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*Research Article*

# A Fixed Point Technique for Solving an Integro-Differential Equation Using Mixed-Monotone Mappings

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The objective of this manuscript is to present new tripled fixed point results for mixed-monotone mappings by a pivotal lemma in the setting of partially ordered complete metric spaces. Our outcomes sum up, enrich, and generalize several results in the current writing. Moreover, some examples have been discussed to strengthen and support our theoretical results. Finally, the theoretical results are applied to study the existence and uniqueness of the solution to an integro-differential equation.

## 1. Brief Introduction

The fixed point (FP) technique is considered a fundamental pillar and a powerful tool in nonlinear analysis because of its many vital applications in many disciplines such as computer science, engineering, economics, biology, chemistry, and physics. In mathematics, this technique plays a prominent role in the study of statistical models, dynamical systems, game-theoretic models, differential equations, and many others. More clearly, for example, this method is mainly applied in finding the analytical solution to some differential and integral equations, fractional equations, integro-differential equations (IDEs), and functional analysis which facilitates the way to find numerical solutions to such problems. These problems were addressed by Fredholm [1], Rus [2], Hammad and De La Sen [3, 4], Ameer et al. [5], Hussain et al. [6, 7] and Younis et al. [8–10]

In [11], the concepts of the coupled FP and a mixed-monotone mapping were initiated, and some exciting work in partially ordered metric spaces (POMSs) have been discussed by the same authors. This idea was investigated by many authors such as Berinde [12], Choudhury and Maity [13], and Aydi et al. [14]. Moreover, in abstract spaces, this concept has many applications in integral and functional equations; see the papers of Cirić et al. [15], Ding et al. [16], Hammad et al. [17, 18], Luong and Thuan [19], Choudhury and Kundu [20], Agarwal et al. [21], Radenović [22], and Hammad et al. [23].

In 2011, coupled FP notions are generalized to tripled fixed points (TFPs) concepts by Berinde and Borcut [24] in the setting of POMSs. Via the mentioned spaces, Borcut and Berinde [25, 26], Karapnar et al. [27] presented pivotal results about TFP theorems and the applications in this direction introduced by Mustafa et al. [28] and Hammad and De la Sen [29, 30].

*Definition 1* [25]. We say that a trio $(\omega, \kappa, \nu) \in \aleph^3$ (where $\aleph \times \aleph \times \aleph = \aleph^3$) is a TFP of a self-mapping $\mho : \aleph^3 \longrightarrow \aleph$ if $\omega = \mho(\omega, \kappa, \nu)$, $\kappa = \mho(\kappa, \nu, \omega)$, and $\nu = \mho(\nu, \omega, \kappa)$.

*Definition 2* [26]. A trio $(\omega, \kappa, \nu) \in \aleph^3$ on a nonempty set $\aleph$ is called a tripled coincidence point of the two self-mappings $\mho : \aleph^3 \longrightarrow \aleph$ and $\beth : \aleph \longrightarrow \aleph$ if $\beth\omega = \mho(\omega, \kappa, \nu)$, $\beth\kappa = \mho(\kappa, \nu, \omega)$, and $\beth\nu = \mho(\nu, \omega, \kappa)$.

*Definition 3* [26]. Assume that $\aleph \neq \varnothing$ is a set. A trio $(\omega, \kappa, \nu) \in \aleph^3$ is called a tripled common FP of $\mho : \aleph^3 \longrightarrow \aleph$ and $\gimel : \aleph \longrightarrow \aleph$, if $\omega = \gimel\omega = \mho(\omega, \kappa, \nu)$, $\kappa = \gimel\kappa = \mho(\kappa, \nu, \omega)$, and $\nu = \gimel\nu = \mho(\nu, \omega, \kappa)$.

*Definition 4* [24]. Assume that $(\aleph, \preccurlyeq)$ is a partially ordered set on the product space $\aleph^3$ defined as follows:

$$\begin{gathered}(\omega, \kappa, \nu), (\omega^*, \kappa^*, \nu^*) \in \aleph^3, \\ (\omega, \kappa, \nu) \preccurlyeq (\omega^*, \kappa^*, \nu^*) \Leftrightarrow \omega \preccurlyeq \omega^*, \\ \kappa^* \preccurlyeq \kappa, \nu \preccurlyeq \nu^*.\end{gathered} \tag{1}$$

Under this partial order, we state the following definitions.

*Definition 5* [24]. A mapping $\mho : \aleph^3 \longrightarrow \aleph$ on a partially ordered set $(\aleph, \preccurlyeq)$ has a mixed-monotone property, if for any $\omega, \kappa, \nu \in \aleph$, we have

$$\begin{gathered}\omega_1, \omega_2 \in \aleph, \omega_1 \preccurlyeq \omega_2 \Rightarrow \mho(\omega_1, \kappa, \nu) \preccurlyeq \mho(\omega_2, \kappa, \nu), \\ \kappa_1, \kappa_2 \in \aleph, \kappa_1 \preccurlyeq \kappa_2 \Rightarrow \mho(\omega, \kappa_1, \nu) \succcurlyeq \mho(\omega, \kappa_2, \nu), \\ \nu_1, \nu_2 \in \aleph, \nu_1 \preccurlyeq \nu_2 \Rightarrow \mho(\omega, \kappa, \nu_1) \preccurlyeq \mho(\omega, \kappa, \nu_2).\end{gathered} \tag{2}$$

*Definition 6* [14]. A mapping $\mho : \aleph^3 \longrightarrow \aleph$ on a partially ordered set $(\aleph, \preccurlyeq)$ has a mixed $\gimel$-monotone property, where $\gimel : \aleph \longrightarrow \aleph$, if for any $\omega, \kappa, \nu \in \aleph$, we have

$$\begin{gathered}\omega_1, \omega_2 \in \aleph, \gimel\omega_1 \preccurlyeq \gimel\omega_2 \Rightarrow \mho(\omega_1, \kappa, \nu) \preccurlyeq \mho(\omega_2, \kappa, \nu), \\ \kappa_1, \kappa_2 \in \aleph, \gimel\kappa_1 \preccurlyeq \gimel\kappa_2 \Rightarrow \mho(\omega, \kappa_1, \nu) \succcurlyeq \mho(\omega, \kappa_2, \nu), \\ \nu_1, \nu_2 \in \aleph, \gimel\nu_1 \preccurlyeq \gimel\nu_2 \Rightarrow \mho(\omega, \kappa, \nu_1) \preccurlyeq \mho(\omega, \kappa, \nu_2).\end{gathered} \tag{3}$$

*Definition 7* [27]. Assume that $\aleph$ is a nonempty set. We say that the mappings $\mho : \aleph^3 \longrightarrow \aleph$ and $\gimel : \aleph \longrightarrow \aleph$ are commutative if $\gimel(\mho(\omega, \kappa, \nu)) = \mho(\gimel\omega, \gimel\kappa, \gimel\nu)$, for all $\omega, \kappa, \nu \in \aleph$.

The first contribution of the TFP for a mixed-monotone mapping in a partially ordered set was presented as follows:

**Theorem 8** [24]. *Let $(\aleph, \zeta, \preccurlyeq)$ be a complete partially ordered metric space (CPOMS). Assume that $\mho : \aleph^3 \longrightarrow \aleph$, so that*

*(i) $\mho$ has a mixed-monotone property*

*(ii) Either $\mho$ is continuous or $\aleph$ has the following properties:*

- *(a) $\omega_n \preccurlyeq \omega$, if the nondecreasing sequence $\omega_n \longrightarrow \omega$*
- *(b) $\nu_n \succcurlyeq \nu$, if the nonincreasing sequence $\nu_n \longrightarrow \nu$, for all $n$*

*(iii) There is $\alpha, \beta, \gamma \geq 0$ with $\alpha + \beta + \gamma < 1$ such that*

$$\zeta(\mho(\omega, \kappa, \nu), \mho(\omega^*, \kappa^*, \nu^*)) \leq \alpha\zeta(\omega, \omega^*) + \beta\zeta(\kappa, \kappa^*) + \gamma\zeta(\nu, \nu^*), \tag{4}$$

*for any $\omega, \kappa, \nu, \omega^*, \kappa^*, \nu^* \in \aleph$, for which $\omega \preccurlyeq \omega^*$, $\kappa^* \preccurlyeq \kappa$, and $\nu \preccurlyeq \nu^*$. If there are $\omega_0, \kappa_0, \nu_0 \in \aleph$ so that $\omega_0 \preccurlyeq \mho(\omega_0, \kappa_0, \nu_0)$, $\kappa_0 \succcurlyeq \mho(\kappa_0, \nu_0, \omega_0)$, and $\nu_0 \preccurlyeq \mho(\nu_0, \omega_0, \kappa_0)$. Then $\mho$ has a TFP.*

In this manuscript, we utilize a pivotal lemma to obtain new TFP results for mixed-monotone mappings in CPOMSs. Our results unify, extend, and generalize the papers [19, 31, 32]. Also, some examples and a corollary are given. Later on, we apply the theoretical results to obtain the solution of a system of IDEs as an application.

## 2. Main Results

We begin this part with the pivotal lemma below.

**Lemma 9.** *Assume that $(\aleph, \preccurlyeq)$ is a partially ordered set and $\mho : \aleph^3 \longrightarrow \aleph$ and $\gimel : \aleph \longrightarrow \aleph$ are two mappings. Suppose that the following assumptions hold:*

*($s_1$) There is $\daleth_0 \in \aleph$, so that for $\rho_1^n\daleth_0, \rho_2^n\daleth_0, \rho_3^n\daleth_0 \in \aleph$, $\rho_1, \rho_2, \rho_3 \in (0, 1)$, and $\rho_1 < \rho_3 < \rho_2$, $n \in \mathbb{N}$*

$$\begin{gathered}\mho(\daleth_0, \daleth_0, \daleth_0) \succcurlyeq \frac{\gimel(\rho_1^n\daleth_0)}{\rho_1^n\daleth_0}, \\ \mho(\daleth_0, \daleth_0, \daleth_0) \preccurlyeq \frac{\gimel(\rho_2^n\daleth_0)}{\rho_2^n\daleth_0}, \\ \mho(\daleth_0, \daleth_0, \daleth_0) \succcurlyeq \frac{\gimel(\rho_3^n\daleth_0)}{\rho_3^n\daleth_0}.\end{gathered} \tag{5}$$

*($s_2$) There is $\Delta : [0,\infty) \longrightarrow [0,\infty)$ with $\Delta(\rho) \in (\rho_1, \rho_2]$, so that for $\rho_1^n\daleth_0, \rho_2^n\daleth_0, \rho_3^n\daleth_0 \in \aleph$, $n \in \mathbb{N}$*

$$\mho(\rho_1^n\daleth_0, \rho_2^n\daleth_0, \rho_3^n\daleth_0) \succcurlyeq \Delta(\rho)\mho\left(\rho_1^{n-1}\daleth_0, \rho_2^{n-1}\daleth_0, \rho_3^{n-1}\daleth_0\right), \tag{6}$$

$$\mho(\rho_2^n\daleth_0, \rho_3^n\daleth_0, \rho_1^n\daleth_0) \preccurlyeq \Delta(\rho)\mho\left(\rho_2^{n-1}\daleth_0, \rho_3^{n-1}\daleth_0, \rho_1^{n-1}\daleth_0\right), \tag{7}$$

$$\mho(\rho_3^n\daleth_0, \rho_1^n\daleth_0, \rho_2^n\daleth_0) \succcurlyeq \Delta(\rho)\mho\left(\rho_3^{n-1}\daleth_0, \rho_1^{n-1}\daleth_0, \rho_2^{n-1}\daleth_0\right). \tag{8}$$

*Then, there is $\omega_0, \kappa_0, \nu_0 \in \aleph$, so that $\mho(\omega_0, \kappa_0, \nu_0) \succcurlyeq \gimel(\omega_0)$, $\mho(\kappa_0, \nu_0, \omega_0) \preccurlyeq \gimel(\kappa_0)$, and $\mho(\nu_0, \omega_0, \kappa_0) \succcurlyeq \gimel(\nu_0)$.*

*Proof.* We split the proof into three steps:

(Step 1) Since $\Delta(\rho) > \rho_1$, then there is a nonnegative integer $\ell = ((\ln(1/\rho_1))/(\ln(\Delta(\rho)/\rho_1))) + 1$, such that for $n \geq \ell$, we get $(\Delta(\rho)/\rho_1)^n \geq (1/\rho_1)$, i.e., $(\Delta(\rho))^n \geq \rho_1^{n-1}$. Take $\omega_0 = \rho_1^n\daleth_0$, $\kappa_0 = \rho_2^n\daleth_0$, and $\nu_0 = \rho_3^n\daleth_0$. By Stipulation $(s_1)$ and (5), we have

$$\begin{aligned}\mho(\omega_0, \kappa_0, \nu_0) &= \mho(\rho_1^n \beth_0, \rho_2^n \beth_0, \rho_3^n \beth_0) \\ &\succcurlyeq \Delta(\rho)\mho\left(\rho_1^{n-1}\beth_0, \rho_2^{n-1}\beth_0, \rho_3^{n-1}\beth_0\right) \\ &\succcurlyeq (\Delta(\rho))^2 \mho\left(\rho_1^{n-2}\beth_0, \rho_2^{n-2}\beth_0, \rho_3^{n-2}\beth_0\right) \succcurlyeq \cdots \\ &\succcurlyeq (\Delta(\rho))^n \mho(\beth_0, \beth_0, \beth_0) \succcurlyeq \rho_1^{n-1}\mho(\beth_0, \beth_0, \beth_0) \\ &\succcurlyeq \gimel(\rho_1^n \beth_0) = \gimel(\omega_0).\end{aligned} \tag{9}$$

(Step 2) Since $\Delta(\rho) \leq \rho_2$, then for each $n \in \mathbb{N}$, we obtain $\rho_2 \leq (\rho_2/\Delta(\rho))^n$, i.e., $(\Delta(\rho))^n \geq \rho_2^{n-1}$. Apply Stipulation $(s_1)$ and (6), we get

$$\begin{aligned}\mho(\kappa_0, \nu_0, \omega_0) &= \mho(\rho_2^n \beth_0, \rho_3^n \beth_0, \rho_1^n \beth_0) \\ &\preccurlyeq \Delta(\rho)\mho\left(\rho_2^{n-1}\beth_0, \rho_3^{n-1}\beth_0, \rho_1^{n-1}\beth_0\right) \\ &\preccurlyeq (\Delta(\rho))^2 \mho\left(\rho_2^{n-2}\beth_0, \rho_3^{n-2}\beth_0, \rho_1^{n-2}\beth_0\right) \preccurlyeq \cdots \\ &\preccurlyeq (\Delta(\rho))^n \mho(\beth_0, \beth_0, \beth_0) \preccurlyeq \rho_2^{n-1}\mho(\beth_0, \beth_0, \beth_0) \\ &\preccurlyeq \gimel(\rho_2^n \beth_0) = \gimel(\kappa_0).\end{aligned} \tag{10}$$

(Step 3) Similar to Step 1, since $\Delta(\rho) > \rho_3$, then we can write $(\Delta(\rho))^n \geq \rho_3^{n-1}$. By Stipulation $(s_1)$ and (7), we have

$$\begin{aligned}\mho(\nu_0, \omega_0, \kappa_0) &= \mho(\rho_3^n \beth_0, \rho_1^n \beth_0, \rho_2^n \beth_0) \\ &\succcurlyeq \Delta(\rho)\mho\left(\rho_3^{n-1}\beth_0, \rho_1^{n-1}\beth_0, \rho_2^{n-1}\beth_0\right) \\ &\succcurlyeq (\Delta(\rho))\mho\left(\rho_3^{n-1}\beth_0, \rho_1^{n-1}\beth_0, \rho_2^{n-1}\beth_0\right) \succcurlyeq \cdots \\ &\succcurlyeq (\Delta(\rho))^n \mho(\beth_0, \beth_0, \beth_0) \succcurlyeq \rho_3^{n-1}\mho(\beth_0, \beth_0, \beth_0) \\ &\succcurlyeq \gimel(\rho_3^n \beth_0) = \gimel(\nu_0).\end{aligned} \tag{11}$$

This completes the proof. □

*Remark 10.* The results (5)-(7) of Lemma 9 still hold if we reserve the symbols "$\succcurlyeq$" and "$\preccurlyeq$," that is, $\omega_0, \kappa_0, \nu_0 \in \aleph$ so that $\mho(\omega_0, \kappa_0, \nu_0) \preccurlyeq \gimel(\omega_0)$, $\mho(\kappa_0, \nu_0, \omega_0) \succcurlyeq \gimel(\kappa_0)$, and $\mho(\nu_0, \omega_0, \kappa_0) \preccurlyeq \gimel(\nu_0)$.

**Theorem 11.** *Let $(\aleph, \zeta, \preccurlyeq)$ be a CPOMS and $\vartheta$ be a zero element in $\aleph$. Assume that $\mho : \aleph^3 \longrightarrow \aleph$ is mixed $\gimel$-monotone mapping, $\gimel : \aleph \longrightarrow \aleph$ is self-mapping, and $\Delta : [0, \infty) \longrightarrow [0, \infty)$, so that $\Delta(\rho) \leq \rho$ for any $\rho \geq 0$. Suppose that the hypotheses below hold:*

*(i)* $\mho(\aleph^3) \subset \gimel(\aleph)$

*(ii)* $\gimel$ *and* $\mho$ *are continuous and commute*

*(iii)* $\mho(\omega, \kappa, \nu)$ *verifies stipulations* $(s_1)$ *and* $(s_2)$ *of Lemma 9*

*(iv) For any $\omega, \kappa, \nu, \omega^*, \kappa^*, \nu^* \in \aleph$ with $\gimel(\omega) \succcurlyeq \gimel(\omega^*)$, $\gimel(\kappa) \preccurlyeq \gimel(\kappa^*)$, $\gimel(\nu) \succcurlyeq \gimel(\nu^*)$, and we have*

$$\begin{aligned}&\zeta(\mho(\omega, \kappa, \nu), \mho(\omega^*, \kappa^*, \nu^*)) \\ &\quad \leq \Delta\left(\max\left\{\frac{1}{2}\zeta(\gimel(\omega), \mho(\omega, \kappa, \nu)), \zeta(\gimel(\omega^*), \mho(\omega^*, \kappa^*, \nu^*))\right\}\right).\end{aligned} \tag{12}$$

*Then, the following conclusions are fulfilled:*

*$(C_1)$ For a triplet $(\omega_0, \kappa_0, \nu_0) \in \aleph$, construct three sequences $\{\omega_n\}$, $\{\kappa_n\}$, and $\{\nu_n\}$ in $\aleph$ verifying*

$$\begin{aligned}\gimel(\omega_n) &= \mho(\omega_{n-1}, \kappa_{n-1}, \nu_{n-1}), \\ \gimel(\kappa_n) &= \mho(\kappa_{n-1}, \nu_{n-1}, \omega_{n-1}), \\ \gimel(\nu_n) &= \mho(\nu_{n-1}, \omega_{n-1}, \kappa_{n-1}),\end{aligned} \tag{13}$$

*for all $n \in \mathbb{N}$. Then, $\gimel(\omega_n) \longrightarrow \omega''$, $\gimel(\kappa_n) \longrightarrow \kappa''$, and $\gimel(\nu_n) \longrightarrow \nu''$, as $n \longrightarrow \infty$*

*$(C_2)$ $\mho$ and $\gimel$ have a tripled coincidence point $(\omega'', \kappa'', \nu'')$. Moreover, assume that $\gimel(\omega_0)$, $\gimel(\kappa_0)$, and $\gimel(\nu_0)$ are comparable, and for each $(\omega, \kappa, \nu) \in \aleph$, $(\gimel(\omega_0), \gimel(\kappa_0), \gimel(\nu_0))$ is comparable to $(\mho(\omega, \kappa, \nu), \mho(\kappa, \nu, \omega), \mho(\nu, \omega, \kappa))$*

*$(C_3)$ $\mho$ and $\gimel$ have a unique common fixed point $\gimel(\vartheta)$, that is, $\gimel(\vartheta) = \gimel(\gimel(\vartheta)) = \mho(\gimel(\vartheta), \gimel(\vartheta), \gimel(\vartheta))$*

*Proof.* We shall prove Conclusion $(C_1)$. By Condition (iii), there are $\omega_0 = \rho_1^n \beth_0$, $\kappa_0 = \rho_2^n \beth_0$, and $\nu_0 = \rho_3^n \beth_0$, so that $\mho(\omega_0, \kappa_0, \nu_0) \succcurlyeq \gimel(\omega_0)$, $\mho(\kappa_0, \nu_0, \omega_0) \preccurlyeq \gimel(\kappa_0)$, and $\mho(\nu_0, \omega_0, \kappa_0) \succcurlyeq \gimel(\nu_0)$, where $\rho_1, \rho_2, \rho_3 \in (0, 1)$. Since $\mho(\aleph^3) \subset \gimel(\aleph)$, by Condition (i), this yields that there exists $\omega_1, \kappa_1, \nu_1 \in \aleph$, so that $\gimel(\omega_1) = \mho(\omega_0, \kappa_0, \nu_0)$, $\gimel(\kappa_1) = \mho(\kappa_0, \nu_0, \omega_0)$, and $\gimel(\nu_1) = \mho(\nu_0, \omega_0, \kappa_0)$. Generally, we can build the three sequences $\{\omega_n\}$, $\{\kappa_n\}$, and $\{\nu_n\}$ in $\aleph$, so that

$$\begin{aligned}\gimel(\omega_n) &= \mho(\omega_{n-1}, \kappa_{n-1}, \nu_{n-1}), \\ \gimel(\kappa_n) &= \mho(\kappa_{n-1}, \nu_{n-1}, \omega_{n-1}), \\ \gimel(\nu_n) &= \mho(\nu_{n-1}, \omega_{n-1}, \kappa_{n-1}), \\ &\text{for } n \in \mathbb{N}.\end{aligned} \tag{14}$$

Since $\mho$ is a mixed $\gimel$-monotone, then we get

$$\begin{aligned}\gimel(\omega_2) &= \mho(\omega_1, \kappa_1, \nu_1) \succcurlyeq \mho(\omega_0, \kappa_0, \nu_0) = \gimel(\omega_1), \\ \gimel(\kappa_2) &= \mho(\kappa_1, \nu_1, \omega_1) \preccurlyeq \mho(\kappa_0, \nu_0, \omega_0) = \gimel(\kappa_1), \\ \gimel(\nu_2) &= \mho(\nu_1, \omega_1, \kappa_1) \succcurlyeq \mho(\nu_0, \omega_0, \kappa_0) = \gimel(\nu_1).\end{aligned} \tag{15}$$

By induction for $n \in \mathbb{N}$, one can write

$$\begin{aligned}\gimel(\omega_n) &\succcurlyeq \gimel(\omega_{n-1}), \\ \gimel(\kappa_n) &\preccurlyeq \gimel(\kappa_{n-1}), \\ \gimel(\nu_n) &\succcurlyeq \gimel(\nu_{n-1}).\end{aligned} \tag{16}$$

It follows from (12) and (13) that for $n \in \mathbb{N}$, we have

$$\begin{aligned}\zeta(\beth(\omega_{n+1}), \beth(\omega_n)) &= \zeta(\mho(\omega_n, \kappa_n, \nu_n), \mho(\omega_{n-1}, \kappa_{n-1}, \nu_{n-1})) \\ &\leq \Delta\left(\max\left\{\frac{1}{2}\zeta(\beth(\omega_n), \mho(\omega_n, \kappa_n, \nu_n)), \right.\right. \\ &\qquad \left.\left. \zeta(\beth(\omega_{n-1}), \mho(\omega_{n-1}, \kappa_{n-1}, \nu_{n-1}))\right\}\right) \\ &\leq \max\left\{\frac{1}{2}\zeta(\beth(\omega_n), \mho(\omega_n, \kappa_n, \nu_n)), \right. \\ &\qquad \left. \zeta(\beth(\omega_{n-1}), \mho(\omega_{n-1}, \kappa_{n-1}, \nu_{n-1}))\right\} \\ &= \max\left\{\frac{1}{2}\zeta(\beth(\omega_n), \beth(\omega_{n+1})), \zeta(\beth(\omega_{n-1}), \beth(\omega_n))\right\}.\end{aligned} \tag{17}$$

If $\zeta(\beth(\omega_{n-1}), \beth(\omega_n)) \leq \zeta(\beth(\omega_n), \beth(\omega_{n+1}))$, then by (16), we find directly that

$$\zeta(\beth(\omega_{n+1}), \beth(\omega_n)) \leq \frac{1}{2}\zeta(\beth(\omega_{n+1}), \beth(\omega_n)), \tag{18}$$

this is a contradiction. So, we should take $\zeta(\beth(\omega_{n+1}), \beth(\omega_n)) = 0$. It is clear that for each $\varepsilon > 0$ and $\sigma \in \mathbb{N}$, we have

$$\zeta(\beth(\omega_n), \beth(\omega_{n+\sigma})) < \varepsilon. \tag{19}$$

Since $\zeta(\beth(\omega_{n-1}), \beth(\omega_n)) > (1/2)\zeta(\beth(\omega_n), \beth(\omega_{n+1}))$, then by (16), we obtain that

$$\zeta(\beth(\omega_{n+1}), \beth(\omega_n)) \leq \zeta(\beth(\omega_n), \beth(\omega_{n-1})), \quad \text{for } n \in \mathbb{N}. \tag{20}$$

Set $\nabla_n = \zeta(\beth(\omega_{n+1}), \beth(\omega_n))$. Using (12)–(17), one can obtain

$$\nabla_n \leq \nabla_{n-1} \leq \nabla_{n-2} \leq \cdots \leq \nabla_0, \quad \text{for } n \in \mathbb{N} \cup \{0\}, \tag{21}$$

thus, we have

$$\zeta(\beth(\omega_{n+1}), \beth(\omega_n)) \leq \zeta(\beth(\omega_1), \beth(\omega_0)). \tag{22}$$

According to the proof of Lemma 9 and Condition (ii) (the continuity), there is $\daleth_0 \in \aleph$, $\rho_1, \rho_2, \rho_3 \in (0, 1)$, so that

$$\begin{aligned}\lim_{n\longrightarrow\infty}\beth(\omega_0) &= \lim_{n\longrightarrow\infty}\beth(\rho_1^n\daleth_0) \preccurlyeq \lim_{n\longrightarrow\infty}\mho(\rho_1^n\daleth_0, \rho_2^n\daleth_0, \rho_3^n\daleth_0) = \lim_{n\longrightarrow\infty}\mho(\omega_0, \kappa_0, \nu_0), \\ \lim_{n\longrightarrow\infty}\beth(\kappa_0) &= \lim_{n\longrightarrow\infty}\beth(\rho_2^n\daleth_0) \succcurlyeq \lim_{n\longrightarrow\infty}\mho(\rho_2^n\daleth_0, \rho_3^n\daleth_0, \rho_1^n\daleth_0) = \lim_{n\longrightarrow\infty}\mho(\kappa_0, \nu_0, \omega_0), \\ \lim_{n\longrightarrow\infty}\beth(\nu_0) &= \lim_{n\longrightarrow\infty}\beth(\rho_3^n\daleth_0) \preccurlyeq \lim_{n\longrightarrow\infty}\mho(\rho_3^n\daleth_0, \rho_1^n\daleth_0, \rho_2^n\daleth_0) = \lim_{n\longrightarrow\infty}\mho(\nu_0, \omega_0, \kappa_0),\end{aligned} \tag{23}$$

that is,

$$\begin{aligned}\beth(\vartheta) &\preccurlyeq \mho(\vartheta, \vartheta, \vartheta), \\ \beth(\vartheta) &\preccurlyeq \mho(\vartheta, \vartheta, \vartheta).\end{aligned} \tag{24}$$

Therefore, $\beth(\vartheta) = \mho(\vartheta, \vartheta, \vartheta)$. By the triangle inequality and (21), one can write for any positive integers $m, n$ with $m > n$, so we have

$$\begin{aligned}\zeta(\beth(\omega_m), \beth(\omega_n)) &\leq \zeta(\beth(\omega_m), \beth(\omega_{m-1})) + \zeta(\beth(\omega_{m-1}), \beth(\omega_{m-2})) \\ &\quad + \cdots + \zeta(\beth(\omega_{n+1}), \beth(\omega_n)) \leq \zeta(\beth(\omega_1), \beth(\omega_0)) \\ &\quad + \zeta(\beth(\omega_1), \beth(\omega_0)) + \cdots + \zeta(\beth(\omega_1), \beth(\omega_0)) \\ &= \zeta(\mho(\omega_0, \kappa_0, \nu_0), \beth(\omega_0)) \\ &\quad + \zeta(\mho(\omega_0, \kappa_0, \nu_0), \beth(\omega_0)) + \cdots \\ &\quad + \zeta(\mho(\omega_0, \kappa_0, \nu_0), \beth(\omega_0)) \\ &= \zeta(\mho(\rho_1^n\daleth_0, \rho_2^n\daleth_0, \rho_3^n\daleth_0), \beth(\rho_1^n\daleth_0)) + \cdots \\ &\quad + \zeta(\mho(\rho_1^n\daleth_0, \rho_2^n\daleth_0, \rho_3^n\daleth_0), \beth(\rho_1^n\daleth_0)).\end{aligned} \tag{25}$$

It follows that $\lim_{m,n\longrightarrow\infty}\zeta(\beth(\omega_m), \beth(\omega_n)) = 0$. Similarly, we can show that $\lim_{m,n\longrightarrow\infty}\zeta(\beth(\kappa_m), \beth(\kappa_n)) = 0$ and $\lim_{m,n\longrightarrow\infty}\zeta(\beth(\nu_m), \beth(\nu_n)) = 0$. This illustrates that $\{\omega_n\}$, $\{\kappa_n\}$, and $\{\nu_n\}$ are Cauchy sequences. The completeness of $\aleph$ leads to the conclusion that there are $\omega'', \kappa'', \nu'' \in \aleph$ so that

$$\begin{aligned}\lim_{n\longrightarrow\infty}\beth(\omega_n) &= \omega'', \\ \lim_{n\longrightarrow\infty}\beth(\kappa_n) &= \kappa'', \\ \lim_{n\longrightarrow\infty}\beth(\nu_n) &= \nu''.\end{aligned} \tag{26}$$

Next, we shall show Conclusion $(C_2)$. Since $\beth$ is continuous, then by (25), we get

$$\begin{aligned}\lim_{n\longrightarrow\infty}\beth(\beth(\omega_n)) &= \beth\left(\omega''\right), \\ \lim_{n\longrightarrow\infty}\beth(\beth(\kappa_n)) &= \beth\left(\kappa''\right), \\ \lim_{n\longrightarrow\infty}\beth(\beth(\nu_n)) &= \beth\left(\nu''\right).\end{aligned} \tag{27}$$

Also, by the commutativity of $\beth$ and $\mho$, we have

$$\begin{aligned}\beth(\beth(\omega_n)) &= \beth(\mho(\omega_{n-1}, \kappa_{n-1}, \nu_{n-1})) = \mho(\beth(\omega_{n-1}), \beth(\kappa_{n-1}), \beth(\nu_{n-1})), \\ \beth(\beth(\kappa_n)) &= \beth(\mho(\kappa_{n-1}, \nu_{n-1}, \omega_{n-1})) = \mho(\beth(\kappa_{n-1}), \beth(\nu_{n-1}), \beth(\omega_{n-1})), \\ \beth(\beth(\nu_n)) &= \beth(\mho(\nu_{n-1}, \omega_{n-1}, \kappa_{n-1})) = \mho(\beth(\nu_{n-1}), \beth(\omega_{n-1}), \beth(\kappa_{n-1})).\end{aligned} \tag{28}$$

By (26) and (27), we deduce that

$$\begin{aligned}\beth\left(\omega''\right) &= \lim_{n\longrightarrow\infty}\beth(\beth(\omega_n)) = \lim_{n\longrightarrow\infty}\mho(\beth(\omega_{n-1}), \beth(\kappa_{n-1}), \beth(\nu_{n-1})) = \mho\left(\omega'', \kappa'', \nu''\right), \\ \beth\left(\kappa''\right) &= \lim_{n\longrightarrow\infty}\beth(\beth(\kappa_n)) = \lim_{n\longrightarrow\infty}\mho(\beth(\kappa_{n-1}), \beth(\nu_{n-1}), \beth(\omega_{n-1})) = \mho\left(\kappa'', \nu'', \omega''\right), \\ \beth\left(\nu''\right) &= \lim_{n\longrightarrow\infty}\beth(\beth(\nu_n)) = \lim_{n\longrightarrow\infty}\mho(\beth(\nu_{n-1}), \beth(\omega_{n-1}), \beth(\kappa_{n-1})) = \mho\left(\nu'', \omega'', \kappa''\right).\end{aligned} \tag{29}$$

Therefore, the trio $(\omega'', \kappa'', \nu'')$ is a tripled coincidence point of $\mho$ and $\beth$.

Finally, to prove Conclusion $(C_2)$, assume that $\beth(\omega_0) \preccurlyeq \beth(\kappa_0)$, $\beth(\kappa_0) \preccurlyeq \beth(\nu_0)$, $\beth(\nu_0) \preccurlyeq \beth(\omega_0)$, and $(\beth(\omega_0), \beth(\kappa_0), \beth(\nu_0)) \preccurlyeq (\mho(\omega, \kappa, \nu), \mho(\kappa, \nu, \omega), \mho(\nu, \omega, \kappa))$, for $(\omega, \kappa, \nu) \in \aleph^3$. Since

$(\omega'', \kappa'', \nu''), (\kappa'', \nu'', \omega''), (\nu'', \omega'', \kappa'') \in \aleph^3$ and by Definition 4, we have

$$(\beth(\omega_0), \beth(\kappa_0), \beth(\nu_0)) \preccurlyeq \left(\mho\left(\omega'', \kappa'', \nu''\right), \mho\left(\kappa'', \nu'', \omega''\right), \mho\left(\nu'', \omega'', \kappa''\right)\right), \tag{30}$$

which yields

$$\begin{aligned} \beth(\omega_0) &\preccurlyeq \mho\left(\omega'', \kappa'', \nu''\right) = \beth\left(\omega''\right), \\ \beth(\kappa_0) &\succcurlyeq \mho\left(\kappa'', \nu'', \omega''\right) = \beth\left(\kappa''\right), \\ \beth(\nu_0) &\preccurlyeq \mho\left(\nu'', \omega'', \kappa''\right) = \beth\left(\nu''\right). \end{aligned} \tag{31}$$

It follows from (13) and (30) and mixed $\beth$-monotonicity of $\mho$ that

$$\begin{aligned} \mho(\omega_0, \kappa_0, \nu_0) &\preccurlyeq \mho\left(\omega'', \kappa'', \nu''\right), \\ \mho(\kappa_0, \nu_0, \omega_0) &\succcurlyeq \mho\left(\kappa'', \nu'', \omega''\right), \\ \mho(\omega_0, \kappa_0, \nu_0) &\preccurlyeq \mho\left(\nu'', \omega'', \kappa''\right). \end{aligned} \tag{32}$$

This leads to

$$\begin{aligned} \beth(\omega_1) &\preccurlyeq \beth\left(\omega''\right), \\ \beth(\kappa_1) &\succcurlyeq \beth\left(\kappa''\right), \\ \beth(\nu_1) &\preccurlyeq \beth\left(\nu''\right). \end{aligned} \tag{33}$$

By induction, for $n \in \mathbb{N}$, we have

$$\begin{aligned} \beth(\omega_n) &\preccurlyeq \beth\left(\omega''\right), \\ \beth(\kappa_n) &\succcurlyeq \beth\left(\kappa''\right), \\ \beth(\nu_n) &\preccurlyeq \beth\left(\nu''\right). \end{aligned} \tag{34}$$

Applying (12), we observe that

$$\begin{aligned} \zeta\left(\beth\left(\omega''\right), \beth(\omega_{n+1})\right) &= \zeta\left(\mho\left(\omega'', \kappa'', \nu''\right), \mho(\omega_n, \kappa_n, \nu_n)\right) \\ &\le \Delta\left(\max\left\{\frac{1}{2}\zeta\left(\beth\left(\omega''\right), \mho\left(\omega'', \kappa'', \nu''\right)\right),\right.\right. \\ &\qquad \left.\left.\zeta(\beth(\omega_n), \mho(\omega_n, \kappa_n, \nu_n))\right\}\right) \\ &= \max\left\{\frac{1}{2}\zeta\left(\beth\left(\omega''\right), \beth\left(\omega''\right)\right),\right. \\ &\qquad \left.\zeta(\beth(\omega_n), \beth(\omega_{n+1}))\right\} \\ &\le \zeta(\beth(\omega_n), \beth(\omega_{n+1})). \end{aligned} \tag{35}$$

In the same manner, we have

$$\begin{aligned} \zeta\left(\beth\left(\kappa''\right), \beth(\kappa_{n+1})\right) &\le \zeta(\beth(\kappa_n), \beth(\kappa_{n+1})), \\ \zeta\left(\beth\left(\nu''\right), \beth(\nu_{n+1})\right) &\le \zeta(\beth(\nu_n), \beth(\nu_{n+1})). \end{aligned} \tag{36}$$

From another direction, since $\beth(\omega_0) \preccurlyeq \beth(\kappa_0)$, and the mixed $\beth$-monotonicity of $\mho$, then $\mho(\omega_0, \kappa_0, \nu_0) \preccurlyeq \mho(\kappa_0, \nu_0, \omega_0)$, i.e., $\beth(\omega_1) \preccurlyeq \beth(\kappa_1)$; similarly, one can obtain $\beth(\kappa_1) \preccurlyeq \beth(\nu_1)$ and $\beth(\nu_1) \preccurlyeq \beth(\omega_1)$. By the proof of Conclusion $(C_1)$, we have

$$\begin{aligned} \beth(\omega_0) \preccurlyeq \cdots \preccurlyeq \beth(\omega_n) \preccurlyeq \beth(\kappa_n) \preccurlyeq \cdots \preccurlyeq \beth(\kappa_0), \\ \beth(\kappa_0) \preccurlyeq \cdots \preccurlyeq \beth(\kappa_n) \preccurlyeq \beth(\nu_n) \preccurlyeq \cdots \preccurlyeq \beth(\nu_0), \\ \beth(\nu_0) \preccurlyeq \cdots \preccurlyeq \beth(\nu_n) \preccurlyeq \beth(\omega_n) \preccurlyeq \cdots \preccurlyeq \beth(\omega_0). \end{aligned} \tag{37}$$

Taking the limit as $n \longrightarrow \infty$, we get

$$\lim_{n \longrightarrow \infty} \beth(\omega_n) = \lim_{n \longrightarrow \infty} \beth(\kappa_n) = \lim_{n \longrightarrow \infty} \beth(\nu_n) = \beth(\vartheta). \tag{38}$$

This implies together with (25) that

$$\omega'' = \kappa'' = \nu'' = \beth(\vartheta). \tag{39}$$

Hence, by (34), (35), (37), and (38), we have

$$\lim_{n \longrightarrow \infty} \zeta\left(\beth\left(\omega''\right), \beth(\omega_{n+1})\right) = \lim_{n \longrightarrow \infty} \zeta\left(\beth\left(\kappa''\right), \beth(\kappa_{n+1})\right) = \lim_{n \longrightarrow \infty} \zeta\left(\beth\left(\nu''\right), \beth(\nu_{n+1})\right) = 0. \tag{40}$$

With the help of (39) and the triangle inequality, we can write

$$\begin{aligned} \zeta\left(\beth\left(\omega''\right), \beth(\vartheta)\right) &\le \zeta\left(\beth\left(\omega''\right), \beth(\omega_{n+1})\right) + \zeta(\beth(\omega_{n+1}), \beth(\vartheta)) \longrightarrow 0 \text{ as } n \longrightarrow \infty, \\ \zeta\left(\beth\left(\kappa''\right), \beth(\vartheta)\right) &\le \zeta\left(\beth\left(\kappa''\right), \beth(\kappa_{n+1})\right) + \zeta(\beth(\kappa_{n+1}), \beth(\vartheta)) \longrightarrow 0 \text{ as } n \longrightarrow \infty, \\ \zeta\left(\beth\left(\nu''\right), \beth(\vartheta)\right) &\le \zeta\left(\beth\left(\nu''\right), \beth(\nu_{n+1})\right) + \zeta(\beth(\nu_{n+1}), \beth(\vartheta)) \longrightarrow 0 \text{ as } n \longrightarrow \infty. \end{aligned} \tag{41}$$

Thus, $\beth(\omega'') = \beth(\kappa'') = \beth(\nu'') = \beth(\vartheta)$. According to (39), we obtain $\beth(\vartheta) = \beth(\beth(\vartheta)) = \mho(\beth(\vartheta), \beth(\vartheta), \beth(\vartheta))$, which proves that $\beth(\vartheta)$ is a common FP of $\mho$ and $\beth$.

To discuss the uniqueness, suppose that $\wp$ is another common FP of $\mho$ and $\beth$, thus $\wp = \beth(\wp) = \mho(\wp, \wp, \wp)$. By the above results and Definition 4, we get

$$(\beth(\omega_0), \beth(\kappa_0), \beth(\nu_0)) \preccurlyeq (\mho(\wp,\wp,\wp), \mho(\wp,\wp,\wp), \mho(\wp,\wp,\wp)). \tag{42}$$

That is, for $\omega_0 = \rho_1^n \eth_0$, $\kappa_0 = \rho_2^n \eth_0$, and $\nu_0 = \rho_3^n \eth_0$, where $\rho_1, \rho_2, \rho_3 \in (0, 1)$, we have

$$\begin{aligned} \beth(\omega_0) \preccurlyeq \mho(\wp,\wp,\wp) \preccurlyeq \beth(\kappa_0), \\ \beth(\nu_0) \preccurlyeq \mho(\wp,\wp,\wp) \preccurlyeq \beth(\kappa_0). \end{aligned} \tag{43}$$

This leads to

$$\lim_{n\longrightarrow\infty} \beth(\omega_0) = \lim_{n\longrightarrow\infty} \beth(\kappa_0) = \lim_{n\longrightarrow\infty} \beth(\nu_0) = \beth(\vartheta). \tag{44}$$

Hence, $\mho(\wp, \wp, \wp) = \beth(\vartheta)$, which leads to $\wp = \beth(\vartheta)$. Therefore, $\beth(\vartheta)$ is a unique common FP of $\mho$ and $\beth$. This finishes the proof. □

Examples below verify the assumptions of Theorem 11.

*Example 1.* Let $\aleph = [0,\infty)$ be endowed with

$$\zeta(\omega, \kappa) = |\kappa - \omega|, \quad \kappa, \omega \in \aleph. \tag{45}$$

Define the order relation $\leq$ by

$$\begin{aligned} &\kappa, \omega \in \aleph, \\ &\omega \preccurlyeq \kappa \Leftrightarrow \omega \leq \kappa. \end{aligned} \tag{46}$$

It is obvious that $(\aleph, \zeta, \preccurlyeq)$ is a CPOMS. Define the mappings $\mho : \aleph^3 \longrightarrow \aleph$ and $\beth : \aleph \longrightarrow \aleph$ by

$$\begin{aligned} \mho(\omega, \kappa, \nu) &= \begin{cases} \dfrac{\omega + \kappa - \nu}{4}, & \text{if } \omega \geq \kappa \geq \nu, \\ 0, & \text{otherwise,} \end{cases} \\ \beth(\omega) &= 2\omega, \end{aligned} \tag{47}$$

respectively. It is clear that $\mho(\aleph^3) \subset \beth(\aleph)$, $\beth$ and $\mho$ are continuous, and $\mho$ have a mixed $\beth$-monotone property.

Now, let us verify Condition (12) of Theorem 11 for all $\omega, \kappa, \nu, \omega^*, \kappa^*, \nu^* \in \aleph$ with $\omega \geq \omega^*$, $\kappa \leq \kappa^*$, and $\nu \geq \nu^*$. Consider that the function $\Delta : [0,+\infty) \longrightarrow [0,+\infty)$ is given by

$$\Delta(\max\{\omega, \kappa\}) = \frac{\omega + \kappa}{2}, \quad \omega, \kappa \in [0,\infty). \tag{48}$$

Now, we consider the cases below:

($\star_1$) If $\omega \geq \kappa \geq \nu$ and $\omega^* \geq \kappa^* \geq \nu^*$, then $\omega \geq \omega^* \geq \kappa^* \geq \kappa \geq \nu$, and we have

$$\begin{aligned} &\zeta(\mho(\omega, \kappa, \nu), \mho(\omega^*, \kappa^*, \nu^*)) \\ &\quad + \left|\frac{\omega^* + \kappa^* - \nu^*}{4}\right| \leq \frac{(\omega + \omega^*) + (\kappa + \kappa^*)}{4} \leq \frac{2\omega + 2\kappa^*}{4} \\ &\leq \frac{\omega + \omega^*}{2} \leq \frac{1}{2}\left(\left(\frac{7\omega}{8} + \frac{|\nu - \kappa|}{8}\right) + \left(\frac{7\omega^*}{4} + \frac{|\nu^* - \kappa^*|}{4}\right)\right) \\ &= \frac{1}{2}\left(\frac{1}{2}\zeta(\beth(\omega), \mho(\omega, \kappa, \nu)) + \zeta(\beth(\omega^*), \mho(\omega^*, \kappa^*, \nu^*))\right) \\ &= \Delta\left(\max\left\{\frac{1}{2}\zeta(\beth(\omega), \mho(\omega, \kappa, \nu)), \zeta(\beth(\omega^*), \mho(\omega^*, \kappa^*, \nu^*))\right\}\right). \end{aligned} \tag{49}$$

($\star_2$) If $\omega \geq \kappa \geq \nu$ and $\omega^* < \kappa^* < \nu^*$, then $\omega \geq \nu \geq \nu^* \geq \omega^* \geq \kappa$, and we have

$$\begin{aligned} &\zeta(\mho(\omega, \kappa, \nu), \mho(\omega^*, \kappa^*, \nu^*)) \\ &= \left|\frac{\omega + \kappa - \nu}{4} - 0\right| \leq \left(\frac{\omega + \kappa}{4}\right) \leq \left(\frac{\omega + \omega^*}{4}\right) \\ &\leq \frac{1}{2}\left(\left(\frac{7\omega}{8} + \frac{|\nu - \kappa|}{8}\right) + 2\omega^*\right) \\ &= \frac{1}{2}\left(\frac{1}{2}\zeta(\beth(\omega), \mho(\omega, \kappa, \nu)) + \zeta(\beth(\omega^*), \mho(\omega^*, \kappa^*, \nu^*))\right) \\ &= \Delta\left(\max\left\{\frac{1}{2}\zeta(\beth(\omega), \mho(\omega, \kappa, \nu)), \zeta(\beth(\omega^*), \mho(\omega^*, \kappa^*, \nu^*))\right\}\right). \end{aligned} \tag{50}$$

($\star_3$) If $\omega < \kappa < \nu$ and $\omega^* \geq \kappa^* \geq \nu^*$, then $\nu \geq \omega \geq \omega^* \geq \kappa^*$, and we have

$$\begin{aligned} &\zeta(\mho(\omega, \kappa, \nu), \mho(\omega^*, \kappa^*, \nu^*)) \\ &= \left|0 - \frac{\omega^* + \kappa^* - \nu^*}{4}\right| = \left|\frac{\omega^* + \kappa^* - \nu^*}{4}\right| \leq \frac{\omega^* + \kappa^*}{4} \\ &\leq \frac{\omega + \omega^*}{4} \leq \frac{1}{2}\left(2\omega + \left(\frac{7\omega^*}{4} + \frac{|\nu^* - \kappa^*|}{4}\right)\right) \\ &= \frac{1}{2}\left(\frac{1}{2}\zeta(\beth(\omega), \mho(\omega, \kappa, \nu)) + \zeta(\beth(\omega^*), \mho(\omega^*, \kappa^*, \nu^*))\right) \\ &= \Delta\left(\max\left\{\frac{1}{2}\zeta(\beth(\omega), \mho(\omega, \kappa, \nu)), \zeta(\beth(\omega^*), \mho(\omega^*, \kappa^*, \nu^*))\right\}\right). \end{aligned} \tag{51}$$

($\star_4$) If $\omega < \kappa < \nu$ and $\omega^* < \kappa^* < \nu^*$, then we have

$$\begin{aligned} &\zeta(\mho(\omega, \kappa, \nu), \mho(\omega^*, \kappa^*, \nu^*)) = 0 \leq \frac{1}{2}(\omega + 2\omega^*) \\ &= \frac{1}{2}\left(\frac{1}{2}\zeta(\beth(\omega), \mho(\omega, \kappa, \nu)) + \zeta(\beth(\omega^*), \mho(\omega^*, \kappa^*, \nu^*))\right) \\ &= \Delta\left(\max\left\{\frac{1}{2}\zeta(\beth(\omega), \mho(\omega, \kappa, \nu)), \zeta(\beth(\omega^*), \mho(\omega^*, \kappa^*, \nu^*))\right\}\right). \end{aligned} \tag{52}$$

The four cases indicate that the requirements of Theorem 11 are fulfilled and $(0, 0, 0)$ is a unique TFP.

*Example 2.* Assume that the first requirements of Example 1 hold with the usual order "≤." Define the mappings $\mho : \aleph^3 \longrightarrow \aleph$ and $\beth : \aleph \longrightarrow \aleph$ by

$$\begin{aligned} \mho(\omega, \kappa, \nu) &= \frac{\omega + \kappa + \nu}{3}, \\ \beth(\omega) &= \omega, \end{aligned} \tag{53}$$

respectively. It is obvious that $(\aleph, \zeta, \leq)$ is a CPOMS, $\mho(\aleph^3) \subset \beth(\aleph)$, $\beth$ and $\mho$ are continuous, and $\mho$ have a mixed $\beth$-monotone property.

Now, let us verify Condition (12) of Theorem 11 for all $\omega, \kappa, \nu, \omega^*, \kappa^*, \nu^* \in \aleph$ with $\omega \leq \omega^*$, $\kappa \geq \kappa^*$, and $\nu \leq \nu^*$. Let $\Delta : [0,+\infty) \longrightarrow [0,+\infty)$ be a function defined by

$$\Delta(\max\{\omega,\kappa\}) = \frac{\omega+\kappa}{2}, \quad \text{for all } \omega,\kappa \in [0,\infty). \tag{54}$$

Now, consider

$$\begin{aligned}
\zeta(\mho(\omega,\kappa,\nu),\mho(\omega^*,\kappa^*,\nu^*)) &= \left|\frac{\omega+\kappa+\nu}{3} - \frac{\omega^*+\kappa^*+\nu^*}{3}\right| \\
&\le \left|\frac{\omega-\omega^*}{3}\right| + \left|\frac{\kappa-\kappa^*}{3}\right| + \left|\frac{\nu-\nu^*}{3}\right| \le \left|\frac{\kappa-\kappa^*}{3}\right| \le \frac{\kappa+\kappa^*}{2} \\
&\le \frac{1}{2}\left(\left(\frac{2\kappa}{6} + \frac{(\omega+\nu)}{6}\right) + \left(\frac{2\kappa^*}{3} + \frac{(\omega^*+\nu^*)}{3}\right)\right) \\
&= \frac{1}{2}\left(\frac{1}{2}\zeta(\daleth(\kappa),\mho(\kappa,\nu,\omega)) + \zeta(\daleth(\kappa^*),\mho(\kappa^*,\nu^*,\omega^*))\right) \\
&= \Delta\left(\max\left\{\frac{1}{2}\zeta(\daleth(\kappa),\mho(\kappa,\nu,\omega)),\zeta(\daleth(\kappa^*),\mho(\kappa^*,\nu^*,\omega^*))\right\}\right).
\end{aligned} \tag{55}$$

Hence, all conditions of Theorem 11 are satisfied and $\mho$ and $\daleth$ have a unique common TFP in $\aleph$ for all $\omega=\kappa=\nu$.

If we set $\daleth = I_{\aleph}$ (the identity mapping on $\aleph$) in Theorem 11, we deduce the result below:

**Corollary 12.** *Let $(\aleph,\zeta,\preccurlyeq)$ be a CPOMS and $\vartheta$ be a zero element in $\aleph$. Assume that $\mho:\aleph^3 \longrightarrow \aleph$ is a mixed-monotone mapping and $\Delta:[0,\infty)\longrightarrow[0,\infty)$ is so that $\Delta(\rho)\le\rho$ for any $\rho\ge 0$. Suppose that the assumptions below are satisfied:*

*(i) $\mho$ is continuous*

*(ii) $\mho(\omega,\kappa,\nu)$ verifies stipulations $(s_1)$ and $(s_2)$ of Lemma 9*

*(iii) For any $\omega,\kappa,\nu,\omega^*,\kappa^*,\nu^*\in\aleph$ with $\omega\succcurlyeq\omega^*$, $\kappa\preccurlyeq\kappa^*$, and $\nu\succcurlyeq\nu^*$, and we have*

$$\begin{aligned}
&\zeta(\mho(\omega,\kappa,\nu),\mho(\omega^*,\kappa^*,\nu^*)) \\
&\quad\le \Delta\left(\max\left\{\frac{1}{2}\zeta(\omega,\mho(\omega,\kappa,\nu)),\zeta(\omega^*,\mho(\omega^*,\kappa^*,\nu^*))\right\}\right).
\end{aligned} \tag{56}$$

*Then, the following conclusions are fulfilled:*

*$(C_1)$ For a triplet $(\omega_0,\kappa_0,\nu_0)\in\aleph$, construct three sequences $\{\omega_n\}$, $\{\kappa_n\}$, and $\{\nu_n\}$ in $\aleph$ verifying*

$$\begin{aligned}
\omega_n &= \mho(\omega_{n-1},\kappa_{n-1},\nu_{n-1}), \\
\kappa_n &= \mho(\kappa_{n-1},\nu_{n-1},\omega_{n-1}), \\
\nu_n &= \mho(\nu_{n-1},\omega_{n-1},\kappa_{n-1}),
\end{aligned} \tag{57}$$

*for $n\in\mathbb{N}$. Then, $\omega_n\longrightarrow\omega''$, $\kappa_n\longrightarrow\kappa''$, and $\nu_n\longrightarrow\nu''$, as $n\longrightarrow\infty$*

*$(C_2)$ $\mho$ has a TFP $(\omega'',\kappa'',\nu'')$. Moreover, assume that $\omega_0$, $\kappa_0$, and $\nu_0$ are comparable and for each $(\omega,\kappa,\nu)\in\aleph$, $(\omega_0,\kappa_0,\nu_0)$ is comparable to $(\mho(\omega,\kappa,\nu),\mho(\kappa,\nu,\omega),\mho(\nu,\omega,\kappa))$*

*$(C_3)$ $\mho$ has a unique FP $\vartheta$, that is $\vartheta=\mho(\vartheta,\vartheta,\vartheta)$*

## 3. An Application to Integro-Differential Equation

In fact, this part is a fundamental pillar of our paper, where the theoretical results presented in the above section are involved in order to obtain the existence of the solution to an IDE of the following form:

$$\begin{aligned}
\omega(\rho) = \frac{1}{2}\int_{\ell_1}^{\rho}\Theta(\rho,\sigma)\Big(&\left[\aleph_1\left(\sigma,\omega(\sigma),\omega'(\sigma)\right) + \mathfrak{R}_1(\sigma,\omega(\sigma))\right] \\
&+ \left[\aleph_2\left(\sigma,\kappa(\sigma),\kappa'(\sigma)\right) + \mathfrak{R}_2(\sigma,\kappa(\sigma))\right] \\
&+ \left[\aleph_3\left(\sigma,\nu(\sigma),\nu'(\sigma)\right) + \mathfrak{R}_3(\sigma,\nu(\sigma))\right]\Big)d\sigma,
\end{aligned} \tag{58}$$

for $\rho\in[\ell_1,\ell_2]$, where $\Theta:[\ell_1,\ell_2]\times[\ell_1,\ell_2]\longrightarrow[1,\infty)$, $\aleph_i:[\ell_1,\ell_2]\times[0,\infty)\times[0,\infty)\longrightarrow[0,\infty)$, and $\mathfrak{R}_i:[\ell_1,\ell_2]\times[0,\infty)\longrightarrow[0,\infty)$ (for $i=1,2,3$) are given continuous functions.

Assume that $\omega(\rho)$, $\kappa(\rho)$, $\nu(\rho)$, $\omega'(\rho)$, $\kappa'(\rho)$, and $\nu'(\rho)$ are nonnegative real continuous functions which are differentiable on $[\ell_1,\ell_2]$, where $\omega'(\rho)$, $\kappa'(\rho)$, and $\nu'(\rho)$ are the first derivative of $\omega(\rho)$, $\kappa(\rho)$, and $\nu(\rho)$ with respect to $\rho$, respectively.

Also, suppose that $\Theta$ is continuously differentiable with respect to its first variable, where $(\partial\Theta/\partial\rho)>0$.

In order to find the existence solution of Problem (57), we shall derive the hypotheses below:

$(\ddagger_i)$ $\aleph_i:[\ell_1,\ell_2]\times[0,\infty)\times[0,\infty)\longrightarrow[0,\infty)$ and $\mathfrak{R}_i:[\ell_1,\ell_2]\times[0,\infty)\longrightarrow[0,\infty)$ (for $i=1,2,3$) are continuous functions so that

$$\begin{aligned}
\mathfrak{R}_1(\rho,\omega(\rho)) &\le \mathfrak{R}_1(\rho,\omega^*(\rho)), \\
\mathfrak{R}_2(\rho,\kappa(\rho)) &\ge \mathfrak{R}_2(\rho,\kappa^*(\rho)), \\
\mathfrak{R}_3(\rho,\nu(\rho)) &\le \mathfrak{R}_3(\rho,\nu^*(\rho)), \\
\aleph_1\left(\rho,\omega(\rho),\kappa'(\rho)\right) &\le \aleph_1\left(\rho,\omega^*(\rho),\kappa^{*'}(\rho)\right), \\
\aleph_2\left(\rho,\kappa(\rho),\nu'(\rho)\right) &\ge \aleph_2\left(\rho,\kappa^*(\rho),\nu^{*'}(\rho)\right), \\
\aleph_3\left(\rho,\nu(\rho),\omega'(\rho)\right) &\le \aleph_3\left(\rho,\nu^*(\rho),\omega^{*'}(\rho)\right),
\end{aligned} \tag{59}$$

for any fixed $\rho\in[\ell_1,\ell_2]$

$(\ddagger_{ii})$ Define the mappings $\mho$ and $\daleth$ by

$$\begin{aligned}
\mho(\omega,\kappa,\nu)(\rho) = \int_{\ell_1}^{\rho}\Theta(\rho,\sigma)\Big(&\aleph_1\left(\sigma,\omega(\sigma),\kappa'(\sigma)\right) \\
&+ \aleph_2\left(\sigma,\kappa(\sigma),\nu'(\sigma)\right) \\
&+ \aleph_3\left(\sigma,\nu(\sigma),\omega'(\sigma)\right)\Big)d\sigma,
\end{aligned} \tag{60}$$

$$(\daleth\omega)(\rho) = \int_{\ell_1}^{\rho}\Theta(\rho,\sigma)(\mathfrak{R}_1(\sigma,\omega(\sigma)) + \mathfrak{R}_2(\sigma,\kappa(\sigma)) + \mathfrak{R}_3(\sigma,\nu(\sigma)))d\sigma, \tag{61}$$

such that

$$\begin{aligned}
&\mathfrak{R}_1(\rho, \mho(\omega,\kappa,\nu)(\rho)) = \aleph_1\left(\rho, \daleth\omega(\rho), (\daleth\kappa)'(\rho)\right),\\
&\mathfrak{R}_2(\rho, \mho(\omega,\kappa,\nu)(\rho)) = \aleph_2\left(\rho, \daleth\kappa(\rho), (\daleth\nu)'(\rho)\right),\\
&\mathfrak{R}_3(\rho, \mho(\omega,\kappa,\nu)(\rho)) = \aleph_3\left(\rho, \daleth\nu(\rho), (\daleth\omega)'(\rho)\right),
\end{aligned} \tag{62}$$

for any fixed $\rho \in [\ell_1, \ell_2]$

($\ddagger_{iii}$) For any fixed $\rho \in [\ell_1, \ell_2]$, there is $\beta \in (0, (1/2))$ so that

$$\begin{aligned}
&\aleph_1\left(\rho, \omega^*(\rho), \kappa^{*'}(\rho)\right) \geq \mathfrak{R}_1(\rho, \omega(\rho)),\\
&\aleph_2\left(\rho, \kappa^*(\rho), \nu^{*'}(\rho)\right) \geq \mathfrak{R}_2(\rho, \kappa(\rho)),\\
&\aleph_3\left(\rho, \nu^*(\rho), \omega^{*'}(\rho)\right) \geq \mathfrak{R}_3(\rho, \nu(\rho)),
\end{aligned} \tag{63}$$

$$\begin{aligned}
&\left|\aleph_1\left(\rho, \omega(\rho), \kappa'(\rho)\right) - \aleph_1\left(\rho, \omega^*(\rho), \kappa^{*'}(\rho)\right)\right|\\
&\quad\leq \frac{\beta}{2}\left(\left|\aleph_1\left(\rho, \omega(\rho), \kappa'(\rho)\right) - \mathfrak{R}_1(\rho, \omega(\rho))\right|\right),\\
&\left|\aleph_2\left(\rho, \kappa(\rho), \nu'(\rho)\right) - \aleph_2\left(\rho, \kappa^*(\rho), \nu^{*'}(\rho)\right)\right|\\
&\quad\leq \frac{\beta}{2}\left(\left|\aleph_2\left(\rho, \kappa(\rho), \nu'(\rho)\right) - \mathfrak{R}_2(\rho, \kappa(\rho))\right|\right),\\
&\left|\aleph_3\left(\rho, \nu(\rho), \omega'(\rho)\right) - \aleph_3\left(\rho, \nu^*(\rho), \omega^{*'}(\rho)\right)\right|\\
&\quad\leq \frac{\beta}{2}\left(\left|\aleph_3\left(\rho, \nu(\rho), \omega'(\rho)\right) - \mathfrak{R}_3(\rho, \nu(\rho))\right|\right),
\end{aligned} \tag{64}$$

($\ddagger_{iv}$) there is $\beth_0(\rho) \in [0,\infty)$, so that for any fixed $\rho \in [\ell_1, \ell_2]$, $\rho_1, \rho_2, \rho_3 \in (0,1)$,

$$\begin{cases}
\mathfrak{R}\left(\rho, \rho_1^j \beth_0(\rho)\right) \leq \aleph\left(\rho, \rho_1^j \beth_0(\rho), \rho_2^j \beth_0'(\rho)\right) \leq \mathfrak{R}\left(\rho, \rho_2^j \beth_0(\rho)\right),\\
\mathfrak{R}\left(\rho, \rho_3^j \beth_0(\rho)\right) \leq \aleph\left(\rho, \rho_3^j \beth_0(\rho), \rho_2^j \beth_0'(\rho)\right) \leq \mathfrak{R}\left(\rho, \rho_2^j \beth_0(\rho)\right).
\end{cases} \tag{65}$$

Now, our main theorem of this section is stated as follows:

**Theorem 13.** *Under hypotheses ($\ddagger_i$)-($\ddagger_{iv}$), System (57) has a unique solution $\daleth(\vartheta)$.*

*Proof.* The proof is splitting into the following steps:

($St_1$) Construct a CPOMS. Assume that $\aleph = (C[\ell_1, \ell_2], \mathbb{R}^+)$ is the set of all nonnegative real continuous functions on $[\ell_1, \ell_2]$, $\vartheta \in \aleph$. Define a metric $\zeta : \aleph \times \aleph \longrightarrow [0,\infty)$ on $\aleph$ by

$$\zeta(\omega,\kappa) = |\omega(\rho) - \kappa(\rho)|_1^2, \quad \forall \omega, \kappa \in \aleph, \rho \in [\ell_1, \ell_2], \tag{66}$$

where $|\omega(\rho)|_1^2 = |\omega(\rho)| + |\omega'(\rho)|$. Define the partial ordered $\preccurlyeq$ by

$$\omega \preccurlyeq \kappa \Leftrightarrow \omega(\rho) \leq \kappa(\rho), \quad \forall \rho \in [\ell_1, \ell_2]. \tag{67}$$

Then, a trio $(\aleph, \zeta, \preccurlyeq)$ is a CPOMS if $\omega \preccurlyeq \omega^*$, $\kappa^* \preccurlyeq \kappa$, and $\nu \preccurlyeq \nu^*$, whenever $\omega(\rho) \leq \omega^*(\rho)$, $\kappa^*(\rho) \leq \kappa(\rho)$, and $\nu(\rho) \leq \nu^*(\rho)$, for all $\omega, \kappa, \nu, \omega^*, \kappa^*, \nu^* \in \aleph$

($St_2$) Construct the mappings $\mho'$ and $\daleth'$. For this purpose, we involve a derivative with respect to $\rho$ on both sides of (59) and (60), respectively, for $\omega, \kappa, \nu \in \aleph$, $\rho \in [\ell_1, \ell_2]$, we get

$$\begin{aligned}
&\mho'(\omega,\kappa,\nu)(\rho)\\
&\quad= \Theta(\rho,\rho)\left(\aleph_1\left(\rho, \omega(\rho), \kappa'(\rho)\right) + \aleph_2\left(\rho, \kappa(\rho), \nu'(\rho)\right)\right.\\
&\qquad\left. + \aleph_3\left(\rho, \nu(\rho), \omega'(\rho)\right)\right) + \int_{\ell_1}^{\rho} \frac{\partial}{\partial\rho}\Theta(\rho,\sigma)\\
&\qquad\cdot \left(\aleph_1\left(\rho, \omega(\rho), \kappa'(\rho)\right) + \aleph_2\left(\rho, \kappa(\rho), \nu'(\rho)\right)\right.\\
&\qquad\left. + \aleph_3\left(\rho, \nu(\rho), \omega'(\rho)\right)\right) d\sigma,
\end{aligned} \tag{68}$$

$$\begin{aligned}
(\daleth\omega)'(\rho) &= \Theta(\rho,\rho)(\mathfrak{R}_1(\rho, \omega(\rho)) + \mathfrak{R}_2(\rho, \kappa(\rho))\\
&\quad + \mathfrak{R}_3(\rho, \nu(\rho))) + \int_{\ell_1}^{\rho} \frac{\partial}{\partial\rho}\Theta(\rho,\sigma)(\mathfrak{R}_1(\sigma, \omega(\sigma))\\
&\quad + \mathfrak{R}_2(\sigma, \kappa(\sigma)) + \mathfrak{R}_3(\sigma, \nu(\sigma)))d\sigma.
\end{aligned} \tag{69}$$

($St_3$) Show that $\mho$ is a mixed $\daleth$-monotone and $\mho$ and $\daleth$ are commuting. If $(\daleth\kappa)'(\rho) \geq (\daleth\kappa^*)'(\rho)$, then by (68), we have

$$\begin{aligned}
&\Theta(\rho,\rho)((\mathfrak{R}_2(\rho, \kappa(\rho)) - \mathfrak{R}_2(\rho, \kappa^*(\rho))) + (\mathfrak{R}_3(\rho, \nu(\rho))\\
&\quad - \mathfrak{R}_3(\rho, \nu^*(\rho))) + (\mathfrak{R}_1(\rho, \omega(\rho)) - \mathfrak{R}_1(\rho, \omega^*(\rho))))\\
&\quad + \int_{\ell_1}^{\rho} \frac{\partial}{\partial\rho}\Theta(\rho,\sigma)((\mathfrak{R}_2(\rho, \kappa(\rho)) - \mathfrak{R}_2(\rho, \kappa^*(\rho)))\\
&\quad + (\mathfrak{R}_3(\rho, \nu(\rho)) - \mathfrak{R}_3(\rho, \nu^*(\rho))) + (\mathfrak{R}_1(\rho, \omega(\rho))\\
&\quad - \mathfrak{R}_1(\rho, \omega^*(\rho))))d\sigma \geq 0,
\end{aligned} \tag{70}$$

since $\Theta(\rho,\rho) \geq 1$, and $(\partial/\partial\rho)\Theta(\rho,\sigma) > 0$, then, we get $\mathfrak{R}_2(\rho, \kappa(\rho)) \geq \mathfrak{R}_1(\rho, \kappa^*(\rho))$. Moreover by hypothesis ($\ddagger_i$), for any fixed $\rho \in [\ell_1, \ell_2]$, $\omega, \kappa, \nu, \omega^*, \kappa^*, \nu^* \in \aleph$. If

$$\begin{aligned}
&(\daleth\omega)(\rho) \leq (\daleth\omega^*)(\rho), (\daleth\kappa)'(\rho) \geq (\daleth\kappa^*)'(\rho),\\
&(\daleth\nu)(\rho) \leq (\daleth\nu^*)(\rho),
\end{aligned} \tag{71}$$

then, we have

$$\mho\left(\omega, \kappa', \nu\right)(\rho) \preccurlyeq \mho\left(\omega^*, \kappa^{*'}, \nu^*\right)(\rho). \tag{72}$$

Similarly, if

$$\begin{aligned}(\gimel\omega)'(\rho)&\le(\gimel\omega^*)'(\rho),\\(\gimel\kappa)(\rho)&\ge(\gimel\kappa^*)(\rho),\\(\gimel\nu)(\rho)\le(\gimel\nu^*)(\rho)&\Rightarrow\mho\left(\omega',\kappa,\nu\right)(\rho)\\&\le\mho\left(\omega'_*,\kappa^*,\nu^*\right)(\rho),\end{aligned}\tag{73}$$

and if

$$\begin{aligned}(\gimel\omega)(\rho)&\le(\gimel\omega^*)(\rho),\\(\gimel\kappa)(\rho)&\ge(\gimel\kappa^*)(\rho),\\(\gimel\nu)'(\rho)\le(\gimel\nu^*)'(\rho)&\Rightarrow\mho\left(\omega,\kappa,\nu'\right)(\rho)\\&\le\mho\left(\omega^*,\kappa^*,\nu'_*\right)(\rho),\end{aligned}\tag{74}$$

then this implies that $\mho$ is a mixed $\gimel$-monotone. By the definition of $\mho$ and $\gimel$, we can write

$$\begin{aligned}\mho(\gimel\omega,\gimel\kappa,\gimel\nu)(\rho)&=\int_{\ell_1}^{\rho}\Theta(\rho,\sigma)\Big(\aleph_1\left(\sigma,\gimel\omega(\sigma),(\gimel\kappa)'(\sigma)\right)\\&\quad+\aleph_2\left(\sigma,\gimel\kappa(\sigma),(\gimel\nu)'(\sigma)\right)+\aleph_3\left(\sigma,\gimel\nu(\sigma),(\gimel\omega)'(\sigma)\right)\Big)d\sigma\\&=\int_{\ell_1}^{\rho}\Theta(\rho,\sigma)(\mathfrak{R}_1(\rho,\mho(\omega,\kappa,\nu)(\rho))+\mathfrak{R}_2(\rho,\mho(\omega,\kappa,\nu)(\rho))\\&\quad+\mathfrak{R}_3(\rho,\mho(\omega,\kappa,\nu)(\rho)))d\sigma=\gimel(\mho(\omega,\kappa,\nu)(\rho)).\end{aligned}\tag{75}$$

($St_4$) Fulfill Condition (12) of Theorem 11. It follows from (60), (63), and (68) that

$$\begin{aligned}&|\mho(\omega,\kappa,\nu)(\rho)-\mho(\omega^*,\kappa^*,\nu^*)(\rho)|\\&=\left|\int_{\ell_1}^{\rho}\Theta(\rho,\sigma)\Big(\left(\aleph_1\left(\sigma,\omega(\sigma),\kappa'(\sigma)\right)-\aleph_1\left(\sigma,\omega^*(\sigma),\kappa^{*'}(\sigma)\right)\right)\right.\\&\quad+\left(\aleph_2\left(\sigma,\kappa(\sigma),\nu'(\sigma)\right)-\aleph_2\left(\sigma,\kappa^*(\sigma),\nu^{*'}(\sigma)\right)\right)\\&\quad\left.+\left(\aleph_3\left(\sigma,\nu(\sigma),\omega'(\sigma)\right)-\aleph_3\left(\sigma,\nu^*(\sigma),\omega^{*'}(\sigma)\right)\right)\Big)d\sigma\right|\\&\le\left|\int_{\ell_1}^{\rho}\Theta(\rho,\sigma)\Big(\frac{\beta}{2}\left(\mathfrak{R}_1(\sigma,\omega(\sigma))-\aleph_1\left(\sigma,\omega(\sigma),\kappa'(\sigma)\right)\right)\right.\\&\quad+\frac{\beta}{2}\left(\mathfrak{R}_2(\sigma,\kappa(\sigma))-\aleph_2\left(\sigma,\kappa(\sigma),\nu'(\sigma)\right)\right)\\&\quad\left.+\frac{\beta}{2}\left(\mathfrak{R}_3(\sigma,\nu(\sigma))-\aleph_3\left(\sigma,\nu(\sigma),\omega'(\sigma)\right)\right)\Big)d\sigma\right|\\&=\frac{\beta}{2}\left|\int_{\ell_1}^{\rho}\Theta(\rho,\sigma)\Big(\left(\mathfrak{R}_1(\sigma,\omega(\sigma))-\aleph_1\left(\sigma,\omega(\sigma),\kappa'(\sigma)\right)\right)\right.\\&\quad+\left(\mathfrak{R}_2(\sigma,\kappa(\sigma))-\aleph_2\left(\sigma,\kappa(\sigma),\nu'(\sigma)\right)\right)\\&\quad\left.+\left(\mathfrak{R}_3(\sigma,\nu(\sigma))-\aleph_3\left(\sigma,\nu(\sigma),\omega'(\sigma)\right)\right)\Big)d\sigma\right|,\end{aligned}\tag{76}$$

$$\begin{aligned}&\left|\mho'(\omega,\kappa,\nu)(\rho)-\mho'(\omega^*,\kappa^*,\nu^*)(\rho)\right|\\&=\Big|\Theta(\rho,\rho)\Big(\left(\aleph_1\left(\rho,\omega(\rho),\kappa'(\rho)\right)-\aleph_1\left(\rho,\omega^*(\rho),\kappa^{*'}(\rho)\right)\right)\\&\quad+\left(\aleph_2\left(\rho,\kappa(\rho),\nu'(\rho)\right)-\aleph_2\left(\rho,\kappa^*(\rho),\nu^{*'}(\rho)\right)\right)\\&\quad+\left(\aleph_3\left(\rho,\nu(\rho),\omega'(\rho)\right)-\aleph_3\left(\rho,\nu^*(\rho),\omega^{*'}(\rho)\right)\right)\Big)\\&\quad+\int_{\ell_1}^{\rho}\frac{\partial}{\partial\rho}\Theta(\rho,\sigma)\Big(\Big(\aleph_1\left(\rho,\omega(\rho),\kappa'(\rho)\right)\\&\quad-\aleph_1\left(\rho,\omega^*(\rho),\kappa^{*'}(\rho)\right)\Big)+\Big(\aleph_2\left(\rho,\kappa(\rho),\nu'(\rho)\right)\\&\quad-\aleph_2\left(\rho,\kappa^*(\rho),\nu^{*'}(\rho)\right)\Big)+\Big(\aleph_3\left(\rho,\nu(\rho),\omega'(\rho)\right)\\&\quad-\aleph_3\left(\rho,\nu^*(\rho),\omega^{*'}(\rho)\right)\Big)\Big)d\sigma\Big|\\&\le\Big|\Theta(\rho,\rho)\Big(\Big(\frac{\beta}{2}\left(\mathfrak{R}_1(\rho,\omega(\rho))-\aleph_1\left(\rho,\omega(\rho),\kappa'(\rho)\right)\right)\\&\quad+\frac{\beta}{2}\left(\mathfrak{R}_2(\rho,\kappa(\rho))-\aleph_2\left(\rho,\kappa(\rho),\nu'(\rho)\right)\right)\\&\quad+\frac{\beta}{2}\left(\mathfrak{R}_3(\rho,\nu(\rho))-\aleph_3\left(\rho,\nu(\rho),\omega'(\rho)\right)\right)\Big)\Big)\\&\quad+\int_{\ell_1}^{\rho}\frac{\partial}{\partial\rho}\Theta(\rho,\sigma)\Big(\frac{\beta}{2}\left(\mathfrak{R}_1(\sigma,\omega(\sigma))-\aleph_1\left(\sigma,\omega(\sigma),\kappa'(\sigma)\right)\right)\\&\quad+\frac{\beta}{2}\left(\mathfrak{R}_2(\sigma,\kappa(\sigma))-\aleph_2\left(\sigma,\kappa(\sigma),\nu'(\sigma)\right)\right)\\&\quad+\frac{\beta}{2}\left(\mathfrak{R}_3(\sigma,\nu(\sigma))-\aleph_3\left(\sigma,\nu(\sigma),\omega'(\sigma)\right)\right)\Big)d\sigma\Big|\\&=\frac{\beta}{2}\Big|\Theta(\rho,\rho)\Big(\Big(\left(\mathfrak{R}_1(\rho,\omega(\rho))-\aleph_1\left(\rho,\omega(\rho),\kappa'(\rho)\right)\right)\\&\quad+\left(\mathfrak{R}_2(\rho,\kappa(\rho))-\aleph_2\left(\rho,\kappa(\rho),\nu'(\rho)\right)\right)\\&\quad+\left(\mathfrak{R}_3(\rho,\nu(\rho))-\aleph_3\left(\rho,\nu(\rho),\omega'(\rho)\right)\right)\Big)\Big)\\&\quad+\int_{\ell_1}^{\rho}\frac{\partial}{\partial\rho}\Theta(\rho,\sigma)\Big(\left(\mathfrak{R}_1(\sigma,\omega(\sigma))-\aleph_1\left(\sigma,\omega(\sigma),\kappa'(\sigma)\right)\right)\\&\quad+\left(\mathfrak{R}_2(\sigma,\kappa(\sigma))-\aleph_2\left(\sigma,\kappa(\sigma),\nu'(\sigma)\right)\right)\\&\quad+\left(\mathfrak{R}_3(\sigma,\nu(\sigma))-\aleph_3\left(\sigma,\nu(\sigma),\omega'(\sigma)\right)\right)\Big)d\sigma\Big|.\end{aligned}\tag{77}$$

From (59), (60), (67)–(76) and by definition of $|.|_1^2$, we conclude that

$$\begin{aligned}&|\mho(\omega,\kappa,\nu)(\rho)-\mho(\omega^*,\kappa^*,\nu^*)(\rho)|\\&=\left|\int_{\ell_1}^{\rho}\Theta(\rho,\sigma)\begin{pmatrix}\left(\aleph_1\left(\sigma,\omega(\sigma),\kappa'(\sigma)\right)-\aleph_1\left(\sigma,\omega^*(\sigma),\kappa^{*'}(\sigma)\right)\right)\\+\left(\aleph_2\left(\sigma,\kappa(\sigma),\nu'(\sigma)\right)-\aleph_2\left(\sigma,\kappa^*(\sigma),\nu^{*'}(\sigma)\right)\right)\\+\left(\aleph_3\left(\sigma,\nu(\sigma),\omega'(\sigma)\right)-\aleph_3\left(\sigma,\nu^*(\sigma),\omega^{*'}(\sigma)\right)\right)\end{pmatrix}d\sigma\right|\\&\le\left|\int_{\ell_1}^{\rho}\Theta(\rho,\sigma)\begin{pmatrix}\frac{\beta}{2}\left(\mathfrak{R}_1(\sigma,\omega(\sigma))-\aleph_1\left(\sigma,\omega(\sigma),\kappa'(\sigma)\right)\right)\\+\frac{\beta}{2}\left(\mathfrak{R}_2(\sigma,\kappa(\sigma))-\aleph_2\left(\sigma,\kappa(\sigma),\nu'(\sigma)\right)\right)\\+\frac{\beta}{2}\left(\mathfrak{R}_3(\sigma,\nu(\sigma))-\aleph_3\left(\sigma,\nu(\sigma),\omega'(\sigma)\right)\right)\end{pmatrix}d\sigma\right|\end{aligned}$$

$$= \frac{\beta}{2}\left| \int_{\ell_1}^{\rho} \Theta(\rho,\sigma) \begin{pmatrix} \left(\mathfrak{R}_1(\sigma,\omega(\sigma)) - \aleph_1\left(\sigma,\omega(\sigma),\kappa'(\sigma)\right)\right) \\ +\left(\mathfrak{R}_2(\sigma,\kappa(\sigma)) - \aleph_2\left(\sigma,\kappa(\sigma),\nu'(\sigma)\right)\right) \\ +\left(\mathfrak{R}_3(\sigma,\nu(\sigma)) - \aleph_3\left(\sigma,\nu(\sigma),\omega'(\sigma)\right)\right) \end{pmatrix} d\sigma \right|, \tag{78}$$

that is

$$\begin{aligned} &\zeta(\mho(\omega,\kappa,\nu)(\rho), \mho(\omega^*,\kappa^*,\nu^*)(\rho)) \\ &\quad \le \beta \times \frac{1}{2}\zeta((\gimel\omega)(\rho), \mho(\omega,\kappa,\nu)(\rho)) \\ &\quad \le \beta \max\left\{\frac{1}{2}\zeta((\gimel\omega)(\rho), \mho(\omega,\kappa,\nu)(\rho)),\right. \\ &\qquad \left.\zeta((\gimel\omega^*)(\rho), \mho(\omega^*,\kappa^*,\nu^*)(\rho))\right\}. \end{aligned} \tag{79}$$

Set $\varDelta(\omega) = \beta\omega$ for $\omega \in [0,\infty), \beta \in (0,(1/2))$. It is clear that $\varDelta(\omega) \le \omega$ for $\omega \ge 0$. Through Inequality (78), Hypothesis (*iv*) of Theorem 11 is verified. Suppose that $\omega_0(\rho) = \rho_1^n \daleth_0$, $\kappa_0(\rho) = \rho_2^n \daleth_0$, and $\nu_0(\rho) = \rho_3^n \daleth_0$, then by ((65)), we have

$$\begin{aligned} &\int_{\ell_1}^{\rho} \Theta(\rho,\sigma)\mathfrak{R}(\rho,\omega_0(\rho))d\sigma \\ &\quad \le \int_{\ell_1}^{\rho} \Theta(\rho,\sigma)\aleph\left(\rho,\omega_0(\rho),\kappa_0'(\rho)\right)d\sigma \\ &\quad \le \int_{\ell_1}^{\rho} \Theta(\rho,\sigma)\mathfrak{R}(\rho,\kappa_0(\rho))d\sigma, \\ &\int_{\ell_1}^{\rho} \Theta(\rho,\sigma)\mathfrak{R}(\rho,\nu_0(\rho))d\sigma \\ &\quad \le \int_{\ell_1}^{\rho} \Theta(\rho,\sigma)\aleph\left(\rho,\nu_0(\rho),\kappa_0'(\rho)\right)d\sigma \\ &\quad \le \int_{\ell_1}^{\rho} \Theta(\rho,\sigma)\mathfrak{R}(\rho,\kappa_0(\rho))d\sigma, \end{aligned} \tag{80}$$

that is

$$\begin{aligned} &(\gimel\omega_0)(\rho) \le \mho(\omega_0,\kappa_0,\nu_0)(\rho) \le (\gimel\kappa_0)(\rho), \\ &(\gimel\nu_0)(\rho) \le \mho(\omega_0,\kappa_0,\nu_0)(\rho) \le (\gimel\kappa_0)(\rho). \end{aligned} \tag{81}$$

This means $\mho(\omega_0,\kappa_0,\nu_0) \succcurlyeq \gimel(\omega_0)$, $\mho(\kappa_0,\nu_0,\omega_0) \preccurlyeq \gimel(\kappa_0)$, and $\mho(\nu_0,\omega_0,\kappa_0) \succcurlyeq \gimel(\nu_0)$. Thus, all requirements of Theorem 11 are fulfilled. So, there is a unique common FP for the mappings $\mho$ and $\gimel$ in the form of $\gimel(\vartheta) \in \aleph$, so that $\gimel(\vartheta) = (1/2)(\mho(\gimel(\vartheta),\gimel(\vartheta),\gimel(\vartheta)) + \gimel(\vartheta))$, which is the unique solution to Problem (57). □

The example below supports our application.

*Example 3.* Consider the following problem:

$$\omega(\rho) = \frac{1}{32}\int_0^{\rho} \rho e^{\rho\sigma}\left(\left(e^{2\omega}+e^{\omega}\right)+\left(e^{2\kappa}+e^{\kappa}\right)+\left(e^{2\nu}+e^{\nu}\right)\right)d\sigma. \tag{82}$$

Problem (80) is a special form of Problem (57), with the following constraints:

$$\begin{aligned} \aleph_1\left(\rho,\omega(\rho),\omega'(\rho)\right) &= \rho\left(\frac{1}{4}e^{\omega}\right)\left(\frac{1}{4}e^{\omega}\right) = \frac{1}{16}\rho e^{2\omega}, \\ \mathfrak{R}_1(\rho,\omega(\rho)) &= \frac{1}{4}\left(\frac{1}{4}\rho e^{\omega}\right) = \frac{1}{16}\rho e^{\omega}, \\ \aleph_2\left(\rho,\kappa(\rho),\kappa'(\rho)\right) &= \frac{1}{16}\rho e^{2\kappa}, \\ \mathfrak{R}_2(\rho,\kappa(\rho)) &= \frac{1}{16}\rho e^{\kappa}, \\ \aleph_3\left(\rho,\nu(\rho),\nu'(\rho)\right) &= \frac{1}{16}\rho e^{2\nu}, \\ \mathfrak{R}_3(\rho,\nu(\rho)) &= \frac{1}{16}\rho e^{\nu}. \end{aligned} \tag{83}$$

$\Theta(\rho,\sigma) = e^{\rho\sigma}$, for all $\rho \in [0,1]$ and $\ell_1 = 0$, $\ell_2 = 1$.

Here, we considered $\omega(\rho) = (1/4)e^{\omega}$, $\kappa(\rho) = (1/4)e^{\kappa}$, and $\nu(\rho) = (1/4)e^{\nu}$. Thus, $\omega'(\rho) = (1/4)e^{\omega}$, $\kappa'(\rho) = (1/4)e^{\kappa}$, and $\nu'(\rho) = (1/4)e^{\nu}$, respectively. It is clear that the function $\Theta : [0,1] \times [0,1] \longrightarrow [1,\infty)$ is continuously differentiable with respect to its first variable, where $(\partial\Theta/\partial\rho) = \sigma e^{\rho\sigma} > 0$. Moreover, $(1/4)e^{\omega}$, $(1/4)e^{\kappa}$, and $(1/4)e^{\nu}$ are nonnegative real continuous functions on $[0,1]$. Also, the two mappings take the following form:

$$\mho(\omega,\kappa,\nu)(\rho) = \frac{1}{16}\int_0^1 \rho e^{\rho\sigma}\left(e^{2\omega}+e^{2\kappa}+e^{2\nu}\right)d\sigma, \tag{84}$$

$$(\gimel\omega)(\rho) = \frac{1}{16}\int_0^1 \rho e^{\rho\sigma}(e^{\omega}+e^{\kappa}+e^{\nu})d\sigma. \tag{85}$$

Now, we are going to satisfy the hypotheses of Theorem 13.

(1) For $\omega \le \omega^*$, $\kappa \ge \kappa^*$, and $\nu \le \nu^*$, the functions $\aleph_i : [0,1] \times [0,\infty) \times [0,\infty) \longrightarrow [0,\infty)$ and $\mathfrak{R}_i : [0,1] \times [0,\infty) \longrightarrow [0,\infty)$ (for $i = 1, 2, 3$) are continuous functions, so that

$$\begin{aligned} \mathfrak{R}_1(\rho,\omega(\rho)) &= \frac{1}{16}\rho e^{\omega} \le \frac{1}{16}\rho e^{\omega^*} = \mathfrak{R}_1(\rho,\omega^*(\rho)), \\ \mathfrak{R}_2(\rho,\kappa(\rho)) &= \frac{1}{16}\rho e^{\kappa} \ge \frac{1}{16}\rho e^{\kappa^*} = \mathfrak{R}_2(\rho,\kappa^*(\rho)), \\ \mathfrak{R}_3(\rho,\nu(\rho)) &= \frac{1}{16}\rho e^{\nu} \le \frac{1}{16}\rho e^{\nu^*} = \mathfrak{R}_3(\rho,\nu^*(\rho)). \end{aligned} \tag{86}$$

This obviously leads to

$$\aleph_1\left(\rho, \omega(\rho), \kappa'(\rho)\right) = \frac{1}{16}\rho e^{\omega+\kappa} \le \frac{1}{16}\rho e^{\omega^*+\kappa^*} = \aleph_1\left(\rho, \omega^*(\rho), \kappa^{*'}(\rho)\right). \quad (87)$$

Similarly, one can write

$$\aleph_2\left(\rho, \kappa(\rho), \nu'(\rho)\right) \ge \aleph_2\left(\rho, \kappa^*(\rho), \nu^{*'}(\rho)\right),$$
$$\aleph_3\left(\rho, \nu(\rho), \omega'(\rho)\right) \le \aleph_3\left(\rho, \nu^*(\rho), \omega^{*'}(\rho)\right). \quad (88)$$

(2) From (82) and (83), and assuming that $(\gimel\kappa)' = (\gimel\kappa)$, we have

$$\begin{aligned}\aleph_1\left(\rho, \gimel\omega(\rho), (\gimel\kappa)'(\rho)\right) &= \frac{1}{16}\rho e^{(\gimel\omega)} e^{(\gimel\kappa)} = \frac{1}{16}\rho e^{\gimel\omega+\gimel\kappa} \\ &= \frac{1}{16}\rho e^{1/16\int_0^1 \rho e^{\rho\sigma}\left(e^{2\omega}+e^{2\kappa}+e^{2\nu}\right)d\sigma} \\ &= \frac{1}{16}\rho e^{\mho(\omega,\kappa,\nu)} = \mathfrak{R}_1(\rho, \mho(\omega, \kappa, \nu)(\rho)).\end{aligned} \quad (89)$$

Similarly, one can write

$$\mathfrak{R}_2(\rho, \mho(\omega, \kappa, \nu)(\rho)) = \aleph_2\left(\rho, \gimel\kappa(\rho), (\gimel\nu)'(\rho)\right),$$
$$\mathfrak{R}_3(\rho, \mho(\omega, \kappa, \nu)(\rho)) = \aleph_3\left(\rho, \gimel\nu(\rho), (\gimel\omega)'(\rho)\right). \quad (90)$$

(3) For any fixed $\rho \in [0, 1]$, there is $\beta = (1/4)$, so that

$$\aleph_1\left(\rho, \omega^*(\rho), \kappa^{*'}(\rho)\right) = \frac{1}{16}\rho e^{\omega^*+\kappa^*} \ge \frac{1}{4}\left(\frac{1}{4}\rho e^{\omega}\right) = \mathfrak{R}_1(\rho, \omega(\rho)). \quad (91)$$

In the same manner, we have

$$\aleph_2\left(\rho, \kappa^*(\rho), \nu^{*'}(\rho)\right) \ge \mathfrak{R}_2(\rho, \kappa(\rho)),$$
$$\aleph_3\left(\rho, \nu^*(\rho), \omega^{*'}(\rho)\right) \ge \beta\mathfrak{R}_3(\rho, \nu(\rho)). \quad (92)$$

Also, we have

$$\begin{aligned}&\left|\aleph_1\left(\rho, \omega(\rho), \kappa'(\rho)\right) - \aleph_1\left(\rho, \omega^*(\rho), \kappa^{*'}(\rho)\right)\right| \\ &= \frac{1}{16}\rho\left|e^{\omega+\kappa} - e^{\omega^*+\kappa^*}\right| \le \frac{1}{8}\rho\left|e^{\omega+\kappa} - e^{\omega}\right|\left(\text{since } e^{\omega^*+\kappa^*} \ge e^{\omega}\right) \\ &= \frac{\beta}{2}\left|\aleph_1\left(\rho, \omega(\rho), \kappa'(\rho)\right) - \mathfrak{R}_1(\rho, \omega(\rho))\right|.\end{aligned} \quad (93)$$

Similarly, with $\beta = (1/4)$, one can write

$$\begin{aligned}&\left|\aleph_2\left(\rho, \kappa(\rho), \nu'(\rho)\right) - \aleph_2\left(\rho, \kappa^*(\rho), \nu^{*'}(\rho)\right)\right| \\ &\quad \le \frac{\beta}{2}\left(\left|\aleph_2\left(\rho, \kappa(\rho), \nu'(\rho)\right) - \mathfrak{R}_2(\rho, \kappa(\rho))\right|\right), \\ &\left|\aleph_3\left(\rho, \nu(\rho), \omega'(\rho)\right) - \aleph_3\left(\rho, \nu^*(\rho), \omega^{*'}(\rho)\right)\right| \\ &\quad \le \frac{\beta}{2}\left(\left|\aleph_3\left(\rho, \nu(\rho), \omega'(\rho)\right) - \mathfrak{R}_3(\rho, \nu(\rho))\right|\right).\end{aligned} \quad (94)$$

(4) There is $\beth_0(\rho) = e^{\rho} \in [0,\infty)$, so that for any fixed $\rho = 1 \in [0, 1]$, $\rho_1 = (1/16)$, $\rho_2 = (1/2)$, and $\rho_3 = (1/20)$, we have

$$\begin{aligned}\frac{1}{64}e &\le \frac{1}{32}e^2 \le \frac{1}{8}e \Rightarrow \frac{1}{4}\left(\frac{1}{16}e^{\rho}\right) \le \frac{1}{16} \times \frac{1}{2}e^{\rho} \\ &\le \frac{1}{4}\left(\frac{1}{2}e^{\rho}\right) \Rightarrow \mathfrak{R}\left(\rho, \frac{1}{16}e^{\rho}\right) \le \aleph\left(\rho, \frac{1}{16}e^{\rho}, \frac{1}{2}e^{\rho}\right) \\ &\le \mathfrak{R}\left(\rho, \frac{1}{2}e^{\rho}\right) \Rightarrow \mathfrak{R}(\rho, \rho_1^n\beth_0(\rho)) \\ &\le \aleph\left(\rho, \rho_1^n\beth_0(\rho), \rho_2^n\beth_0'(\rho)\right) \le \mathfrak{R}(\rho, \rho_2^n\beth_0(\rho)), \\ \frac{1}{80}e &\le \frac{1}{40}e^2 \le \frac{1}{8}e \Rightarrow \frac{1}{4}\left(\frac{1}{20}e^{\rho}\right) \le \frac{1}{20} \times \frac{1}{2}e^{\rho} \\ &\le \frac{1}{4}\left(\frac{1}{2}e^{\rho}\right) \Rightarrow \mathfrak{R}\left(\rho, \frac{1}{20}e^{\rho}\right) \le \aleph\left(\rho, \frac{1}{20}e^{\rho}, \frac{1}{2}e^{\rho}\right) \\ &\le \mathfrak{R}\left(\rho, \frac{1}{2}e^{\rho}\right) \Rightarrow \mathfrak{R}(\rho, \rho_3^n\beth_0(\rho)) \\ &\le \aleph\left(\rho, \rho_3^n\beth_0(\rho), \rho_2^n\beth_0'(\rho)\right) \le \mathfrak{R}(\rho, \rho_2^n\beth_0(\rho)).\end{aligned} \quad (95)$$

Hence, we find that all hypotheses of Theorem 13 are fulfilled. So, the Problem (80) has a unique solution.

## 4. Conclusion

Due to the multiple applications of fixed point theory, it has become widespread in many scientific disciplines, especially in nonlinear analysis. It contributes significantly to the study of the existence and uniqueness of the solution to many differential and integral equations, as well as integro-differential equations. So, the main objectives of this paper have been to present some new tripled fixed point results for mixed monotone mappings in the framework of partially ordered metric spaces, and these new results have extended to a lot of papers in the literature. Furthermore, to support the proposed results, some illustrative examples have been given and the existence and uniqueness of the solution to the integro-differential equation have been obtained.

## Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

## Authors' Contributions

All authors contributed equally and significantly in writing this article.

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