Hindawi
Journal of Function Spaces
Volume 2021, Article ID 9943199, 11 pages
https://doi.org/10.1155/2021/9943199

*Research Article*

# Characterization and Stability of Multimixed Additive-Quartic Mappings: A Fixed Point Application

**Abasalt Bodaghi**,[1] **Idham Arif Alias**,[2] **Lida Mousavi**,[3] **and Sedigheh Hosseini**[4]

[1]*Department of Mathematics, Garmsar Branch, Islamic Azad University, Garmsar, Iran*
[2]*Department of Mathematics, Faculty of Science, Universiti Putra Malaysia, 43400 UPM Serdang, Selangor Darul Ehsan, Malaysia*
[3]*Department of Mathematics, Yadegar-e-Imam Khomeini (RAH), Shahr-e-Rey Branch, Islamic Azad University, Tehran, Iran*
[4]*Department of Mathematics, Kermanshah Branch, Islamic Azad University, Kermanshah, Iran*

Correspondence should be addressed to Abasalt Bodaghi; abasalt.bodaghi@gmail.com
and Idham Arif Alias; idham_aa@upm.edu.my

Received 25 March 2021; Accepted 1 November 2021; Published 12 November 2021

Academic Editor: Zoran Mitrovic



In this article, we introduce the multi-additive-quartic and the multimixed additive-quartic mappings. We also describe and characterize the structure of such mappings. In other words, we unify the system of functional equations defining a multi-additive-quartic or a multimixed additive-quartic mapping to a single equation. We also show that under what conditions, a multimixed additive-quartic mapping can be multiadditive, multiquartic, and multi-additive-quartic. Moreover, by using a fixed point technique, we prove the Hyers-Ulam stability of multimixed additive-quartic functional equations thus generalizing some known results.

## 1. Introduction

Let $V$ be a commutative group, $W$ be a linear space over rational numbers, and $n$ be an integer with $n \geq 2$. A mapping $f : V^n \longrightarrow W$ is called

(i) *Multiadditive* if it satisfies the Cauchy's functional equation $A(x+y) = A(x) + A(y)$ in each variable [1]

(ii) *Multiquadratic* if it fulfills quadratic functional equation $Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y)$ in each variable [2, 3]

(iii) *Multicubic* if it satisfies the cubic equation $C(2x+y) + C(2x-y) = 2C(x+y) + 2C(x-y) + 12C(x)$ in each variable [4, 5]

(iv) *Multiquartic* if it satisfies the quartic equation

$$\mathfrak{Q}(x+2y) + \mathfrak{Q}(x-2y) = 4\mathfrak{Q}(x+y) + 4\mathfrak{Q}(x-y) - 6\mathfrak{Q}(x) + 24\mathfrak{Q}(y), \quad (1)$$

in each variable [6, 7].

We have the following observations about a several variables mapping $f : V^n \longrightarrow W$.

(i) $f$ is multiadditive [1] if and only if it satisfies

$$f(x_1 + x_2) = \sum_{j_1,\cdots,j_n \in \{1,2\}} f\left(x_{1j_1}, \cdots, x_{nj_n}\right). \quad (2)$$

(ii) $f$ is multiquadratic [8] if and only if it satisfies

$$\sum_{s\in\{-1,1\}^n} f(x_1+sx_2)=2^n\sum_{j_1,\cdots,j_n\in\{1,2\}} f\left(x_{1j_1},\cdots,x_{nj_n}\right), \quad (3)$$

where $x_j=(x_{1j},x_{2j},\cdots,x_{nj})\in V^n$ with $j\in\{1,2\}$. More information about the structure of multiadditive and multiquadratic mappings, we refer for instance to [9, 10].

Bodaghi et al. [4] (resp., [6]) provided a characterization of multicubic (resp., multiquartic) mappings, and they showed that every multicubic (resp., multiquartic) mapping can be shown a single functional equation and vice versa.

Lee et al. [11] introduced and obtained the general solution of the quartic functional equation which somewhat different from (1) as follows:

$$\mathfrak{Q}(2x+y)+\mathfrak{Q}(2x-y)=4\mathfrak{Q}(x+y)+4\mathfrak{Q}(x-y)+24\mathfrak{Q}(x)-6\mathfrak{Q}(y). \quad (4)$$

For the generalized forms of the quartic functional, equations (1) and (4) refer to [12, 13]. Recently, in [14] and motivated by (4), a new form of multiquartic mappings was introduced, and the structure of such mappings was described.

Speaking of the stability of a functional equation, we follow the question raised in 1940 by Ulam [15] for group homomorphisms. Hyers [16] presented a partial solution to the problem of Ulam. Later, Hyers' theorem was extended and generalized in various forms by many mathematicians such as Aoki [17] and Rassias [18]. Recall that a functional equation $\mathfrak{F}$ is said to be *stable* if any mapping $\phi$ fulfilling $\mathfrak{F}$ approximately; then, it is near to an exact solution of $\mathfrak{F}$. Next, several stability problems of various functional equations and mappings have been investigated by many mathematicians which can be found in literatures.

In the last two decades, the stability problem for several variable mappings such as multiadditive, multi-Jensen, multiquadratic, multicubic, and multiquartic mappings by applying direct and fixed point methods has been studied by a number of authors which are available for example in [1, 2, 4, 8, 9, 19–26].

In [27], Eshaghi Gordji introduced and obtained the general solution of the following mixed type additive and quartic functional equation

$$\begin{aligned} f(2x+y)+f(2x-y)&=4[(f(x+y)+f(x-y)]-\frac{3}{7}\\ &\quad\cdot(f(2y)-2f(y))+2f(2x)-8f(x). \end{aligned} \quad (5)$$

He also established the Hyers-Ulam Rassias stability of the above functional equation in real normed spaces. The stability of (5) in non-Archimedean orthogonality spaces is studied in [28]. A different and equivalent form of mixed type additive and quartic functional equation from (5) was introduced by the first author in [29] as follows:

$$\begin{aligned} &f(x+2y)-4f(x+y)-4f(x-y)+f(x-2y)\\ &\quad=\frac{12}{7}(f(2y)-2f(y))-6f(x). \end{aligned} \quad (6)$$

It is easily verified that the function $f(x)=\alpha x^4+\beta x$ is a solution of equations (5) and (6); the generalized version of equation (6) can be found in [30].

This paper is organized as follows: In the second section, we firstly define multi-additive-quartic mappings and include a characterization of such mappings. In fact, we prove that every multi-additive-quartic mapping can be shown a single functional equation and vice versa (under some extra conditions). Section 3 is devoted to the study of stricture of multimixed additive-quartic mappings. In other words, motivated by equation (6), we introduce the multimixed additive-quartic mappings and reduce the system of $n$ equations defining the multimixed additive-quartic mappings to a single equation, namely, the multimixed additive-quartic functional equation. In Section 4, we prove the Hyers-Ulam stability for the multi-additive-quartic and the multimixed additive-quartic mappings in the setting of Banach spaces by applying a fixed point method [31]. As an application of this result, we establish the stability of multi-additive-quartic mappings. Finally, we show that under some mild conditions every multiadditive and multiquartic functional equations are $\delta$-stable for a small positive number $\delta$.

## 2. Characterization of Multi-Additive-Quartic Mappings

Throughout this paper, $\mathbb{N}$ and $\mathbb{Q}$ stand for the set of all positive integers and the rational numbers, respectively, $\mathbb{N}_0:=\mathbb{N}\cup\{0\}$, $\mathbb{R}_+:=[0,\infty)$. For any $l\in\mathbb{N}_0$, $m\in\mathbb{N}$, $t=(t_1,\cdots,t_m)\in\{-1,1\}^m$, and $x=(x_1,\cdots,x_m)\in V^m$, we write $lx:=(lx_1,\cdots,lx_m)$ and $tx:=(t_1x_1,\cdots,t_mx_m)$, where $ra$ stands, as usual, for the $r$th power of an element $a$ of the commutative group $V$.

Let $V$ and $W$ be linear spaces, $n\in\mathbb{N}$ and $k\in\{0,\cdots,n\}$. A mapping $f: V^n\longrightarrow W$ is called $k$-additive and $n-k$-quartic (briefly, multi-additive-quartic) if $f$ is additive in each of some $k$ variables and satisfies (4) in each of the other variables. In what follows, for simplicity, it is assume that $f$ is additive in each of the first $k$ variables. Moreover, for $k=n$ ($k=0$), the above definition leads to the so-called multiadditive (multiquartic) mappings.

In the sequel, we assume that $V$ and $W$ are vector spaces over $\mathbb{Q}$. Moreover, we identify $x=(x_1,\cdots,x_n)\in V^n$ with $(x^k,x^{n-k})\in V^k\times V^{n-k}$, where $x^k:=(x_1,\cdots,x_k)$ and $x^{n-k}:=(x_{k+1},\cdots,x_n)$. Let $n\in\mathbb{N}$ with $n\geq 2$ and $x_i^n=(x_{i1},x_{i2},\cdots,x_{in})\in V^n$, where $i\in\{1,2\}$. Throughout, we shall denote $x_i^n$ by $x_i$ if there is no risk of mistake. Put also $x_i^k=(x_{i1},\cdots,x_{ik})\in V^k$ and $x_i^{n-k}=(x_{i,k+1}\cdots,x_{in})\in V^{n-k}$. For $x_1,x_2\in V^n$ and $p_i\in\mathbb{N}_0$ with $0\leq p_i\leq n$ and $0\leq k\leq n-1$, set $\mathcal{N}^{n-k}=\{\mathfrak{N}_{n-k}=(N_{k+1},\cdots,N_n)\mid N_j\in\{x_{1j}\pm x_{2j},x_{1j},x_{2j}\}\}$, where $j\in\{k+1,\cdots,n\}$. Consider the subset $\mathcal{N}^{n-k}_{(p_1,p_2)}$ of $\mathcal{N}$ as follows:

$$\mathcal{N}_{(p_1,p_2)}^{n-k} := \left\{ \mathfrak{N}_{n-k} \in \mathcal{N}^{n-k} \mid \operatorname{Card}\{N_j : N_j = x_{ij}\} = p_i (i \in \{1,2\}) \right\}. \tag{7}$$

To achieve our aims, for the multi-additive-quartic mappings, we use the oncoming notations:

$$f\left(\mathcal{N}_{(p_1,p_2)}^{n-k}\right) := \sum_{\mathfrak{N}_{n-k} \in \mathcal{N}_{(p_1,p_2)}^{n-k}} f(\mathfrak{N}_n), \tag{8}$$

$$f\left(z, \mathcal{N}_{(p_1,p_2)}^{n-k}\right) := \sum_{\mathfrak{N}_{n-k} \in \mathcal{N}_{(p_1,p_2)}^{n-k}} f(z, \mathfrak{N}_{n-k})(z \in V). \tag{9}$$

For each $x_1, x_2 \in V^n$, we consider the equation

$$\begin{aligned} &\sum_{t \in \{-1,1\}^{n-k}} f\left(x_1^k + x_2^k, 2x_1^{n-k} + t x_2^{n-k}\right) \\ &= \sum_{p_1=0}^{n-k} \sum_{p_2=0}^{n-k-p_1} \sum_{i \in \{1,2\}} 4^{n-k-p_1-p_2} 24^{p_1} (-6)^{p_2} f\left(x_i^k, \mathcal{N}_{(p_1,p_2)}^{n-k}\right), \end{aligned} \tag{10}$$

for all $x_i^k = (x_{i1}, \cdots, x_{ik}) \in V^k$ and $x_i^{n-k} = (x_{i,k+1} \cdots, x_{\text{in}}) \in V^{n-k}$ where $i \in \{1,2\}$.

It is shown in Proposition 2.2 in [14] that if a mapping $f : V^n \longrightarrow W$ is multiquartic, then it satisfies the equation

$$\sum_{t \in \{-1,1\}^n} f(2x_1 + t x_2) = \sum_{p_1=0}^{n} \sum_{p_2=0}^{n-p_1} 4^{n-p_1-p_2} 24^{p_1} (-6)^{p_2} f\left(\mathcal{N}_{(p_1,p_2)}^{n-k}\right). \tag{11}$$

The next proposition shows that the system of $n$ equations defining a multi-additive-quartic mapping can be reduced to (10).

**Proposition 1.** *Let $n \in \mathbb{N}$ and $k \in \{0, \cdots, n\}$. Suppose that a mapping $f : V^n \longrightarrow W$ is $k$-additive and $n-k$-quartic (multi-additive-quartic) mapping. Then, $f$ fulfills equation (10).*

*Proof.* For $k \in \{0, n\}$, the result follows from Proposition 2.2 in [14] and Theorem 2 in [1], and so we prove the assertion for the case that $k \in \{1, \cdots, n-1\}$. For any $x^{n-k} \in V^{n-k}$, consider the mapping $g_{x^{n-k}} : V^k \longrightarrow W$ defined by $g_{x^{n-k}}(x^k) := f(x^k, x^{n-k})$ for $x^k \in V^k$. The assumption shows that $g_{x^{n-k}}$ is $k$-additive, and thus, we can obtain from Theorem 2 in [1] that

$$g_{x^{n-k}}\left(x_1^k + x_2^k\right) = \sum_{j_1,j_2,\cdots,j_k \in \{1,2\}} g_{x^{n-k}}\left(x_{j_1 1}, x_{j_2 2}, \cdots, x_{j_k k}\right), \left(x_1^k, x_2^k \in V^k\right). \tag{12}$$

The above equality implies that

$$f\left(x_1^k + x_2^k, x^{n-k}\right) = \sum_{j_1,\cdots,j_k \in \{1,2\}} f\left(x_{j_1 1}, \cdots, x_{j_k k}, x^{n-k}\right), \tag{13}$$

for all $x_1^k, x_2^k \in V^k$ and $x^{n-k} \in V^{n-k}$. Repeat the above method, and for any $x^k \in V^k$, define the mapping $h_{x^k} : V^{n-k} \longrightarrow W$ via $h_{x^k}(x^{n-k}) := f(x^k, x^{n-k}), x^{n-k} \in V^{n-k}$. This mapping is $n-k$-quartic, and hence, by Proposition 2.2 from [14], we have

$$\begin{aligned} &\sum_{t \in \{-1,1\}^{n-k}} h_{x^k}\left(2x_1^{n-k} + t x_2^{n-k}\right) \\ &= \sum_{p_1=0}^{n-k} \sum_{p_2=0}^{n-k-p_1} 4^{n-k-p_1-p_2} 24^{p_1} (-6)^{p_2} h_{x^k}\left(\mathcal{N}_{(p_1,p_2)}^{n-k}\right), \end{aligned} \tag{14}$$

for all $x_1^{n-k}, x_2^{n-k} \in V^{n-k}$. On the other hand, by the definition of $h_{x^k}$, relation (14) converts to

$$\begin{aligned} &\sum_{t \in \{-1,1\}^{n-k}} f\left(x^k, 2x_1^{n-k} + t x_2^{n-k}\right) \\ &= \sum_{p_1=0}^{n-k} \sum_{p_2=0}^{n-k-p_1} 4^{n-k-p_1-p_2} 24^{p_1} (-6)^{p_2} f\left(x^k, \mathcal{N}_{(p_1,p_2)}^{n-k}\right), \end{aligned} \tag{15}$$

for all $x_1^{n-k}, x_2^{n-k} \in V^{n-k}$ and $x^k \in V^k$. It now follows between (13) and (15) that

$$\begin{aligned} &\sum_{t \in \{-1,1\}^{n-k}} f\left(x_1^k + x_2^k, 2x_1^{n-k} + t x_2^{n-k}\right) \\ &= \sum_{p_1=0}^{n-k} \sum_{p_2=0}^{n-k-p_1} 4^{n-k-p_1-p_2} 24^{p_1} (-6)^{p_2} f\left(x_1^k + x_2^k, \mathcal{N}_{(p_1,p_2)}^{n-k}\right) \\ &= \sum_{p_1=0}^{n-k} \sum_{p_2=0}^{n-k-p_1} \sum_{j_1,j_2,\cdots,j_n \in \{1,2\}} 4^{n-k-p_1-p_2} 24^{p_1} (-6)^{p_2} f \\ &\quad \cdot \left(x_{j_1 1}, x_{j_2 2}, \cdots, x_{j_n n}, \mathcal{N}_{(p_1,p_2)}^{n-k}\right) \\ &= \sum_{p_1=0}^{n-k} \sum_{p_2=0}^{n-k-p_1} \sum_{i \in \{1,2\}} 4^{n-k-p_1-p_2} 24^{p_1} (-6)^{p_2} f\left(x_i^k, \mathcal{N}_{(p_1,p_2)}^{n-k}\right), \end{aligned} \tag{16}$$

for all $x_i^k = (x_{i1}, \cdots, x_{ik}) \in V^k$ and $x_i^{n-k} = (x_{i,k+1} \cdots, x_{\text{in}}) \in V^{n-k}$. This finishes the proof. □

By Proposition 6, it is easily verified that the mapping $f(z_1, \cdots, z_n) = c\prod_{i=1}^k z_i \prod_{j=k+1}^n z_j^4$ satisfies (10), and so this equation is said to be *multi-additive-quartic functional equation*.

*Definition 2.* Let $r \in \mathbb{N}$. Consider a mapping $f : V^n \longrightarrow W$. We say $f$

(i) Satisfies (has) the *r -power condition* in the *j*th variable if

$$f\left(z_1,\cdots,z_{j-1},2z_j,z_{j+1},\cdots,z_n\right)=2^r f\left(z_1,\cdots,z_{j-1},z_j,z_{j+1},\cdots,z_n\right), \tag{17}$$

for all $z_1,\cdots,z_n\in V^n$. Sometimes 4-power condition is called *quartic condition*.

(ii) Has *zero condition* if $f(x)=0$ for any $x\in V^n$ with at least one component which is equal to zero

We remember that the binomial coefficient for all $n,r\in\mathbb{N}_0$ with $n\geq r$ is defined and denoted by $\begin{pmatrix} n \\ r \end{pmatrix}:=n!/r!(n-r)!$.

We wish to show that if a mapping satisfies equation (10), then it is multi-additive-quartic. For doing it, we need the upcoming lemma. The method of the proof of Lemma 3 is similar to the proof of ([14], Lemma 2.5) and so we include lemma without the proof.

**Lemma 3.** *Suppose that a mapping $f:V^n\longrightarrow W$ satisfies equation (10). Under one of the following assumptions, f satisfying zero condition.*

*(i) $f$ satisfies the quartic condition in the last $n-k$ variables*

*(ii) $f$ is even in the last $n-k$ variables*

**Theorem 4.** *Suppose that a mapping $f:V^n\longrightarrow W$ fulfilling equation (10). Under one of the hypothesis of Lemma 3, $f$ is multi-additive-quartic.*

*Proof.* It follows from Lemma 3; $f$ satisfies zero condition. Putting $x_2^{n-k}=(0,\cdots,0)$ in the left side of (10) and applying the hypothesis, we obtain

$$2^{n-k}\times 2^{4(n-k)}f\left(x_1^k+x_2^k,x_1^{n-k}\right)=2^{5(n-k)}f\left(x_1^k+x_2^k,2x_1^{n-k}\right). \tag{18}$$

On the other hand, by using Lemma 3, the right side of (10) converts to

$$\begin{aligned}&\sum_{p_1=0}^{n-k}\begin{pmatrix} n-k \\ p_1 \end{pmatrix}4^{n-k-p_1}24^{p_1}2^{n-k-p_1}f\left(x_{j_1 1},x_{j_2 2},\cdots,x_{j_k k},x_1^{n-k}\right)\\&=\sum_{p_1=0}^{n-k}\begin{pmatrix} n-k \\ p_1 \end{pmatrix}8^{n-k-p_1}24^{p_1}f\left(x_{j_1 1},x_{j_2 2},\cdots,x_{j_k k},x_1^{n-k}\right)\\&=2^{5(n-k)}\sum_{j_1,j_2,\cdots,j_k\in\{1,2\}}f\left(x_{j_1 1},x_{j_2 2},\cdots,x_{j_k k},x_1^{n-k}\right).\end{aligned} \tag{19}$$

Now, relations (18) and (19) necessitate that

$$f\left(x_1^k+x_2^k,x_1^{n-k}\right)=\sum_{j_1,j_2,\cdots,j_n\in\{1,2\}}f\left(x_{j_1 1},x_{j_2 2},\cdots,x_{j_n n},x_1^{n-k}\right), \tag{20}$$

for all $x_1^k,x_2^k\in V^n$ and $x_1^{n-k}\in V^{n-k}$. In light of Theorem 2 in [1], we see that $f$ is additive in each of the $k$ first variables. In addition, by considering $x_2^k=(0,\cdots,0)$ in (10) and applying again Lemma 3, we have

$$\begin{aligned}&\sum_{t\in\{-1,1\}^{n-k}}f\left(x_1^k,2x_1^{n-k}+tx_2^{n-k}\right)\\&=\sum_{p_1=0}^{n-k}\sum_{p_2=0}^{n-k-p_1}4^{n-k-p_1-p_2}24^{p_1}(-6)^{p_2}f\left(x_1^k,\mathcal{N}^{n-k}_{(p_1,p_2)}\right),\end{aligned} \tag{21}$$

for all $x_1^k\in V^k$ and $x_1^{n-k},x_2^{n-k}\in V^{n-k}$, and thus, by Theorem 2.6 in [14], $f$ is quartic in each of the last $n-k$ variables. The proof of second part is similar. □

## 3. Characterization of Multimixed Additive-Quartic Mappings

In this section, we introduce the multimixed additive-quartic mappings and then characterize them as an equation. We start this section with the definition of such mappings.

*Definition 5.* Let $V$ and $W$ be vector spaces over $\mathbb{Q}$, $n\in\mathbb{N}$. A mapping $f:V^n\longrightarrow W$ is called *n -multimixed additive-quartic* or briefly *multimixed additive-quartic* if $f$ satisfies mixed additive-quartic equation (6) in each variable.

Let $n\in\mathbb{N}$ with $n\geq 2$ and $x_i^n=(x_{i1},x_{i2},\cdots,x_{in})\in V^n$, where $i\in\{1,2\}$. For $x_1,x_2\in V^n$ and $q\in\mathbb{N}_0$ with $0\leq q\leq n$, put

$$\mathcal{M}=\left\{\mathfrak{M}_n=(M_1,\cdots,M_n)\mid M_j\in\left\{x_{1j}\pm 2x_{2j},2x_{2j}\right\},j\in\{1,\cdots,n\}\right\}. \tag{22}$$

Consider the subset $\mathcal{M}_q^n$ of $\mathcal{M}$ as follows:

$$\mathcal{M}_q^n:=\left\{\mathfrak{M}_n\in\mathcal{M}\mid \mathrm{Card}\left\{M_j:M_j=2x_{2j}\right\}=q\right\}. \tag{23}$$

Hereafter, for the multimixed additive-quartic mappings, we use the following notations:

$$f\left(\mathcal{M}_q^n\right):=\sum_{\mathfrak{M}_n\in\mathcal{M}_q^n}f(\mathfrak{M}_n), \tag{24}$$

$$f\left(\mathcal{M}_q^n,z\right):=\sum_{\mathfrak{M}_n\in\mathcal{M}_q^n}f(\mathfrak{M}_n,z)(z\in V). \tag{25}$$

Next, we reduce the system of $n$ equations defining the multimixed additive-quartic mapping to obtain a single functional equation.

**Proposition 6.** *If a mapping $f: V^n \longrightarrow W$ is multimixed additive-quartic, it satisfies the equation*

$$\sum_{q=0}^{n}\left(-\frac{12}{7}\right)^{q} f\left(\mathscr{M}_q^n\right) = \sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1} 4^{n-p_1-p_2}(-6)^{p_1}\left(-\frac{24}{7}\right)^{p_2} f\left(\mathscr{N}_{(p_1,p_2)}^n\right), \tag{26}$$

*where $f(\mathscr{M}_q^n)$ and $f(\mathscr{N}_{(p_1,p_2)}^n)$ are defined in (24) and (8), respectively.*

*Proof.* The proof is based on induction for $n$. For $n=1$, it is obvious that $f$ satisfies (6). Assume that (26) holds for some positive integer $n>1$. Then

$$\begin{aligned}
&\sum_{q=0}^{n+1}\left(-\frac{12}{7}\right)^{q+1} f\left(\mathscr{M}_q^{n+1}\right) \\
&= \sum_{q=0}^{n}\sum_{t\in\{1,-1\}}\left(-\frac{12}{7}\right)^{q} f\left(\mathscr{M}_q^n, x_{1n+1}+2tx_{2,n+1}\right) \\
&\quad - \frac{12}{7}\sum_{q=0}^{n}\left(-\frac{12}{7}\right)^{q} f\left(\mathscr{M}_q^n, 2x_{2,n+1}\right) \\
&= \sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1}\sum_{t\in\{1,-1\}} 4^{n-p_1-p_2}(-6)^{p_1}\left(-\frac{24}{7}\right)^{p_2} f \\
&\quad \cdot\left(\mathscr{N}_{(p_1,p_2)}^n, x_{1,n+1}+2tx_{2,n+1}\right) - \frac{12}{7}\sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1} 4^{n-p_1-p_2}(-6)^{p_1} \\
&\quad \cdot\left(-\frac{24}{7}\right)^{p_2} f\left(\mathscr{N}_{(p_1,p_2)}^n, 2x_{2,n+1}\right) \\
&= 4\sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1}\sum_{s\in\{1,-1\}} 4^{n-p_1-p_2}(-6)^{p_1}\left(-\frac{24}{7}\right)^{p_2} f \\
&\quad \cdot\left(\mathscr{N}_{(p_1,p_2)}^n, x_{1,n+1}+sx_{1,n+1}\right) - 6\sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1} 4^{n-p_1-p_2}(-6)^{p_1} \\
&\quad \cdot\left(-\frac{24}{7}\right)^{p_2}\left(f\left(\mathscr{N}_{(p_1,p_2)}^n, x_{1,n+1}\right) - \frac{24}{7}\sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1} 4^{n-p_1-p_2}(-6)^{p_1}\right. \\
&\quad \cdot\left(-\frac{24}{7}\right)^{p_2} f\left(\mathscr{N}_{(p_1,p_2)}^n, x_{2,n+1}\right) = \sum_{p_1=0}^{n+1}\sum_{p_2=0}^{n+1-p_1} 4^{n+1-p_1-p_2}(-6)^{p_1} \\
&\quad \cdot\left(-\frac{24}{7}\right)^{p_2} f\left(\mathscr{N}_{(p_1,p_2)}^{n+1}\right).
\end{aligned} \tag{27}$$

The assertion is now proved. □

Since the mapping $f(z_1,\cdots,z_n)=\prod_{j=1}^{n}(a_j z_j^4 + b_j z_j)$ is multimixed additive-quartic, it satisfies (26) by proposition above, and so this equation is called *multimixed additive-quartic functional equation*.

Here, we bring an elementary lemma without proof.

**Lemma 7.** *Let $n, k, p_l \in \mathbb{N}_0$, such that $k+\sum_{l=1}^{m} p_l \le n$, where $l \in \{1,\cdots,m\}$. Then*

$$\begin{aligned}
&\binom{n-k}{n-k-\sum_{l=1}^{m} p_l}\binom{\sum_{l=1}^{m} p_l}{\sum_{l=1}^{m-1} p_l}\cdots\binom{p_1+p_2}{p_1} \\
&= \binom{n-k}{p_1}\binom{n-k-p_1}{p_2}\cdots\binom{n-k-\sum_{l=1}^{m-1} p_l}{p_m}.
\end{aligned} \tag{28}$$

Similar to Lemma 2.1 from [6], we need the following lemma in obtaining our goal in this section. The proof is similar, but we include some parts for the sake of completeness.

**Lemma 8.** *If a mapping $f: V^n \longrightarrow W$ satisfies equation (26), then it has zero condition.*

*Proof.* Putting $x_1 = x_2 = (0,\cdots,0)$ in (26), we have

$$\begin{aligned}
&\left[\sum_{q=0}^{n}\binom{n}{n-q}\left(-\frac{12}{7}\right)^{q} 2^{n-q}\right] f(0,\cdots,0) \\
&= \left[\sum_{p_2=0}^{n}\sum_{p_1=0}^{n-p_2}\binom{n}{n-p_1-p_2}\binom{p_1+p_2}{p_1} 2^{n-p_1-p_2} 4^{n-p_1-p_2}(-6)^{p_1}\right. \\
&\quad \left.\cdot\left(-\frac{24}{7}\right)^{p_2}\right] f(0,\cdots,0).
\end{aligned} \tag{29}$$

Using Lemma 7 for $k=0$ and $p_1, p_2$, the right side of (29) will be as follows:

$$\begin{aligned}
&\left[\sum_{p_2=0}^{n}\sum_{p_1=0}^{n-p_2}\binom{n}{n-p_1-p_2}\binom{p_1+p_2}{p_1} 2^{n-p_1-p_2} 4^{n-p_1-p_2}(-6)^{p_1}\left(-\frac{24}{7}\right)^{p_2}\right] f(0,\cdots,0) \\
&= 2^n\left[\sum_{p_2=0}^{n}\binom{n}{p_2}\left(-\frac{12}{7}\right)^{p_2}\sum_{p_1=0}^{n-p_2}\binom{n-p_2}{p_1} 4^{n-p_1-p_2}(-3)^{p_1}\right] f(0,\cdots,0) \\
&= 2^n\left[\sum_{p_2=0}^{n}\binom{n}{p_2}\left(-\frac{12}{7}\right)^{p_2}(4-3)^{n-p_2}\right] f(0,\cdots,0) = 2^n\left(-\frac{12}{7}+1\right)^{n} f(0,\cdots,0) \\
&= \left(-\frac{10}{7}\right)^{n} f(0,\cdots,0).
\end{aligned} \tag{30}$$

On the other hand, by a simple computation, the left side of (29) is

$$\left(\frac{2}{7}\right)^{n} f(0,\cdots,0). \tag{31}$$

It follows from relations (29), (30), and (31) that $f(0,\cdots,0)=0$. One can continue this method to show that $f$ has zero condition. □

*Definition 9.* A mapping $f : V^n \longrightarrow W$ is

(iii) Odd in the $j$th variable if

$$f(z_1, \cdots, z_{j-1}, -z_j, z_{j+1}, \cdots, z_n) = -f(z_1, \cdots, z_{j-1}, z_j, z_{j+1}, \cdots, z_n). \tag{32}$$

(iv) Even in the $j$th variable if

$$f(z_1, \cdots, z_{j-1}, -z_j, z_{j+1}, \cdots, z_n) = f(z_1, \cdots, z_{j-1}, z_j, z_{j+1}, \cdots, z_n). \tag{33}$$

**Proposition 10.** *Suppose that a mapping $f : V^n \longrightarrow W$ satisfies equation (26). Then, it is multimixed additive-quartic. Moreover,*

*(i) If $f$ is odd in a variable, then it is additive in the same variable*

*(ii) If $f$ is even in a variable, then it is quartic in the same variable*

*Proof.* Let $j \in \{1, \cdots, n\}$ be arbitrary and fixed. Set

$$f_j^*(z) := f(z_1, \cdots, z_{j-1}, z, z_{j+1}, \cdots, z_n). \tag{34}$$

Putting $x_{2k} = 0$ for all $k \in \{1,\cdots,n\} \setminus \{j\}$ in (26) and using Lemma 8, we get

$$\begin{aligned}
&2^{n-1}\left[f_j^*(z+2w) + f_j^*(z-2w) - \frac{12}{7} f_j^*(2w)\right] \\
&= \left[\sum_{p_1=0}^{n-1} \binom{n-1}{p_1} 4^{n-p_1}(-6)^{p_1} 2^{n-p_1-1}\right]\left(f_j^*(z+w) + f_j^*(z-w)\right) \\
&\quad + \left[\sum_{p_1=1}^{n} \binom{n-1}{p_1-1} 4^{n-p_1}(-6)^{p_1} 2^{n-p_1}\right] f_j^*(z) \\
&\quad + \left[\sum_{p_1=0}^{n} \binom{n-1}{p_1} 4^{n-p_1-1}(-6)^{p_1}\left(-\frac{24}{7}\right) 2^{n-p_1-1}\right] f_j^*(w) \\
&= 4\left[2^{n-1}\sum_{p_1=0}^{n-1} \binom{n-1}{p_1} 4^{n-1-p_1}(-3)^{p_1}\right]\left(f_j^*(z+w) + f_j^*(z-w)\right) \\
&\quad - 6\left[2^{n-1}\sum_{p_1=0}^{n-1} \binom{n-1}{p_1} 4^{n-1-p_1}(-3)^{p_1}\right] f_j^*(z) - \frac{24}{7} \\
&\quad \cdot \left[2^{n-1}\sum_{p_1=0}^{n-1} \binom{n-1}{p_1} 4^{n-1-p_1}(-3)^{p_1}\right] f_j^*(w) \\
&= 4 \times 2^{n-1}\left(f_j^*(z+w) + f_j^*(z-w)\right) - 6 \times 2^{n-1} f_j^*(z) \\
&\quad - \frac{24}{7} \times 2^{n-1} f_j^*(w).
\end{aligned} \tag{35}$$

The above equalities show that

$$\begin{aligned}
&f_j^*(z+2w) + f_j^*(z-2w) - \frac{12}{7} f_j^*(2w) \\
&\quad = 4\left[f_j^*(z+w) + f_j^*(z-w)\right] - 6 f_j^*(z) - \frac{24}{7} f_j^*(w).
\end{aligned} \tag{36}$$

In other words, (6) is true for $f_j^*$. Since $j$ is arbitrary, $f$ is a multimixed additive-quartic mapping.

(i) Repeating the proof of Lemma 2.1 (i) from [29] for $f_j^*$, we see that $f_j^*(z+w) = f_j^*(z) + f_j^*(w)$. This means that $f$ is additive in the $j$th variable

(ii) Similar to the previous part, it follows from the proof of part (ii) of Lemma 2.1 in [29] that

$$\begin{aligned}
f_j^*(2z+w) + f_j^*(2z-w) &= 4\left[f_j^*(z+w) + f_j^*(z-w)\right] \\
&\quad + 24 f_j^*(z) - 6 f_j^*(w).
\end{aligned} \tag{37}$$

Therefore, $f$ is quartic in the $j$th variable. □

**Corollary 11.** *Suppose a mapping $f : V^n \longrightarrow W$ satisfies equation (26).*

*(i) If $f$ is odd in each variable, then it is multiadditive. Moreover, it satisfies (2)*

*(ii) If $f$ is even in each variable, then it is multiquartic. In particular, it fulfills (11)*

*(iii) If $f$ is odd in each of some $k$ variables and is even in each of the other variables, then it is multi-additive-quartic. In addition, (10) is valid for $f$*

## 4. Various Stability Results

In this section, we prove some Hyers-Ulam stability results by a fixed point method in the setting of Banach spaces. In what follows, we denote the set of all mappings from $E$ to $F$ by $F^E$. We remember the following theorem which is an essential result in fixed point theory ([23], Theorem 1). This achievement is a key tool in obtaining our aim in this section.

**Theorem 12.** *Let the hypotheses*

*(A1) $Y$ is a Banach space, $E$ is a nonempty set, $j \in \mathbb{N}$, $g_1, \cdots, g_j : E \longrightarrow E$, and $L_1, \cdots, L_j : E \longrightarrow \mathbb{R}_+$*

*(A2) $\mathcal{T} : Y^E \longrightarrow Y^E$ is an operator satisfying the inequality*

$$\|\mathcal{T}\lambda(x) - \mathcal{T}\mu(x)\| \le \sum_{i=1}^{j} L_i(x)\|\lambda(g_i(x)) - \mu(g_i(x))\|, \lambda, \mu \in Y^E, x \in E, \tag{38}$$

*(A3)* $\Lambda : \mathbb{R}_+^E \longrightarrow \mathbb{R}_+^E$ *is an operator defined through*

$$\Lambda\delta(x) := \sum_{i=1}^{j} L_i(x)\delta(g_i(x))\delta \in \mathbb{R}_+^E, x \in E, \tag{39}$$

*hold, and a function* $\theta : E \longrightarrow \mathbb{R}_+$ *and a mapping* $\phi : E \longrightarrow Y$ *fulfill the following two conditions:*

$$\|\mathcal{T}\phi(x) - \phi(x)\| \le \theta(x), \theta^*(x) := \sum_{l=0}^{\infty} \Lambda^l \theta(x) < \infty (x \in E). \tag{40}$$

*Then, there exists a unique fixed point* $\psi$ *of* $\mathcal{T}$ *such that*

$$\|\phi(x) - \psi(x)\| \le \theta^*(x)(x \in E). \tag{41}$$

*Moreover,* $\psi(x) = \lim_{l\longrightarrow\infty} \mathcal{T}^l \phi(x)$ *for all* $x \in E$.

For the rest of this paper and for each mapping $f : V^n \longrightarrow W$, we consider the difference operator $\Gamma_{AQ} f : V^n \times V^n \longrightarrow W$ defined via

$$\Gamma_{AQ} f(x_1, x_2) := \sum_{q=0}^{n} \left(-\frac{12}{7}\right)^q f\left(\mathcal{M}_q^n\right) - \sum_{p_1=0}^{n} \sum_{p_2=0}^{n-p_1} 4^{n-p_1-p_2} (-6)^{p_1} \cdot \left(-\frac{24}{7}\right)^{p_2} f\left(\mathcal{N}_{(p_1,p_2)}^n\right), \tag{42}$$

where $f(\mathcal{M}_q^n)$ and $f(\mathcal{N}_{(p_1,p_2)}^n)$ are defined in (24) and (8), respectively. In the sequel, all mappings $f : V^n \longrightarrow W$ are assumed that satisfy zero condition. With this assumption, we have the next stability result for functional equation (26) in the odd case.

**Theorem 13.** *Let* $\beta \in \{-1, 1\}$ *be fixed,* $V$ *be a linear space, and* $W$ *be a Banach space. Suppose that* $\phi : V^n \times V^n \longrightarrow \mathbb{R}_+$ *is a mapping satisfying*

$$\lim_{l\longrightarrow\infty} \left(\frac{1}{2^{n\beta}}\right)^l \phi\left(2^{\beta l} x_1, 2^{\beta l} x_2\right) = 0, \tag{43}$$

*for all* $x_1, x_2 \in V^n$ *and*

$$\Phi(x) =: \left(\frac{7}{12}\right)^n \frac{1}{2^{n((\beta+1)/2)}} \sum_{l=0}^{\infty} \left(\frac{1}{2^{n\beta}}\right)^l \phi\left(0, 2^{\beta l + ((\beta-1)/2)} x\right) < \infty, \tag{44}$$

*for all* $x \in V^n$. *Assume also* $f : V^n \longrightarrow W$ *is a mapping fulfilling the inequality*

$$\left\|\Gamma_{AQ} f(x_1, x_2)\right\| \le \phi(x_1, x_2), \tag{45}$$

*for all* $x_1, x_2 \in V^n$. *If* $f$ *is odd in each variable, then there exists a unique multiadditive mapping* $\mathcal{A} : V^n \longrightarrow W$ *such that*

$$\|f(x) - \mathcal{A}(x)\| \le \Phi(x), \tag{46}$$

*for all* $x \in V^n$.

*Proof.* Replacing $(x_1, x_2)$ by $(0, x_1)$ in (45) and using the assumptions, we have

$$\left\|\left(-\frac{12}{7}\right)^n f(2x) - \left(-\frac{24}{7}\right)^n f(x)\right\| \le \phi(0, x), \tag{47}$$

for all $x = x_1 \in V^n$ (here and the rest of the proof) and so

$$\left\|f(x) - \frac{1}{2^n} f(2x)\right\| \le \left(\frac{7}{24}\right)^n \phi(0, x). \tag{48}$$

Set

$$\theta(x) := \left(\frac{7}{12}\right)^n \frac{1}{2^{n((\beta+1)/2)}} \phi\left(0, 2^{(\beta-1)/2} x\right), \text{and } \mathcal{T}\theta(x) := \frac{1}{2^{n\beta}} \theta\left(2^\beta x\right) \left(\theta \in W^{V^n}\right). \tag{49}$$

Then, relation (48) can be modified as

$$\|f(x) - \mathcal{T} f(x)\| \le \theta(x)(x \in V^n). \tag{50}$$

Define $\Lambda\eta(x) := (1/2^{n\beta})\eta(2^\beta x)$ for all $\eta \in \mathbb{R}_+^{V^n}$. It is seen that $\Lambda$ has the form (A3) of Theorem 12 for which $E = V^n$, $g_1(x) = 2^\beta x$, and $L_1(x) = 1/2^{n\beta}$. Furthermore, for each $\lambda, \mu \in W^{V^n}$, we get

$$\begin{aligned}\|\mathcal{T}\lambda(x) - \mathcal{T}\mu(x)\| &= \left\|\frac{1}{2^{n\beta}}\left[\lambda\left(2^\beta x\right) - \mu\left(2^\beta x\right)\right]\right\| \\ &\le L_1(x)\|\lambda(g_1(x)) - \mu(g_1(x))\|.\end{aligned} \tag{51}$$

The above relation portrays that the hypothesis (A2) holds. By induction on $l$, one can check that for any $l \in \mathbb{N}_0$, we have

$$\Lambda^l \theta(x) := \left(\frac{1}{2^{n\beta}}\right)^l \theta\left(2^{\beta l} x\right) = \left(\frac{7}{12}\right)^n \left(\frac{1}{2^{n((\beta+1)/2)}}\right)^l \phi\left(0, 2^{\beta l + ((\beta-1)/2)} x\right). \tag{52}$$

Now, relations (44) and (52) necessitate that all assumptions of Theorem 12 are satisfied. Hence, there exists a unique mapping $\mathcal{A} : V^n \longrightarrow W$ such that

$$\mathcal{A}(x) = \lim_{l\longrightarrow\infty} \left(\mathcal{T}^l f\right)(x) = \frac{1}{2^{n\beta}} \mathcal{A}\left(2^\beta x\right) (x \in V^n), \tag{53}$$

and (46) holds. In continuation, we prove that

$$\left\|\Gamma_{AQ}\left(\mathcal{T}^l f\right)(x_1, x_2)\right\| \le \left(\frac{1}{2^{n\beta}}\right)^l \phi\left(2^{\beta l}x_1, 2^{\beta l}x_2\right), \tag{54}$$

for all $x_1, x_2 \in V^n$ and $l \in \mathbb{N}_0$. We argue by induction on $l$. Clearly, inequality (54) is valid for $l = 0$ by (45). Assume that (54) is true for an $l \in \mathbb{N}_0$. Then

$$\begin{aligned}
&\|\Gamma_{AQ}\left(\mathcal{T}^{l+1}f\right)(x_1, x_2)\| \\
&= \left\|\sum_{q=0}^{n}\left(-\frac{12}{7}\right)^q \left(\mathcal{T}^{l+1}f\right)\left(\mathcal{M}_q^n\right) - \sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1} 4^{n-p_1-p_2}(-6)^{p_1} \cdot\left(-\frac{24}{7}\right)^{p_2}\left(\mathcal{T}^{l+1}f\right)\left(\mathcal{N}^n_{(p_1,p_2)}\right)\right\| \\
&= \frac{1}{2^{n\beta}}\left\|\sum_{q=0}^{n}\left(-\frac{12}{7}\right)^q \left(\mathcal{T}^{l+1}f\right)\left(2^\beta\mathcal{M}_q^n\right) - \sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1} 4^{n-p_1-p_2}(-6)^{p_1} \cdot\left(-\frac{24}{7}\right)^{p_2}\left(\mathcal{T}^{l+1}f\right)\left(2^\beta\mathcal{N}^n_{(p_1,p_2)}\right)\right\| \\
&= \frac{1}{2^{n\beta}}\left\|\Gamma_{AQ}\left(\mathcal{T}^l f\right)\left(2^\beta x_1, 2^\beta x_2\right)\right\| \\
&\le \left(\frac{1}{2^{n\beta}}\right)^{l+1}\phi\left(2^{\beta(l+1)}x_1, 2^{\beta(l+1)}x_2\right),
\end{aligned} \tag{55}$$

for all $x_1, x_2 \in V^n$. Letting $l \longrightarrow \infty$ in (54) and applying (43), we arrive at $\Gamma_{AQ}\mathcal{A}(x_1, x_2) = 0$ for all $x_1, x_2 \in V^n$. This means that the mapping $\mathcal{A}$ satisfies (26), and so it is multi-additive by Corollary 11. This finishes the proof. □

Here, in analogy with Theorem 13, we bring the next stability result for functional equation (26) in the even case.

**Theorem 14.** *Let $\beta \in \{-1, 1\}$ be fixed, $V$ be a linear space, and $W$ be a Banach space. Suppose that $\phi : V^n \times V^n \longrightarrow \mathbb{R}_+$ is a mapping satisfying*

$$\lim_{l\longrightarrow\infty}\left(\frac{1}{2^{4n\beta}}\right)^l \phi\left(2^{\beta l}x_1, 2^{\beta l}x_2\right) = 0, \tag{56}$$

*for all $x_1, x_2 \in V^n$ and*

$$\Psi(x) =: \left(\frac{7}{2}\right)^n \frac{1}{2^{4n((\beta+1)/2)}}\sum_{l=0}^{\infty}\left(\frac{1}{2^{4n\beta}}\right)^l \phi\left(2^{\beta l+((\beta+1)/2)}x, 0\right) < \infty, \tag{57}$$

*for all $x \in V^n$. Assume also $f : V^n \longrightarrow W$ is a mapping fulfilling the inequality*

$$\left\|\Gamma_{AQ}f(x_1, x_2)\right\| \le \phi(x_1, x_2), \tag{58}$$

*for all $x_1, x_2 \in V^n$. If $f$ is even in each variable, then there exists a unique solution $\mathcal{Q} : V^n \longrightarrow W$ of (26) such that*

$$\|f(x) - \mathcal{Q}(x)\| \le \Psi(x), \tag{59}$$

*for all $x \in V^n$. In particular, if $\mathcal{Q}$ is even mapping in each variable, then it is multiquartic.*

*Proof.* Replacing $(x_1, x_2)$ by $(0, x_1)$ in (58) and applying the hypotheses, we obtain

$$\begin{aligned}
&\left\|\sum_{q=0}^{n}\binom{n}{q}\left(-\frac{12}{7}\right)^q 2^{n-q}f(2x) - \sum_{p_2=0}^{n}\binom{n}{p_2}\left(-\frac{24}{7}\right)^{p_2}4^{n-p_2}\times 2^{n-p_2}f(x)\right\| \\
&\le \phi(0, x),
\end{aligned} \tag{60}$$

where $x = x_1 \in V^n$ (here and the rest of the proof). On the other hand

$$\begin{aligned}
\sum_{q=0}^{n}\binom{n}{q}\left(-\frac{12}{7}\right)^q 2^{n-q} &= \left(-\frac{12}{7}+2\right)^n = \left(\frac{2}{7}\right)^n, \\
\sum_{p_2=0}^{n}\binom{n}{p_2}\left(-\frac{24}{7}\right)^{p_2}4^{n-p_2}\times 2^{n-p_2} &= \left(-\frac{24}{7}+8\right)^n = \left(\frac{32}{7}\right)^n.
\end{aligned} \tag{61}$$

By the relations above (60) will be

$$\left\|\left(\frac{2}{7}\right)^n f(2x) - \left(\frac{32}{7}\right)^n f(x)\right\| \le \phi(0, x), \tag{62}$$

and so

$$\left\|f(x) - \frac{1}{2^{4n}}f(2x)\right\| \le \left(\frac{7}{32}\right)^n \phi(0, x). \tag{63}$$

One can rewrite (63) as

$$\|f(x) - \mathcal{T}f(x)\| \le \theta(x)(x \in V^n), \tag{64}$$

where

$$\begin{aligned}
\theta(x) &:= \left(\frac{7}{2}\right)^n \frac{1}{2^{4n((\beta+1)/2)}}\phi\left(0, 2^{(\beta-1)/2}x\right), \\
\mathcal{T}\theta(x) &:= \frac{1}{2^{4n\beta}}\theta\left(2^\beta x\right)\left(\theta \in W^{V^n}\right).
\end{aligned} \tag{65}$$

Similar to the proof of Theorem 13, consider $\Lambda\eta(x) := (1/2^{4n\beta})\eta(2^\beta x)$ for all $\eta \in \mathbb{R}_+^{V^n}$, and hence, $\Lambda$ satisfies (A3) of Theorem 12 with $E = V^n$, $g_1(x) = 2^\beta x$, and $L_1(x) = 1/2^{4n\beta}$. Moreover, for each $\lambda, \mu \in W^{V^n}$, we obtain

$$\begin{aligned}
\|\mathcal{T}\lambda(x) - \mathcal{T}\mu(x)\| &= \left\|\frac{1}{2^{4n\beta}}\left[\lambda\left(2^\beta x\right) - \mu\left(2^\beta x\right)\right]\right\| \\
&\le L_1(x)\|\lambda(g_1(x)) - \mu(g_1(x))\|.
\end{aligned} \tag{66}$$

The last relation implies that the hypothesis (A2) is true. It is easily checked by induction on $l$ that for any $l \in \mathbb{N}_0$ and $x \in V^n$

$$\Lambda^l\theta(x) := \left(\frac{1}{2^{4n\beta}}\right)^l \theta\left(2^{\beta l}x\right) = \left(\frac{7}{32}\right)^n \left(\frac{1}{2^{4n((\beta+1)/2)}}\right)^l \psi \cdot \left(0, 2^{\beta l+((\beta-1)/2)}x\right). \tag{67}$$

It now follows between (57) and (67) that all assumptions of Theorem 12 hold, and thus, there exists a unique mapping $\mathcal{Q} : V^n \longrightarrow W$ such that

$$\mathcal{Q}(x) = \lim_{l\longrightarrow\infty}\left(\mathcal{T}^l f\right)(x) = \frac{1}{2^{4n\beta}}\mathcal{Q}\left(2^\beta x\right)(x \in V^n), \tag{68}$$

and (46) is valid. Similar to the proof of Theorem 13, one can show that

$$\left\|\Gamma_{AQ}\left(\mathcal{T}^l f\right)(x_1, x_2)\right\| \le \left(\frac{1}{2^{4n\beta}}\right)^l \psi\left(2^{\beta l}x_1, 2^{\beta l}x_2\right), \tag{69}$$

for all $x_1, x_2 \in V^n$ and $l \in \mathbb{N}_0$. Letting $l \longrightarrow \infty$ in (69) and applying (56), we arrive at $\Gamma_{AQ}\mathcal{Q}(x_1, x_2) = 0$ for all $x_1, x_2 \in V^n$, and therefore, the mapping $\mathcal{Q}$ satisfies (26). The last part follows from part (ii) of Corollary 11. □

Here and subsequently, it is assumed that $V$ is a normed space and $W$ is a Banach space unless otherwise stated explicitly. In the following corollary, we show that the multiadditive and multiquartic mappings are stable. Since the proof is routine, we include it without proof.

**Corollary 15.** *Given $\alpha \in \mathbb{R}$. Suppose that $f : V^n \longrightarrow W$ is a mapping satisfying the inequality*

$$\|\Gamma_{AQ}f(x_1, x_2)\| \le \sum_{i=1}^{2}\sum_{j=1}^{n}\|x_{ij}\|^\alpha, \tag{70}$$

*for all $x_1, x_2 \in V^n$.*

*(i) If $\alpha \ne n$ and $f$ is odd in each variable, then there exists a unique multiadditive mapping $\mathcal{A} : V^n \longrightarrow W$ such that*

$$\|f(x) - \mathcal{A}(x)\| \le \left(\frac{7}{12}\right)^n \frac{1}{|2^\alpha - 2^n|}\sum_{j=1}^{n}\|x_{1j}\|^\alpha. \tag{71}$$

*(ii) If $\alpha \ne 4n$ and $f$ is even in each variable, then there exists a unique solution $\mathcal{Q} : V^n \longrightarrow W$ of (26) such that*

$$\|f(x) - \mathcal{Q}(x)\| \le \left(\frac{7}{2}\right)^n \frac{1}{|2^\alpha - 2^{4n}|}\sum_{j=1}^{n}\|x_{1j}\|^\alpha, \tag{72}$$

*for all $x = x_1 \in V^n$. Moreover, if $\mathcal{Q}$ is even mapping in each variable, then it is multiquartic.*

The upcoming corollaries are direct consequences of Theorems 13 and 14 when the functional equation (26) is controlled by a small positive number $\delta$.

**Corollary 16.** *Let $\delta > 0$ and $f : V^n \longrightarrow W$ be a mapping satisfying the inequality*

$$\|\Gamma_{AQ}f(x_1, x_2)\| \le \delta, \tag{73}$$

*for all $x_1, x_2 \in V^n$.*

*(i) If $f$ is odd in each variable, then there exists a unique multiadditive mapping $\mathcal{A} : V^n \longrightarrow W$ such that*

$$\|f(x) - \mathcal{A}(x)\| \le \left(\frac{7}{12}\right)^n \frac{\delta}{2^n - 1}, \tag{74}$$

*for all $x \in V^n$*

*(ii) If $f$ is even in each variable, then there exists a unique solution $\mathcal{Q} : V^n \longrightarrow W$ of (26) such that*

$$\|f(x) - \mathcal{Q}(x)\| \le \left(\frac{7}{2}\right)^n \frac{\delta}{2^{4n} - 1}, \tag{75}$$

*for all $x \in V^n$. Furthemore, if $\mathcal{Q}$ is even mapping in each variable, then it is multiquartic.*

*Proof.* Letting the constant function $\phi(x_1, x_2) = \delta$ for all $x_1, x_2 \in V^n$ and using Theorem 13 and Theorem 14 in the case $\beta = 1$, one can obtain the desired result. □

Given the mapping $f : V^n \longrightarrow W$, we define the operator $\Gamma f : V^n \times V^n \longrightarrow W$ through

$$\begin{aligned}\Gamma f(x_1, x_2) &:= \sum_{t\in\{-1,1\}^{n-k}} f\left(x_1^k + x_2^k, 2x_1^{n-k} + tx_2^{n-k}\right) \\ &- \sum_{p_1=0}^{n-k}\sum_{p_2=0}^{n-k-p_1}\sum_{i\in\{1,2\}} 4^{n-k-p_1-p_2}24^{p_1}(-6)^{p_2} f \\ &\cdot \left(x_i^k, \mathcal{N}_{(p_1,p_2)}^{n-k}\right),\end{aligned} \tag{76}$$

for all $x_i^k = (x_{i1}, \cdots, x_{ik}) \in V^k$ and $x_i^{n-k} = (x_{i,k+1} \cdots, x_{in}) \in V^{n-k}$ where $i \in \{1, 2\}$ and $f(\mathcal{N}_{(p_1,p_2)}^{n-k})$ are defined in (8).

In the next result, we show that the functional equation (10) can be stable.

**Theorem 17.** *Let $\beta \in \{-1, 1\}$ be fixed, $V$ be a linear space, and $W$ be a Banach space. Suppose that $\phi : V^n \times V^n \longrightarrow \mathbb{R}_+$ is a mapping satisfying the inequality*

$$\sum_{l=0}^{\infty} \left(\frac{1}{2^{(4n-3k)\beta}}\right)^{l} \phi\left(2^{\beta l-(|\beta-1|/2)} x_1, 2^{\beta l-(|\beta-1|/2)} x_2\right) < \infty, \quad (77)$$

*for all $x_1, x_2 \in V^n$. Assume also $f : V^n \longrightarrow W$ is a mapping satisfying the inequality*

$$\|\Gamma f(x_1, x_2)\| \le \phi(x_1, x_2), \quad (78)$$

*for all $x_1, x_2 \in V^n$. Then, there exists a unique solution $\mathcal{F} : V^n \longrightarrow W$ of (10) such that*

$$\|f(x) - \mathcal{F}(x)\| \le \Phi(x), \quad (79)$$

*for all $x = (x^k, x^{n-k}) \in V^n$, where*

$$\begin{aligned}\Phi(x) = {} & \frac{1}{2^{(4n-3k)(|\beta+1|/2)+n-k}} \sum_{l=0}^{\infty} \left(\frac{1}{2^{(4n-3k)\beta}}\right)^{l} \phi \\ & \cdot \left(2^{\beta l-(|\beta-1|/2)} x, \left(2^{\beta l-(|\beta-1|/2)} x^k, 0\right)\right).\end{aligned} \quad (80)$$

*Proof.* Putting $x_1^k = x_2^k = x^k$ and $x_1^{n-k} = x^{n-k}, x_2^{n-k} = 0$ in (79), we have

$$\begin{aligned}&\left\|2^{n-k} f(2x) - \sum_{p_1=0}^{n} \binom{n-k}{p_1} 2^k 4^{n-k-p_1} 24^{p_1} 2^{n-k-p_1} f(x)\right\| \\ &\quad \le \phi\left(x, \left(x^k, 0\right)\right),\end{aligned} \quad (81)$$

in which $x = (x^k, x^{n-k})$. A computation shows that (81) can be rewritten as follows:

$$\left\|2^{n-k} f(2x) - 2^{5n-4k} f(x)\right\| \le \phi\left(x, \left(x^k, 0\right)\right), \quad (82)$$

and so

$$\left\|f(2x) - 2^{4n-3k} f(x)\right\| \le \frac{1}{2^{n-k}} \phi\left(x, \left(x^k, 0\right)\right). \quad (83)$$

Set

$$\xi(x) \coloneqq \frac{1}{2^{(4n-3k)(|\beta+1|/2)+n-k}} \phi\left(\frac{x}{2^{(|\beta-1|/2)}}, \left(\frac{x^k}{2^{(|\beta-1|/2)}}, 0\right)\right), \quad (84)$$

and $\mathcal{T}\xi(x) \coloneqq (1/2^{(4n-3k)\beta})\xi(2^\beta x)$ where $\xi \in W^{V^n}$. Then, relation (83) can be modified as

$$\|f(x) - \mathcal{T} f(x)\| \le \xi(x) (x \in V^n). \quad (85)$$

Define $\Lambda\eta(x) \coloneqq (1/2^{(4n-3k)\beta})\eta(2^\beta x)$ for all $\eta \in \mathbb{R}_+^{V^n}$, $x = (x^k, x^{n-k}) \in V^n$. The rest of the proof is similar to the proof of Theorem 13. □

**Corollary 18.** *Let $\delta > 0$. If $f : V^n \longrightarrow W$ is a mapping satisfying the inequality*

$$\|\Gamma f(x_1, x_2)\| \le \delta, \quad (86)$$

*for all $x_1, x_2 \in V^n$, then there exists a unique solution $\mathcal{F} : V^n \longrightarrow W$ of (10) such that*

$$\|f(x) - \mathcal{F}(x)\| \le \frac{\delta}{2^{n-k}\left(2^{4n-3k} - 1\right)}, \quad (87)$$

*for all $x \in V^n$.*

*Proof.* Setting the constant function $\phi(x_1, x_2) = \delta$ for all $x_1$, $x_2 \in V^n$ and applying Theorem 17 in the case $\beta = 1$, the result can be found. □

## 5. Conclusion

In the present paper, we introduced the multi-additive-quartic and multimixed additive-quartic mappings. Indeed, we characterized the mentioned mappings and then unified the system of functional equations defining a multi-additive-quartic or a multimixed additive-quartic mapping to a single equation. We also showed that under which conditions a multimixed additive-quartic mapping is multiadditive, multiquartic, and multi-additive-quartic. Finally, we applied a fixed point theorem to establish the Hyers-Ulam stability of multi-additive-quartic mappings and multimixed additive-quartic functional equations.

## Data Availability

All results are obtained without any software and found by manual computations.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

## Authors' Contributions

All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

## Acknowledgments

The present research was fully supported by the Journal Publication Fund of Universiti Putra Malaysia, Serdang, Selangor, Malaysia.

## References


[1] K. Ciepliński, "Generalized stability of multi-additive mappings," *Applied Mathematics Letters*, vol. 23, no. 10, pp. 1291–1294, 2010.

[2] K. Ciepliński, "On the generalized Hyers-Ulam stability of multi-quadratic mappings," *Computers & Mathematcs with Applications*, vol. 62, no. 9, pp. 3418–3426, 2011.

[3] F. Skof, "Proprieta locali e approssimazione di operatori," *Rendiconti del Seminario Matematico e Fisico di Milano*, vol. 53, no. 1, pp. 113–129, 1983.

[4] A. Bodaghi and B. Shojaee, "On an equation characterizing multi-cubic mappings and its stability and hyperstability," *Fixed Point Theory*, vol. 22, no. 1, pp. 83–92, 2021.

[5] K. W. Jun and H. M. Kim, "The generalized Hyers-Ulam-Rassias stability of a cubic functional equation," *Journal of Mathematical Analysis and Applications*, vol. 274, no. 2, pp. 867–878, 2002.

[6] A. Bodaghi, C. Park, and O. T. Mewomo, "Multiquartic functional equations," *Advances in Difference Equations*, vol. 2019, no. 1, 2019.

[7] J. M. Rassias, "Solution of the Ulam stability problem for quartic mappings," *Glasnik Matematicki Series III*, vol. 34, no. 2, pp. 243–252, 1999.

[8] X. Zhao, X. Yang, and C.-T. Pang, "Solution and stability of the multiquadratic functional equation," *Abstract and Applied Analysis*, vol. 2013, Article ID 415053, 2013.

[9] A. Bodaghi, "Functional inequalities for generalized multi-quadratic mappings," *Journal of Inequalities and Applications*, vol. 2021, no. 1, 2021.

[10] M. Kuczma, "An introduction to the theory of functional equations and inequalities," in *Cauchy's Equation and Jensen's Inequality*, Birkhauser, Verlag, Basel, 2009.

[11] S. Lee, S. Im, and I. Hwang, "Quartic functional equations," *Journal of Mathematical Analysis and Applications*, vol. 307, no. 2, pp. 387–394, 2005.

[12] A. Bodaghi, "Intuitionistic fuzzy stability of the generalized forms of cubic and quartic functional equations," *Journal of Intelligent & Fuzzy Systems*, vol. 30, no. 4, pp. 2309–2317, 2016.

[13] D. Kang, "On the stability of generalized quartic mappings in quasi-$\beta$-normed spaces," *Journal of Inequalities and Applications*, vol. 2010, Article ID 198098, 2010.

[14] Z. Abbasbeygi, A. Bodaghi, and A. Gharibkhajeh, "On an equation characterizing multi-quartic mappings and its stability," *International Journal of Nonlinear Analysis and Applications*, vol. 13, no. 1, pp. 991–1002, 2022.

[15] S. M. Ulam, *Problems in Modern Mathematics, Chapter VI, Science Ed.*, Wiley, New York, 1940.

[16] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, no. 4, pp. 222–224, 1941.

[17] T. Aoki, "On the stability of the linear transformation in Banach spaces," *Journal of the Mathematical Society of Japan*, vol. 2, no. 1-2, pp. 64–66, 1950.

[18] T. M. Rassias, "On the stability of the linear mapping in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 72, no. 2, pp. 297–300, 1978.

[19] A. Bahyrycz, K. Cieplinski, and J. Olko, "On an equation characterizing multi-Cauchy-Jensen mappings and its Hyers-Ulam stability," *Acta Mathematica Scientia. Series B. English Edition*, vol. 35, no. 6, pp. 1349–1358, 2015.

[20] A. Bahyrycz, K. Ciepliński, and J. Olko, "On an equation characterizing multi-additive-quadratic mappings and its Hyers-Ulam stability," *Applied Mathematics and Computation*, vol. 265, pp. 448–455, 2015.

[21] A. Bahyrycz and J. Olko, "On stability and hyperstability of an equation characterizing multi-Cauchy-Jensen mappings," *Results in Mathematics*, vol. 73, no. 2, 2018.

[22] A. Bodaghi, "Approximation on the multi-m-Jensen-quadratic mappings and a fixed point approach," *Mathematica Slovaca*, vol. 71, no. 1, pp. 117–128, 2021.

[23] J. Brzdęk, J. Chudziak, and Z. Páles, "A fixed point approach to stability of functional equations," *Nonlinear Anal.*, vol. 74, no. 17, pp. 6728–6732, 2011.

[24] M. B. Ghaemi, M. Majani, and M. E. Gordji, "General system of cubic functional equations in non-Archimedean spaces," *Tamsui Oxford Journal of Information and Mathematical Sciences*, vol. 28, no. 4, pp. 407–423, 2012.

[25] C.-G. Park, "Multi-quadratic mappings in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 131, no. 8, pp. 2501–2504, 2003.

[26] T. Z. Xu, "Stability of multi-Jensen mappings in non-Archimedean normed spaces," *Journal of Mathematical Physics*, vol. 53, no. 2, article 023507, 2012.

[27] M. Eshaghi Gordji, "Stability of a functional equation deriving from quartic and additive functions," *Bulletin of the Korean Mathematical Society*, vol. 47, no. 3, pp. 491–502, 2010.

[28] H. Lee, S. W. Kim, B. J. Son, D. H. Lee, and S. Y. Kang, "Additive-quartic functional equation in non-Archimedean orthogonality spaces," *Korean Journal of Mathematics*, vol. 20, no. 1, pp. 33–46, 2012.

[29] A. Bodaghi, "Stability of a mixed type additive and quartic functional equation," *Filomat*, vol. 28, no. 8, pp. 1629–1640, 2014.

[30] A. Bodaghi, "Approximate mixed type additive and quartic functional equation," *Boletim da Sociedade Paranaense de Matemática*, vol. 35, no. 1, pp. 43–56, 2017.

[31] J. Brzdek, "Stability of the equation of the p-Wright affine functions," *Aequationes mathematicae*, vol. 85, no. 3, pp. 497–503, 2013.