Research Article

Barrier Solutions of Elliptic Differential Equations in Musielak-Orlicz-Sobolev Spaces

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In this paper, we study the solution set of the following Dirichlet boundary equation: $-\operatorname{div}(a_1(x, u, Du)) + a_0(x, u) = f(x, u, Du)$ in Musielak-Orlicz-Sobolev spaces, where $a_1 : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$, $a_0 : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$, and $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ are all Carathéodory functions. Both a_1 and f depend on the solution u and its gradient Du . By using a linear functional analysis method, we provide sufficient conditions which ensure that the solution set of the equation is nonempty, and it possesses a greatest element and a smallest element with respect to the ordering “ $\leq$,” which are called barrier solutions.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary. Fan [1] established a sub-supersolution method for the following differential equations of divergence form:

$$-\operatorname{div}(a_1(x, Du)) + a_0(x, u) = f(x, u), \quad (1)$$

in Ω coupled with Neumann or Dirichlet boundary condition in reflexive Musielak-Orlicz-Sobolev spaces $W_0^1 L_\Phi(\Omega)$ and obtained the existence of greatest and smallest solutions within the order interval $[\underline{u}, \bar{u}]$, where $\underline{u} := \max\{\underline{u}_1, \underline{u}_2, \dots, \underline{u}_k\}$, $\bar{u} := \min\{\bar{u}_1, \bar{u}_2, \dots, \bar{u}_m\}$, and $\underline{u}_1, \underline{u}_2, \dots, \underline{u}_k$ and $\bar{u}_1, \bar{u}_2, \dots, \bar{u}_m$ are subsolutions and supersolutions, respectively. Dong and Fang [2] studied the existence of weak solutions for the following equations:

$$-\operatorname{div}(a_1(x, Du)) + a_0(x, u) = f(x, u, Du), \quad (2)$$

in Ω coupled with Neumann or Dirichlet boundary condition in separable Musielak-Orlicz-Sobolev spaces and obtained the existence of greatest and smallest solutions within the order interval $[\underline{u}, \bar{u}]$. Carl and Le [3] established

the enclosure of solutions, i.e., $[\underline{u}, \bar{u}]$, for multivalued elliptic variational inequalities by using the sub-supersolution method in Sobolev spaces. However, Carl [4] pointed out that there may exist some more solutions of some elliptic equations which are not within the corresponding interval $[\underline{u}, \bar{u}]$. For example, the solution set $\mathcal{S}$ of the following Dirichlet boundary value problem

$$\begin{cases} -\Delta u - \lambda u = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (3)$$

is given by $\mathcal{S} = \{u = c\varphi_1 : c \in \mathbb{R}\}$, where φ_1 is the eigenfunction belonging to λ_1 , and $\lambda_1 > 0$ is the first eigenvalue of the negative Laplacian $-\Delta$ (see [4]). Then, $\mathcal{S} \neq \emptyset$ neither is norm bounded nor has the extreme solutions. Therefore, Carl [4] provided some sufficient conditions which ensure that the solution set of an elliptic variational inequality possesses a greatest element u^* and a smallest element u_* with respect to the natural partial ordering of functions in Sobolev spaces. Here, u^* and u_* are called barrier solutions (see [4]). In [5], Chipot et al.

established the existence of the minimal solution to some elliptic variational inequalities in Sobolev spaces.

It is well known that different classes of differential equations correspond to different function space settings. The Musielak-Orlicz-Sobolev spaces are the extensions of Sobolev spaces, variable exponent Sobolev spaces, and Orlicz-Sobolev spaces. Gwiazda et al. [6] proved the existence of renormalized solutions to the following nonlinear elliptic equations:

$$-\operatorname{div} A(x, Du) = f \quad (4)$$

in Ω coupled with Dirichlet boundary condition in Musielak-Orlicz-Sobolev spaces, where $A : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a Carathéodory function and $f \in L^1(\Omega)$. The growth of A is controlled by Musielak-Orlicz function Φ :

$$A(x, \xi) \xi \geq c_A (\Phi(x, |\xi|) + \bar{\Phi}(x, |A(x, \xi)|)), \quad (5)$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^N$, where $\bar{\Phi}$ is a complementary function to Φ , the constant $c_A \in (0, 1]$. Although they did not require $\Phi \in \Delta_2$ nor $\bar{\Phi} \in \Delta_2$, condition (5) is stronger than the growth condition (3.1) and the coercive condition (3.2) in [2]. In [7], Chlebicka and Karppinen studied removable sets for elliptic equations of the form

$$-\operatorname{div} A(x, Du) = 0 \quad (6)$$

in Ω , where A satisfies the nonstandard growth condition and the coercivity condition expressed by means of a Musielak-Orlicz function.

The purpose of this paper is to provide sufficient conditions which ensure that the solution set $\mathcal{S}$ of the following Dirichlet boundary problem

$$\begin{cases} -\operatorname{div} (a_1(x, u, Du)) + a_0(x, u) = f(x, u, Du), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (7)$$

is nonempty and possesses barrier solutions in Musielak-Orlicz-Sobolev spaces, where $a_1 : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$, $a_0 : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$, and $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ are Carathéodory functions. Both a_1 and f depend on the solution u and its gradient Du .

This paper is organized as follows: Section 2 contains some preliminaries and some technical lemmas which will be needed. In Section 3, we first give the sufficient conditions which ensure the existence of weak solutions for (7) in Musielak-Orlicz-Sobolev spaces. Some examples are given for the application of our results. We point that (5) is strong. Then, we provide some sufficient conditions which ensure that the solution set of (7) is bounded, compact, and direct. Finally, we provide some sufficient conditions which ensure that the solution set of (7) possesses barrier solutions in Musielak-Orlicz-Sobolev spaces.

We refer to some results in variable exponent Sobolev or Orlicz-Sobolev spaces [8–22].

2. Preliminaries

Now, we list briefly some definitions and facts about Musielak-Orlicz-Sobolev spaces; for more details, see [1, 2, 23–25].

$\Phi : \Omega \times \mathbb{R}_+ = [0, +\infty) \rightarrow \mathbb{R}_+$ is called a generalized N -function (i.e., a Musielak-Orlicz function), denoted by $\Phi \in N(\Omega)$, if it satisfies the following conditions:

- (i) $\Phi(x, u)$ is an N -function of the variable $u \geq 0$ for every $x \in \Omega$, i.e., it is a convex, nondecreasing, and continuous function of u such that $\Phi(x, 0) = 0$, $\Phi(x, u) > 0$ for $u > 0$, and there hold the conditions

$$\begin{aligned} \lim_{u \rightarrow 0^+} \sup_{x \in \Omega} \frac{\Phi(x, u)}{u} &= 0, \\ \lim_{u \rightarrow +\infty} \inf_{x \in \Omega} \frac{\Phi(x, u)}{u} &= +\infty \end{aligned} \quad (8)$$

- (ii) $\Phi(x, u)$ is a measurable function of x for all $u \geq 0$

Equivalently, for all $x \in \Omega$ and all $u \geq 0$, Φ admits the representation $\Phi(x, u) = \int_0^u \varphi(x, \tau) d\tau$, where $\varphi(x, t)$ is the right-hand derivative of $\Phi(x, \cdot)$ at t , for a fixed $x \in \Omega$ and all $t \geq 0$. Then, for every $x \in \Omega$, $\varphi(x, t)$ is a right-continuous and nondecreasing function of $t \geq 0$, $\varphi(x, 0) = 0$, $\varphi(x, t) > 0$ for $t > 0$, and $\varphi(x, t) \rightarrow +\infty$ as $t \rightarrow +\infty$ for every $x \in \Omega$. The complementary function $\bar{\Phi}$ to Φ is defined as follows:

$$\bar{\Phi}(x, v) = \sup_{u \geq 0} \{uv - \Phi(x, u)\} \text{ for all } v \geq 0, x \in \Omega. \quad (9)$$

Then, $\bar{\Phi} \in N(\Omega)$ and Φ is also the complementary function to $\bar{\Phi}$. Equivalently, $\bar{\Phi}$ admits the representation $\bar{\Phi}(x, v) = \int_0^v \phi(x, \sigma) d\sigma$, where ϕ is given by $\phi(x, \sigma) = \sup \{\tau : \varphi(x, \tau) \leq \sigma\}$, for all $x \in \Omega$.

Moreover, one has the Young inequality: $uv \leq \Phi(x, u) + \bar{\Phi}(x, v)$, for all $u, v \geq 0, x \in \Omega$.

Φ is said to satisfy the Δ_2 condition ($\Phi \in \Delta_2$, for short), if there exists a positive constant $K > 1$ and a nonnegative function $h \in L^1(\Omega)$ such that $\Phi(x, 2u) \leq K\Phi(x, u) + h(x)$, for all $u \geq 0$ and a.e. $x \in \Omega$.

Φ is called locally integrable, if $\int_\Omega \Phi(x, u) dx < \infty$ for every $u > 0$.

The following assumptions will be used.

(Φ_1) $\inf_{x \in \Omega} \Phi(x, 1) = c_1 > 0$.

(Φ_2) For every $t_0 > 0$, there exists a $c = c(t_0) > 0$ such that

$$\frac{\Phi(x, t)}{t} \geq c, \quad (10)$$

$$\frac{\bar{\Phi}(x, t)}{t} \geq c, \quad (11)$$

for all $t \geq t_0$, and all $x \in \Omega$.

Clearly, (10) implies (Φ_1).

Let $\Phi \in N(\Omega)$. We denote by $L^0(\Omega)$ the set of all (equivalence classes of) Lebesgue measurable functions from Ω to $\mathbb{R}$. The Musielak-Orlicz space (i.e., the generalized Orlicz space) $L_\Phi(\Omega)$ is defined by

$$L_\Phi(\Omega) = \left\{ u \in L^0(\Omega) : \rho_\Phi\left(\frac{u}{\lambda}\right) < +\infty, \text{ for some } \lambda > 0 \right\}, \quad (12)$$

with the (Luxemburg) norm

$$\|u\|_\Phi = \inf \left\{ \lambda > 0 : \rho_\Phi\left(\frac{u}{\lambda}\right) \leq 1 \right\}. \quad (13)$$

Moreover, the set

$$K_\Phi(\Omega) = \{ u \in L^0(\Omega) : \rho_\Phi(u) < +\infty \} \quad (14)$$

will be called the Musielak-Orlicz class (i.e., the generalized Orlicz class). A function $u \in L^0(\Omega)$ will be called a finite element of $L_\Phi(\Omega)$, if $\lambda u \in K_\Phi(\Omega)$ for every $\lambda > 0$. The space of all finite elements of $L^0(\Omega)$ will be denoted by $E_\Phi(\Omega)$. Then, $K_\Phi(\Omega)$ is a convex subset of $L_\Phi(\Omega)$, $L_\Phi(\Omega)$ is the smallest vector subspace of $L^0(\Omega)$ containing $K_\Phi(\Omega)$, and $E_\Phi(\Omega)$ is the largest vector subspace of $L^0(\Omega)$ contained in $K_\Phi(\Omega)$.

If Φ is locally integrable, then $E_\Phi(\Omega)$ is a separable space, and $E_\Phi(\Omega) = K_\Phi(\Omega) = L_\Phi(\Omega)$ if and only if $\Phi \in \Delta_2$.

If Φ is locally integrable and satisfy (10), then $(E_\Phi(\Omega))^* = L_{\bar{\Phi}}(\Omega)$. Moreover, if $\bar{\Phi}$ is also locally integrable satisfying (11), and $\Phi, \bar{\Phi} \in \Delta_2$, then $L_\Phi(\Omega)$ is reflexive.

The Musielak-Orlicz-Sobolev space $W^1 L_\Phi(\Omega)$ is defined by

$$W^1 L_\Phi(\Omega) = \{ u \in L_\Phi(\Omega) : \forall |\alpha| \leq 1, D^\alpha u \in L_\Phi(\Omega) \}, \quad (15)$$

where $\alpha = (\alpha_1, \dots, \alpha_N)$ with nonnegative integers $\alpha_i, i = 1, \dots, N$, $|\alpha| = |\alpha_1| + |\alpha_2| + \dots + |\alpha_N|$ and $D^\alpha u$ denote the distributional derivatives.

Let

$$\rho_\Phi(u) = \sum_{|\alpha| \leq 1} \int_\Omega \Phi(x, |D^\alpha u(x)|) dx, \quad (16)$$

$$\|u\|_{\Phi, \Omega} = \inf \left\{ \lambda > 0 : \rho_\Phi\left(\frac{u}{\lambda}\right) \leq 1 \right\},$$

for $u \in W^1 L_\Phi(\Omega)$. Then, $\|u\|_{\Phi, \Omega}$ is a norm on $W^1 L_\Phi(\Omega)$. The pair $(W^1 L_\Phi(\Omega), \|u\|_{\Phi, \Omega})$ is a Banach space if Φ is locally integrable and satisfies (Φ_1) .

If $\Phi(x, u) = \Phi(u)$ is independent of x , then $W^1 L_\Phi(\Omega)$ becomes the Orlicz-Sobolev space. Let $p \in C_+(\bar{\Omega})$, where $C_+(\bar{\Omega}) = \{ p : \bar{\Omega} \rightarrow \mathbb{R} \text{ is continuous, } p(x) > 1, \forall x \in \bar{\Omega} \}$. If

$\Phi(x, |u|) = |u|^{p(x)}$, then $W^1 L_\Phi(\Omega)$ becomes the variable exponent Sobolev space

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : Du \in \left(L^{p(x)}(\Omega) \right)^N \right\}, \quad (17)$$

where $L^{p(x)}(\Omega) = \{ u \in L^0(\Omega) : \int_\Omega |u(x)|^{p(x)} dx < +\infty \}$. $L^{p(x)}(\Omega)$ is equipped with the Luxemburg norm

$$\|u\|_{L^{p(x)}(\Omega)} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\} \left(u \in L^{p(x)}(\Omega) \right), \quad (18)$$

and $W^1 L_\Phi(\Omega)$ is equipped with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{L^{p(x)}(\Omega)} + \|Du\|_{L^{p(x)}(\Omega)} \left(u \in W^{1,p(x)}(\Omega) \right). \quad (19)$$

It is easy to see that

$$W^1 L_\Phi(\Omega) = \{ u \in L_\Phi(\Omega) : |Du| \in L_\Phi(\Omega) \}. \quad (20)$$

Denote $\|Du\|_\Phi = \| |Du| \|_\Phi$ and $\|u\|_{1,\Phi} = \|u\|_\Phi + \|Du\|_\Phi$. Clearly, $\|u\|_{1,\Phi}$ is equivalent to $\|u\|_{\Phi, \Omega}$.

The space $W^1 L_\Phi(\Omega)$ can be identified with a subspace of the product of $N+1$ copies of $L_\Phi(\Omega)$. Denote the product by $\prod L_\Phi$. The subspace is $\sigma(\prod L_\Phi, \prod E_{\bar{\Phi}})$ closed. Let $W_0^1 L_\Phi(\Omega)$ be the $\sigma(\prod L_\Phi, \prod E_{\bar{\Phi}})$ closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1 L_\Phi(\Omega)$.

The space $W^1 E_\Phi(\Omega)$ can be identified with a subspace of the product of $N+1$ copies of $E_\Phi(\Omega)$. Denote the product by $\prod E_\Phi$. Let $W_0^1 E_\Phi(\Omega)$ be the (norm) closure of $\mathcal{D}(\Omega)$ in $W^1 L_\Phi(\Omega)$.

Lemma 1 (see [23], Lemma 14.4). *Let $\Phi \in N(\Omega)$, $\bar{\Phi}$ be a complementary function to Φ , and $\Phi^{-1}, \bar{\Phi}^{-1}$ be the inverse functions to $\Phi, \bar{\Phi}$, respectively. Then,*

$$u \leq \Phi^{-1}(x, u) \bar{\Phi}^{-1}(x, u) \leq 2u, \text{ for every } u \geq 0, x \in \Omega. \quad (21)$$

By Lemma 1, we can deduce the following lemma.

Lemma 2. *Let $\Phi \in N(\Omega)$, $\bar{\Phi}$ be a complementary function to Φ and $\Phi^{-1}, \bar{\Phi}^{-1}$ be the inverse functions to $\Phi, \bar{\Phi}$, respectively. Then,*

$$\frac{\Phi(x, u)}{u} \leq \bar{\Phi}^{-1}(x, \Phi(x, u)) \leq \frac{2\Phi(x, u)}{u}, \text{ for every } u > 0, x \in \Omega. \quad (22)$$

Lemma 3 (see [26], Lemma 2.1). *If $u \in W^1 L_\Phi(\Omega)$, then $u^+, u^- \in W^1 L_\Phi(\Omega)$ and*

$$\begin{aligned} Du^+ &= \begin{cases} Du, & \text{if } u > 0, \\ 0, & \text{if } u \leq 0, \end{cases} \\ Du^- &= \begin{cases} 0, & \text{if } u \geq 0, \\ -Du, & \text{if } u < 0. \end{cases} \end{aligned} \quad (23)$$

Here, $u^+ = \max\{u, 0\}$, $u^- = -\min\{u, 0\}$.

3. Existence Theorems

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary. Let $\Phi \in N(\Omega)$, $\bar{\Phi}$ be a complementary function to Φ with $\bar{\Phi} \in \Delta_2$, Φ and $\bar{\Phi}$ be locally integrable. Suppose that the embedding $W_0^1 L_\Phi(\Omega) \hookrightarrow L_\Phi(\Omega)$ is compact.

For $u, v \in L^0(\Omega)$, we use the standard notations: $u \wedge v = \min\{u, v\}$, $u \vee v = \max\{u, v\}$, $u^+ := u \vee 0$, $u^- := -u \wedge 0$, $u \leq v \Leftrightarrow u(x) \leq v(x)$ for a.e. $x \in \Omega$. According to [1] (Remark 3.1), $u \vee v, u \wedge v \in W^1 L_\Phi(\Omega)$, for any $u, v \in W^1 L_\Phi(\Omega)$. Clearly, $u \vee v, u \wedge v \in W_0^1 L_\Phi(\Omega)$, if $u, v \in W_0^1 L_\Phi(\Omega)$.

Let $a_1 : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ be a Carathéodory function satisfying the following conditions:

(A₁) For a.e. $x \in \Omega$, all $s \in \mathbb{R}$ and all $\xi, \eta \in \mathbb{R}^N$,

$$\begin{aligned} |a_1(x, s, \xi)| &\leq g_1(x) + b_1 \bar{\Phi}^{-1}(x, \Phi(x, k_1 |s|)) \\ &\quad + b_2 \bar{\Phi}^{-1}(x, \Phi(x, k_2 |\xi|)), \end{aligned} \quad (24)$$

$$a_1(x, s, \xi) \xi \geq b_3 \Phi(x, |\xi|) - g_2(x), \quad (25)$$

$$[a_1(x, s, \xi) - a_1(x, s, \eta)](\xi - \eta) > 0, \xi \neq \eta, \quad (26)$$

where $b_1, b_2, b_3, k_1, k_2 > 0$, $g_1 \in E_{\bar{\Phi}}(\Omega)$, $g_1 \geq 0$, $g_2 \in L^1(\Omega)$, and $g_2 \geq 0$.

Let $a_0 : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function satisfying the following conditions:

(A₀) For a.e. $x \in \Omega$ and all $s \in \mathbb{R}$,

$$|a_0(x, s)| \leq g_1(x) + b_1 \bar{\Phi}^{-1}(x, \Phi(x, k_1 |s|)), \quad (27)$$

$$a_0(x, s) s \geq b_3 \Phi(x, |s|) - g_2(x), \quad (28)$$

where $b_1, b_3, k_1 > 0$, $g_1 \in E_{\bar{\Phi}}(\Omega)$, $g_1 \geq 0$, $g_2 \in L^1(\Omega)$, and $g_2 \geq 0$.

Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ be a Carathéodory function satisfying the following growth condition:

(F) For a.e. $x \in \Omega$, all $s \in \mathbb{R}$, and all $\xi \in \mathbb{R}^N$,

$$|f(x, s, \xi)| \leq q(x) + b_4 \bar{\Phi}^{-1}(x, \Phi(x, |s|)) + b_5 \bar{\Phi}^{-1}(x, \Phi(x, |\xi|)), \quad (29)$$

where $q \in E_{\bar{\Phi}}(\Omega)$, $b_4, b_5 > 0$. Denote by F the Nemytskii operator associated to f , that is,

$$F(u)(x) = f(x, u(x), Du(x)), \text{ for } x \in \Omega. \quad (30)$$

Consider the following Dirichlet boundary value problem:

$$\begin{cases} -\operatorname{div}(a_1(x, u, Du)) + a_0(x, u) = f(x, u, Du), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \quad (31)$$

A function u is called a weak solution of (31) if $u \in W_0^1 L_\Phi(\Omega)$, $F(u) \in L_{\bar{\Phi}}(\Omega)$, and u satisfies the equation

$$\begin{aligned} \int_{\Omega} a_1(x, u, Du) Dv dx + \int_{\Omega} a_0(x, u) v dx \\ = \int_{\Omega} f(x, u, Du) v dx, \text{ for all } v \in W_0^1 L_\Phi(\Omega). \end{aligned} \quad (32)$$

A function u is called a subsolution (resp., supersolution) of (31) if $u \in W_0^1 L_\Phi(\Omega)$, $F(u) \in L_{\bar{\Phi}}(\Omega)$, and (32) holds with “=” replaced by “ $\leq$ ” (resp., “ $\geq$ ”) for every nonnegative functions v in $W_0^1 L_\Phi(\Omega)$ (see [1]).

Theorem 4. *Let (A_1) , (A_0) , and (F) hold. If*

$$2b_4 + b_5 < b_3, \quad (33)$$

then there exists at least one weak solution of (31).

Proof. Denote $V = W_0^1 E_\Phi(\Omega)$. Define $\Gamma : V \rightarrow V^*$,

$$\begin{aligned} (\Gamma u, v) &:= \int_{\Omega} a_1(x, u, Du) Dv dx + \int_{\Omega} a_0(x, u) v dx \\ &\quad - \int_{\Omega} F(u) v dx, \end{aligned} \quad (34)$$

$\forall v \in V$. Then, Γ is well defined.

Since V is separable, there is a sequence $\{w_n\}_{n=1}^\infty \subset \mathcal{D}(\Omega)$ such that $\{w_n\}_{n=1}^\infty$ is dense in V . Let $V_m = \operatorname{span}\{w_1, \dots, w_m\}$ and consider $\Gamma|_{V_m}$.

Similar to the proof of Theorem 3.1 in [27], $u \rightarrow \Gamma|_{V_m} u : V_m \rightarrow V_m^*$ is continuous. $\square$

In view of (29), for every $u \in V$, we have

$$\begin{aligned} \left| \int_{\Omega} F(u) u dx \right| &\leq C_1^* \|q\|_{\bar{\Phi}} \|u\|_{1, \Phi} + (2b_4 + b_5) \int_{\Omega} \Phi(x, |u|) dx \\ &\quad + b_5 \int_{\Omega} \Phi(x, |Du|) dx, \end{aligned} \quad (35)$$

for some constant $C_1^* > 0$.

Combining (25), (28), (33), and (35), we obtain

$$\begin{aligned} (\Gamma u, u) &\geq (b_3 - b_5) \int_{\Omega} \Phi(x, |Du|) dx \\ &\quad + (b_3 - 2b_4 - b_5) \int_{\Omega} \Phi(x, |u|) dx - C - C_1^* \|q\|_{\bar{\Phi}} \|u\|_{1,\Phi} \\ &\geq (b_3 - b_5) \int_{\Omega} \Phi(x, |Du|) dx - C - C_1^* \|q\|_{\bar{\Phi}} \|u\|_{1,\Phi}, \end{aligned} \quad (36)$$

for all $u \in V$, and some constant $C > 0$ independent of u . Consequently,

$$\begin{aligned} \frac{(\Gamma u, u)}{\|u\|_{1,\Phi}} &\geq \frac{(b_3 - b_5) \int_{\Omega} \Phi(x, |Du|) dx}{\|u\|_{1,\Phi}} - \frac{C}{\|u\|_{1,\Phi}} - C_1^* \|q\|_{\bar{\Phi}} \\ &\geq (b_3 - b_5) \frac{\int_{\Omega} \Phi(x, |Du|) dx}{C_2^* \|Du\|_{L^1(\Omega)}} - \frac{C}{\|u\|_{1,\Phi}} - C_1^* \|q\|_{\bar{\Phi}}, \end{aligned} \quad (37)$$

for all $u \in V_m$, and some constant $C_2^* > 0$. Using ([2], Lemma 1), we have

$$\frac{(\Gamma u, u)}{\|u\|_{1,\Phi}} \longrightarrow +\infty, \quad \text{as } \|u\|_{1,\Phi} \longrightarrow +\infty. \quad (38)$$

By [27] (Remark 2.1), there exists a Galerkin solution $u_m \in V_m$ for every $m \in \mathbb{N}$ such that

$$(\Gamma u_m, v) = 0, \quad \forall v \in V_m. \quad (39)$$

Using the density of $\{w_n\}$, we deduce that

$$(\Gamma u_m, v) = 0, \quad \forall v \in V. \quad (40)$$

Therefore, the sequence $\{u_m\}$ is bounded in V . Consequently, there exists $u_0 \in V$, and we can extract a subsequence $\{u_n\}$ of $\{u_m\}$ such that

$$u_n \rightharpoonup u_0 \text{ weakly in } V \text{ for } \sigma\left(\prod L_{\Phi}, \prod E_{\bar{\Phi}}\right), \quad (41)$$

$$u_n \longrightarrow u_0 \text{ strongly in } L_{\Phi}(\Omega), \quad (42)$$

$$u_n(x) \longrightarrow u_0(x) \text{ for a.e. } x \in \Omega, \quad (43)$$

as $n \longrightarrow \infty$. By (24), $\{a_1(x, u_n, Du_n)\}$ is bounded in $(L_{\bar{\Phi}}(\Omega))^N$, and $\{a_0(x, u_n)\}$ and $\{f(x, u_n, Du_n)\}$ are bounded in $L_{\bar{\Phi}}(\Omega)$. Thanks to (42),

$$\begin{aligned} \int_{\Omega} a_0(x, u_n)(u_n - u_0) dx &\longrightarrow 0, \\ \int_{\Omega} f(x, u_n, Du_n)(u_n - u_0) dx &\longrightarrow 0, \end{aligned} \quad (44)$$

as $n \longrightarrow \infty$. It follows that

$$\int_{\Omega} a_1(x, u_n, Du_n)(Du_n - Du_0) dx \longrightarrow 0, \quad (45)$$

as $n \longrightarrow \infty$.

Therefore, similar to [27], we can deduce that there exists a subsequence of $\{u_n\}$ still denoted by $\{u_n\}$ such that

$$Du_n(x) \longrightarrow Du_0(x) \text{ for a.e. } x \in \Omega, \quad (46)$$

as $n \longrightarrow \infty$. Combining (43) and (46), we obtain that

$$\begin{aligned} a_1(x, u_n, Du_n) &\longrightarrow a_1(x, u_0, Du_0) \text{ a.e. in } \Omega, \\ a_0(x, u_n) &\longrightarrow a_0(x, u_0) \text{ a.e. in } \Omega, \\ f(x, u_n, Du_n) &\longrightarrow f(x, u_0, Du_0) \text{ a.e. in } \Omega, \end{aligned} \quad (47)$$

as $n \longrightarrow \infty$.

Using [25] (Lemma 1) and letting $n \longrightarrow \infty$, passing to a subsequence if necessary, we can deduce from (40) that

$$(\Gamma u_0, v) = 0, \quad \forall v \in V. \quad (48)$$

By [28] (Lemma 2.5) and [25] (Lemma 7),

$$(\Gamma u_0, v) = 0, \quad \forall v \in W_0^1 L_{\Phi}(\Omega). \quad (49)$$

Thus, there exists at least one weak solution u_0 of (31).

Example 1. Let $\Phi(x, |t|) = e^{p(x)|t|} - p(x)|t| - 1$, for $x \in \Omega$ and $t \in \mathbb{R}$, where $p : \Omega \longrightarrow \mathbb{R}$ is a measurable function such that $1 < p_- := \inf_{x \in \Omega} p(x) \leq p(x) \leq p_+ := \sup_{x \in \Omega} p(x) < +\infty$. Then, $\Phi \notin \Delta_2$.

Example 2. Put $a_1(x, s, \xi) = a(x, s)\varphi(x, |\xi|)\xi/|\xi|$ if $\xi \in \mathbb{R}^N \setminus \{0\}$ and $a_1(x, s, \xi) = 0$ if $\xi = 0$, for a.e. $x \in \Omega$, all $s \in \mathbb{R}$, where a is a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying $\alpha \leq a(x, s) \leq \beta$ with $\alpha > 0$ and $\beta > 0$ for all $s \in \mathbb{R}$, $\varphi(x, s)$ is the right-hand derivative of $\Phi(x, \cdot)$ at s , and 0 is the zero vector in $\mathbb{R}^N$. Let $a_0(x, s) = \alpha\varphi(x, |s|)s/|s|$, and $f(x, s, \xi) = b(x, s)\varphi(x, |\xi|)|s|$, for a.e. $x \in \Omega$, all $s \in \mathbb{R}$ and all $\xi \in \mathbb{R}^N$, where b is a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying $|b(x, s)| \leq \gamma$ with $0 < \gamma < \alpha/3$. Then, a_1 , a_0 , and f satisfy conditions (A_1) , (A_0) , and (29), respectively.

Remark 5. Clearly, condition (5) implies that

$$\begin{aligned} A(x, \xi)\xi &\geq c_A \Phi(x, |\xi|), \\ A(x, \xi)\xi &\geq c_A \bar{\Phi}(x, |A(x, \xi)|), \end{aligned} \quad (50)$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^N$. By Lemma 2, we have

$$\frac{|\xi|}{c_A} \geq \frac{\bar{\Phi}(x, |A(x, \xi)|)}{|A(x, \xi)|} \geq \frac{1}{2} \bar{\Phi}^{-1}(x, \bar{\Phi}(x, |A(x, \xi)|)). \quad (51)$$

It implies that

$$\bar{\Phi}(x, |A(x, \xi)|) \leq \Phi\left(x, \frac{2}{c_A} \xi\right). \quad (52)$$

Consequently,

$$|A(x, \xi)| \leq \bar{\Phi}^{-1}\left(x, \Phi\left(x, \frac{2}{c_A} \xi\right)\right). \quad (53)$$

Hence, condition (5) is stronger than not only the growth condition (3.1) but also the coercive condition (3.2) in [2].

Let $\mathcal{S}$ be the set of all weak solutions of (31). Under the assumptions of Theorem 4, the solution set $\mathcal{S}$ of (31) is nonempty.

Theorem 6. Assume that the assumptions of Theorem 4 hold. Then $\mathcal{S}$ is bounded and closed in $W_0^1 L_\Phi(\Omega)$. Moreover, $\mathcal{S}$ is compact in $W_0^1 L_\Phi(\Omega)$.

Proof. Denote $V = W_0^1 L_\Phi(\Omega)$. For $u \in V$, define $\rho(u) = \int_\Omega [\Phi(x, |u|) + \Phi(x, |Du|)] dx$, and $\|u\|_\rho = \inf \{\lambda > 0 : \rho(u/\lambda) \leq 1\}$, then $\|u\|_\rho$ is a norm of V equivalent to $\|u\|_{1,\Phi}$ (see [1]). Taking $\alpha_0 = \min \{b_3 - b_5, b_3 - 2b_4 - b_5\}$ in the first inequality of (36) and by [2] (Proposition 3.1), there exists $h \in L^1(\Omega)$, such that, for every $\varepsilon > 0$,

$$\begin{aligned} (\Gamma u, u) &\geq \alpha_0 (\|u\|_\rho - \varepsilon) G(\|u\|_\rho - \varepsilon) \\ &\quad - \alpha_0 (\|u\|_\rho - \varepsilon) \int_\Omega |h(x)| dx - C_1 - C_2 \|u\|_{1,\Phi}, \end{aligned} \quad (54)$$

for all $\forall u \in V$, as $\|u\|_{1,\Phi}$ is large enough, where the constants C_1 and C_2 are independent of u . Consequently, we can deduce that $\mathcal{S}$ is bounded in $W_0^1 L_\Phi(\Omega)$.

Next, we prove that $\mathcal{S}$ is closed. Indeed, assume $\{u_n\}_{n=1}^\infty \subset \mathcal{S}$, and $u_n \rightharpoonup u_0$ in $W_0^1 L_\Phi(\Omega)$ as $n \rightarrow \infty$. Since $\mathcal{S}$ is bounded, $\{u_n\}_{n=1}^\infty$ is bounded. Therefore, by passing to a subsequence if necessary, we can assume without loss of generality that

$$u_n \rightharpoonup u_0 \text{ weakly in } V \text{ for } \sigma(PL_\Phi, PE_\Phi), \quad (55)$$

as $n \rightarrow \infty$. Similar to the proof of Theorem 4, we can obtain that u_0 is a weak solution of (31), i.e., $u_0 \in \mathcal{S}$. Consequently, $\mathcal{S}$ is closed in V . $\square$

For any subsequence $\{u_n\}_{n=1}^\infty$ in $\mathcal{S}$, $\{u_n\}_{n=1}^\infty$ is bounded. Repeating the proof of Theorem 4, by passing to a subsequence if necessary, we can obtain that $\{u_n\}_{n=1}^\infty$ is convergent to u_0 , and u_0 is a weak solution of (31), i.e., $u_0 \in \mathcal{S}$. Consequently, $\mathcal{S}$ is sequentially compact. Since $\mathcal{S}$ is closed, $\mathcal{S}$ is compact.

Definition 7 ([4], Definition 2.1). Let $(\mathcal{P}, \leq)$ be a partially ordered set. A subset $\mathcal{C}$ of $\mathcal{P}$ is said to be upward directed if for each pair $x, y \in \mathcal{C}$ there is a $z \in \mathcal{C}$ such that $x \leq z$ and $y \leq z$. Similarly, $\mathcal{C}$ is downward directed if for each pair $x, y \in \mathcal{C}$ there is $w \in \mathcal{C}$ such that $w \leq x$ and $w \leq y$. If $\mathcal{C}$ is both upward and downward directed, it is called directed.

If there exist $u_*, u^* \in \mathcal{S}$ such that $u_* \leq u^*$, and $u_* \leq u \leq u^*$, for any $u \in \mathcal{S}$, then u_* and u^* are called barrier solutions of (31) (see [4]). Apparently, u_* and u^* are the least element and the greatest element with respect to the ordering “ $\leq$,” respectively.

To show that $\mathcal{S}$ is directed with respect to the ordering “ $\leq$,” the following modulus of continuity condition on the coefficient $a_1 : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is required:

(A₁') For a.e. $x \in \Omega$, all $s, s' \in \mathbb{R}$ and all $\xi \in \mathbb{R}^N$,

$$\begin{aligned} &|a_1(x, s, \xi) - a_1(x, s', \xi)| \\ &\leq \left[g_3(x) + \bar{\Phi}^{-1}(x, \Phi(x, |s|)) + \bar{\Phi}^{-1}(x, \Phi(x, |s'|)) \right. \\ &\quad \left. + \bar{\Phi}^{-1}(x, \Phi(x, |\xi|)) \right] \omega(|s - s'|), \end{aligned} \quad (56)$$

where $g_3 \in E_\Phi(\Omega)$, $g_3 \geq 0$, and $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a continuous function such that

$$\int_0^\varepsilon \frac{1}{\omega(r)} dr = +\infty, \quad (57)$$

for every $\varepsilon > 0$.

Theorem 8. Assume that the assumptions of Theorem 4 and (A₁') hold. Let $u_1, u_2 \in \mathcal{S}$. Then $u_1 \vee u_2$ and $u_1 \wedge u_2$ are the subsolution and supersolution of (31), respectively.

Proof. Denote $V = W_0^1 L_\Phi(\Omega)$. Let $u_1, u_2 \in \mathcal{S}$. In view of assumption (A₁'), for any $\varepsilon > 0$, there exists $\delta(\varepsilon) \in (0, \varepsilon)$ such that $\int_{\delta(\varepsilon)}^\varepsilon 1/\omega(r) dr = 1$. Define the function $\theta_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}_+$, which is defined by

$$\theta_\varepsilon(s) = \begin{cases} 0, & \text{if } s < \delta(\varepsilon), \\ \int_{\delta(\varepsilon)}^s \frac{1}{\omega(r)} dr, & \text{if } \delta(\varepsilon) \leq s \leq \varepsilon, \\ 1, & \text{if } s > \varepsilon. \end{cases} \quad (58)$$

Then, for every $\varepsilon > 0$, the function θ_ε is continuous, piecewise differentiable, and the derivative is nonnegative and bounded. Consequently, θ_ε is Lipschitz continuous and nondecreasing, and it satisfies $\theta_\varepsilon \rightarrow \chi_{\{s>0\}}$ as $\varepsilon \rightarrow 0$. Moreover, $\theta'_\varepsilon(s) = 1/\omega(s)$ if $\delta(\varepsilon) \leq s \leq \varepsilon$, and $\theta'_\varepsilon(s) = 0$ if $s \notin [\delta(\varepsilon), \varepsilon]$ (see [29]). $\square$

Let $\psi \in V$. Replacing u with u_1, u_2 and taking the test function $v = \psi(1 - \theta_\varepsilon(u_2 - u_1))$, $v = \psi\theta_\varepsilon(u_2 - u_1)$ in (32), respectively, and adding the resulting equations, we obtain

$$\begin{aligned}
& \int_{\Omega} a_1(x, u_1, Du_1) D\psi dx \\
& + \int_{\Omega} [a_1(x, u_2, Du_2) - a_1(x, u_1, Du_1)] D(\psi\theta_{\varepsilon}(u_2 - u_1)) dx \\
& + \int_{\Omega} a_0(x, u_1) \psi dx \\
& + \int_{\Omega} [a_0(x, u_2) - a_0(x, u_1)] \psi\theta_{\varepsilon}(u_2 - u_1) dx \\
& = \int_{\Omega} f(x, u_1, Du_1) \psi dx \\
& + \int_{\Omega} [f(x, u_2, Du_2) - f(x, u_1, Du_1)] \psi\theta_{\varepsilon}(u_2 - u_1) dx.
\end{aligned} \tag{59}$$

By applying (26), (A'_1) , and the properties of θ_{ε} , we have

$$\begin{aligned}
& \int_{\Omega} [a_1(x, u_2, Du_2) - a_1(x, u_1, Du_1)] D(\psi\theta_{\varepsilon}(u_2 - u_1)) dx \\
& = \int_{\Omega} [a_1(x, u_2, Du_2) - a_1(x, u_1, Du_1)] \theta_{\varepsilon}(u_2 - u_1) D\psi dx \\
& + \int_{\Omega} [a_1(x, u_2, Du_2) - a_1(x, u_2, Du_1)] \psi\theta'_{\varepsilon}(u_2 - u_1) D(u_2 - u_1) dx \\
& + \int_{\Omega} [a_1(x, u_2, Du_1) - a_1(x, u_1, Du_1)] \psi\theta'_{\varepsilon}(u_2 - u_1) D(u_2 - u_1) dx \\
& \geq \int_{\Omega} [a_1(x, u_2, Du_2) - a_1(x, u_1, Du_1)] \theta_{\varepsilon}(u_2 - u_1) D\psi dx \\
& - \int_{\{\delta(\varepsilon) < u_2 - u_1 < \varepsilon\}} \left[g_3(x) + \bar{\Phi}^{-1}(x, \Phi(x, |u_1|)) \right. \\
& \left. + \bar{\Phi}^{-1}(x, \Phi(x, |u_2|)) + \bar{\Phi}^{-1}(x, \Phi(x, |Du_1|)) \right] |\psi| |D(u_2 - u_1)| dx.
\end{aligned} \tag{60}$$

The last term on the right-hand side of (60) tends to 0, as $\varepsilon \rightarrow 0$. Using Lebesgue's dominated convergence theorem, we obtain that

$$\begin{aligned}
& \int_{\Omega} [a_1(x, u_2, Du_2) - a_1(x, u_1, Du_1)] \theta_{\varepsilon}(u_2 - u_1) D\psi dx \\
& \longrightarrow \int_{\Omega} [a_1(x, u_2, Du_2) - a_1(x, u_1, Du_1)] \chi_{\{u_2 > u_1\}} D\psi dx,
\end{aligned} \tag{61}$$

$$\begin{aligned}
& \int_{\Omega} [a_0(x, u_2) - a_0(x, u_1)] \psi\theta_{\varepsilon}(u_2 - u_1) dx \\
& \longrightarrow \int_{\Omega} [a_0(x, u_2) - a_0(x, u_1)] \chi_{\{u_2 > u_1\}} \psi dx,
\end{aligned} \tag{62}$$

$$\begin{aligned}
& \int_{\Omega} [f(x, u_2, Du_2) - f(x, u_1, Du_1)] \psi\theta_{\varepsilon}(u_2 - u_1) dx \\
& \longrightarrow \int_{\Omega} [f(x, u_2, Du_2) - f(x, u_1, Du_1)] \chi_{\{u_2 > u_1\}} \psi dx,
\end{aligned} \tag{63}$$

as $\varepsilon \rightarrow 0$.

Combining (59)–(63), we get that

$$\begin{aligned}
& \int_{\Omega} a_1(x, u_1, Du_1) D\psi dx \\
& + \int_{\Omega} [a_1(x, u_2, Du_2) - a_1(x, u_1, Du_1)] \chi_{\{u_2 > u_1\}} D\psi dx \\
& + \int_{\Omega} a_0(x, u_1) \psi dx + \int_{\Omega} [a_0(x, u_2) - a_0(x, u_1)] \chi_{\{u_2 > u_1\}} \psi dx \\
& \leq \int_{\Omega} f(x, u_1, Du_1) \psi dx \\
& + \int_{\Omega} [f(x, u_2, Du_2) - f(x, u_1, Du_1)] \chi_{\{u_2 > u_1\}} \psi dx,
\end{aligned} \tag{64}$$

which yields that $u_1 \vee u_2$ is a subsolution of (31), i.e., $u_1 \vee u_2 \in \mathcal{S}$. Analogously, we can deduce that $u_1 \wedge u_2$ is a supersolution of (31).

Theorem 9. Assume that the assumptions of Theorem 4 and (A'_1) hold. Then, $\mathcal{S}$ is directed downward and upward; that is, for all $u_1, u_2 \in \mathcal{S}$, there exist $u, v \in \mathcal{S}$ such that $v \leq u_1 \wedge u_2$ and $u_1 \vee u_2 \leq u$.

Proof. Denote $V = W_0^1 L_{\Phi}(\Omega)$. For $u_1, u_2 \in \mathcal{S}$, let $u_0 = u_1 \vee u_2$. We will prove that there exists $\tilde{u} \in \mathcal{S}$ such that $u_0 \leq \tilde{u}$.

Define the cutoff function $b : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$b(x, s) = \begin{cases} 0, & \text{if } s > u_0(x), \\ -\bar{\Phi}^{-1}(x, \Phi(x, u_0(x) - s)), & \text{if } s \leq u_0(x), \end{cases} \tag{65}$$

for $x \in \Omega, s \in \mathbb{R}$. Then for any $u \in L_{\Phi}(\Omega)$, $b(x, u) \in L_{\Phi}(\Omega)$. $\square$

Denote $\{u \leq u_0\} := \{x \in \Omega : u(x) \leq u_0(x)\}$. For any $u \in V$, by Young inequality, one has

$$\begin{aligned}
\int_{\Omega} b(x, u) u dx & \geq \int_{\{u \leq u_0\}} \Phi(x, u_0 - u) dx \\
& - \int_{\{u \leq u_0\}} \frac{1}{2} [\Phi(x, u_0 - u) + \Phi(x, 2|u_0|)] dx \\
& \geq -\frac{1}{2} \int_{\Omega} \Phi(x, 2|u_0|) dx.
\end{aligned} \tag{66}$$

Define $Tu = u \vee u_0$. By Remark 3.1 in [1], $T : V \rightarrow V$ is continuous. According to the definition of T , T is bounded.

Thanks to (29), and by Young inequality, we have

$$\begin{aligned}
\left| \int_{\Omega} F(Tu) u dx \right| & \leq (2b_4 + b_5) \int_{\{u > u_0\}} \Phi(x, |u|) dx \\
& + b_5 \int_{\Omega} \Phi(x, |Du|) dx + C_1 \|u\|_{1, \Phi},
\end{aligned} \tag{67}$$

for all $u \in V$, and some constant $C_1 > 0$ independent of u . From (27) and (28), we can deduce that

$$\begin{aligned} \int_{\Omega} a_0(x, Tu)u dx &\geq b_3 \int_{\{u > u_0\}} \Phi(x, |u|) dx \\ &\quad - \int_{\{u > u_0\}} g_2(x) dx - C_2 \|u\|_{1,\Phi}, \end{aligned} \quad (68)$$

for all $u \in V$, and some constant $C_2 > 0$ independent of u .

Let us consider the following auxiliary equation of finding $u \in V$ such that

$$\begin{aligned} \int_{\Omega} a_1(x, Tu, Du) Dv dx + \int_{\Omega} a_0(x, Tu) v dx + \int_{\Omega} b(x, u) v dx \\ = \int_{\Omega} F(Tu) v dx. \end{aligned} \quad (69)$$

Define $\Gamma_T : V \longrightarrow V^*$,

$$\begin{aligned} (\Gamma_T u, v) &:= \int_{\Omega} a_1(x, Tu, Du) Dv dx + \int_{\Omega} a_0(x, Tu) v dx \\ &\quad + \int_{\Omega} b(x, u) v dx - \int_{\Omega} F(Tu) v dx, \end{aligned} \quad (70)$$

$\forall v \in V$. Then, Γ_T is well defined.

Combining (25), (28), (29), (33), (66), (67), and (68), we have

$$(\Gamma_T u, u) \geq (b_2 - b_4) \int_{\Omega} \Phi(x, |Du|) dx - C_3 \|u\|_{1,\Phi} - C_4, \quad (71)$$

$\forall u \in V$, where $C_3, C_4 > 0$ are two constants independent of u .

Similar to the proof of Theorem 4, we can deduce that there exists $\tilde{u} \in V$ such that

$$(\Gamma_T \tilde{u}, v) = 0, \quad \forall v \in V, \quad (72)$$

i.e., $\tilde{u}$ is a weak solution of (69).

Denote $\{u_0 > \tilde{u}\} = \{x \in \Omega : u_0(x) > \tilde{u}(x)\}$. Replacing u with $\tilde{u}$ in (69), and taking $v = (u_0 - \tilde{u})^+ \in V$, we can obtain that

$$\begin{aligned} \int_{\Omega} a_1(x, T\tilde{u}, D\tilde{u}) D(u_0 - \tilde{u})^+ dx \\ + \int_{\Omega} a_0(x, T\tilde{u}) (u_0 - \tilde{u})^+ dx \\ + \int_{\Omega} b(x, \tilde{u}) (u_0 - \tilde{u})^+ dx \\ = \int_{\Omega} F(T\tilde{u}) (u_0 - \tilde{u})^+ dx. \end{aligned} \quad (73)$$

By Theorem 8, we know

$$\int_{\Omega} a_1(x, u_0, Du_0) Dv dx + \int_{\Omega} a_0(x, u_0) v dx \leq \int_{\Omega} F(u_0) v dx, \quad (74)$$

for any $v \in V$ with $v \geq 0$.

Taking $v = (u_0 - \tilde{u})^+ \in V$ in (74), we have

$$\begin{aligned} \int_{\Omega} a_1(x, u_0, Du_0) D(u_0 - \tilde{u})^+ dx + \int_{\Omega} a_0(x, u_0) (u_0 - \tilde{u})^+ dx \\ \leq \int_{\Omega} F(u_0) (u_0 - \tilde{u})^+ dx. \end{aligned} \quad (75)$$

Subtracting (73) from (75), we obtain that

$$\begin{aligned} \int_{\Omega} [a_1(x, u_0, Du_0) - a_1(x, T\tilde{u}, D\tilde{u})] D(u_0 - \tilde{u})^+ dx \\ + \int_{\Omega} [a_0(x, u_0) - a_0(x, T\tilde{u})] (u_0 - \tilde{u})^+ dx \\ - \int_{\Omega} b(x, \tilde{u}) (u_0 - \tilde{u})^+ dx \\ \leq \int_{\Omega} [F(u_0) - F(T\tilde{u})] (u_0 - \tilde{u})^+ dx. \end{aligned} \quad (76)$$

Thanks to (26), the first term of the left-hand side of (76) is nonnegative. Clearly, both the second term of the left-hand side and the right-hand side of (76) equal 0. Indeed,

$$\begin{aligned} \int_{\Omega} [a_0(x, u_0) - a_0(x, T\tilde{u})] (u_0 - \tilde{u})^+ dx \\ = \int_{\{u_0 > \tilde{u}\}} [a_0(x, u_0) - a_0(x, u_0)] (u_0 - \tilde{u}) dx = 0, \end{aligned} \quad (77)$$

$$\begin{aligned} \int_{\Omega} [F(u_0) - F(T\tilde{u})] (u_0 - \tilde{u})^+ dx \\ = \int_{\{u_0 > \tilde{u}\}} [F(u_0) - F(u_0)] (u_0 - \tilde{u}) dx = 0. \end{aligned}$$

Therefore,

$$\int_{\Omega} b(x, \tilde{u}) (u_0 - \tilde{u})^+ dx \geq 0. \quad (78)$$

On the other hand,

$$\begin{aligned} \int_{\Omega} b(x, \tilde{u}) (u_0 - \tilde{u})^+ dx &= - \int_{\{u_0 > \tilde{u}\}} \bar{\Phi}^{-1}(x, \Phi(x, u_0 - \tilde{u})) (u_0 - \tilde{u}) dx \\ &\leq - \int_{\{u_0 > \tilde{u}\}} \Phi(x, u_0 - \tilde{u}) dx \leq 0. \end{aligned} \quad (79)$$

It follows that $u_0 \leq \tilde{u}$.

From the definitions of l , T , we deduce that

$$\begin{aligned} a_1(x, T\tilde{u}(x), D\tilde{u}(x)) &= a_1(x, \tilde{u}(x), D\tilde{u}(x)), \\ a_0(x, T\tilde{u}(x)) &= a_0(x, \tilde{u}(x)), \\ b(x, \tilde{u}(x)) &= 0, f(x, T\tilde{u}(x), DT\tilde{u}(x)) = f(x, \tilde{u}(x), D\tilde{u}(x)), \end{aligned} \quad (80)$$

for a.e. $x \in \Omega$. We note that (69) reduces to (32). Hence, $\tilde{u}$ is a weak solution of (31). Consequently, $\tilde{u} \in \mathcal{S}$, and $u_1 \vee u_2 \leq \tilde{u}$.

Similarly, there exists $v \in \mathcal{S}$, such that $v \leq u_1 \wedge u_2$. Therefore, $\mathcal{S}$ is a directed set.

Similar to step (c) of the proof of Theorem 3.1 in [4], we have the following result.

Theorem 10. Assume that the assumptions of Theorem 4 and (A'_1) hold. Let $\Phi \in \Delta_2$. Then (31) admits barrier solutions, i.e., there exist $u_*, u^* \in \mathcal{S}$ such that $u_* \leq u \leq u^*$, $\forall u \in \mathcal{S}$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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