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Research Article

Coincidence Point Results on Relation Theoretic $(F_w, \mathscr{R})_g$ -Contractions and Applications

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Motivated by the ideas of F-weak contractions and $(F, \mathscr{R})_g$-contractions, the notion of $(F_w, \mathscr{R})_g$-contractions is introduced and studied in the present paper. The idea is to establish some interesting results for the existence and uniqueness of a coincidence point for these contractions. Further, using an additional condition of weakly compatible mappings, a common fixed-point theorem and a fixed-point result are proved for $(F_w, \mathscr{R})_g$-contractions in metric spaces equipped with a transitive binary relation. The results are elaborated by illustrative examples. Some consequences of these results are also deduced in ordered metric spaces and metric spaces endowed with graph. Finally, as an application, the existence of the solution of certain Voltera type integral equations is investigated.

1. Introduction and Preliminaries

In the development of the metric fixed-point theory, one of the main pillar is the Banach contraction principle [1], which states that every contraction on a complete metric space has a unique fixed point. Due to its extensive application potential, this concept has been observed in various forms over the years (see [2–9]).

The concept of F-contractions was introduced by Wardowski [10]. He proved some new fixed-point results for such kind of contractions. He built these results in a different way rather than traditional ways as done by many authors. Later on, fixed points for F-contractions were proved by Secelean [11] using an iterated function. Abbas et al. [12] extended the work of Wardowski and established various results of fixed points using F-contraction mappings. For further related works on F-contractions, see [13–16].

The idea of $(F, \mathscr{R})$-contractions was established by Sawangsup et al. [17]. They used this idea to demonstrate some fixed-point consequences using a binary relation. It is further investigated by Imdad et al. [18]. In present paper, we study the results presented by Alfaqih et al. [19] and we define $(F_w, \mathscr{R})_g$-contractions. We also prove similar results for $(F_w, \mathscr{R})_g$-contractions.

Recall that a binary relation $\mathscr{R}$ on nonempty set X is said to be a partial order if it is reflexive, antisymmetric, and transitive. Moreover, the inverse or transpose or dual relation of $\mathscr{R}$, denoted by $\mathscr{R}^{-1}$, is defined by

$$\mathscr{R}^{-1} = \left\{ (x, y) \in X^2 : (y, x) \in \mathscr{R} \right\}. \tag{1}$$

The symmetric closure of $\mathscr{R}$, denoted by $\mathscr{R}^s$, is defined as the set $\mathscr{R} \cup \mathscr{R}^{-1}$, that is, $\mathscr{R}^s := \mathscr{R} \cup \mathscr{R}^{-1}$. In fact, $\mathscr{R}^s$ is the smallest symmetric relation on X containing $\mathscr{R}$.

Notice that there is another binary relation $\mathscr{R}^{\sharp} \subseteq \mathscr{R}$ on X, which is defined by $k\mathscr{R}^{\sharp}\ell$, whenever $k\mathscr{R}\ell$ and $k \neq \ell$.

Definition 1 [10]. Let $\mathbb{F}$ be the set of functions $F : (0,\infty) \longrightarrow \mathbb{R}$ such that

(F_1) F is strictly increasing;

(F_2) For every sequence $\{\beta_n\} \subset (0,\infty)$, $\lim_{n\longrightarrow\infty}\beta_n = 0$ iff$\lim_{n\longrightarrow\infty}F(\beta_n) = -\infty$;

(F_3) There is $k \in (0,1)$ so that $\lim_{\beta\longrightarrow 0^+}\beta^k F(\beta) = 0$.

The following functions are in $\mathbb{F}$:

$$\begin{aligned} F(\beta) &= \ln(\beta), \\ F(\beta) &= \ln(\beta) + \beta, \\ F(\beta) &= \ln\left(\beta^2 + \beta\right), \\ F(\beta) &= -\left(\frac{1}{\sqrt{\beta}}\right). \end{aligned} \tag{2}$$

Many papers in literature deal with the concept of F-contractions (see [20–22]). Throughout this work, the set of all continuous functions verifying (F_2) is denoted by $\mathscr{F}$.

Definition 2. Let $X \neq \phi$ and $\mathscr{R}$ be a binary relation on X. A sequence $\{\varsigma_n\} \subseteq X$ is such that $\varsigma_n \mathscr{R} \varsigma_{n+1}$ for all $n \in \mathbb{N}_0$, then it is called an $\mathscr{R}$ preserving sequence.

Definition 3. Consider a metric space (X,d) with a binary relation $\mathscr{R}$. Then, X is called $\mathscr{R}$ complete if each $\mathscr{R}$ preserving Cauchy sequence is convergent in X.

Definition 4 [23]. Let (X,d) be a metric space and $\mathscr{R}$ be a binary relation on X, $T : X \longrightarrow X$ and $x \in X$. We say that T is $\mathscr{R}$-continuous at x if for each $\mathscr{R}$-preserving sequence $\{\varsigma_n\} \subseteq X$ so that $\varsigma_n \longrightarrow x$, we have $T\varsigma_n \longrightarrow Tx$. Also, T is named to be $\mathscr{R}$-continuous if it is $\mathscr{R}$-continuous at any element of X.

Definition 5 [23]. Let (X,d) be a metric space and $\mathscr{R}$ be a binary relation on X and $T, g : X \longrightarrow X$ and $x \in X$. We say that T is $(g,\mathscr{R})$-continuous at x if for each sequence $\{\varsigma_n\} \subseteq X$ so that $\{g\varsigma_n\}$ is $\mathscr{R}$-preserving and $g\varsigma_n \longrightarrow gx$, we have $T\varsigma_n \longrightarrow Tx$. Also, T is named to be $(g,\mathscr{R})$-continuous if it is $(g,\mathscr{R})$-continuous at any element of X.

Definition 6 [24]. For $x, y \in X$, a path of length $p(p \in \mathbb{N})$ in $\mathscr{R}$ from x to y is a finite sequence $\{u_0, u_1, \cdots\cdots, u_p\} \subseteq X$ such that $u_0 = x$, $u_p = y$, and $(u_i, u_{i+1}) \in \mathscr{R}$ for every $i \in \{0, 1, \ldots\cdots, p-1\}$. Also, a subset $L \subseteq X$ is called $\mathscr{R}$ connected if for any two elements $x, y \in L$, there is a path from x to y in $\mathscr{R}$.

Definition 7 [23]. Let (X,d) be a metric space and $\mathscr{R}$ be a binary relation on X and $T, g : X \longrightarrow X$. The pair (T,g) is $\mathscr{R}$-compatible if for each sequence $\{\varsigma_n\} \subseteq X$ so that $\{T\varsigma_n\}$ and $\{g\varsigma_n\}$ are $\mathscr{R}$-preserving and $\lim_{n\longrightarrow\infty} g\varsigma_n = \lim_{n\longrightarrow\infty} T\varsigma_n = x \in X$,

$$\lim_{n\longrightarrow\infty} d(gT\varsigma_n, Tg\varsigma_n) = 0. \tag{3}$$

Definition 8. Let f and g be self-maps of a set X. If $x = fx = gx$ for some $x \in X$, then x is said to a common fixed point of f and g.

Definition 9 [25]. Let $f, g : X \longrightarrow X$. If $w = fx = gx$ for some $x \in X$, then x is said to be a coincidence point of f and g, and w is said to be a point of coincidence of f and g.

f and g are said to be weakly compatible if they commute at their coincidence point, i.e., if $fx = gx$ for some $x \in X$, then $fgx = gfx$.

Definition 10 [26]. Let (M,d) be a metric space endowed with a binary relation $\mathscr{R}$. Such a $\mathscr{R}$ is named to be d-self closed if for each $\mathscr{R}$-preserving sequence $\{\varsigma_n\} \subseteq M$ so that $\{\varsigma_n\} \longrightarrow x$, there is $\{\varsigma_{n_k}\}$ of $\{\varsigma_n\}$ so that $[\varsigma_{n_k}, x] \in \mathscr{R} \forall k \in \mathbb{N}_0$.

Definition 11 [23]. Let M be a nonempty set and $T, g : M \longrightarrow M$. A binary relation $\mathscr{R}$ on M is called (T,g) closed if for any $x, y \in M$, $gx\mathscr{R}gy$ yields that $Tx\mathscr{R}Ty$.

Lemma 12 [27, 28]. *Consider a metric space (X,d) and a sequence $\{k_m\}$ in X. If $\{k_m\}$ is not Cauchy in X, then are $\varepsilon > 0$ and $\{k_{m(j)}\}$ and $\{k_{t(j)}\}$ of $\{k_m\}$ so that*

$$j \leq m(j) \leq t(j), d\left(k_{m(j)}, k_{t(j)-1}\right) \leq \varepsilon < d\left(k_{m(j)}, k_{t(j)}\right) \forall j \in N_0. \tag{4}$$

Moreover, if $\{k_m\}$ is so that $\lim_{m\longrightarrow\infty} d(k_m, k_{m+1}) = 0$, then

$$\lim_{j\longrightarrow\infty} d\left(k_{m(j)}, k_{t(j)}\right) = \lim_{j\longrightarrow\infty} d\left(k_{m(j)-1}, k_{t(j)-1)}\right) = \varepsilon. \tag{5}$$

Lemma 13 [29]. *Let X be a nonempty set and $g : X \longrightarrow X$. Then, there is a subset $E \subseteq X$ so that $g(E) = g(X)$ and $g : E \longrightarrow E$ is one to one.*

2. Main Results

We begin this section by introducing the idea of $(F_w, \mathscr{R})_g$-contractions as follows.

Definition 14. Consider a metric space (X,d) endowed with a transitive binary relation $\mathscr{R}$ on X and $Q, g : X \longrightarrow X$. Then, T is called an $(F_w, \mathscr{R})_g$-contractions if there exist $F \in \mathscr{F}$ and $\tau > 0$ such that

$$\begin{aligned} &\tau + F((d(Qk, Q\ell)) \\ &\quad \leq F\left(\max\left\{d(gk, g\ell), d(gk, Qk), d(g\ell, Q\ell), \frac{d(gk, Q\ell) + d(g\ell, Qk)}{2}\right\}\right), \end{aligned} \tag{6}$$

for all $k, l \in X$ with $gk\mathscr{R}^{\sharp}g\ell$ and $Qk\mathscr{R}^{\sharp}Q\ell$.

Remark 15. Every $(F, \mathscr{R})_g$ contraction is an $(F_w, \mathscr{R})_g$ contraction, but the converse of statement is not true.

The following result is easy to prove. We omit it.

Proposition 16. *Let (X, d) be a metric space endowed with a transitive binary relation $\mathscr{R}$. Given $Q, g : X \longrightarrow X$. Then, for each $F \in \mathscr{F}$, we have equivalence of the two following statements:*

(a) *$\forall k, l \in X$ so that $(gk, g\ell) \in \mathscr{R}^{\sharp}$ and $(Qk, Q\ell) \in \mathscr{R}^{\sharp}$*

$$\tau + F(d(Qk, Q\ell)) \leq F \cdot \left(\max\left\{d(gk, g\ell), d(gk, Qk), (g\ell, Q\ell), \frac{d(gk, Q\ell) + d(g\ell, Qk)}{2}\right\}\right). \tag{7}$$

(b) *$\forall k, l \in X$ such that either $(gk, g\ell), (Qk, Q\ell) \in \mathscr{R}^{\sharp}$ or $(g\ell, gk), (Q\ell, Qk) \in \mathscr{R}^{\sharp}$*

$$\tau + F(d(Qk, Q\ell)) \leq F \cdot \left(\max\left\{d(gk, g\ell), d(gk, Qk), (g\ell, Q\ell), \frac{d(gk, Q\ell) + d(g\ell, Qk)}{2}\right\}\right). \tag{8}$$

Theorem 17. *Consider a metric space (X, d) equipped with $\mathscr{R}$ (where $\mathscr{R}$ is a transitive binary relation) and $Q, g : X \longrightarrow X$. Assume that:*

(1) *there exists $k_0 \in X$ such that $gk_0 \mathscr{R} Qk_0$*

(2) *$\mathscr{R}$ is (Q, g)-closed*

(3) *Q is an $(F_w, \mathscr{R})_g$-contraction*

(4)

  (a) *A subset K of X exists such that $Q(X) \subseteq K \subseteq g(X)$ and K is $\mathscr{R}$-complete*

  (b) *One of the subsequent conditions is fulfilled:*

   (i) *Q is $(g, \mathscr{R})$-continuous, or*

   (ii) *Q and g are continuous, or*

   (iii) *$\mathscr{R} \mid K$ is d-self closed in condition that (6) holds for all $k, l \in X$ with $gk \mathscr{R} g\ell$ and $Qk \mathscr{R}^{\sharp} Q\ell$*

or on the other hand:

(α) (α_1) $\exists$ a subset L of X such that $Q(X) \subseteq g(X) \subseteq L$ and L is $\mathscr{R}$- complete,

(α_2) (Q, g) is an $\mathscr{R}$-compatible pair,

(α_3) Q and g are $\mathscr{R}$-continuous.

Then, the pair (Q, g) admits a coincidence point.

Proof. In the above two cases (11) and (α), note that $Q(X) \subseteq g(X)$. Using assumption (6), we get $gk_0 \mathscr{R} Qk_0$. If $Qk_0 = gk_0$, then a coincidence point of (Q, g) is k_0. This completes the proof. Suppose that $Qk_0 \neq gk_0$. Since $Q(X) \subseteq g(X)$, there must exist $k_1 \in X$ such that $gk_1 = Qk_0$. Similarly, there is $k_2 \in X$ such that $gk_2 = Qk_1$. Proceeding in this way, we can construct a sequence $\{k_m\} \subseteq X$ such that

$$gk_{m+1} = Qk_m \forall m \in \mathbb{N}_0. \tag{9}$$

Now, we will prove $\{gk_m\}$ is an $\mathscr{R}$-preserving sequence, that is,

$$gk_m \mathscr{R} gk_{m+1} \forall m \in \mathbb{N}_0. \tag{10}$$

By using induction, we will prove this claim. If we put $m = 0$ in (9) and use condition (6), we get $gk_o \mathscr{R} gk_1$. This implies that the above statement holds for $m = 0$. Suppose that (10) is accurate for $m = j \geq 1$, that is, $gk_j \mathscr{R} gk_{j+1}$. Since $\mathscr{R}$ is (Q, g)-closed, we get $Qk_j \mathscr{R} Qk_{j+1}$, and so $gk_{j+1} \mathscr{R} gk_{j+2}$.

Hence, our claim is true for all $m \in \mathbb{N}_0$. By using (9) and (10), we can conclude that $\{Qk_m\}$ is also an $\mathscr{R}$-preserving sequence, that is,

$$Qk_m \mathscr{R} Qk_{m+1} \forall m \in \mathbb{N}_0. \tag{11}$$

If $Qk_{m_0} = Qk_{m_{0+1}}$ for some $m_0 \in \mathbb{N}_0$, then k_{m_0} is a coincidence point of (Q, g).

Suppose on the contrary that $Qk_m \neq Qk_{m+1}$ for al $m \in \mathbb{N}_0$. With the help of (9), (10), (11), and condition (10), we can see that

$$\begin{aligned}
&\tau + F(d(Qk_{m-1}, Qk_m)) \leq F \\
&\cdot \Bigg(\max\Bigg\{d(gk_{m-1}, gk_m), d(gk_{m-1}, Qk_{m-1}), d(gk_m, Qk_m), \\
&\cdot \frac{d(gk_{m-1}, Qk_m) + d(gk_m, Qk_{m-1})}{2}\Bigg\}\Bigg) \forall m \in \mathbb{N}_0. = F \\
&\cdot \left(\max\left\{d(gk_{m-1}, Qk_{m-1}), d(gk_m, Qk_m), \frac{d(gk_{m-1}, Qk_m)}{2}\right\}\right). \leq F \\
&\cdot \Bigg(\max\Bigg\{d(gk_{m-1}, Qk_{m-1}), d(gk_m, Qk_m), \\
&\cdot \frac{d(gk_{m-1}, gk_m) + d(gk_m, Qk_m)}{2}\Bigg\}\Bigg). = F \\
&\cdot (\max\{d(gk_{m-1}, gk_m), d(gk_m, Qk_m)\}).
\end{aligned} \tag{12}$$

Now, $\max\{d(gk_{m-1}, gk_m), d(gk_m, Qk_m)\}$ cannot be $d(gk_m, Qk_m)$. Otherwise,

$$\tau + F(d(gk_m, gk_{m+1})) \leq d(gk_m, gk_{m+1}), \tag{13}$$

which is a contradiction. Hence, $\max\{d(gk_{m-1}, gk_m), d(gk_m, Qk_m)\} = d(gk_{m-1}, gk_m)$. Therefore,

$$\begin{aligned}
\tau + F(d(Qk_{m-1}, Qk_m)) \leq F(d(gk_{m-1}, gk_m)) \Rightarrow F(d(Qk_{m-1}, Qk_m)) \\
\leq F(d(gk_{m-1}, gk_m))) - \tau.
\end{aligned} \tag{14}$$

Take $\gamma_m = d(gk_m, gk_{m+1})$. With the help of above condition, we obtain

$$F(\gamma_m) \le F(\gamma_{m-1}) - \tau \le F(\gamma_{m-2}) - 2\tau .. \cdots \le F(\gamma_0) - m\tau (\forall m \in \mathbb{N}). \quad (15)$$

By using (F_2) and taking $m \longrightarrow \infty$ in above inequality, we obtain

$$\lim_{m \longrightarrow \infty} F(\gamma_m) = -\infty. \quad (16)$$

This together with (F_2) imply that

$$\lim_{m \longrightarrow \infty} \gamma_m = d(gk_m, gk_{m+1}) = 0. \quad (17)$$

Now, we will show that $\{gk_m\}$ is a Cauchy sequence. We argue by contradiction. In this case, Lemma 12 guarantees the existence of $\varepsilon > 0$ and two subsequences $\{gk_{m_j}\}$ and $\{gk_{t_j}\}$ of $\{gk_m\}$ such that

$$d\left(gk_{m(j)}, gk_{t(j-1)}\right) \le \varepsilon < d\left(gk_{m(j)}, gk_{t(j)}\right), \quad (18)$$

with

$$j \le m(j) \le t(j), \quad \forall j \in \mathbb{N}_0, \quad (19)$$

$$\lim_{j \longrightarrow \infty} d\left(gk_{m(j)}, gk_{t(j)}\right) = d\left(gk_{m(j)-1}, gk_{t(j)-1}\right) = \varepsilon. \quad (20)$$

This implies that there is $j_0 \in \mathbb{N}_0$ so that $d(gk_{m(j)-1}, gk_{t(j)-1}) > 0 \forall j \ge j_0$.

Since $\mathscr{R}$ is transitive, one writes

$$gk_{m(j)-1} \mathscr{R}^{\sharp} gk_{t(j)-1} \text{ and } Qk_{m(j)-1} \mathscr{R}^{\sharp} Qk_{t(j)-1} \forall j \ge j_0. \quad (21)$$

Using condition (10), we have for all $j \ge j_0$,

$$\tau + F\left(d\left(Qk_{m(j)-1}, Qk_{t(j)-1}\right)\right) \le F \max \cdot \left(d\left(gk_{m(j)-1}, gk_{t(j)-1}\right), d\left(gk_{m(j)-1}, Qk_{m(j)-1}\right), d\left(gk_{t(j)-1}, Qk_{t(j)-1}\right), \cdot \frac{d\left(gk_{m(j)-1}, Qk_{t(j)-1}\right) + d\left(gk_{t(j)-1}, Qk_{m(j)-1}\right)}{2} \right). \quad (22)$$

Denote

$$\max \left\{ d\left(gk_{m(j)-1}, gk_{t(j)-1}\right), d\left(gk_{m(j)-1}, Qk_{m(j)-1}\right), d\left(gk_{t(j)-1}, Qk_{t(j)-1}\right), \cdot \frac{d\left(gk_{m(j)-1}, Qk_{t(j)-1}\right) + d\left(gk_{t(j)-1}, Qk_{m(j)-1}\right)}{2} \right\} = \mathscr{D}\left(k_{m(j)-1}, k_{t(j)-1}\right). \quad (23)$$

If $\mathscr{D}(k_{m(j)-1}, k_{t(j)-1}) = d(gk_{m(j)-1}, gk_{t(j)-1})$ or it is equal to $(d(gk_{m(j)-1}, Qk_{t(j)-1}) + d(gk_{t(j)-1}, Qk_{m(j)-1}))/2$ then taking $j \longrightarrow \infty$ and using (20), we get

$$\lim_{j \longrightarrow \infty} \mathscr{D}\left(k_{m(j)-1}, k_{t(j)-1}\right) = \varepsilon. \quad (24)$$

Since F is continuous, letting $m \longrightarrow \infty$ in (22) and using (20) and (24), we get

$$\tau + F(\varepsilon) \le F(\varepsilon), \quad (25)$$

which is a contradiction. On the other hand, if $\mathscr{D}(k_{m(j)-1}, k_{t(j)-1}) = d(gk_{m(j)-1}, Qk_{m(j)-1})$ or it is equal to $d(gk_{t(j)-1}, Qk_{t(j)-1})$ then letting $m \longrightarrow \infty$ in (22), using continuity of F and (20) together with condition F_2, we get $\tau + F(\varepsilon) \le -\infty$, which is again a contradiction. Thus, $\{gk_m\}$ is a Cauchy sequence.

Let the condition (11) hold. With the help of (9), we obtain $gk_m \subseteq Q(X)$. Therefore, $\{gk_m\}$ is $\mathscr{R}$-preserving Cauchy in K. By utilizing $\mathscr{R}$-completeness of K, there is $l \in K$ so that $gk_m \longrightarrow l$. As $K \subseteq g(X)$, there is $v \in X$ so that $l = gv$. Hence, by using (2),

$$\lim_{m \longrightarrow \infty} gk_m = \lim_{m \longrightarrow \infty} Qk_m = gv. \quad (26)$$

In order to prove that v is coincidence point of (Q, g), we will use three different cases of condition (b). First of all, suppose that Q is $(g, \mathscr{R})$-continuous. By utilizing (10) and (26), we get

$$\lim_{m \longrightarrow \infty} Qk_m = Qv. \quad (27)$$

By utilizing (26) and (27), we get $Qv = gv$. This shows that v is a coincidence point of (Q, g).

Now, suppose the second case of (b), that is, Q and g are continuous. Since $X \ne \phi$ and $g : X \longrightarrow X$, by using Lemma 13, there is $B \subseteq X$ so that $g(B) = g(X)$ and $g : B \longrightarrow B$ is one-one. Define a mapping $f : g(B) \longrightarrow g(X)$ by

$$f(gb) = Q(b) \forall gb \in g(B) \text{ where } b \in B. \quad (28)$$

Recall that g is one-one and $Q(X) \subseteq g(X)$, so f is well-defined mapping. As Q and g are continuous, f is also continuous. Now, utilizing the fact that $g(X) = g(B)$, we can rewrite condition (a) as $Q(X) \subseteq K \subseteq g(B)$, so that, without loss of generality, we can select a sequence $\{k_m\}$ in B and $v \in B$. By using (26), (28), and continuity of f, we have

$$Qv = f(gv) = f\left(\lim_{m \longrightarrow \infty} gk_m\right) = \lim_{m \longrightarrow \infty} f(gk_m) = \lim_{m \longrightarrow \infty} Qk_m = gv. \quad (29)$$

Finally, assume that condition (iii) of (b) holds, which implies that $\mathscr{R} \mid K$ is d-self closed and (2.1) detain $\forall k, l \in X$, with $gk \mathscr{R} g\ell$ and $Qk \mathscr{R}^{\sharp} Q\ell$. As $\{gk_m\} \subseteq K$, $\{gk_m\}$ is $\mathscr{R} \mid K$ preserving due to (10) and with the help of (26) $gk_m \longrightarrow gv$. So, there is a subsequence $\{gk_{m_j}\} \subseteq \{gk_m\}$ such that

$$\left[gk_{m_j}, gv\right] \in \mathscr{R}|K \subseteq \mathscr{R} \forall j \in \mathbb{N}_0. \quad (30)$$

Utilizing condition (b) and (30), one writes

$$\left[Qk_{m_j}, Qv\right] \in \mathscr{R}|K \subseteq \mathscr{R} \forall j \in \mathbb{N}_0. \quad (31)$$

Now, let $q = \{j \in \mathbb{N} : Qk_{m_j} = Qv\}$. If the set q is infinite, then $\{Qk_{m_j}\}$ has a subsequence $\{Qk_{m_{j_p}}\}$, such that $Qk_{m_{j_p}} = Qv$. This implies that $\lim_{p\longrightarrow\infty} Qk_{m_{j_p}} = Qv \forall p \in \mathbb{N}$. By using (26), we have $\lim_{m\longrightarrow\infty} Qk_m = gv$. So we obtain $Qv = gv$.

If the set q is finite, then $\{Qk_{m_j}\}$ has a subsequence $\{Qk_{m_{j_p}}\}$ such that $Qk_{m_{j_p}} \neq Qv \forall p \in \mathbb{N}$. Next, we will show that $\lim_{p\longrightarrow\infty} Qk_{m_{j_p}} = Qv$. With the help of (30), (31) and $k_{m_{j_p}} \neq Qv \forall p \in \mathbb{N}$, we have

$$\left[gk_{m_{j_p}}, gv\right] \in \mathscr{R}|K \subseteq \mathscr{R} \forall p \in \mathbb{N}_0, \quad (32)$$

$$\left[Qk_{m_{j_p}}, Qv\right] \in \mathscr{R}|K \subseteq \mathscr{R} \text{ and } Qk_{m_{j_p}} \neq Qv \forall p \in \mathbb{N}_0. \quad (33)$$

Now, with the help of (32), (33), Proposition 16 and the fact that (2.1) is satisfied, we get

$$F\left(d\left(Qk_{m_{j_p}}, Qv\right)\right) \leq F\left(\max\left\{d\left(gk_{m_{j_p}}, gv\right), d\left(gk_{m_{j_p}}, Qk_{m_{j_p}}\right), d(gv, Qv), \cdot \frac{d\left(gk_{m_{j_p}}, Qv\right) + d\left(gv, Qk_{m_{j_p}}\right)}{2}\right\}\right) - \tau. \quad (34)$$

Denote

$$\max\left\{d\left(gk_{m_{j_p}}, gv\right), d\left(gk_{m_{j_p}}, Qk_{m_{j_p}}\right), d(gv, Qv), \cdot \frac{d\left(gk_{m_{j_p}}, Qv\right) + d\left(gv, Qk_{m_{j_p}}\right)}{2}\right\} = \mathscr{D}\left(k_{m_{j_p}}, v\right). \quad (35)$$

If $\mathscr{D}(k_{m_{j_p}}, v) = d(gk_{m_{j_p}}, gv)$then, we have

$$\tau + F\left(d\left(Qk_{m_{j_p}}, Qv\right)\right) \leq F\left(d\left(gk_{m_{j_p}}, gv\right)\right) \Rightarrow F\left(d\left(Qk_{m_{j_p}}, Qv\right)\right) \leq F\left(d\left(gk_{m_{j_p}}, gv\right)\right) - \tau. \quad (36)$$

By using (26), (F_2) and taking $p \longrightarrow \infty$, we get $\lim_{p\longrightarrow\infty} Qk_{m_{j_p}} = Qv$. If $\mathscr{D}(k_{m_{j_p}}, v) = d(gk_{m_{j_p}}, gv)$ then, we have

$$\tau + F\left(d\left(Qk_{m_{j_p}}, Qv\right)\right) \leq F\left(d\left(gk_{m_{j_p}}, gv\right)\right) \Rightarrow F\left(d\left(Qk_{m_{j_p}}, Qv\right)\right) \leq F\left(d\left(gk_{m_{j_p}}, gv\right)\right) + Ld\left(gk_{m_{j_p}}, gk_{m_{j_{p-1}}}\right) - \tau. \quad (37)$$

By using (26), (F_2) and taking $p \longrightarrow \infty$, we get $\lim_{p\longrightarrow\infty} Qk_{m_{j_p}} = Qv$. Now, if $\mathscr{D}(k_{m_{j_p}}, v) = d(gv, Qv)$, then

$$F\left(d\left(Qk_{m_{j_p}}, Qv\right)\right) \leq F\left(d\left(gk_{m_{j_p}}, Qk_{m_{j_p}}\right)\right) - \tau. \quad (38)$$

By using (26), (F_2) and taking $p \longrightarrow \infty$, we get $\lim_{p\longrightarrow\infty} Qk_{m_{j_p}} = Qv$. If $\mathscr{D}(k_{m_{j_p}}, v) = (d(gk_{m_{j_p}}, Qv) + d(gv, Qk_{m_{j_p}}))/2$, then, we have

$$\begin{aligned}\tau + F\left(d\left(Qk_{m_{j_p}}, Qv\right)\right) &\leq F\left(\frac{d\left(gk_{m_{j_p}}, Qv\right) + d\left(gv, Qk_{m_{j_p}}\right)}{2}\right)\\ &\Rightarrow \lim_{p\longrightarrow\infty} F\left(d\left(Qk_{m_{j_p}}, Qv\right)\right)\\ &\leq \lim_{p\longrightarrow\infty} F\left(\frac{d\left(gk_{m_{j_p}}, Qk_{m_{j_p}}\right) + d\left(gk_{m_{j_p}}, Qk_{m_{j_p}}\right)}{2}\right) - \tau\\ &= \lim_{p\longrightarrow\infty} F\left(\frac{d\left(gk_{m_{j_p}}, gk_{m_{j_{p-1}}}\right) + d\left(gk_{m_{j_p}}, gk_{m_{j_{p-1}}}\right)}{2}\right) - \tau.\end{aligned} \quad (39)$$

By using (26), (F_2) and taking $p \longrightarrow \infty$, we get

$$\lim_{p\longrightarrow\infty} Qk_{m_{j_p}} = Qv. \quad (40)$$

From (26) and (40), we obtain $Qv = gv$. Hence, when the set q is finite or infinite, v is a coincidence point of Q and g. Now, if (α) holds, then $gk_m \subseteq L$, and hence $\{gk_m\}$ is an $\mathscr{R}$-preserving Cauchy sequence in L. Since L is $\mathscr{R}$-complete, there is $u \in L$ so that

$$\lim_{m\longrightarrow\infty} gk_m = u. \quad (41)$$

Using Equations (9) and (41), one gets

$$\lim_{m\longrightarrow\infty} Qk_m = u. \quad (42)$$

Now, with the help of (10), (41), and continuity of g, we have

$$\lim_{m\longrightarrow\infty} g(gk_m) = g\left(\lim_{m\longrightarrow\infty} gk_m\right) = gu. \quad (43)$$

Utilizing (11), (42) and continuity of g to find

$$\lim_{m\longrightarrow\infty} g(Qk_m) = g\left(\lim_{m\longrightarrow\infty} Qk_m\right) = gu. \quad (44)$$

As Qk_m and gk_m are $\mathscr{R}$-preserving due to (10), (11) and

$$\lim_{m\longrightarrow\infty} Qk_m = \lim_{m\longrightarrow\infty} gk_m = u. \quad (45)$$

Now, using (41), (42), and condition (α_2),

$$\lim_{m\longrightarrow\infty} d(gQk_m, Qgk_m) = 0. \quad (46)$$

Next, we will demonstrate that u is a coincidence point of (Q, g). Making use of (10), (41) and the $\mathscr{R}$-continuity of Q, we get

$$\lim_{m\longrightarrow\infty} Q(gk_m) = Q\left(\lim_{m\longrightarrow\infty} gk_m\right) = Qu. \tag{47}$$

With the use of (44), (46), and (47), we get

$$\begin{aligned} d(gu, Qu) &= d\left(\lim_{m\longrightarrow\infty} gQk_m, \lim_{m\longrightarrow\infty} Qgk_m\right) \\ &= \lim_{m\longrightarrow\infty} d(gQk_m, Qgk_m) = 0 \Rightarrow Qu = gu. \end{aligned} \tag{48}$$

This implies that u is a coincidence point of (Q, g). □

Theorem 17 does not guarantee the uniqueness of a coincidence point. The following theorem guarantees that coincidence point is unique.

Theorem 18. *Suppose all hypothesis of Theorem 17 are true except* (α) *and assume that gu and gv are* $\mathscr{R}$*-comparable for all* $u \neq v \in$ *coin* (Q, g)*, and one of Q or g is one-one, then there is a unique coincidence point of* (Q, g).

Proof. The set coin(Q, g) is nonempty, because of Theorem 17. Consider two elements $u, v \in \text{coin}(Q, g)$, then by definition of coin(Q, g), we have $[gv, gu] \in \mathscr{R}$ and $Qu = gu$, $Qv = gv$. This implies $[Qu, Qv] \in \mathscr{R}$.

Now, if $gu = gv$, we obtain $Qv = gv = gu = Qu$, and hence, $v = u$, because one of Q and g is one-one.

If $gu \neq gv$, then by utilizing condition (10) and Proposition 16, we get

$$\begin{aligned} &\tau + F(d(Qu, Qv)) \le F \\ &\cdot\left(d(gu, gv), d(gu, Qv), d(gv, Qv), \frac{d(gu, Qv) + d(gv, Qv)}{2}\right) \\ &= F(d(Qu, Qv)). \end{aligned} \tag{49}$$

Since $\tau > 0$, our assumption is false. Therefore, a unique coincidence point of (Q, g) exists. □

Theorem 19. *Consider above theorem and add a condition that* (Q, g) *is a weakly compatible pair, then a unique common fixed point of* (Q, g) *exists.*

Proof. Above theorem assures that the pair (Q, g) has a unique coincidence point. Let v be the common coincidence point and suppose $z \in X$ be such that

$$z = Qv = gv. \tag{50}$$

The weak compatibility of Q and g leads to $Qz = Qgv = gQv = gz$. That is, z is a coincidence point of Q and g. Since v is unique, one writes $z = v$. That is, the uniqueness of a common fixed point. Since all the assumptions of Theorem 18 are true, the set coin(Q, g) is nonempty. □

Example 1. Let $X = [0,\infty)$ and define $d : X \times X \longrightarrow \mathbb{R}$ *by* $d(x, y) = |x - y|$. Then, (X, d) is a complete metric space.

Consider the sequence $\{\varsigma_n\} \subseteq X$ which is defined by $\{\varsigma_n = (n(n+1)(4n-1))/3, n \ge 1\}$.

Define the binary relation $\mathscr{R}$ on X by

$$\mathscr{R} = \{(\varsigma_i, \varsigma_i), (\varsigma_i, \varsigma_{i+1}) \text{ such that } \varsigma_i \le \varsigma_{i+1}\}. \tag{51}$$

Define $Q, g : X \longrightarrow X$ by

$$Qx = \begin{cases} x, & \text{if } 0 \le x \le \varsigma_1, \\ \varsigma_2, & \text{if } \varsigma_1 < x \le \varsigma_2, \\ \varsigma_i + \dfrac{\varsigma_{i+1} - \varsigma_i}{\varsigma_{i+2} - \varsigma_{i+1}}(x - \varsigma_{i+1}), & \text{if } \varsigma_{i+1} \le x \le \varsigma_{i+2}, \end{cases} \tag{52}$$

and

$$gx = \left\{\varsigma_i + \frac{\varsigma_{i+1} - \varsigma_i}{\varsigma_{i+2} - \varsigma_{i+1}}(x - \varsigma_i), \quad \text{if } \varsigma_i \le x \le \varsigma_{i+1}, i = 1, 2, \cdots. \right. \tag{53}$$

Observe that if $gx\mathscr{R}^\sharp gy$ and $Qx\mathscr{R}^\sharp Qy$, then $x = \varsigma_i$ and $y = \varsigma_{i+1}$ for $i \in \mathbb{N} - 1$. Further, by choosing $F(\alpha) = \ln \alpha$ and $\alpha \in (0,+\infty)$, we have

$$\begin{aligned} &F(d(Q\varsigma_i, Q\varsigma_{i+1})) = F(|\varsigma_{i-1} - \varsigma_i|) = F(|\varsigma_i - \varsigma_{i-1}|) = \ln|\varsigma_i - \varsigma_{i-1}|, \\ &F\left(\max\left\{d(g\varsigma_i, g\varsigma_{i+1}), d(g\varsigma_i, Q\varsigma_i), d(g\varsigma_{i+1}, Q\varsigma_{i+1}),\right.\right. \\ &\left.\left.\cdot\frac{d(g\varsigma_i, Q\varsigma_{i+1}) + d(Q\varsigma_i, g\varsigma_{i+1})}{2}\right\}\right) \\ &= F\left(\max\left\{|\varsigma_{i+1} - \varsigma_i|, |\varsigma_i - \varsigma_{i-1}|, |\varsigma_{i+1} - \varsigma_i|, \frac{|\varsigma_{i+1} - \varsigma_{i-1}|}{2}\right\}\right) \\ &= F(\varsigma_{i+1} - \varsigma_i) = \ln d(g\varsigma_{i+1}, Qxi + 1). \end{aligned} \tag{54}$$

Now, for $n = 2, 3, \cdots$ and for $\tau = \ln 3$, we have

$$\tau + \ln|(\varsigma_i - \varsigma_{i-1}| \le |\varsigma_{i+1} - \varsigma_i|. \tag{55}$$

Therefore,

$$\begin{aligned} &\ln(3) + F(d(Q\varsigma_i, Q\varsigma_{i+1})) \le Fd(g\varsigma_{i+1}, Qxi + 1), \\ &\forall x, y \in X \text{ such that } gx\mathscr{R}^\sharp gy \text{ and } Qx\mathscr{R}^\sharp Qy. \end{aligned} \tag{56}$$

Moreover, all the assumptions of Theorem 19 are true, and ς_1 is the unique common fixed point of (Q, g).

On setting $g = I$ in Theorem 19, we obtain the following result.

Theorem 20. *Consider a self-mapping* $Q : X \longrightarrow X$ *and let* (X, d) *be a metric space with a transitive binary relation* $\mathscr{R}$*. Assume that:*

(1) $\exists k_0 \in X$ *such that* $k_0 \mathscr{R} Q k_0$

(2) $\mathscr{R}$ *is Q-closed*

(3) *Q is an* $(F_w, \mathscr{R})$*-contraction*

(4) (α) ∃ *a subset K of X such that* $Q(X) \subseteq K$ *and K is* $\mathscr{R}$ *-complete,*

(η) *one of these conditions hold:*

(i) *Q is* $\mathscr{R}$*-continuous, or*

(ii) $\mathscr{R} | K$ *is d-self closed on condition that (1.1) with binary relation holds* $\forall k, l \in X$ *with* $k \mathscr{R} l$ *and* $Qk \mathscr{R}^{\sharp} Q\ell$

Then, a fixed point of Q exists. Furthermore, if
(e) $[u, v] \in Fix(Q) \Rightarrow [u, v] \in \mathscr{R}$.
Then, such fixed point of Q is unique.

Theorem 21. *Replace condition (e) of above theorem by:*
(e ∗) $Fix(Q)$ *is* $\mathscr{R}^s$*-connected,*
then Q has a unique fixed point.

Proof. Assume on contrary that Q has more than one fixed point, say u and v with $u \neq v$. Then, there exists a path $\mathscr{R}^s \subseteq Fix(Q)$. As it is from v to u of length q, let us denote the path by $\{v_0, \cdots v_q\}$ such that $v_p \neq v_{p+1}$ for each p where $0 \leq p \leq j-1$. If $v = u$, it is a contradiction. Hence,

$$v_0 = v, v_q = u \text{ and } \left[v_p, v_{p+1}\right] \in \mathscr{R} \text{ for each } p(0 \leq p \leq q-1). \tag{57}$$

As $v_p \in Fix(Q)$, so $Q(v_p) = v_p$ for each $p \in \{0, 1, \cdots, q\}$. With the help of condition (c), we obtain

$$\tau + F\left(d\left(v_p, v_{p+1}\right)\right) \leq F\left(\max\left\{d\left(v_p, v_{p+1}\right), d\left(v_p, v_{p+1}\right), d\left(v_{p+1}, v_{p+1}\right), \cdot \frac{d\left(v_p, v_{p+1}\right) + d\left(v_{p+1}, v_p\right)}{2}\right\}\right). \tag{58}$$

That is,

$$\tau + F\left(d\left(v_p, v_{p+1}\right)\right) \leq F\left(d\left(v_p, v_{p+1}\right)\right). \tag{59}$$

Since $\tau > 0$, our supposition is not true. Hence, Q has a unique fixed point. □

In the next section, we are presenting a significance of our results in ordered metric spaces.

3. Some Consequences in Ordered Metric Spaces

Definition 22. Let (X, d) be a metric space and $(X, \preccurlyeq)$ be an ordered set, then the triplet $(X, d, \preccurlyeq)$ is known as an ordered metric space.

Definition 23. Consider self-mappings $Q, g : X \longrightarrow X$ and an ordered set $(X, \preccurlyeq)$. If, for any $k, l \in X$, $gk \preccurlyeq g\ell$ implies that $Qk \preccurlyeq Q\ell$. Then, Q is g-increasing.

Remark 24. Notice that the notion of Q is g-increasing is equal to say that $\preccurlyeq$ is (Q, g)-closed.

Taking $\mathscr{R} = \preccurlyeq$ in Theorem 17 to 19 and with the help of Remark 24, we state the following result.

Corollary 25. *Consider self-mappings* $Q, g : X \longrightarrow X$ *and an ordered metric space* $(X, d, \preccurlyeq)$*. Assume that:*

(a) $\exists k_0 \in X$ *such that* $gk_0 \preccurlyeq Qk_0$

(b) *Qisg-increasing*

(c) *There are* $\tau > 0$ *and* $F \in \mathscr{F}$ *so that*

$$\tau + F(d(Qk, Q\ell)) \leq F\left(\max\left\{(d(gk, g\ell), d(gk, Qk), d(g\ell, Q\ell), \cdot \frac{d(gk, Q\ell) + d(g\ell, Qk)}{2}\right\}\right), \tag{60}$$

(d) ∃ *a subset K of X such that* $Q(X) \subseteq K \subseteq g(X)$ *and K is* $\preccurlyeq$ *-complete*

(e) *Either Q and g are continuous, or Q is* $(g, \preccurlyeq)$*-continuous. Then, a coincidence point of* (Q, g) *exists. Additionally, we suppose that*

(f) *Qu and gv are* $\preccurlyeq$*-comparable for all distinct coincidence points* $u, v \in coin(Q, g)$*, then pair* (Q, g) *has a unique coincidence point*

Furthermore, if Q and g are weakly compatible, then (Q, g) *has a unique common fixed point.*
Taking $\mathscr{R} = \preccurlyeq$ *in Theorem 20 and with the help of Remark 24, we conclude the result given below.*

Corollary 26. *Consider an ordered metric space* $(X, d, \preccurlyeq)$ *and mapping* $Q : X \longrightarrow X$*. Suppose the that conditions given below are fulfilled:*

(a) $\exists k_0 \in X$ *such that* $k_0 \preccurlyeq Qk_0$

(b) *Q is* $\preccurlyeq$*-increasing*

(c) $\exists \tau > 0$ *and* $F \in \mathscr{F}$ *such that*

$$\tau + F(d(Qk, Q\ell)) \leq F\left(\max\left\{(d(k, l), d(k, Qk), (l, Q\ell), \cdot \frac{d(k, Q\ell) + d(l, Qk)}{2}\right\}\right), \tag{61}$$

(d) *A subset K of X exists such that* $Q(X) \subseteq K$ *and K is* $\preccurlyeq$ *-complete*

(e) *Q is ≼-continuous. Then a fixed point of Q exists. Furthermore,*

(f) *if for any two fixed points $u, v \in Q$ we have $[u, v] \in \preccurlyeq$, then Q has a unique fixed point*

4. Applications to Metric Spaces Endowed with a Graph

Jachymski [30] in 2008 has instituted the idea of metric spaces endowed with a graph in order to generalize the idea of a partial ordering and specified the Banach contraction principle in metric spaces and partially ordered metric spaces. In this section, we are going to present an application of our results in the situating of complete metric spaces endowed with a graph.

Corollary 27. *Consider self-mappings $Q, g : X \longrightarrow X$ on a metric space (X, d) endowed with a graph $G = (V(G), E(G))$. Define ≼ on X as $u \preccurlyeq v$ if and only if there is an edge between u and v. Assume that all the conditions given in Corollary 25 are satisfied. Then a coincidence point of (Q, g) exists. Further, if we suppose that Qu and gv are comparable on edges for all distinct coincidence points $u, v \in coin(Q, g)$, then the pair (Q, g) has a unique coincidence point.*

Furthermore, a unique common fixed point of (Q, g) exists if Q and g are weakly compatible.

Corollary 28. *Consider a metric space (X, d) endowed with a graph G and a mapping $Q : X \longrightarrow X$. Define ≼ on X as $u \preccurlyeq v$ if and only if there is an edge between u and v. Suppose that conditions given in Corollary 27 are fulfilled. Then, a fixed point of Q exists. Furthermore, if $u, v \in Fix(Q)$ are such that there is an edge between u and v, then a unique fixed point of Q exists.*

5. Applications to Integral Equations

In this section, we present an application of Theorem 21 by finding a solution of the integral equation of Volterra type given below:

$$u(t) = \int_0^t K(t, s, u(s))ds + h(t), \qquad t \in [0, 1]. \tag{62}$$

Here, $K : [0, 1] \times [0, 1] \times \mathbb{R} \longrightarrow \mathbb{R}$ and $h : [0, 1] \longrightarrow \mathbb{R}$.

Let X be the Banach space of all continuous functions $u : [0, 1] \longrightarrow \mathbb{R}$. Define a norm on X as follows.

$\|u\| = \max_{t\in[0,1]} |u(t)|$. Then, the metric d on X is defined as $d(u, v) = \|u - v\| \forall u, v \in X$.

Definition 29. A function $\alpha \in X$ such that

$$\alpha(t) \le \int_0^t K(t, s, \alpha(s))ds + h(t), \qquad t \in [0, 1], \tag{63}$$

is called a lower solution for (62).

Definition 30. A function $\beta \in X$ such that

$$\beta(t) \ge \int_0^t K(t, s, \beta(s))ds + h(t), \qquad t \in [0, 1], \tag{64}$$

is called an upper solution for (62).

Now, we have enough material to prove the following results.

Theorem 31. *Assume that in third variable K is nondecreasing and there is $\tau > 0$ such that*

$$|K(t, s, u) - K(t, s, v)| \le \frac{|u - v|}{\tau \mathcal{D}(u, v) + 1}, \tag{65}$$

for all $t, s \in [0, 1]$ and $u, v \in X$, where $\mathcal{D}(u, v) = \max\{d(u, v), d(u, Qu), (v, Qv), ((d(u, Qv) + d(v, Qu))/2)\}$. Then, the existence of a unique solution of the integral Equation (62) follows from the existence of lower solution of (62).

Proof. Let $Q(u(t)) = \int_0^t K(t, s, u(s))ds + h(t)$ for all $u \in X$, be a self operator on X. It is clear that u is a fixed point of the operator Q if and only if it is solution of the Equation (62).

Let $\mathcal{R}$ be the binary relation on X defined by

$$\mathcal{R} = \{(u, v) \in X \times X : u(t) \le v(t) \text{ for all t} \in [0, 1]\}. \tag{66}$$

Now, for any $u, v \in \mathcal{R}$ and for all $t \in [0, 1]$

$$\begin{aligned} Q(u(t)) &= \int_0^t K(t, s, u(s))ds + h(t) \le \int_0^t K(t, s, v(s))ds + h(t) \\ &= Q(v(t)). \end{aligned} \tag{67}$$

This implies that $(Qu, Qv) \in \mathcal{R}$. That is, $\mathcal{R}$ is Q closed.

Now, let $(u, v) \in \mathcal{R}$ and consider

$$\begin{aligned} |Q(u(t)) - Q(v(t))| &= \left| \int_0^t (K(t, s, u(s)) - K(t, s, v(s)))ds \right| \\ &\le \int_0^t |K(t, s, u(s)) - K(t, s, v(s))|ds \\ &\le \int_0^t \frac{|u - v|}{\tau\mathcal{D}(u, v) + 1} ds \le \frac{1}{\tau\mathcal{D}(u, v) + 1} \\ &\quad \cdot \int_0^t \max_{t\in[0,1]} |u(t) - v(t)| \Big) ds \\ &= \frac{1}{\tau\mathcal{D}(u, v) + 1} \int_0^t d(u, v)ds \le \frac{1}{\tau\mathcal{D}(u, v) + 1} \\ &\quad \cdot \int_0^t \mathcal{D}(u, v)ds = \frac{\mathcal{D}(u, v)}{\tau\mathcal{D}(u, v) + 1} t, \\ &\le \frac{\mathcal{D}(u, v)}{\tau\mathcal{D}(u, v) + 1} \text{ since } t \in [0, 1]. \end{aligned} \tag{68}$$

Therefore, we have

$$|Q(u(t)) - Q(v(t))| \leq \frac{\mathscr{D}(u, v)}{\tau\mathscr{D}(u, v) + 1}, \quad \forall t \in [0, 1]. \tag{69}$$

On taking supremum on both sides of above inequality, we obtain

$$\|Q(u) - Q(v)\| \leq \frac{\mathscr{D}(u, v)}{\tau\mathscr{D}(u, v) + 1}. \tag{70}$$

It yields that

$$\tau - \frac{1}{\|Q(u) - Q(v)\|} \leq \frac{-1}{\mathscr{D}(u, v)}, \tag{71}$$

or

$$\tau - \frac{1}{d(Q(u), Q(v))} \leq \frac{-1}{\mathscr{D}(u, v)}. \tag{72}$$

By choosing $F(\mu) = -1/\mu$, $\mu > 0$, from the above inequality, we get

$$\tau + F(d(Q(u), Q(v))) \leq F(\mathscr{D}(u, v)). \tag{73}$$

Hence, inequality (6) is satisfied. We have defined binary relation $\mathscr{R}$ on X by $u\mathscr{R}v$ if and only if $u(t) \leq v(t)$ for all $t \in [0, 1]$. Now, consider an $\mathscr{R}$-preserving sequence $\{u_n\}$ in $C[0, 1]$ which converges to $u \in X$. Then, we have

$$u_0(t) \leq u_1(t) \leq \cdots u_n(t) \leq u_{t+1}(s) \leq \cdots, \tag{74}$$

which gives us $u_n(t) \leq u(t) \forall t \in [0, 1]$. Therefore, $\mathscr{R}$ is d self closed on X. To show that $\text{Fix}(Q)$ is $\mathscr{R}^s$-connected, if $u, v \in \text{Fix}(Q)$, then $w = \max\{u, v\} \in C[0, 1]$. Since $u \leq w$ and $v \leq w$, thus $u\mathscr{R}w$ and $v\mathscr{R}w$. Therefore, all conditions of Theorem 21 are true. Hence, the conclusion holds. □

Now, in the situation where upper solution is presented, we have the following result.

Theorem 32. *Consider that in third variable K is nonincreasing and there is $\tau > 0$ such that*

$$|K(t, s, u) - K(t, s, v)| \leq \frac{|u(t) - v(t)|}{\tau\mathscr{D}(u, v) + 1}, \tag{75}$$

for all $t, s \in [0, 1]$ and $u, v \in X$, where $\mathscr{D}(u, v) = \max\{(d(u, v), d(u, Qu), d(v, Qv), ((d(u, Qv) + d(v, Qu))/2)\}$. Then, the existence of a unique solution of the integral Equation (62) follows from the existence of an upper solution of (62).

Proof. Let the binary relation on X be defined by

$$\mathscr{R} = \{(u, v) \in X \times X : u(s) \geq v(s) \text{ for all } t \in [0, 1]\}. \tag{76}$$

Now, proceeding as in Theorem 31, we can conclude that all the assumptions of Theorem 21 are satisfied and it guarantees the existence of a unique solution of (62). □

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally in writing this article. All authors read and approved the final manuscript.

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