Research Article

Solvability for a New Class of Moore-Gibson-Thompson Equation with Viscoelastic Memory, Source Terms, and Integral Condition

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This paper deals with the existence and uniqueness of solutions of a new class of Moore-Gibson-Thompson equation with respect to the nonlocal mixed boundary value problem, source term, and nonnegative memory kernel. Galerkin's method was the main used tool for proving our result. This work is a generalization of recent homogenous work.

1. Introduction

In this contribution, we are interested to study the existence and uniqueness of solutions of the following problem

$$\begin{cases} \mathcal{L}u(x, t) = au_{ttt} + \beta u_{tt} - c^2 \Delta u - b \Delta u_t - \int_0^t h(t-\sigma) \Delta u(\sigma) d\sigma = F(x, t), \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), u_{tt}(x, 0) = u_2(x), \\ \frac{\partial u}{\partial \eta} = \int_0^t \int_{\Omega} (\xi, \tau) d\xi d\tau, x \in \partial\Omega. \end{cases} \quad (1)$$

Here, a and β are physical parameters, and c is the speed of sound. The convolution term $\int_0^t h(t-s) \Delta u(s) ds$ reflects the memory effect of materials due to viscoelasticity, F is a given function, and h is the relaxation function satisfying

(H1) $h \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ is a nonincreasing function satisfying

$$h(0) > 0, 1 - h_0 = l > 0, \quad (2)$$

where $h_0 = G(\infty) = \int_0^\infty h(s) ds > 0$, $G(t) = \int_0^t h(s) ds$, and $h'' > 0$.

(H2) $\exists \zeta > 0$ satisfying

$$h'(t) \leq -\zeta h(t), t \geq 0. \quad (3)$$

(H3)

$$\beta - a > 0 \quad (4)$$

The phenomena resulting from sound waves (diffraction, interference, reflection) in terms of modeling are very important. As the existence of the third derivative is very important, especially in the field of thermodynamics (EIT), the study of these models is considered the beginning of an in-depth understanding of both convergent and good behavior. From the results extracted, the equation of MGT resulted in nonlinear acoustics, for much depth, see ([1–7]) and especially [8] where equation of MGT appeared for the first time. Also, nonlinear problems of great importance can be considered [9], where Galerkin's method was applied in solving them, for more depth ([2, 3, 10–13]). Recently, in [14], the authors studied the equation of MGT with memory. Likewise, in [1], the authors used Galerkin's method to demonstrate the ability to solve a mixed problem of MGT equation in the absence of viscous elasticity and memory. Based on work [9] and the works we mentioned earlier, we want to prove the existence and uniqueness of a weak solution to the problem (1).

We divide this paper into the following: in the second part, we put some definitions and appropriate spaces. Then, we apply Galerkin's method to prove the existence, and in the fourth part, we demonstrate the uniqueness.

2. Preliminaries

We will define the spaces: $V(Q_T)$ and $W(Q_T)$ by

$$\begin{aligned} V(Q_T) &= \{u \in W_2^1(Q_T): u_t \in W_2^1(Q_T), u, \nabla u \in L_h^2(Q_T)\}, \\ W(Q_T) &= \{u \in V(Q_T): u(x, T) = 0\}, \\ L_h^2(Q_T) &= \left\{u \in V(Q_T): \int_0^T h \circ u(t) dt < \infty\right\}, \end{aligned} \quad (5)$$

where

$$h \circ u(t) = \int_{\Omega} \int_0^t h(t-\sigma)(u(t) - u(\sigma))^2 d\sigma dx. \quad (6)$$

Consider the equation

$$\begin{aligned} a(u_{tt}, v)_{L^2(Q_T)} + \beta(u_t, v)_{L^2(Q_T)} - c^2(\Delta u, v)_{L^2(Q_T)} \\ - b(\Delta u_t, v)_{L^2(Q_T)} - (\Delta w, v)_{L^2(Q_T)} = (F, v)_{L^2(Q_T)}, \end{aligned} \quad (7)$$

where

$$w(x, t) = \int_0^t h(t-\sigma)u(x, \sigma) d\sigma, \quad (8)$$

and $(\cdot, \cdot)_{L^2(Q_T)}$ stands for the inner product in $L^2(Q_T)$, u is supposed to be a solution of (1) and $v \in W(Q_T)$. Evaluation of the inner product in [9] gives

$$\begin{aligned} & -a(u_{tt}, v_t)_{L^2(Q_T)} - \beta(u_t, v_t)_{L^2(Q_T)} + c^2(\nabla u, \nabla v)_{L^2(Q_T)} \\ & + b(\nabla u_t, \nabla v)_{L^2(Q_T)} + (\nabla w, \nabla v)_{L^2(Q_T)} \\ & = (F, v)_{L^2(Q_T)} + c^2 \int_0^T \int_{\partial\Omega} v \left(\int_0^t \int_{\Omega} v(\xi, \tau) d\xi d\tau \right) ds_x dt \\ & + b \int_0^T \int_{\partial\Omega} v \int_{\Omega} u(\xi, t) d\xi ds_x dt - b \int_0^T \int_{\partial\Omega} v \int_{\Omega} u_0(\xi) d\xi ds_x dt \\ & + a(u_2(x), v(x, 0))_{L^2(\Omega)} + \beta(u_1(x), v(x, 0))_{L^2(\Omega)} \\ & + \int_0^T \int_{\partial\Omega} v \left(\int_0^t \int_{\Omega} w(\xi, \tau) d\xi d\tau \right) ds_x dt. \end{aligned} \quad (9)$$

We give two useful inequalities:

- (i) Gronwall inequality. Let the nonnegative integrable functions $\varphi(t)$, $\phi(t)$ on the interval I with the nondecreasing function $h(t)$. If $\forall t \in I$, we have

$$\phi(t) \leq \varphi(t) + c \int_0^t \phi(s) ds, \quad (10)$$

where $c > 0$, hence,

$$\phi(t) \leq \varphi(t) \exp(ct). \quad (11)$$

- (ii) Trace inequality (see [15]). If $\Phi \in W_1^2(\Omega)$ where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, then for any $\varepsilon > 0$,

$$\|\Phi\|_{L^2(\partial\Omega)}^2 \leq \varepsilon \|\nabla \Phi\|_{L^2(\Omega)}^2 + l(\varepsilon) \|\Phi\|_{L^2(\Omega)}^2, \quad (12)$$

where $l(\varepsilon) > 0$.

Definition 1. We call a generalized solution to the problem (1) for each function $u \in V(Q_T)$ that fulfills the equation (9) for each $v \in W(Q_T)$.

3. Solvability of the Problem

In this section, we use Galerkin's method to prove the existence of a generalized solution of our problem.

Theorem 2. If $u_0 \in W_2^1(\Omega)$, $u_1 \in W_2^1(\Omega)$, $u_2 \in L^2(\Omega)$, and $F \in L^2(Q_T)$, then there is at least one generalized solution in $V(Q_T)$ to problem (1).

Proof. Let $\{Z_k(x)\}_{k \geq 1}$ be a fundamental system in $W_2^1(\Omega)$, such that $(Z_k, Z_l)\Omega = \delta_{k,l}$.

First, we will give an approximate solution of the problem (1) in the form

$$u^N(x, t) = \sum_{k=1}^N C_k(t) Z_k(x), \quad (13)$$

where $C_k(t)$ are constants given by the conditions, for $k = 1, \dots, N$,

$$(\mathcal{L}u(x, t), Z_l(x))_{L^2(\Omega)} = (F(x, t), Z_l(x))_{L^2(\Omega)} \quad (14)$$

and can be determined from the relations

$$\begin{aligned} & a(u_{ttt}^N, Z_l(x))_{L^2(\Omega)} + \beta(u_{tt}^N, Z_l(x))_{L^2(\Omega)} + c^2(\nabla u^N, \nabla Z_l(x))_{L^2(\Omega)} \\ & + b(\nabla u_t^N, \nabla Z_l(x))_{L^2(\Omega)} + (\nabla w^N, \nabla Z_l(x))_{L^2(\Omega)} \\ & = (F(x, t), Z_l(x))_{L^2(\Omega)} + c^2 \int_{\partial\Omega} Z_l(x) \left(\int_0^t \int_{\Omega} u^N(\xi, \tau) d\xi d\tau \right) ds_x \\ & + b \int_{\partial\Omega} Z_l(x) \left(\int_0^t \int_{\Omega} u^N(\xi, \tau) d\xi d\tau \right) ds_x \\ & + \int_{\partial\Omega} Z_l(x) \left(\int_0^t \int_{\Omega} w^N(\xi, \tau) d\xi d\tau \right) ds_x, \end{aligned} \quad (15)$$

substitution of (13) into (15), and we find for $l = 1, \dots, N$.

$$\begin{aligned} & \int_{\Omega} \sum_{k=1}^N \left\{ aC_k'''(t) Z_k(x) Z_l(x) \beta C_k'''(t) Z_k(x) Z_l(x) \right. \\ & + c^2 C_k(t) \nabla Z_k(x) \cdot \nabla Z_l(x) + bC_k'(t) \nabla Z_k(x) \cdot \nabla Z_l(x) \\ & + \left. \left(\int_0^t h(t-\sigma) C_k(\sigma) d\sigma \right) \nabla Z_k(x) \cdot \nabla Z_l(x) \right\} dx \\ & = (F(x, t), Z_l(x))_{L^2(\Omega)} + c^2 \sum_{k=1}^N \int_0^t C_k(\tau) \left(\int_{\partial\Omega} Z_l(x) \int_{\Omega} Z_k(\xi) d\xi ds_x \right) d\tau \\ & + b \sum_{k=1}^N \int_0^t C_k'(\tau) \left(\int_{\partial\Omega} Z_l(x) \int_{\Omega} Z_k(\xi) d\xi ds_x \right) d\tau \\ & + \sum_{k=1}^N \int_0^t \left(\int_0^\tau h(\tau-\sigma) C_k(\sigma) d\sigma \right) \left(\int_{\partial\Omega} Z_l(x) \int_{\Omega} Z_k(\xi) d\xi ds_x \right) d\tau \end{aligned} \quad (16)$$

From (15) it follows that

$$\begin{aligned} & \sum_{k=1}^N \left\{ aC_k'''(t) (Z_k(x), Z_l(x))_{L^2(\Omega)} + \beta C_k'''(t) (Z_k(x), Z_l(x))_{L^2(\Omega)} \right. \\ & + c^2 C_k(t) (\nabla Z_k(x), \nabla Z_l(x))_{L^2(\Omega)} \\ & + bC_k'(t) (\nabla Z_k(x), \nabla Z_l(x))_{L^2(\Omega)} \\ & + \left. \left(\int_0^t h(t-\sigma) C_k(\sigma) d\sigma \right) (\nabla Z_k(x), \nabla Z_l(x))_{L^2(\Omega)} \right\} \\ & = (F(x, t), Z_l(x))_{L^2(\Omega)} + c^2 \sum_{k=1}^N \int_0^t C_k(\tau) \left(\int_{\partial\Omega} Z_l(x) \int_{\Omega} Z_k(\xi) d\xi ds_x \right) d\tau \\ & + b \sum_{k=1}^N \int_0^t C_k'(\tau) \left(\int_{\partial\Omega} Z_l(x) \int_{\Omega} Z_k(\xi) d\xi ds_x \right) d\tau \\ & + \sum_{k=1}^N \int_0^t \left(\int_0^\tau h(\tau-\sigma) C_k(\sigma) d\sigma \right) \left(\int_{\partial\Omega} Z_l(x) \int_{\Omega} Z_k(\xi) d\xi ds_x \right) d\tau \end{aligned} \quad (17)$$

Let

$$(Z_k, Z_l)_{L^2(\Omega)} = \delta_{kl} = \begin{cases} 1, & k = l \\ 0, & k \neq l \end{cases},$$

$$(\nabla Z_k, \nabla Z_l)_{L^2(\Omega)} = \gamma_{kl}, \quad (18)$$

$$\int_{\partial\Omega} Z_l(x) \int_{\Omega} Z_k(\xi) d\xi ds = \chi_{kl},$$

$$(F(x, t), Z_l(x))_{L^2(\Omega)} = F_l(t).$$

Then, (17) can be written as

$$\begin{aligned} & \sum_{k=1}^N \left\{ a\delta_{kl} C_k'''(t) + \beta\delta_{kl} C_k'''(t) + c^2 C_k(t) \gamma_{kl} \right. \\ & + \left. \left(\int_0^t h(t-\sigma) C_k(\sigma) d\sigma \right) \gamma_{kl} + bC_k'(t) \gamma_{kl} \right. \\ & - \left. \int_0^t \left[c^2 C_k(\tau) \chi_{kl} - \left(\int_0^\tau h(\tau-\sigma) C_k(\sigma) d\sigma \right) \chi_{kl} \right] d\tau \right\} = F_l(t). \end{aligned} \quad (19)$$

By differentiating (two times) with respect to t , it gives

$$\begin{aligned} & \sum_{k=1}^N \left\{ a\delta_{kl} C_k'''(t) + \beta\delta_{kl} C_k'''(t) + c^2 C_k''(t) \gamma_{kl} + h(0) C_k'(t) \gamma_{kl} \right. \\ & + \left. bC_k''(t) \gamma_{kl} - c^2 C_k(t) \chi_{kl} - bC_k''(t) \chi_{kl} - h(0) C_k(t) \chi_{kl} \right\} = F_l''(t). \end{aligned} \quad (20)$$

$$\sum_{k=1}^N \left\{ a\delta_{kl} C_k'''(0) + \beta\delta_{kl} C_k'''(0) \right\}. \quad (21)$$

We find a system of differential equations of fifth order with respect to t , constant coefficients, and the initial conditions (21). Hence, we obtain a Cauchy problem of linear differential equations with smooth coefficients that is uniquely solvable. Thus, $\forall n, \exists u^N(x)$ satisfying (15).

Now, we prove that u^N is sequence bounded. To do this, we multiply each equation of (15) by the appropriate $C_k'(t)$ summing over k from 1 to N . Hence, by integration the result equality with respect to t from 0 to τ , and $\tau \leq T$, it gives

$$\begin{aligned} & a(u_{ttt}^N, u_t^N)_{L^2(Q_\tau)} + \beta(u_{ttt}^N, u_t^N)_{L^2(Q_\tau)} + c^2(\nabla u^N, \nabla u_t^N)_{L^2(Q_\tau)} \\ & + b(\nabla u_t^N, \nabla u_t^N)_{L^2(Q_\tau)} + (\nabla w^N, \nabla u_t^N)_{L^2(Q_\tau)} \\ & = (F, u_t^N)_{L^2(Q_\tau)} + c^2 \int_0^\tau \int_0^\tau u_t^N(x, t) \left(\int_0^t \int_{\Omega} (\xi, \eta) d\xi d\eta \right) ds_x dt \\ & + b \int_0^\tau \int_{\partial\Omega} u_t^N(x, t) \left(\int_0^t \int_{\Omega} u_t^N(\xi, \eta) d\xi d\eta \right) ds_x dt \\ & + \int_0^\tau \int_{\partial\Omega} u_t^N(x, t) \left(\int_0^t \int_{\Omega} w^N(\xi, \eta) d\xi d\eta \right) ds_x dt. \end{aligned} \quad (22)$$

Evaluation of the terms on the LHS of (22) gives

$$\begin{aligned} a(u_{tt}^N, u_t^N)_{L^2(Q_\tau)} &= a(u_{\tau\tau}^N, u_\tau^N)_{L^2(\Omega)} - a(u_{tt}^N(x, 0), u_t^N(x, 0))_{L^2(\Omega)} \\ &\quad - a \int_0^\tau \|u_{tt}(x, t)\|_{L^2(\Omega)}^2 dt, \end{aligned} \quad (23)$$

$$\beta(u_{tt}^N, u_t^N)_{L^2(Q_\tau)} = \frac{\beta}{2} \|u_\tau^N(x, \tau)\|_{L^2(\Omega)}^2 - \frac{\beta}{2} \|u_t^N(x, 0)\|_{L^2(\Omega)}^2, \quad (24)$$

$$\begin{aligned} c^2(\nabla u^N, \nabla u_t^N)_{L^2(Q_\tau)} &= \frac{c^2}{c} \|\nabla u^N(x, \tau)\|_{L^2(\Omega)}^2 \\ &\quad - \frac{c^2}{c} \|\nabla u^N(x, 0)\|_{L^2(\Omega)}^2, \end{aligned} \quad (25)$$

$$b(\nabla u_t^N, \nabla u_t^N)_{L^2(Q_\tau)} = b \int_0^\tau \|\nabla u_t^N(x, t)\|_{L^2(\Omega)}^2 dt, \quad (26)$$

$$\begin{aligned} (\nabla w^N, \nabla u_t^N)_{L^2(Q_\tau)} &= \frac{1}{2} h \circ \nabla u^N(\tau) - \frac{1}{2} G(\tau) \|\nabla u^N(x, \tau)\|_{L^2(\Omega)}^2 \\ &\quad - \frac{1}{2} \int_0^\tau h'^N(t) dt \\ &\quad + \frac{1}{2} \int_0^\tau h(t) \|\nabla u^N(x, \tau)\|_{L^2(\Omega)}^2 dt, \end{aligned} \quad (27)$$

$$\begin{aligned} c^2 \int_0^\tau \int_{\partial\Omega} u_t^N \left(\int_0^t \int_{\Omega} u^N(\xi, \eta) d\xi d\eta \right) ds_x dt \\ = c^2 \int_{\partial\Omega} u^N(x, \tau) \int_0^\tau \int_{\Omega} u^N(\xi, t) d\xi dt ds_x \\ - c^2 \int_{\partial\Omega} \int_0^\tau u^N(x, t) \int_{\Omega} u^N(\xi, t) d\xi dt ds_x, \end{aligned} \quad (28)$$

So,

$$\begin{aligned} b \int_0^\tau \int_{\partial\Omega} u_t^N \left(\int_0^t \int_{\Omega} u_t^N(\xi, \eta) d\xi d\eta \right) ds_x dt \\ = b \int_{\partial\Omega} \int_0^\tau u_t^N(x, t) \int_{\Omega} u^N(\xi, t) d\xi dt ds_x \\ - b \int_{\partial\Omega} \int_0^\tau u_t^N(x, t) \int_{\Omega} u^N(\xi, 0) d\xi dt ds_x. \end{aligned} \quad (29)$$

Thus,

$$\begin{aligned} \int_0^\tau \int_{\partial\Omega} u_t^N \left(\int_0^t \int_{\Omega} w^N(\xi, \eta) d\xi d\eta \right) ds_x dt \\ = \int_{\partial\Omega} u^N(x, \tau) \int_0^\tau \int_{\Omega} u^N(\xi, t) d\xi dt ds_x \\ - \int_{\partial\Omega} \int_0^\tau u^N(x, t) \int_{\Omega} w^N(\xi, t) d\xi dt ds_x. \end{aligned} \quad (30)$$

Taking into account the equalities (23)–(30) in (22), we end up with

$$\begin{aligned} a(u_{\tau\tau}^N, u_\tau^N)_{L^2(\Omega)} + \frac{\beta}{2} \|u_\tau^N(x, \tau)\|_{L^2(\Omega)}^2 \\ + \frac{c^2}{2} \|\nabla u^N(x, \tau)\|_{L^2(\Omega)}^2 - \frac{1}{2} h \circ \nabla u^N(\tau) \\ - \frac{1}{2} G(\tau) \|\nabla u^N(x, \tau)\|_{L^2(\Omega)}^2 \\ = a(u_{tt}^N(x, 0), u_t^N(x, 0))_{L^2(\Omega)} + \frac{\beta}{2} \|u_\tau^N(x, 0)\|_{L^2(\Omega)}^2 \\ + \frac{c^2}{2} \|\nabla u^N(x, 0)\|_{L^2(\Omega)}^2 + a \int_0^\tau \|u_{tt}(x, t)\|_{L^2(\Omega)}^2 dt \\ + b \int_0^\tau \|\nabla u_t^N(x, t)\|_{L^2(\Omega)}^2 dt + \frac{1}{2} \int_0^\tau h'^N(t) dt \\ - \frac{1}{2} \int_0^\tau h(t) \|\nabla u^N(x, t)\|_{L^2(\Omega)}^2 dt \\ + c^2 \int_{\partial\Omega} u^N(x, \tau) \int_0^\tau \int_{\Omega} u^N(\xi, t) d\xi dt ds_x \\ - c^2 \int_{\partial\Omega} \int_0^\tau u^N(x, t) \int_{\Omega} u^N(\xi, t) d\xi dt ds_x \\ + b \int_{\partial\Omega} \int_0^\tau u_t^N(x, t) \int_{\Omega} u^N(\xi, t) d\xi dt ds_x \\ - b \int_{\partial\Omega} \int_0^\tau u_t^N(x, t) \int_{\Omega} u^N(\xi, 0) d\xi dt ds_x \\ + \int_{\partial\Omega} u^N(x, \tau) \int_0^\tau \int_{\Omega} u^N(\xi, t) d\xi dt ds_x \\ - \int_{\partial\Omega} \int_0^\tau u^N(x, t) \int_{\Omega} w^N(\xi, t) d\xi dt ds_x + (F, u_t^N)_{L^2(Q_\tau)} \end{aligned} \quad (31)$$

Now, multiplying the equations of (15) by $C_k''(t)$, collect them from 1 to N and integrating the result with respect to t from 0 to τ , and $\tau \leq T$, we find

$$\begin{aligned} a(u_{ttt}^N, u_{tt}^N)_{L^2(Q_\tau)} + \beta(u_{tt}^N, u_{tt}^N)_{L^2(Q_\tau)} + c^2(\nabla u^N, \nabla u_{tt}^N)_{L^2(Q_\tau)} \\ + b(\nabla u_t^N, \nabla u_{tt}^N)_{L^2(Q_\tau)} + (\nabla w^N, \nabla u_{tt}^N)_{L^2(Q_\tau)} \\ = (F, u_{tt}^N)_{L^2(Q_\tau)} + c^2 \int_0^\tau \int_{\partial\Omega} u_{tt}^N(x, t) \left(\int_0^t \int_{\Omega} u^N(\xi, \eta) d\xi d\eta \right) ds_x dt \\ + b \int_0^\tau \int_{\partial\Omega} u_{tt}^N(x, t) \left(\int_0^t \int_{\Omega} u_\eta^N(\xi, \eta) d\xi d\eta \right) ds_x dt \\ - \int_0^\tau \int_{\partial\Omega} u_{tt}^N(x, t) \left(\int_0^t \int_{\Omega} w^N(\xi, \eta) d\xi d\eta \right) ds_x dt \end{aligned} \quad (32)$$

With the same reasoning in (22), we find

$$a(u_{ttt}^N, u_{tt}^N)_{L^2(Q_\tau)} = \frac{a}{2} \|u_{\tau\tau}^N(x, \tau)\|_{L^2(\Omega)}^2 - \frac{a}{2} \|u_{tt}^N(x, 0)\|_{L^2(\Omega)}^2, \quad (33)$$

$$\beta(u_{tt}^N, u_{tt}^N)_{L^2(Q_\tau)} = \beta \int_0^\tau \|u_{tt}^N(x, t)\|_{L^2(\Omega)}^2 dt, \quad (34)$$

$$\begin{aligned} c^2(\nabla u^N, \nabla u_{tt}^N)_{L^2(Q_\tau)} &= c^2(\nabla u^N(x, \tau), \nabla u_\tau^N(x, \tau))_{L^2(Q_\tau)} \\ &\quad - c^2(\nabla u^N(x, 0), \nabla u_\tau^N(x, 0))_{L^2(\Omega)} \\ &\quad - c^2 \int_0^\tau \|\nabla u_t^N(x, t)\|_{L^2(\Omega)}^2 dt, \end{aligned} \quad (35)$$

$$\begin{aligned} b(\nabla u_t^N, \nabla u_{tt}^N)_{L^2(Q_\tau)} &= \frac{b}{2} \|\nabla u_\tau^N(x, \tau)\|_{L^2(\Omega)}^2 \\ &\quad - \frac{b}{2} \|\nabla u_t^N(x, 0)\|_{L^2(\Omega)}^2, \end{aligned} \quad (36)$$

$$\begin{aligned} (\nabla w^N, \nabla u_{tt}^N)_{L^2(Q_\tau)} &= \frac{1}{2} \left\{ h'^N(\tau) \|\nabla u^N(x, \tau)\|_{L^2(\Omega)}^2 - 2(\nabla w^N(\tau), \nabla u_\tau^N)_{L^2(\Omega)} \right\} \\ &\quad + \frac{1}{2} \int_0^\tau h''^N(t) dt - \frac{1}{2} \int_0^\tau h'(t) \|\nabla u^N(x, t)\|_{L^2(\Omega)}^2 dt, \end{aligned} \quad (37)$$

$$\begin{aligned} c^2 \int_0^\tau \int_{\partial\Omega} u_{tt}^N \left(\int_0^\tau \int_\Omega u^N(\xi, \eta) d\xi d\eta \right) ds_x dt \\ = c^2 \int_{\partial\Omega} u_\tau^N(x, \tau) \int_0^\tau \int_\Omega u^N(\xi, t) d\xi dt ds_x \\ - c^2 \int_{\partial\Omega} \int_0^\tau u_t^N(x, t) \int_\Omega u^N(\xi, t) d\xi dt ds_x, \end{aligned} \quad (38)$$

$$\begin{aligned} b \int_0^\tau \int_{\partial\Omega} u_{tt}^N(x, t) \left(\int_0^t \int_\Omega u_\eta^N(\xi, \eta) d\xi d\eta \right) ds_x dt \\ = b \int_{\partial\Omega} u_\tau^N(x, \tau) \int_\Omega u^N(\xi, \tau) d\xi ds_x \\ - b \int_{\partial\Omega} u_\tau^N(x, \tau) \int_\Omega u^N(\xi, 0) d\xi ds_x \\ - b \int_{\partial\Omega} \int_0^\tau u_t^N(x, t) \int_\Omega u_t^N(\xi, t) d\xi dt ds_x. \end{aligned} \quad (39)$$

$$\begin{aligned} \int_0^\tau \int_{\partial\Omega} u_{tt}^N \left(\int_0^t \int_\Omega w^N(\xi, \eta) d\xi d\eta \right) ds_x dt \\ = \int_{\partial\Omega} u_\tau^N(x, \tau) \int_0^\tau \int_\Omega w^N(\xi, t) d\xi dt ds_x \\ - \int_{\partial\Omega} \int_0^\tau u_t^N(x, t) \int_\Omega w^N(\xi, t) d\xi dt ds_x. \end{aligned} \quad (40)$$

A substitution of equalities (33)–(40) in (22) gives

$$\begin{aligned} &\frac{a}{2} \|u_{\tau\tau}^N(x, \tau)\|_{L^2(\Omega)}^2 + \frac{b}{2} \|\nabla u_\tau^N(x, \tau)\|_{L^2(\Omega)}^2 \\ &\quad + c^2(\nabla u^N(x, \tau), \nabla u_\tau^N(x, \tau))_{L^2(\Omega)} \\ &\quad - \frac{1}{2} \left\{ h'^N(\tau) + h(\tau) \|\nabla u^N(x, \tau)\|_{L^2(\Omega)}^2 \right. \\ &\quad \left. - 2(\nabla w^N(\tau), \nabla u_\tau^N)_{L^2(\Omega)} \right\} \\ &= \frac{a}{2} \|u_{tt}^N(x, 0)\|_{L^2(\Omega)}^2 + c^2(\nabla u^N(x, 0), \nabla u_t^N(x, 0))_{L^2(\Omega)} \\ &\quad + \frac{b}{2} \|\nabla u_t^N(x, 0)\|_{L^2(\Omega)}^2 - \beta \int_0^\tau \|u_{tt}^N(x, t)\|_{L^2(\Omega)}^2 dt \\ &\quad + c^2 \int_0^\tau \|\nabla u_t(x, t)\|_{L^2(\Omega)}^2 dt - \frac{1}{2} \int_0^\tau h''^N(t) dt \\ &\quad + \frac{1}{2} \int_0^\tau h'(t) \|\nabla u^N(x, t)\|_{L^2(\Omega)}^2 dt \\ &\quad + c^2 \int_{\partial\Omega} u_\tau^N(x, \tau) \int_0^\tau \int_\Omega u^N(\xi, t) d\xi dt ds_x \\ &\quad - c^2 \int_{\partial\Omega} \int_0^\tau u_t^N(x, t) \int_\Omega u^N(\xi, t) d\xi dt ds_x \\ &\quad + b \int_{\partial\Omega} u_\tau^N(x, \tau) \int_\Omega u^N(\xi, \tau) d\xi ds_x \\ &\quad - b \int_{\partial\Omega} u_\tau^N(x, \tau) \int_\Omega u^N(\xi, 0) d\xi ds_x \\ &\quad - b \int_{\partial\Omega} \int_0^\tau u_t^N(x, t) \int_\Omega u_t^N(\xi, t) d\xi dt ds_x \\ &\quad \cdot \int_{\partial\Omega} u_\tau^N(x, t) \int_0^\tau \int_\Omega w^N(\xi, t) d\xi dt ds_x \\ &\quad - \int_{\partial\Omega} \int_0^\tau u_t^N(x, t) \int_\Omega w^N(\xi, t) d\xi dt ds_x + (F, u_{tt}^N)_{L^2(Q_\tau)}. \end{aligned} \quad (41)$$

Multiplying (32) by λ and using (41), we get

$$\begin{aligned} \lambda a(u_{\tau\tau}^N, u_\tau^N)_{L^2(\Omega)} + \frac{\lambda\beta}{2} \|u_\tau^N(x, \tau)\|_{L^2(\Omega)}^2 \\ + \frac{\lambda c^2}{2} \|\nabla u^N(x, \tau)\|_{L^2(\Omega)}^2 \frac{\lambda}{2} h \circ \nabla u^N(\tau) \\ - \frac{\lambda}{2} G(\tau) \|\nabla u^N(x, \tau)\|_{L^2(\Omega)}^2 \frac{a}{2} \|u_{\tau\tau}^N(x, \tau)\|_{L^2(\Omega)}^2 \\ + \frac{b}{2} \|\nabla u_\tau^N(x, \tau)\|_{L^2(\Omega)}^2 + c^2(\nabla u^N(x, \tau), \nabla u_\tau^N(x, \tau))_{L^2(\Omega)} \\ - \frac{1}{2} \left\{ h'^N(\tau) + h(\tau) \|\nabla u^N(x, \tau)\|_{L^2(\Omega)}^2 \right. \\ \left. - 2(\nabla w^N(\tau), \nabla u_\tau^N)_{L^2(\Omega)} \right\} \\ = \lambda(F, u_t^N)_{L^2(Q_\tau)} + (F, u_{tt}^N)_{L^2(Q_\tau)} \\ + \lambda a(u_{tt}^N(x, 0), u_t^N(x, 0))_{L^2(\Omega)} \\ + (\lambda a - \beta) \int_0^\tau \|u_{tt}(x, t)\|_{L^2(\Omega)}^2 dt \end{aligned}$$

$$\begin{aligned}
& + (c^2 - \lambda b) \int_0^\tau \|\nabla u_t^N(x, t)\|_{L^2(\Omega)}^2 dt + \frac{\lambda}{2} \int_0^\tau h'^N(t) dt \\
& - \frac{\lambda}{2} \int_0^\tau h(t) \|\nabla u^N(x, t)\|_{L^2(\Omega)}^2 dt \\
& + \lambda c^2 \int_{\partial\Omega} u^N(x, \tau) \int_0^\tau \int_\Omega u^N(\xi, t) d\xi dt ds_x \\
& + \frac{\lambda c^2}{2} \|\nabla u^N(x, 0)\|_{L^2(\Omega)}^2 \\
& - \lambda c^2 \int_{\partial\Omega} \int_0^\tau u^N(x, t) \int_\Omega u^N(\xi, t) d\xi dt ds_x \\
& + \frac{\lambda \beta}{2} \|u_t^N(x, 0)\|_{L^2(\Omega)}^2 \\
& + (\lambda b - c^2) \int_{\partial\Omega} \int_0^\tau u_t^N(x, t) \int_\Omega u^N(\xi, t) d\xi dt ds_x \\
& - \lambda b \int_{\partial\Omega} \int_0^\tau u_t^N(x, t) \int_\Omega u^N(\xi, 0) d\xi dt ds_x \\
& + \lambda \int_{\partial\Omega} u^N(x, \tau) \int_0^\tau \int_\Omega w^N(\xi, t) d\xi dt ds_x \\
& - \lambda \int_{\partial\Omega} \int_0^\tau u^N(x, t) \int_\Omega w^N(\xi, t) d\xi dt ds_x \\
& + \frac{a}{2} \|u_t^N(x, 0)\|_{L^2(\Omega)}^2 + c^2 (\nabla u^N(x, 0), \nabla u_t^N(x, 0))_{L^2(\Omega)} \\
& + \frac{b}{2} \|\nabla u_t^N(x, 0)\|_{L^2(\Omega)}^2 - \frac{1}{2} \int_0^\tau h''^N(t) dt \\
& + \frac{1}{2} \int_0^\tau h'(t) \|\nabla u^N(x, t)\|_{L^2(\Omega)}^2 dt \\
& + c^2 \int_{\partial\Omega} u_\tau^N(x, \tau) \int_0^\tau \int_\Omega u^N(\xi, t) d\xi dt ds_x \\
& + b \int_{\partial\Omega} u_\tau^N(x, \tau) \int_\Omega u^N(x, \tau) d\xi ds_x \\
& - b \int_{\partial\Omega} u_\tau^N(x, \tau) \int_\Omega u^N(\xi, 0) d\xi ds_x \\
& - b \int_{\partial\Omega} \int_0^\tau u_t^N(\xi, t) d\xi dt ds_x \int_{\partial\Omega} \\
& \cdot u_\tau^N(x, \tau) \int_0^\tau \int_\Omega w^N(\xi, t) d\xi dt ds_x \\
& - \int_{\partial\Omega} \int_0^\tau u_\tau^N(x, t) \int_\Omega w^N(\xi, t) d\xi dt ds_x. \tag{42}
\end{aligned}$$

where $0 < \lambda < 1$.

With the help of Cauchy and the trace inequalities, we can estimate all the terms in the RHS of (42) that gives

$$\begin{aligned}
& c^2 \lambda \int_{\partial\Omega} u^N(x, \tau) \int_0^\tau \int_\Omega u^N(\xi, t) d\xi dt ds_x \\
& \leq \frac{c^2 \lambda}{2 \varepsilon_1} \left(\varepsilon \|\nabla u^N(x, \tau)\|_{L^2(\Omega)}^2 + l(\varepsilon) \|u^N(x, \tau)\|_{L^2(\Omega)}^2 \right) \\
& + \frac{c^2 \lambda}{2} \varepsilon_1 T |\Omega| |\partial\Omega| \int_0^\tau \|u^N(x, t)\|_{L^2(\Omega)}^2 dt, \tag{43}
\end{aligned}$$

$$\begin{aligned}
& - c^2 \lambda \int_{\partial\Omega} \int_0^\tau u^N(x, t) \int_\Omega u^N(\xi, t) d\xi dt ds_x \\
& \leq \frac{c^2 \lambda}{2} \varepsilon \int_0^\tau \|\nabla u^N(x, t)\|_{L^2(\Omega)}^2 dt \\
& + \frac{c^2 \lambda}{2} (l(\varepsilon) + |\Omega| |\partial\Omega|) \int_0^\tau \|u^N(x, t)\|_{L^2(\Omega)}^2 dt, \tag{44}
\end{aligned}$$

$$\begin{aligned}
& (b\lambda - c^2) \int_{\partial\Omega} \int_0^\tau u_t^N(x, t) \int_\Omega u^N(\xi, \tau) d\xi dt ds_x \\
& \leq \frac{(b\lambda - c^2)}{2} \left(\varepsilon \int_0^\tau \|\nabla u_t^N(x, t)\|_{L^2(\Omega)}^2 dt \right. \\
& \quad \left. + l(\varepsilon) \int_0^\tau \|\nabla u_t^N(x, t)\|_{L^2(\Omega)}^2 dt \right) \\
& + \frac{(b\lambda - c^2)}{2} |\Omega| |\partial\Omega| \int_0^\tau \|u^N(x, t)\|_{L^2(\Omega)}^2 dt, \tag{45}
\end{aligned}$$

$$\begin{aligned}
& - b\lambda \int_{\partial\Omega} \int_0^\tau u_t^N(x, t) \int_\Omega u^N(\xi, 0) d\xi dt ds_x \\
& \leq \frac{b\lambda}{2} \left(\varepsilon \int_0^\tau \|\nabla u_t^N(x, t)\|_{L^2(\Omega)}^2 dt + l(\varepsilon) \int_0^\tau \|u_t^N(x, t)\|_{L^2(\Omega)}^2 dt \right) \\
& + \frac{b\lambda}{2} |\Omega| |\partial\Omega| T \|u^N(x, 0)\|_{L^2(\Omega)}^2, \tag{46}
\end{aligned}$$

$$\begin{aligned}
& c^2 \int_{\partial\Omega} u_\tau^N(x, \tau) \int_0^\tau \int_\Omega u^N(\xi, t) d\xi dt ds_x \\
& \leq \frac{c^2}{2} \left(\frac{\varepsilon}{\varepsilon_2} \|\nabla u_\tau^N(x, \tau)\|_{L^2(\Omega)}^2 + \frac{l(\varepsilon)}{\varepsilon_2} \|u_\tau^N(x, \tau)\|_{L^2(\Omega)}^2 \right) \\
& + \frac{c^2}{2} \varepsilon_2 |\Omega| |\partial\Omega| T \int_0^\tau \|u^N(x, t)\|_{L^2(\Omega)}^2 dt, \tag{47}
\end{aligned}$$

$$\begin{aligned}
& + b \int_{\partial\Omega} u_\tau^N(x, \tau) \int_\Omega u^N(\xi, \tau) d\xi ds_x \\
& \leq \frac{b}{2 \varepsilon_3} \left(\varepsilon \|\nabla u_\tau^N(x, \tau)\|_{L^2(\Omega)}^2 + l(\varepsilon) \|u_\tau^N(x, \tau)\|_{L^2(\Omega)}^2 \right) \\
& + \frac{b}{2} \varepsilon_3 |\Omega| |\partial\Omega| \|u^N(x, \tau)\|_{L^2(\Omega)}^2, \tag{48}
\end{aligned}$$

$$\begin{aligned}
& - b \int_{\partial\Omega} u_\tau^N(x, \tau) \int_\Omega u^N(\xi, 0) d\xi ds_x \\
& \leq \frac{b}{2 \varepsilon_4} \left(\varepsilon \|\nabla u_\tau^N(x, \tau)\|_{L^2(\Omega)}^2 + l(\varepsilon) \|u_\tau^N(x, \tau)\|_{L^2(\Omega)}^2 \right) \\
& + \frac{b}{2} \varepsilon_4 |\Omega| |\partial\Omega| \|u^N(x, \tau)\|_{L^2(\Omega)}^2, \tag{49}
\end{aligned}$$

$$\begin{aligned}
& - b \int_{\partial\Omega} u_\tau^N(x, t) \int_\Omega u_\tau^N(\xi, \tau) d\xi dt ds_x \\
& \leq \frac{b}{2} \int_0^\tau \|\nabla u_\tau^N(x, \tau)\|_{L^2(\Omega)}^2 dt \\
& + \frac{b}{2} (l(\varepsilon) |\Omega| |\partial\Omega|) \int_0^\tau \|u_\tau^N(x, t)\|_{L^2(\Omega)}^2 dt, \tag{50}
\end{aligned}$$

$$-\frac{\lambda a}{2} \|u_{\tau\tau}^N(x, \tau)\|_{L^2(\Omega)}^2 - \frac{\lambda a}{2} \|u_{\tau\tau}^N(x, \tau)\|_{L^2(\Omega)}^2 \leq \lambda a (u_{\tau\tau}^N(x, \tau), u_{\tau}^N(x, \tau))_{L^2(\Omega)}, \quad (51)$$

$$-\frac{c^2 \varepsilon_7}{2} \|\nabla u^N(x, \tau)\|_{L^2(\Omega)}^2 - \frac{c^2}{2\varepsilon_7} \|\nabla u_{\tau}^N(x, \tau)\|_{L^2(\Omega)}^2 \leq c^2 (u^N(x, \tau), \nabla u_{\tau}^N(x, \tau))_{L^2(\Omega)}, \quad (52)$$

$$\lambda a (u_{tt}^N(x, 0), u_{\tau}^N(x, 0)) \leq \frac{\lambda a}{2} \|u_{tt}^N(x, 0)\|_{L^2(\Omega)}^2 + \frac{\lambda a}{2} \|u_{\tau}^N(x, 0)\|_{L^2(\Omega)}^2, \quad (53)$$

$$c^2 (\nabla u^N(x, 0), u_{\tau}^N(x, 0))_{L^2(\Omega)} \leq \frac{c^2}{2} \|\nabla u^N(x, 0)\|_{L^2(\Omega)}^2 + \frac{c^2}{2} \|\nabla u_{\tau}^N(x, 0)\|_{L^2(\Omega)}^2. \quad (54)$$

$$-\frac{G(\tau)}{2} \left(\frac{1}{\varepsilon_8} + \frac{1}{\varepsilon_9} \right) \|\nabla u^N(x, \tau)\|_{L^2(\Omega)}^2 - \frac{\varepsilon_8}{2} h \circ \nabla u^N(\tau) - \frac{\varepsilon_9}{2} G(\tau) \|\nabla u^N(x, \tau)\|_{L^2(\Omega)}^2 \leq 2 (\nabla w^N(x, \tau), \nabla u_{\tau}^N(x, \tau))_{L^2(\Omega)}, \quad (55)$$

$$\int_{\partial\Omega} u_{\tau}^N(x, \tau) \int_0^{\tau} \int_{\Omega} w^N(\xi, t) d\xi dt ds_x \leq \left(\frac{1}{2\varepsilon_5} + \frac{1}{2\varepsilon_5'} \right) \left(\varepsilon \|\nabla u_{\tau}^N(x, \tau)\|_{L^2(\Omega)}^2 + l(\varepsilon) \|u_{\tau}^N(x, \tau)\|_{L^2(\Omega)}^2 \right) + |\Omega| |\partial\Omega| T \left(\frac{\varepsilon_5}{2} \int_0^{\tau} h \circ u^N(t) dt + \frac{\varepsilon_5'}{2} h_0 \int_0^{\tau} \|u^N(x, t)\|_{L^2(\Omega)}^2 dt \right), \quad (56)$$

$$\lambda \int_{\partial\Omega} u^N(x, \tau) \int_0^{\tau} \int_{\Omega} w^N(\xi, t) d\xi dt ds_x \leq \lambda \left(\frac{1}{2\varepsilon_6} + \frac{1}{2\varepsilon_6'} \right) \left(\varepsilon \|\nabla u^N(x, \tau)\|_{L^2(\Omega)}^2 + l(\varepsilon) \|u^N(x, \tau)\|_{L^2(\Omega)}^2 \right) + \lambda |\Omega| |\partial\Omega| T \left(\frac{\varepsilon_6}{2} \int_0^{\tau} h \circ u^N(t) dt + \frac{\varepsilon_6'}{2} h_0 \int_0^{\tau} \|u^N(x, \tau)\|_{L^2(\Omega)}^2 dt \right), \quad (57)$$

$$\int_{\partial\Omega} \int_0^{\tau} u_t^N(x, \tau) \int_{\Omega} w^N(\xi, t) d\xi dt ds_x \leq \left(\varepsilon \int_0^{\tau} \|\nabla u_t^N(x, t)\|_{L^2(\Omega)}^2 dt + l(\varepsilon) \int_0^{\tau} \|\nabla u_t^N(x, t)\|_{L^2(\Omega)}^2 dt \right) + \frac{1}{2} |\Omega| |\partial\Omega| T \left(\int_0^{\tau} h \circ u^N(t) dt + h_0 \int_0^{\tau} \|u^N(x, t)\|_{L^2(\Omega)}^2 dt \right), \quad (58)$$

$$\lambda \int_{\partial\Omega} \int_0^{\tau} u^N(x, t) \int_{\Omega} u^N(\xi, t) d\xi dt ds_x \leq \lambda \left(\varepsilon \int_0^{\tau} \|\nabla u^N(x, t)\|_{L^2(\Omega)}^2 dt + l(\varepsilon) \int_0^{\tau} \|u^N(x, t)\|_{L^2(\Omega)}^2 dt \right) + \frac{\lambda}{2} |\Omega| |\partial\Omega| T \left(\int_0^{\tau} h \circ u^N(t) dt + h_0 \int_0^{\tau} \|u^N(x, t)\|_{L^2(\Omega)}^2 dt \right), \quad (59)$$

$$\lambda (F, u_{\tau}^N)_{L^2(Q_{\tau})} \leq \frac{\lambda}{2} \int_0^{\tau} \|F(x, t)\|_{L^2(\Omega)}^2 dt + \frac{1}{2} \int_0^{\tau} \|u_{tt}(x, t)\|_{L^2(\Omega)}^2 dt. \quad (60)$$

Combining inequalities (45)-(60) and equality (44) and make use of the following inequality

$$m_1 \|u^N(x, \tau)\|_{L^2(\Omega)}^2 \leq m_1 \|u^N(x, t)\|_{L^2(\Omega)}^2 + m_1 \|u_{\tau}^N(x, t)\|_{L^2(\Omega)}^2 + m_1 \|u^N(x, 0)\|_{L^2(\Omega)}^2,$$

$$m_2 \|u_t^N(x, \tau)\|_{L^2(\Omega)}^2 \leq m_2 \|u_t^N(x, t)\|_{L^2(\Omega)}^2 + m_2 \|u_{tt}^N(x, t)\|_{L^2(\Omega)}^2 + m_2 \|u_t^N(x, 0)\|_{L^2(\Omega)}^2,$$

$$m_3 \|\nabla u^N(x, \tau)\|_{L^2(\Omega)}^2 \leq m_3 \|\nabla u^N(x, t)\|_{L^2(Q_{\tau})}^2 + m_3 \|\nabla u_t^N(x, t)\|_{L^2(Q_{\tau})}^2 + m_3 \|\nabla u^N(x, 0)\|_{L^2(\Omega)}^2,$$

$$h \circ u^N(\tau) \leq \frac{1}{2} h \circ u^N(t) dt + \frac{h_0}{2} \|u_t^N(x, t)\|_{L^2(Q_{\tau})}^2 - \int_0^{\tau} h'^N(t) dt,$$

$$h \circ \nabla u^N(\tau) \leq \frac{1}{2} \int_0^{\tau} h \circ \nabla u^N(t) dt + \frac{h_0}{2} \|\nabla u_t^N(x, t)\|_{L^2(Q_{\tau})}^2 - \int_0^{\tau} h'^N(t) dt, \quad (61)$$

where

$$m_1 := \frac{c^2 \lambda}{\varepsilon_1} l(\varepsilon) + \frac{b}{2} \varepsilon_3 |\Omega| |\partial\Omega| + \lambda \left(\frac{1}{2\varepsilon_6} + \frac{1}{2\varepsilon_6'} \right) l(\varepsilon),$$

$$m_2 := l(\varepsilon) \left(\frac{c^2}{2\varepsilon_2} + \frac{b}{2\varepsilon_2} + \frac{b}{2\varepsilon_4} + \left(\frac{1}{2\varepsilon_5} + \frac{1}{2\varepsilon_5'} \right) \right),$$

$$m_3 := \frac{c^2 \varepsilon_7}{2} + h(0) + \frac{c^2 \lambda}{2\varepsilon_1} + \frac{c^2 \varepsilon_9}{2} + \lambda \left(\frac{1}{2\varepsilon_6} + \frac{1}{2\varepsilon_6'} \right) \varepsilon, \quad (62)$$

and we have

$$\begin{aligned}
& \frac{c^2 \lambda}{2\varepsilon_1} l(\varepsilon) \|u^N(x, \tau)\|_{L^2(\Omega)}^2 + \frac{\lambda}{2} (\beta - a) \|u_t^N(x, \tau)\|_{L^2(\Omega)}^2 \\
& + \frac{\lambda}{2} (c^2 - G(\tau)) \|\nabla u^N(x, \tau)\|_{L^2(\Omega)}^2 \\
& + \left\{ \frac{b}{2} - \frac{c^2 \varepsilon}{2\varepsilon_2} - \frac{b\varepsilon}{2\varepsilon_3} - \frac{b\varepsilon}{2\varepsilon_4} - \frac{c^2}{2\varepsilon_7} - \frac{\varepsilon}{2\varepsilon_5} \right. \\
& \left. - \frac{\varepsilon}{2\varepsilon_5'} - \frac{G(\tau)}{2} \left(\frac{1}{\varepsilon_8} + \frac{1}{\varepsilon_9} \right) \right\} \|\nabla u_\tau^N(x, \tau)\|_{L^2(\Omega)}^2 \\
& + \left(\frac{a}{2} - \frac{\lambda a}{2} \right) \|u_{\tau\tau}^N(x, \tau)\|_{L^2(\Omega)}^2 \\
& + \left(\frac{\lambda + 1}{2} \right) h \circ \nabla u^N(\tau) + h \circ u^N(\tau) \\
& \leq \gamma_1 \int_0^\tau \|u^N(x, t)\|_{L^2(\Omega)}^2 dt + \gamma_2 \int_0^\tau \|u_t^N(x, t)\|_{L^2(\Omega)}^2 dt \\
& + \underbrace{\left\{ \frac{c^2 \lambda}{2} \varepsilon + \lambda \varepsilon + m_3 \right\}}_{\alpha_1} \int_0^\tau \|u^N(x, t)\|_{L^2(\Omega)}^2 dt \\
& + \gamma_3 \int_0^\tau \|\nabla u_\tau^N(x, t)\|_{L^2(\Omega)}^2 dt + \underbrace{\left(\frac{\lambda + 3}{2} \right) \xi}_{\alpha_2} \int_0^\tau h \circ \nabla u^N(\tau) \\
& + \gamma_4 \int_0^\tau h \circ u^N(\tau) + \underbrace{\left\{ \frac{\lambda a + \beta + m_2}{2} \right\}}_{\alpha_3} \int_0^\tau \|u_{tt}^N(x, t)\|_{L^2(\Omega)}^2 dt \\
& + \underbrace{\left\{ \frac{b\lambda}{2} |\Omega| |\partial\Omega| T + \frac{b}{2} \varepsilon_4 |\Omega| |\partial\Omega| + m_1 \right\}}_{\alpha_4} \|u^N(x, 0)\|_{L^2(\Omega)}^2 \\
& + \underbrace{\left\{ \frac{\lambda a}{2} + \frac{\lambda \beta}{2} + m_2 \right\}}_{\alpha_5} \|u_t^N(x, 0)\|_{L^2(\Omega)}^2 \\
& + \left(\frac{\lambda + 1}{2} \right) \int_0^\tau \|F(x, t)\|_{L^2(\Omega)}^2 dt \\
& + \underbrace{\left\{ \frac{\lambda}{2} c^2 + \frac{c^2}{2} + \frac{1}{2} h(0) + m_3 \right\}}_{\alpha_6} \|\nabla u^N(x, 0)\|_{L^2(\Omega)}^2 \\
& + \underbrace{\left\{ \frac{b}{2} + \frac{c^2}{2} \right\}}_{\alpha_7} \|\nabla u_t^N(x, 0)\|_{L^2(\Omega)}^2 \\
& + \underbrace{\left\{ \frac{\lambda a}{2} + \frac{a}{2} \right\}}_{\alpha_7} \|\nabla u_{tt}^N(x, 0)\|_{L^2(\Omega)}^2,
\end{aligned} \tag{63}$$

where

$$\begin{aligned}
\gamma_1 := & \left\{ \frac{c^2 \lambda}{2} \varepsilon_1 T |\partial\Omega| |\Omega| + \frac{c^2 \lambda}{2} (l(\varepsilon) + |\partial\Omega| |\Omega|) \right. \\
& + \frac{(b\lambda + c^2)}{2} |\partial\Omega| |\Omega| + \left(\frac{c^2}{2} \varepsilon_2 + \frac{c^2}{2} \varepsilon_5' h_0 + \frac{\lambda + 1}{2} \right) \\
& \left. \cdot |\partial\Omega| |\Omega| T + \lambda l(\varepsilon) + m_1 \right\},
\end{aligned}$$

$$\begin{aligned}
\gamma_2 := & \left\{ \frac{(b\lambda + c^2)}{2} l(\varepsilon) + \frac{b\lambda}{2} l(\varepsilon) + \frac{b}{2} (l(\varepsilon) + |\partial\Omega| |\Omega|) + l(\varepsilon) \right. \\
& \left. + \frac{h_0}{2\xi} + \frac{\lambda}{2} + m_2 \right\},
\end{aligned} \tag{64}$$

$$\begin{aligned}
\gamma_3 := & \left\{ c^2 + \lambda b + \frac{(b\lambda + c^2)}{2} \varepsilon + \frac{b(\lambda + 1)}{2} \varepsilon \right. \\
& \left. + \lambda \varepsilon + \frac{h_0 \lambda}{2} + \frac{h_0}{2\xi} + m_3 \right\},
\end{aligned}$$

$$\gamma_4 := \left\{ \left(\frac{\varepsilon_5}{2} + \frac{\lambda + 1}{2} \right) |\partial\Omega| |\Omega| T + \frac{3\xi}{2} \right\}. \tag{65}$$

Choosing $\varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_5', \varepsilon_7, \varepsilon_8$ and ε_9 sufficiently large

$$\begin{aligned}
\alpha_0 = & \frac{b}{2} - \frac{c^2 \varepsilon}{2\varepsilon_2} - \frac{b\varepsilon}{2\varepsilon_3} - \frac{b\varepsilon}{2\varepsilon_4} - \frac{c^2}{2\varepsilon_7} - \frac{\varepsilon}{2\varepsilon_5} \\
& - \frac{\varepsilon}{2\varepsilon_5'} - \frac{G(\tau)}{2} \left(\frac{1}{\varepsilon_8} + \frac{1}{\varepsilon_9} \right) > 0.
\end{aligned} \tag{66}$$

By using (2)-(4), the relation (64) reduces to

$$\begin{aligned}
& \|u^N(x, \tau)\|_{L^2(\Omega)}^2 + \|\nabla u^N(x, \tau)\|_{L^2(\Omega)}^2 \\
& + \|u_\tau^N(x, \tau)\|_{L^2(\Omega)}^2 + \|\nabla u_\tau^N(x, \tau)\|_{L^2(\Omega)}^2 \\
& + \|u_{\tau\tau}^N(x, \tau)\|_{L^2(\Omega)}^2 + h \circ u^N(\tau) + h \circ \nabla u^N(\tau) \\
& \leq D \int_0^\tau \left\{ \|u^N(x, t)\|_{L^2(\Omega)}^2 + \|\nabla u^N(x, t)\|_{L^2(\Omega)}^2 \right. \\
& + \|u_t^N(x, t)\|_{L^2(\Omega)}^2 + \|\nabla u_t^N(x, t)\|_{L^2(\Omega)}^2 + \|u_{tt}^N(x, t)\|_{L^2(\Omega)}^2 \\
& + h \circ u^N(t) + h \circ \nabla u^N(t) + \|F\|_{L^2(\Omega)}^2 \Big\} dt \\
& + D \left(\|u^N(x, 0)\|_{W_2^1(\Omega)}^2 + \|u_t^N(x, 0)\|_{W_2^1(\Omega)}^2 \right. \\
& \left. + \|u_{tt}^N(x, 0)\|_{L^2(\Omega)}^2 \right), \\
& \leq D \int_0^\tau \left\{ \|u^N(x, t)\|_{L^2(\Omega)}^2 + \|u_{tt}^N(x, t)\|_{L^2(\Omega)}^2 \right. \\
& + h \circ u^N(t) + h \circ \nabla u^N(t) + \|F\|_{L^2(\Omega)}^2 \Big\} dt \\
& + D \left(\|u^N(x, 0)\|_{W_2^1(\Omega)}^2 + \|u_t^N(x, 0)\|_{W_2^1(\Omega)}^2 \right. \\
& \left. + \|u_{tt}^N(x, 0)\|_{L^2(\Omega)}^2 + h \circ u^N(0) + h \circ \nabla u^N(0) \right),
\end{aligned} \tag{67}$$

where

$$D := \frac{\max \{\alpha_i, i = 1 \dots 8\}}{\min \{(c^2 \lambda / 2 \varepsilon_1) l(\varepsilon), (\lambda / 2)(\beta - a), (\lambda / 2)(c^2 - G(\tau)), (a/2)(1 - \lambda), (\lambda + 1/2), 1, \alpha_0\}}. \quad (68)$$

Using the inequality of Gronwall to (67) and integrating the result from 0 to τ that gives

$$\begin{aligned} & \|u^N(x, t)\|_{W_2^1(Q_\tau)}^2 + \|u_t^N(x, t)\|_{W_2^1(Q_\tau)}^2 \\ & + \|u^N(x, t)\|_{h, W_2^1(Q_\tau)}^2 + \|u_{tt}^N(x, t)\|_{L^2(Q_\tau)}^2 \\ & \leq De^{DT} \left\{ \|u_0(x)\|_{W_2^1(\Omega)}^2 + \|u_1(x)\|_{W_2^1(\Omega)}^2 \right. \\ & \quad \left. + \|u_2(x)\|_{L^2(\Omega)}^2 + \|F\|_{L^2(\Omega)}^2 \right\}. \end{aligned} \quad (69)$$

where

$$\|u(x, t)\|_{h, W_2^1(Q_\tau)}^2 := \int_0^\tau h \circ u(t) dt + \int_0^\tau h \circ \nabla u(t) dt, \quad (70)$$

$$\|u(x, 0)\|_{h, W_2^1(Q_\tau)}^2 = 0.$$

We deduce from (69) that

$$\begin{aligned} & \|u^N(x, t)\|_{W_2^1(Q_\tau)}^2 + \|u_t^N(x, t)\|_{W_2^1(Q_\tau)}^2 \\ & + \|u^N(x, t)\|_{h, W_2^1(Q_\tau)}^2 + \|u_{tt}^N(x, t)\|_{L^2(Q_\tau)}^2 \leq A. \end{aligned} \quad (71)$$

Hence, $\{u^N\}_{N \geq 1}$ is sequence bounded in $V(Q_T)$, and we can extract from it a subsequence for which we use the same notation which converges weakly in $V(Q_T)$ to a limit function $u(x, t)$, and we have to show that $u(x, t)$ is a generalized solution of (1). Since $u^N(x, t) \rightharpoonup u(x, t)$ in $L^2(Q_T)$ and $u^N(x, 0) \rightharpoonup \zeta(x)$ in $L^2(\Omega)$, then $u(x, 0) = \zeta(x)$.

Now to prove that (15) holds, we multiply each of the relations (15) by a function $p_l(t) \in W_2^1(0, T)$, $p_l(T) = 0$. Hence, collect them the obtained equalities ranging from $l = 1$ to $l = N$ and integrating the result over t on $(0, T)$. If we let $\eta^N = \sum_{k=1}^N p_l(t) Z_k(x)$, then we have

$$\begin{aligned} & a(u_{tt}^N, \eta_t^N)_{L^2(Q_T)} + \beta(u_t^N, \eta_t^N)_{L^2(Q_T)} - c^2(\nabla u^N, \nabla \eta^N)_{L^2(Q_T)} \\ & - b(\nabla u_t^N, \nabla \eta^N)_{L^2(Q_T)} - (\nabla w^N, \nabla \eta^N)_{L^2(Q_T)} \\ & = a(u_{tt}^N(x, 0), \eta^N(0))_{L^2(\Omega)} + \beta(u_t^N(x, 0), \eta^N(0))_{L^2(\Omega)} \\ & + c^2 \int_{\partial\Omega} \int_0^T \eta^N(x, t) \left(\int_0^t u^N(\xi, \tau) d\xi d\tau \right) dt ds_x \\ & + b \int_{\partial\Omega} \int_0^T \eta^N(x, t) \int_{\Omega} u^N(\xi, \tau) d\xi dt ds_x \\ & - b \int_{\partial\Omega} \int_0^T \eta^N(x, t) \int_{\Omega} u^N(\xi, 0) d\xi dt ds_x \end{aligned}$$

$$\begin{aligned} & + \int_{\partial\Omega} \int_0^T \eta^N(x, t) \left(\int_0^t \int_{\Omega} w^N(\xi, \tau) d\xi d\tau \right) dt ds_x \\ & + (F, \eta_t^N)_{L^2(Q_T)}, \end{aligned} \quad (72)$$

for all η^N of the form $\sum_{k=1}^N p_l(t) Z_k(x)$ and $\alpha > 0$.
Since

$$\begin{aligned} & \int_0^t \int_{\Omega} ((u^N(\xi, \tau) - u(\xi, \tau)) d\xi d\tau) \leq \sqrt{T|\Omega|} \|u^N - u\|_{L^2(Q_T)}, \\ & \|u^N - u\|_{L^2(Q_T)} \rightarrow 0 \text{ as } N \rightarrow \infty. \end{aligned} \quad (73)$$

Thus, the limit function u satisfies (15) for every $\eta^N = \sum_{k=1}^N p_l(t) Z_k(x)$.

We define the totality of all functions of the form $\eta^N = \sum_{k=1}^N p_l(t) Z_k(x)$ by $\mathcal{Q}_N$, with $p_l(t) \in W_2^1(0, T)$, $p_l(T) = 0$.

But $\cup_{l=1}^N \mathcal{Q}_N$ is dense in $W(Q_T)$, hence the relation (15) holds $\forall u \in W(Q_T)$. Then, we have shown that the limit function $u(x, t)$ is a generalized solution of problem (1) in $V(Q_T)$.

4. Uniqueness of the Problem

Theorem 3. *The problem (1) cannot have more than one generalized solution in $V(Q_T)$.*

Proof. Suppose that $\exists_{u_1, u_2} \in V(Q_T)$ two different generalized solutions for the problem (1). Hence, the difference $U = u_1 - u_2$ solves

$$\begin{cases} aU_{ttt} + \beta U_{tt} - c^2 \Delta U - b \Delta U_t - \int_0^t h(t-s) \Delta U(s) ds = 0, \\ U(x, 0) = U_t(x, 0) = U_{tt}(x, 0) = 0, \\ \frac{\partial U}{\partial \eta} = \int_0^t \int_{\Omega} U(\xi, \tau) d\xi d\tau, x \in \partial\Omega, \end{cases} \quad (74)$$

and (9) gives

$$\begin{aligned}
& a(U_{tt}, v_t)_{L^2(Q_T)} + \beta(U_t, v_t)_{L^2(Q_T)} + c^2(\nabla U, \nabla v)_{L^2(Q_T)} \\
& + b(\nabla U_t, \nabla v)_{L^2(Q_T)} + (\nabla W_t, \nabla v)_{L^2(Q_T)} \\
& = -c^2 \int_0^T \int_{\partial\Omega} v \left(\int_0^t U(\xi, \tau) d\xi d\tau \right) ds_x dt \\
& - b \int_0^T \int_{\partial\Omega} v \int_{\Omega} U(\xi, \tau) d\xi ds_x dt \\
& - \int_0^T \int_{\partial\Omega} v_t \left(\int_{\Omega} W(\xi, \tau) d\xi \right) ds_x dt.
\end{aligned} \tag{75}$$

where

$$W(x, t) = \int_0^t h(t - \sigma) U(x, \sigma) d\sigma. \tag{76}$$

Let the function

$$v(x, t) = \begin{cases} \int_t^\tau U(x, s) ds & 0 \leq t \leq \tau, \\ 0, & \tau \leq t \leq T. \end{cases} \tag{77}$$

It is obvious that $v \in W(Q_T)$ and $v_t(x, t) = -U(x, t)$ for all $t \in [0, \tau]$. By integration by parts in the LHS of (75) that yields

$$\begin{aligned}
-a(U_{tt}, v_t)_{L^2(Q_T)} &= a(U_\tau(x, \tau), U(x, \tau))_{L^2(\Omega)} \\
&- a \int_0^\tau \|U_t(x, t)\|_{L^2(\Omega)}^2 dt,
\end{aligned} \tag{78}$$

$$-\beta(U_t, v_t)_{L^2(Q_T)} = \frac{\beta}{2} \|U(x, \tau)\|_{L^2(\Omega)}^2, \tag{79}$$

$$c^2(\nabla U, \nabla v)_{L^2(Q_T)} = \frac{c^2}{2} \|\nabla v(x, 0)\|_{L^2(\Omega)}^2, \tag{80}$$

$$\begin{aligned}
(\nabla W, \nabla v)_{L^2(Q_T)} &\leq h_0 \int_0^\tau \|\nabla v(x, t)\|_{L^2(\Omega)}^2 dt \\
&+ \frac{h_0}{2} \int_0^\tau \|\nabla U(x, t)\|_{L^2(\Omega)}^2 dt \\
&+ \frac{h_0}{2} \int_0^\tau h_0 \nabla U(t) dt,
\end{aligned} \tag{81}$$

$$b(\nabla U_t, \nabla v)_{L^2(Q_T)} = b \int_0^\tau \|\nabla v_t(x, t)\|_{L^2(\Omega)}^2 dt. \tag{82}$$

Plugging (78)–(82) into (75), we obtain

$$\begin{aligned}
& a(U_\tau(x, \tau), U(x, \tau))_{L^2(\Omega)} + \frac{\beta}{2} \|U(x, \tau)\|_{L^2(\Omega)}^2 + \frac{c^2}{2} \|\nabla v(x, 0)\|_{L^2(\Omega)}^2 \\
& = a \int_0^\tau \|U_\tau(x, t)\|_{L^2(\Omega)}^2 dt - b \int_0^\tau \|v_t(x, t)\|_{L^2(\Omega)}^2 dt h_0 \\
& \quad \cdot \int_0^\tau \|\nabla v(x, t)\|_{L^2(\Omega)}^2 dt + \frac{h_0}{2} \int_0^\tau \|\nabla U(x, t)\|_{L^2(\Omega)}^2 dt \\
& \quad + \frac{h_0}{2} h_0 \nabla U(t) dt - \int_0^T \int_{\partial\Omega} v \left(\int_0^t W(\xi, \tau) d\xi d\tau \right) ds_x dt \\
& \quad - c^2 \int_0^T \int_{\partial\Omega} v \left(\int_0^t U(\xi, \tau) d\xi d\tau \right) ds_x dt \\
& \quad - b \int_0^T \int_{\partial\Omega} v \int_{\Omega} U(\xi, \tau) d\xi ds_x dt.
\end{aligned} \tag{83}$$

Now since

$$v^2(x, t) = \left(\int_t^\tau U(x, s) ds \right)^2 \leq \tau \int_0^\tau U^2(x, s) ds, \tag{84}$$

then

$$\|v\|_{L^2(Q_T)}^2 \leq \tau^2 \|U\|_{L^2(Q_T)}^2 \leq T^2 \|U\|_{L^2(Q_T)}^2. \tag{85}$$

Applying the inequality of the trace, the RHS of (83) gives

$$\begin{aligned}
& c^2 \int_0^T \int_{\partial\Omega} v \left(\int_0^t U(\xi, \tau) d\xi d\tau \right) ds_x dt \\
& \leq \frac{c^2}{2} T^2 \{l(\varepsilon) + |\Omega| |\partial\Omega|\} \int_0^\tau \|U(x, t)\|_{L^2(\Omega)}^2 dt \\
& \quad + \frac{c^2}{2} \varepsilon \int_0^\tau \|\nabla v(x, t)\|_{L^2(\Omega)}^2 dt,
\end{aligned} \tag{86}$$

$$\begin{aligned}
& b \int_0^T \int_{\partial\Omega} v \int_{\Omega} U(\xi, t) d\xi ds_x dt \\
& \leq \frac{b}{2} \{T^2 l(\varepsilon) + |\Omega| |\partial\Omega|\} \int_0^\tau \|U(x, t)\|_{L^2(\Omega)}^2 dt \\
& \quad + \frac{b}{2} \varepsilon \int_0^\tau \|\nabla v(x, t)\|_{L^2(\Omega)}^2 dt,
\end{aligned} \tag{87}$$

$$\begin{aligned}
& \int_0^T \int_{\partial\Omega} v \left(\int_0^t W(\xi, \tau) d\xi d\tau \right) ds_x dt \\
& \leq \frac{1}{2} T^2 \{l(\varepsilon) + |\Omega| |\partial\Omega|\} \int_0^\tau h \circ U(t) dt \\
& \quad + \frac{1}{2} T^2 h \circ \{l(\varepsilon) + |\Omega| |\partial\Omega|\} \int_0^\tau \|U(x, t)\|_{L^2(\Omega)}^2 dt \\
& \quad + \varepsilon \int_0^\tau \|\nabla v(x, t)\|_{L^2(\Omega)}^2 dt.
\end{aligned} \tag{88}$$

Combining the relations (86)–(83) and (87)–(88), we get

$$\begin{aligned}
 & a(U_\tau(x, \tau), U(x, \tau))_{L^2(\Omega)} + \frac{\beta}{2} \|U(x, \tau)\|_{L^2(\Omega)}^2 \\
 & + \frac{c^2}{2} \|\nabla v(x, 0)\|_{L^2(\Omega)}^2 \\
 & \leq \left\{ \frac{c^2}{2} T^2 (l(\varepsilon) + |\Omega| |\partial\Omega|) + \frac{(b + h_0)}{2} (T^2 l(\varepsilon) + |\Omega| |\partial\Omega|) \right\} \\
 & \cdot \int_0^\tau \|U(x, t)\|_{L^2(\Omega)}^2 dt + \left(\left(\frac{c^2 + b}{2} + 1 \right) \varepsilon + h(0) \right) \\
 & \cdot \int_0^\tau \|\nabla v(x, t)\|_{L^2(\Omega)}^2 dt + a \int_0^\tau \|U_t(x, t)\|_{L^2(\Omega)}^2 dt \\
 & + \frac{h_0}{2} \int_0^\tau \|\nabla U(x, t)\|_{L^2(\Omega)}^2 dt + \frac{h_0}{2} \int_0^\tau h \circ \nabla U(t) dt \\
 & + \frac{1}{2} T^2 (l(\varepsilon) + |\Omega| |\partial\Omega|) \int_0^\tau h \circ U(t) dt.
 \end{aligned} \tag{89}$$

Next, multiplying (74) by U_{tt} and integrating the result over $Q_\tau = \Omega \times (0, \tau)$, we find

$$\begin{aligned}
 & a(U_{ttt}, U_{tt})_{L^2(Q_\tau)} + \beta(U_{tt}, U_{tt})_{L^2(Q_\tau)} - c^2(\Delta U, U_{tt})_{L^2(Q_\tau)} \\
 & - b(\Delta U_t, U_{tt})_{L^2(Q_\tau)} - (\Delta W, U_{tt})_{L^2(Q_\tau)} = 0.
 \end{aligned} \tag{90}$$

An integration by parts in (91) yields

$$a(U_{ttt}, U_{tt})_{L^2(Q_\tau)} = \frac{a}{2} \|U_{\tau\tau}(x, \tau)\|_{L^2(\Omega)}^2, \tag{91}$$

$$\beta(U_{tt}, U_{tt})_{L^2(Q_\tau)} = \beta \int_0^\tau \|U_{tt}(x, t)\|_{L^2(\Omega)}^2 dt, \tag{92}$$

$$\begin{aligned}
 -c^2(\Delta U, U_{tt})_{L^2(Q_\tau)} &= c^2(\nabla U_\tau(x, \tau), \nabla U(x, \tau))_{L^2(\Omega)} \\
 &- c^2 \int_0^\tau \|\nabla U_t(x, t)\|_{L^2(\Omega)}^2 dt \\
 &- c^2 \int_{\partial\Omega} U_\tau(x, \tau) \left(\int_0^\tau \int_\Omega U(\xi, \eta) d\xi d\eta \right) ds_x \\
 &+ c^2 \int_{\partial\Omega} \int_0^\tau U_t(x, t) \int_\Omega U(\xi, t) d\xi dt ds_x,
 \end{aligned} \tag{93}$$

$$\begin{aligned}
 -b(\Delta U_t, U_{tt})_{L^2(Q_\tau)} &= \frac{b}{2} \|\nabla U_t(x, \tau)\|_{L^2(\Omega)}^2 \\
 &- b \int_{\partial\Omega} U_t(x, \tau) \int_\Omega U(\xi, \tau) d\xi ds_x \\
 &+ b \int_0^\tau \int_{\partial\Omega} U_t(x, t) \int_\Omega U_t(\xi, t) d\xi ds_x dt,
 \end{aligned} \tag{94}$$

$$\begin{aligned}
 (\nabla W, \nabla U_{tt})_{L^2(Q_\tau)} &= -\frac{1}{2} h' \circ \nabla U(\tau) - \frac{1}{2} h(\tau) \|\nabla U(x, \tau)\|_{L^2(Q_\tau)}^2 \\
 &+ 2(\nabla W(\tau), \nabla U_\tau)_{L^2(\Omega)} \\
 &- \frac{1}{2} \int_0^\tau h'(t) \|\nabla U(x, t)\|_{L^2(\Omega)}^2 dt \\
 &+ \frac{1}{2} \int_0^\tau h'' \circ \nabla U(t) dt \\
 &+ \int_{\partial\Omega} U_\tau(x, \tau) \int_0^\tau \int_\Omega W(\xi, \sigma) d\xi d\sigma ds_x \\
 &- \int_0^\tau \int_{\partial\Omega} U_t(x, t) \int_\Omega W(\xi, t) d\xi ds_x dt.
 \end{aligned} \tag{95}$$

Substitution (91)–(95) into (90), we get the equality

$$\begin{aligned}
 & \frac{a}{2} \|U_{\tau\tau}(x, \tau)\|_{L^2(\Omega)}^2 + c^2(\nabla U_\tau(x, \tau), \nabla U(x, \tau))_{L^2(\Omega)} \\
 & + \frac{b}{2} \|\nabla U_\tau(x, \tau)\|_{L^2(\Omega)}^2 - \frac{1}{2} h' \circ \nabla U(\tau) \\
 & - \frac{1}{2} h(\tau) \|\nabla U(x, \tau)\|_{L^2(\Omega)}^2 + 2(\nabla W(\tau), \nabla U_\tau)_{L^2(\Omega)} \\
 & = -\beta \int_0^\tau \|U_{tt}(x, t)\|_{L^2(\Omega)}^2 dt + c^2 \int_0^\tau \|\nabla U_t(x, t)\|_{L^2(\Omega)}^2 dt \\
 & + c^2 \int_{\partial\Omega} U_\tau(x, \tau) \left(\int_0^\tau \int_\Omega U(\xi, t) d\xi dt \right) ds_x \\
 & - c^2 \int_{\partial\Omega} \int_0^\tau U_t(x, t) \int_\Omega U(\xi, t) d\xi dt ds_x \\
 & + b \int_{\partial\Omega} U_\tau(x, \tau) \int_\Omega U(\xi, \tau) d\xi ds_x \\
 & - b \int_0^\tau \int_{\partial\Omega} U_t(x, t) \int_\Omega W(\xi, t) d\xi ds_x dt \\
 & + \int_{\partial\Omega} U_\tau(x, \tau) \int_0^\tau \int_\Omega W(\xi, \sigma) d\xi d\sigma ds_x \\
 & - \int_0^\tau \int_{\partial\Omega} U_t(x, t) \int_\Omega W(\xi, t) d\xi ds_x dt \\
 & + \frac{1}{2} \int_0^\tau h'(t) \|\nabla U(x, t)\|_{L^2(\Omega)}^2 dt \\
 & - \frac{1}{2} \int_0^\tau h'' \circ \nabla U(t) dt.
 \end{aligned} \tag{96}$$

The RHS of (96) can be bounded as follows

$$\begin{aligned}
 & c^2 \int_{\partial\Omega} U_\tau(x, \tau) \left(\int_0^T \int_\Omega U(\xi, t) d\xi dt \right) ds_x \\
 & \leq \frac{c^2}{2\varepsilon_1'} \left(\varepsilon \|\nabla U_\tau(\xi, \tau)\|_{L^2(\Omega)}^2 + l(\varepsilon) \|U_\tau(x, \tau)\|_{L^2(\Omega)}^2 \right) \\
 & + \frac{c^2}{2\varepsilon_1'} T |\partial\Omega| |\Omega| \int_0^T \|U(x, t)\|_{L^2(\Omega)}^2 dt,
 \end{aligned} \tag{97}$$

$$\begin{aligned}
& -c^2 \int_{\partial\Omega} \int_0^\tau U_t(x, t) \int_{\Omega} U(\xi, \tau) d\xi dt ds_x \\
& \leq \frac{c^2}{2} \int_0^\tau \left\{ \varepsilon \|\nabla U_t(x, t)\|_{L^2(\Omega)}^2 + l(\varepsilon) \|U_t(x, t)\|_{L^2(\Omega)}^2 \right\} dt \\
& \quad + \frac{c^2}{2} |\Omega| |\partial\Omega| \int_0^\tau \|U(x, t)\|_{L^2(\Omega)}^2 dt,
\end{aligned} \tag{98}$$

$$\begin{aligned}
& b \int_{\partial\Omega} U_\tau(x, \tau) \left\| \int_{\Omega} U(\xi, \tau) d\xi ds_x \right\| \\
& \leq \frac{b}{2\varepsilon_2'} \left(\varepsilon \|\nabla U_\tau(x, \tau)\|_{L^2(\Omega)}^2 + l(\varepsilon) \|U_\tau(x, \tau)\|_{L^2(\Omega)}^2 \right) \\
& \quad + \frac{b}{2} \varepsilon_2' T |\Omega| |\partial\Omega| \|U(x, \tau)\|_{L^2(\Omega)}^2,
\end{aligned} \tag{99}$$

$$\begin{aligned}
& -b \int_0^\tau \int_{\partial\Omega} U_t(x, t) \int_{\Omega} U(\xi, t) d\xi ds_x dt \\
& \leq \frac{b}{2} \{ l(\varepsilon) + T |\Omega| |\partial\Omega| \} \int_0^\tau \|U_t(x, t)\|_{L^2(\Omega)}^2 dt \\
& \quad + \frac{b}{2} \varepsilon \int_0^\tau \|\nabla U_t(x, t)\|_{L^2(\Omega)}^2 dt,
\end{aligned} \tag{100}$$

$$\begin{aligned}
& \int_{\partial\Omega} U_\tau(x, \tau) \left(\int_0^\tau \int_{\Omega} W(\xi, \tau) d\xi dt \right) ds_x \\
& \leq \left(\frac{1}{2\varepsilon_3} + \frac{1}{2\varepsilon_4} \right) \left(\varepsilon \|\nabla U_\tau(x, \tau)\|_{L^2(\Omega)}^2 + l(\varepsilon) \|U_\tau(x, \tau)\|_{L^2(\Omega)}^2 \right) \\
& \quad + \frac{1}{2} \varepsilon_3 T h_0 |\partial\Omega| |\Omega| \int_0^\tau \|U(x, t)\|_{L^2(\Omega)}^2 dt \\
& \quad + \frac{1}{2} \varepsilon_4 T |\partial\Omega| |\Omega| \int_0^\tau h \circ U(t) dt,
\end{aligned} \tag{101}$$

$$\begin{aligned}
& - \int_{\partial\Omega} \int_0^\tau U_t(x, t) \int_{\Omega} W(\xi, t) d\xi dt ds_x \\
& \leq \int_0^\tau \left\{ \varepsilon \|\nabla U_t(x, t)\|_{L^2(\Omega)}^2 + l(\varepsilon) \|U_t(x, t)\|_{L^2(\Omega)}^2 \right\} dt \\
& \quad + \frac{1}{2} h_0 |\Omega| |\partial\Omega| \int_0^\tau \|U(x, t)\|_{L^2(\Omega)}^2 dt \\
& \quad + \frac{1}{2} |\Omega| |\partial\Omega| \int_0^\tau h \circ U(t) dt,
\end{aligned} \tag{102}$$

So, combining inequalities (97)–(102), we obtain

$$\begin{aligned}
& \frac{a}{2} \|U_{\tau\tau}(x, \tau)\|_{L^2(\Omega)}^2 - \frac{1}{2} h' \circ \nabla U(\tau) - \frac{1}{2} h(\tau) \|\nabla U(x, \tau)\|_{L^2(\Omega)}^2 \\
& \quad + c^2 (\nabla U_\tau(x, \tau), \nabla U(x, \tau))_{L^2(\Omega)} \\
& \quad + 2 (\nabla W(x, \tau), \nabla U_\tau(x, \tau))_{L^2(\Omega)} \\
& \quad + \left\{ \frac{b}{2} - \frac{c^2}{2\varepsilon_1'} \varepsilon - \frac{b}{2\varepsilon_2'} \varepsilon - \left(\frac{1}{2\varepsilon_3} + \frac{1}{2\varepsilon_4} \right) \varepsilon \right\} \|\nabla U_\tau(x, \tau)\|_{L^2(\Omega)}^2
\end{aligned}$$

$$\begin{aligned}
& - \left\{ \frac{c^2 l(\varepsilon)}{2\varepsilon_1'} + \frac{bl(\varepsilon)}{2\varepsilon_2'} + \left(\frac{1}{2\varepsilon_3} + \frac{1}{2\varepsilon_4} \right) l(\varepsilon) \|U_\tau(x, \tau)\|_{L^2(\Omega)}^2 \right. \\
& \quad \left. - \frac{b}{2} \varepsilon_2' T |\Omega| |\partial\Omega| \|U(x, \tau)\|_{L^2(\Omega)}^2 \right\} \\
& \leq \beta \int_0^\tau \|U_{tt}(x, t)\|_{L^2(\Omega)}^2 dt + \frac{1}{2} T |\Omega| (\varepsilon_3 + 1) \int_0^\tau h \circ U(t) dt \\
& \quad + \frac{1}{2} \int_0^\tau h'' \circ U(t) dt - \frac{1}{2} \int_0^\tau h'(t) \|U_\tau(x, \tau)\|_{L^2(\Omega)}^2 dt \\
& \quad + \left\{ c^2 + \frac{c^2}{2} \varepsilon + \frac{b}{2} \varepsilon \right\} \int_0^\tau \|\nabla U_t(x, t)\|_{L^2(\Omega)}^2 dt \\
& \quad + \left\{ \frac{c^2}{2} \varepsilon_1' T |\Omega| |\partial\Omega| + \frac{1}{2} (c^2 + h_0 + T h_0 \varepsilon_4) |\Omega| |\partial\Omega| \right\} \\
& \quad \cdot \int_0^\tau \|U(x, t)\|_{L^2(\Omega)}^2 dt \\
& \quad + \left(\frac{b}{2} \{ l(\varepsilon) + T |\Omega| |\partial\Omega| \} + \frac{1}{2} (c^2 + 1) l(\varepsilon) \right) \\
& \quad \cdot \int_0^\tau \|U_t(x, t)\|_{L^2(\Omega)}^2 dt.
\end{aligned} \tag{103}$$

Adding side to side (89) and (103) that gives

$$\begin{aligned}
& a(U_\tau(x, \tau), U(x, \tau))_{L^2(\Omega)} \\
& \quad + \left\{ \frac{\beta}{2} - \frac{b}{2} \varepsilon_2' T |\Omega| |\partial\Omega| \right\} \|U(x, \tau)\|_{L^2(\Omega)}^2 \\
& \quad \cdot \frac{c^2}{2} \|\nabla v(x, 0)\|_{L^2(\Omega)}^2 + \frac{a}{2} \|U_{\tau\tau}(x, \tau)\|_{L^2(\Omega)}^2 \\
& \quad + c^2 (\nabla U_\tau(x, \tau), \nabla U(x, \tau))_{L^2(\Omega)} \\
& \quad + \left\{ \frac{b}{2} - \frac{c^2}{2\varepsilon_1'} \varepsilon - \frac{b}{2\varepsilon_2'} \varepsilon - \left(\frac{1}{2\varepsilon_3} + \frac{1}{2\varepsilon_4} \right) \varepsilon \right\} \\
& \quad \cdot \|\nabla U_\tau(x, \tau)\|_{L^2(\Omega)}^2 - \left\{ \frac{c^2}{2\varepsilon_1'} + \frac{b}{2\varepsilon_2'} + \left(\frac{1}{2\varepsilon_3} + \frac{1}{2\varepsilon_4} \right) l(\varepsilon) \right\} \\
& \quad \cdot l(\varepsilon) \|U_\tau(x, \tau)\|_{L^2(\Omega)}^2 - \frac{1}{2} h' \circ \nabla U(\tau) \\
& \quad + 2 (\nabla W(x, \tau), \nabla U_\tau(x, \tau))_{L^2(\Omega)} \\
& \leq \gamma_4 \int_0^\tau \|U(x, t)\|_{L^2(\Omega)}^2 dt + \left\{ a + \frac{b}{2} (l(\varepsilon) + T |\Omega| |\partial\Omega|) \right. \\
& \quad + \left(\frac{c^2}{2} + 1 \right) l(\varepsilon) \left. \right\} \int_0^\tau \|U_t(x, t)\|_{L^2(\Omega)}^2 dt \\
& \quad + \left(\left(\frac{c^2 + b}{2} + 1 \right) \varepsilon + h_0 \right) \int_0^\tau \|\nabla v(x, t)\|_{L^2(\Omega)}^2 dt \\
& \quad + \left\{ c^2 + \frac{c^2}{2} \varepsilon + \frac{b}{2} \varepsilon + \frac{h_0}{4} \right\} \int_0^\tau \|\nabla U_t(x, t)\|_{L^2(\Omega)}^2 dt^2 \\
& \quad \cdot \left(\frac{1}{2} |\Omega| |\partial\Omega| (\varepsilon_3 T + 1 + T^2) + \frac{1}{2} T^2 l(\varepsilon) \right) \int_0^\tau h \circ U(t) dt \\
& \quad - \frac{1}{2} \int_0^\tau h'' \circ \nabla U(t) dt + \frac{h_0}{2} \int_0^\tau h \circ \nabla U(t) dt \\
& \quad + \left(\frac{h(0)}{4} + \frac{h_0}{2} \right) \int_0^\tau \|\nabla U(x, t)\|_{L^2(\Omega)}^2 dt,
\end{aligned} \tag{104}$$

where

$$\begin{aligned} \gamma_4 := & \left\{ \frac{c^2}{2} T^2 (l(\varepsilon) + |\Omega| |\partial\Omega|) + \frac{b}{2} (T^2 l(\varepsilon) + |\Omega| |\partial\Omega|) \right. \\ & + \frac{1}{2} T^2 (l(\varepsilon) + |\Omega| |\partial\Omega| h_0) + \frac{c^2}{2\varepsilon_1} \varepsilon'_1 T |\Omega| |\partial\Omega| \\ & \left. + \frac{c^2}{2} |\Omega| |\partial\Omega| + \frac{1}{2} h_0 |\Omega| |\partial\Omega| (\varepsilon_4 T^2 + 1) \right\}. \end{aligned} \quad (105)$$

Now, the last term on the RHS of (104), we give the function $\theta(x, t)$ by

$$\theta(x, t) := \int_0^t U(x, s) ds. \quad (106)$$

Hence, we use (77), and we get

$$\begin{aligned} v(x, t) &= \theta(x, \tau) - \theta(x, t), \nabla v(x, 0) = \nabla \theta(x, \tau), \\ \|\nabla v\|_{L^2(Q_\tau)}^2 &= \|\nabla \theta(x, \tau) - \nabla \theta(x, t)\|_{L^2(\Omega)}^2 \\ &\leq 2 \left(\tau \|\nabla \theta(x, \tau)\|_{L^2(\Omega)}^2 + \|\nabla \theta(x, t)\|_{L^2(Q_\tau)}^2 \right). \end{aligned} \quad (107)$$

And using the inequalities

$$\begin{aligned} m'_1 \|U(x, \tau)\|_{L^2(\Omega)}^2 &\leq m'_1 \|U(x, t)\|_{L^2(Q_\tau)}^2 + m_1 \|U_t(x, t)\|_{L^2(Q_\tau)}^2, \\ m'_2 \|U_\tau(x, \tau)\|_{L^2(\Omega)}^2 &\leq m'_2 \|U_t(x, t)\|_{L^2(Q_\tau)}^2 + m_2 \|U_{tt}(x, t)\|_{L^2(Q_\tau)}^2, \\ m'_3 \|\nabla U(x, \tau)\|_{L^2(\Omega)}^2 &\leq m'_3 \|\nabla U(x, t)\|_{L^2(Q_\tau)}^2 + m_3 \|\nabla U_t(x, t)\|_{L^2(Q_\tau)}^2, \\ m'_4 h \circ U(\tau) &\leq m'_4 \left(1 + \frac{\zeta}{2} \right) \int_0^\tau h' \circ U(t) dt + m_4 \|\nabla U_t(x, \tau)\|_{L^2(Q_\tau)}^2, \\ -\frac{a}{2} \|U_\tau(x, \tau)\|_{L^2(\Omega)}^2 - \frac{a}{2} \|U(x, \tau)\|_{L^2(\Omega)}^2 &\leq a(U_\tau(x, \tau), U(x, \tau))_{L^2(\Omega)}, \\ -\frac{c^2}{2\varepsilon'_3} \|\nabla U_\tau(x, \tau)\|_{L^2(\Omega)}^2 - \frac{c^2}{2} \varepsilon'_3 \|\nabla U(x, \tau)\|_{L^2(\Omega)}^2 &\leq c^2 (\nabla U_\tau(x, \tau), \nabla U(x, \tau))_{L^2(\Omega)}, \\ -\left(\frac{1}{4\varepsilon_9} + \frac{h_0}{4\varepsilon_8} \right) \|\nabla U_\tau(x, \tau)\|_{L^2(\Omega)}^2 &\quad - h_0 \varepsilon_8 \|\nabla U(x, \tau)\|_{L^2(\Omega)}^2 - \varepsilon_9 h_0 \nabla U(\tau) \\ &\leq 2(\nabla W(x, \tau), \nabla U_\tau(x, \tau))_{L^2(\Omega)}. \end{aligned} \quad (108)$$

Let

$$\begin{cases} m'_1 := \frac{a}{2} + \frac{b}{2} \varepsilon'_2 T |\Omega| |\partial\Omega|, \\ m'_2 := 1 + \frac{a}{2} - \frac{c^2}{2\varepsilon'_3} - \left\{ \frac{c^2}{2\varepsilon'_1} + \frac{b}{2\varepsilon'_2} + \frac{1}{2\varepsilon_3} + \frac{1}{2\varepsilon_4} \right\} l(\varepsilon), \\ m'_3 := \frac{c^2}{2} \varepsilon'_3 + h_0 \varepsilon_8, \end{cases} \quad (109)$$

and we choose $\varepsilon'_1, \varepsilon'_2, \varepsilon'_3, \varepsilon_8$ and ε_9 sufficiently large

$$\frac{c^2}{2\varepsilon'_1} \varepsilon + \frac{b}{2\varepsilon'_2} \varepsilon + \frac{c^2}{2\varepsilon'_3} + \left(\frac{1}{4\varepsilon_9} + \frac{h_0}{4\varepsilon_9} \right) \varepsilon < \frac{b}{2}. \quad (110)$$

As τ is arbitrary, and we get

$$A := \frac{c^2}{2} - 2\tau \varepsilon \left((c^2 + b)/2 + 1 + h_0 \right) > 0. \quad (111)$$

Thus, inequality (104) takes the form

$$\begin{aligned} & \frac{\beta}{2} \|U(x, \tau)\|_{L^2(\Omega)}^2 + \|U_\tau(x, \tau)\|_{L^2(\Omega)}^2 + \|\nabla U(x, \tau)\|_{L^2(\Omega)}^2 \\ & + \underbrace{\left\{ \frac{b}{2} - \frac{c^2}{2\varepsilon'_1} \varepsilon - \frac{b}{2\varepsilon'_2} \varepsilon - \frac{c^2}{2\varepsilon'_3} + \frac{1}{4\varepsilon_9} + \frac{h_0}{4\varepsilon_8} \right\}}_{\gamma_6} \\ & \cdot \|\nabla U_\tau(x, \tau)\|_{L^2(\Omega)}^2 - \frac{1}{2} h' \circ \nabla U(\tau) - h' \circ U(\tau) \\ & + \frac{a}{2} \|U_{\tau\tau}(x, \tau)\|_{L^2(\Omega)}^2 + A \|\nabla \theta(x, \tau)\|_{L^2(\Omega)}^2 \\ & \leq (\gamma_4 + m'_1) \int_0^\tau \|U(x, t)\|_{L^2(\Omega)}^2 dt + \gamma_5 \int_0^\tau \|U_t(x, t)\|_{L^2(\Omega)}^2 dt \\ & + \left(m'_3 + \frac{h_0}{4} \right) \int_0^\tau \|\nabla U(x, t)\|_{L^2(\Omega)}^2 dt \\ & + (\beta + m'_2) \int_0^\tau \|U_{tt}(x, t)\|_{L^2(\Omega)}^2 dt \\ & + \underbrace{\left\{ c^2 + \frac{c^2}{2} \varepsilon + \frac{b}{2} \varepsilon + \frac{h_0}{4} + \frac{h(0)}{2} + m'_3 \right\}}_{\gamma_7} \\ & \cdot \int_0^\tau \|\nabla U_t(x, t)\|_{L^2(\Omega)}^2 dt + 2\varepsilon \underbrace{\left(\frac{(c^2 + b)}{2} + 1 + h_0 \right)}_{\gamma_8} \\ & \cdot \int_0^\tau \|\nabla \theta(x, t)\|_{L^2(\Omega)}^2 dt - \frac{1}{2} \int_0^\tau h' \circ \nabla U(t) dt \\ & - \left(\frac{\varepsilon_9}{\zeta} + 1 \right) \int_0^\tau h' \circ \nabla U(t) dt \end{aligned}$$

$$-\underbrace{\frac{1}{\zeta} \left(\frac{1}{2} |\Omega| |\partial\Omega| (\varepsilon_3 T + 1 + T^2) + \frac{1}{2} T^2 l(\varepsilon) + 1 \right)}_{\gamma_9} \cdot \int_0^\tau h' \circ U(t) dt, \quad (112)$$

where

$$\gamma_5 := a + \frac{b}{2} (l(\varepsilon) + T |\Omega| |\partial\Omega|) + \left(\frac{c^2}{2} + 1 \right) l(\varepsilon) + \frac{h(0)}{2} + m'_1 + m'_2. \quad (113)$$

We get

$$\begin{aligned} & \|U(x, \tau)\|_{L^2(\Omega)}^2 + \|U_\tau(x, \tau)\|_{L^2(\Omega)}^2 + \|\nabla U(x, \tau)\|_{L^2(\Omega)}^2 \\ & - h' \circ U(\tau) + \|\nabla U_\tau(x, \tau)\|_{L^2(\Omega)}^2 + \|U_{\tau\tau}(x, \tau)\|_{L^2(\Omega)}^2 \\ & + \|\nabla \theta(x, \tau)\|_{L^2(\Omega)}^2 - h' \circ \nabla U(\tau) \\ & \leq D' \int_0^\tau \left\{ \|U(x, t)\|_{L^2(\Omega)}^2 + \|U_t(x, t)\|_{L^2(\Omega)}^2 \right. \\ & + \|\nabla U(x, t)\|_{L^2(\Omega)}^2 - h' \circ U(t) + \|\nabla U_t(x, t)\|_{L^2(\Omega)}^2 \\ & \left. + \|U_{tt}(x, t)\|_{L^2(\Omega)}^2 + \|\nabla \theta(x, t)\|_{L^2(\Omega)}^2 - h' \circ \nabla U(t) \right\} dt, \end{aligned} \quad (114)$$

where

$$D' := \frac{\max \{ (\beta/2), 1, (\eta^2/2), \gamma_6, (a/2), A \}}{\min \left\{ (\gamma_4 + m'_1), \gamma_5, \left(m'_3 + (h_0/4) \right), \gamma_7, \left(\beta + m'_2 \right), ((\varepsilon_9/\zeta) + 1), \gamma_8, \gamma_9 \right\}}. \quad (115)$$

Hence, applying Gronwall's lemma to (114) gives

$$\begin{aligned} & \|U(x, \tau)\|_{L^2(\Omega)}^2 + \|U_\tau(x, \tau)\|_{L^2(\Omega)}^2 + \|\nabla U_\tau(x, \tau)\|_{L^2(\Omega)}^2 \\ & - h' \circ U(\tau) + \|U_{\tau\tau}(x, \tau)\|_{L^2(\Omega)}^2 + \|\nabla U(x, \tau)\|_{L^2(\Omega)}^2 \\ & + \|\nabla \theta(x, \tau)\|_{L^2(\Omega)}^2 - h' \circ \nabla U(\tau) \leq 0, \end{aligned} \quad (116)$$

for all $\tau \in [0, (c^2/4\varepsilon(((c^2 + b)/2) + 1 + h_0))]$.

For the intervals, we use the same method

$$\tau \in \left[\frac{(m-1)c^2}{4\varepsilon(((c^2 + b)/2) + 1 + h_0)}, \frac{mc^2}{4\varepsilon(((c^2 + b)/2) + 1 + h_0)} \right] \quad (117)$$

to cover the whole interval $[0, T]$ and thus proving that $U(x, \tau) = 0, \forall \tau$ in $[0, T]$. Hence, the uniqueness is proved.

5. Conclusion

The objective of this work is the study of solvability of the Moore-Gibson-Thompson equation with viscoelastic memory term and integral condition by using the Galerkin method. The MGT equation is a nonlinear partial differential equation that arises in hydrodynamics and some physical applications. Recent developments in numerical schemes for solving MGT have placed immense interest in nonlinear dispersive wave models. In the next work, we will try to use the same method with Boussinesque and Hall-MHD equations which are nonlinear partial differential equation that arises in hydrodynamics and some physical applications. It was subsequently applied to problems in the percolation of water in porous subsurface strata (see [6, 15–24], for

example, [10, 11, 25, 26]) by using some famous algorithms (see [27–29]).

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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