t) = \left( \zeta^{n+1} \mathcal{E}_{s} \mathcal{L} + c \mathcal{E} \right) \sim \mathcal{E}(t).
$$

(62)

By using (61) and $\zeta'(t) \leq 0$, we arrive at

$$
\mathcal{L}_{1}'(t) = \left( \zeta^{n+1} \mathcal{E}_{s} \mathcal{L} + c \mathcal{E} \right) \leq -m' \zeta^{n}(t) \mathcal{E}_{s+1}(t).
$$

(63)

Integration over $(t_{0}, t)$ leads to for some constant $m' > 0$ such that

$$
\mathcal{L}_{1}(t) \leq m' \left( 1 + \int_{t_{0}}^{t} \zeta^{n+1}(p) dp \right)^{-1/\zeta}.
$$

(64)

The equivalence of $\mathcal{L}_{1}(t)$ and $\mathcal{E}$ completes Proof of Theorem 8.

**Remark 12.**

1. We mention here that we have coupled our system without the classical way, i.e., our idea is not to couple equations in the logarithmic nonlinear terms.

2. Most contribution here is to obtain our nonexistence result with less conditions on the viscoelastic terms.

**Data Availability**

No data were used in this study.

**Conflicts of Interest**

The authors declare that they have no competing interests.

**Authors’ Contributions**

The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

**References**


