Research Article

Blow-Up for a Stochastic Viscoelastic Lamé Equation with Logarithmic Nonlinearity

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In this paper, we consider an initial boundary value problem of stochastic viscoelastic wave equation with nonlinear damping and logarithmic nonlinear source terms. We proved a blow-up result for the solution with decreasing kernel.

1. Introduction

In recent years, stochastic partial differential equations in a separable Hilbert space have been studied by many authors, and various results on the existence, uniqueness, stability, blow-up, and other quantitative and qualitative properties of solutions have been established.

In this work, we consider the following problem of stochastic wave equation:

\[
\begin{cases}
  u_{tt} - \mu \Delta u - (\lambda + \mu) \nabla (\text{div} u) + \int_0^t h(t-s) \Delta u(s) \, ds + |u_t|^{q-2} u_t = u|u|^{p-2} \ln |u|^k + \varepsilon \sigma(x, t) W_t(x, t) \text{ in } D \times [0, +\infty), \\
  u(x, t) = 0 \text{ on } \partial D \times [0, +\infty), \\
  u(x, 0) = u_0(x), u_t(x, 0) = u_1(x) \text{ in } D,
\end{cases}
\]

(1)

where \( D \) is a bounded domain in \( IR^n \), \( n \in IN^* \), with a smooth boundary \( \partial D \); \( \mu, \lambda \) are the Lamé constants which satisfy \( \mu > 0 \), \( \lambda + \mu \geq 0 \); \( h \) is a positive function, \( p > q \geq 2 \); the constant \( k \) is a small nonnegative real number; and \( L^2(D) \) is the set of square integrable function on \( D \) equipped with the inner product \( \langle \cdot, \cdot \rangle \) and its norm \( \| \cdot \|_2 \).

\( W(x, t) \) is an infinite dimensional Wiener process, \( \sigma(x, t) \) is \( L^2(D) \) valued progressively measurable, and \( \varepsilon \) is a positive constant which measures the strength of noise.

It is common to observe a wave motion as a physical phenomenon which is mathematically modeled by a partial differential equation of hyperbolic type. Much has been
written about such equations regarding their widespread applications to engineering and sciences. However, for more realistic models, the random fluctuation had been taken into consideration which led to introduced stochastic wave equation in 1960’s. Several examples of linear stochastic wave propagation and applications can be found in [1]. Mueller [2] was the first who investigate the existence of explosive solutions for some stochastic wave equation. Motivated by Mueller [2], Chow [3] was interested by knowing how does a random perturbation affect the solution behavior for a wave equation with a polynomial nonlinearity. He was concerned with the existence of local and global solutions of the stochastic equation:

\[ \begin{cases} u_{tt} = \Delta u + f(u) + \sigma(u) W_t(x, t) \text{ in } x \in \mathbb{R}^d, & t > 0, \\ u(x, 0) = g(x), u_t(x, 0) = h(x), \end{cases} \]

where the initial data \( g \) and \( h \) are given functions and the nonlinear terms \( f(u) \) and \( \sigma(u) \) are assumed to be polynomials in \( u \). Four years later, he [4] established an energy inequality and the exponential bound for a linear stochastic equation and gave the existence theorem for a unique global solution for the randomly perturbed wave equation:

\[ \begin{cases} u_{tt} + 2\alpha u_t - A(x, \partial x) u(x, t) = f(x, t) + \sigma(x, t) W_t(x, t) \text{ in } x \in \mathbb{R}^d, & t > 0, \\ u(x, 0) = u_0(x), u_t(x, 0) = v_0(x). \end{cases} \]  

In 2009, Chow [5] studied the problem of explosive solutions for a class of nonlinear stochastic wave equation in a domain \( \mathcal{D} \subset \mathbb{R}^d \) for \( d \geq 3 \):

\[ \begin{cases} u_{tt} = (\varepsilon^2 \Delta - a) u + f(u) + \sigma(u, x, t) W_t(x, t) \text{ in } x \in \mathcal{D}, & t > 0, \\ u(x, 0) = g(x), u_t(x, 0) = h(x). \end{cases} \]  

We can mention some other works such as Cheng et al. [6] who studied the existence of a global solution and blow-up solutions for the nonlinear stochastic viscoelastic wave equation with nonlinear damping and source terms:

\[ \begin{cases} u_{tt} - \Delta u + \int_0^t h(t-s) \Delta u(s) ds + |u_t|^{q-2} u_t = u|u|^{p-2} + \varepsilon \sigma(x, t) W_t(x, t) \text{ in } \mathcal{D} \times [0, +\infty[, \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x) \text{ in } \mathcal{D}. \end{cases} \]  

Moreover, Kim et al. [7] considered the stochastic quasilinear viscoelastic wave equation with nonlinear damping and source terms:

\[ \begin{cases} |u_t|^p u_{tt} - \Delta u - \Delta u_t + \int_0^t h(t-s) \Delta u(s) ds + |u_t|^{q-2} u_t = u|u|^{p-2} + \varepsilon \sigma(x, t) W_t(x, t) \text{ in } \mathcal{D} \times [0, +\infty[, \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x) \text{ in } \mathcal{D}. \end{cases} \]  

They showed the existence of a global solution and blow-up in finite time.

Recently, Yang et al. [8] treated the following stochastic nonlinear viscoelastic wave equation:

\[ \begin{cases} |u_t|^p u_{tt} - \Delta u - \Delta u_t + \int_0^t h(t-s) \Delta u(s) ds = \sigma(x, t) W_t(x, t) \text{ in } \mathcal{D} \times [0, +\infty[, \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x) \text{ in } \mathcal{D}. \end{cases} \]
They established the existence of global solution and asymptotic stability of the solution by using some properties of the convex function.

However, it was noticed that the logarithmic nonlinearity appears naturally in many branches of physics such as nuclear physics, optics, and geophysics (see [9, 10]). These specific applications in physics and other fields attract a lot of mathematicians to work with such problems. In the deterministic case, Al-Gharabli [11] investigated the stability of the solution of a viscoelastic plate equation with a logarithmic nonlinearity source term for the following problem:

\[ \begin{align*}
  u_{tt} + \Delta^2 u + u + \int_0^t h(t-s)\Delta u^2(s)ds &= u \ln |u|^{\beta} in \mathcal{D} \times [0, +\infty], \\
  u = \frac{\partial u}{\partial \nu} &= 0 in \partial \mathcal{D} \times [0, +\infty], \\
  u(x, 0) = u_0(x), u_t(x, 0) = u_1(x) in \mathcal{D},
\end{align*} \]

where \( \mathcal{D} \subseteq \mathbb{R}^2 \) is a bounded domain with a smooth boundary \( \partial \mathcal{D} \). The vector \( \nu \) is the unit outer normal to \( \partial \mathcal{D} \), and \( h \) is the nondecreasing nonnegative function.

Mezouar et al. [12] treated a more general problem where they considered the following nonlinear viscoelastic Kirchhoff equation with a time-varying delay term:

\[ \begin{align*}
  |u_t|^\beta |u_t|^2 (\|\nabla u\|^2) \Delta u - \Delta u_{tt} + \int_0^t h(t-s)\Delta u(s)ds + \mu_1 g_1(u_t(x, t)) + \mu_2 g_2(u_t(x, t - \tau(t))) &= ku \ln |u| in \mathcal{D} \times [0, +\infty], \\
  u(x, t) &= 0 on \partial \mathcal{D} \times [0, +\infty], \\
  u(x, 0) &= u_0(x), u_t(x, 0) = u_1(x) in \mathcal{D},
\end{align*} \]

The paper is organized as follows: in Section 2, we introduce some basic definitions, necessary assumptions, and lemmas that are helpful in proving our main result. Section 3 is devoted to show the blow-up of the solution of our problem.

2. Preliminaries

Let \( (\Omega, \mathcal{F}, P) \) be a complete probability space for which a filtration \( \{ \mathcal{F}_t, t \geq 0 \} \) of increasing sub \( \sigma \) – fields \( \mathcal{F}_t \) is given and \( W(x, t) \) be a continuous Wiener random field in this space with a mean zero and the covariance operator \( Q \) satisfying

\[ Tr(Q) = \sum_{\ell \geq 1} \lambda_{\ell} < \infty. \]

\( W(x, t) \) is defined by

\[ W(x, t) = \sum_{j=1}^{\infty} \sqrt{\lambda_j} \beta_j(t) e_j(t), \quad j \in \mathbb{N}^+, t \geq 0, \]

where \( \beta_j(t) \) is a sequence of real-valued standard Brownian motions mutually independent on the probability space \( (\Omega, \mathcal{F}, P) \), \( \lambda_j \) are the eigenvalues of \( Q \), and \( e_j \) are the corresponding eigenvectors. That is,

\[ Qe_j = \lambda_j e_j. \]

Note \( E(.) \) stands for expectation with respect to probability measure \( P \). Let \( \mathcal{H} \) be the set of \( L_2^0 = L^2(\Omega; L^2(V, V)) \)-valued processes with the norm

\[ \|\phi(t)\|_{\mathcal{H}}^2 = E \int_0^t \|\phi(s)\|_{L_2^0}^2 ds = E \int_0^t Tr(\phi(s)Q^{\ast}(s))ds < \infty, \]

where \( \phi^{\ast}(s) \) denotes the adjoint operator of \( \phi(s) \) and \( V = H^1_0(\mathcal{D}) \) which is equivalent to \( H^1(\mathcal{D}) \). For any process \( \phi(s) \in \mathcal{H} \), we can define the stochastic integral with respect to the \( Q \)-Wiener process as \( \int_0^t \phi(s)dW(s) \) which is a martingale. For more details about the infinite dimension Wiener process and stochastic integral, we refer to Da Prato and Zabczyk (pp. 90-96, [13]).

To state and prove our result, we need some assumptions.

A1. Assume that \( h : IR^+ \rightarrow IR^+ \) is a \( C^1 \) nonincreasing function satisfying

\[ h(0) > 0, \mu - \int_0^\infty h(s)ds = l > 0, \]

and there exist tow nonnegative constants \( \varsigma_1 \) and \( \varsigma_2 \) such that

\[ -\varsigma_1 h(t) \leq h'(t) \leq -\varsigma_2 h(t), \quad t \geq 0. \]
Theorem 1. Assume that (A1) and (A3) hold. If established by combining the proof given in [6, 12].

Lemma 2 [14] (Sobolev-Poincaré’s inequality). Let m be an integer with

\[ 2 \leq m \leq +\infty \] (n = 1, 2) \quad (23) \]

\[ 2 \leq m \leq 2n \] \quad \[ (n - 2)(n \geq 3) \]. \quad (24)
Then there exists a constant \( C_* = C_*(\mathcal{D}, m) \) such that
\[
\|u\|_m \leq C_* \|\nabla u\|_2, \quad \text{for } u \in H^1_0(\mathcal{D}).
\] (25)

Lemma 3 [15]. For \( h, \varphi \in C^1([0, \infty), IR) \), we have
\[
\int_{\mathcal{D}} h \ast \varphi \varphi, dx = -\frac{1}{2} h(t) \|\varphi(t)\|_2^2 + \frac{1}{2} \left( h' \varphi \right)(t) - \frac{1}{2} \frac{d}{dt} \left( \int_0^t h(s) ds \right) \|\varphi\|^2.
\] (26)

Lemma 4. Let \((u, v)\) be a solution of the problem (21) with the initial data \( (u_0, v_0) \in H^1_0(\mathcal{D}) \times L^2(\mathcal{D}) \). Then there exists a constant \( C_\ast \) such that for each \( x \in \mathcal{D} \),
\[
e
\int_{\mathcal{D}} h \ast \varphi \varphi, dx + \frac{1}{2} \int_0^t \left( h'(s) \varphi^2(s) ds + \int_0^t \langle \varphi(s), \varphi(s) \rangle dW_s \right) + \frac{\varepsilon^2}{2} \sum_{j=1}^\infty \int_{\mathcal{D}} \lambda_j \varphi^2(x) \sigma^2(x, s) dx ds.
\] (27)

Proof. We can apply the Itô's formula to (21) for each \( x \in \mathcal{D} \) after integrating the above equation over \( \mathcal{D} \) to get
\[
\|v(t)\|_2^2 = \|v(0)\|_2^2 + 2 \int_0^t \langle v(s), \mu \Delta u(t) + (\lambda + \mu) \nabla (\text{div } u) \rangle dx ds - \int_0^t h(s - \tau) \Delta u(t) d\tau - |v|^2 v + u|u|^2 \ln |u|^4 ds dx + 2 \int_0^t \langle v(s), \varphi(x, s) \rangle dW_s,
\] (28)

By using integration by parts, we get
\[
\mu \int_{\mathcal{D}} \int_0^t \nabla u(t) \nabla v dx ds = -\mu \int_{\mathcal{D}} \int_0^t \text{div } u(t) \nabla v dx ds = -\frac{\mu}{2} \|\nabla u(t)\|_2^2 - \|\nabla u(0)\|_2^2.
\] (29)

By applying Lemma 3, we have
\[
\int_0^t \int_0^t h(s - \tau) \varphi \varphi, d\tau v(s) dr dx ds
\] (31)

We have
\[
\int_0^t \int_0^t h(s - \tau) \varphi \varphi, d\tau v(s) dr dx ds = -\int_0^t \int_0^t h(s - \tau) \nabla u(t) \nabla v(s) dr dx ds + \int_0^t \left( \frac{1}{2} h(s) \|\nabla u(s)\|_2^2 - \frac{1}{2} \left( h' \varphi\right)(s) + \frac{1}{2} \frac{d}{ds} \left( \int_0^t h(r) dr \right) \|\nabla u(s)\|_2^2 \right) ds.
\]

3. Blow-Up

We prove our main result for \( p > q \); we purpose
\[
E \int_0^\infty \int_{\mathcal{D}} \sigma^2(x, t) dx dt < \infty,
\] (33)

\[
G(t) = \frac{\varepsilon^2}{2} \sum_{j=1}^\infty E \int_{\mathcal{D}} \lambda_j \varphi_j^2(x) \sigma^2(x, s) dx ds,
\] (34)

By replacing (29)–(32) in (28) and multiplying equation (28) by 1/2, we arrive at (27).

\[
E \int_0^\infty \int_{\mathcal{D}} \sigma^2(x, s) dx ds = E_1 < \infty,
\] (35)

where
\[
Tr(Q) = \sum_{j=1}^\infty \lambda_j < \text{coand } c_0 = \sup_{j=1} E_\infty \|e\|_\infty < \infty.
\] (36)
Lemma 5. Let \((u, v)\) be a solution of system (21) with initial data \((u_0, v_0) \in H_0^1(\Omega) \times L^2(\Omega)\). Then, we have

\[
\frac{d}{dt} E(t) = -E\|v(t)\|_q^q - \frac{1}{2} h(t) E\|\nabla u(t)\|_2^2 + \frac{1}{2} E \left( h' \nabla u \right)(t) + \frac{1}{2} \sum_{j=1}^{\infty} \int_{\Omega} \lambda_j \xi_j^2(x) \sigma^2(x, t) dx,
\]

\[
(37)
\]

\[
E(u(t), v(t)) = E(u_0, v_0) - \mu \int_0^t E\|\nabla u(s)\|_2^2 ds
- (\lambda + \mu) \int_0^t E\|\text{div } u(s)\|_2^2 ds
+ E \int_0^t \left( h(s) - r \right) \langle \nabla u(r), \nabla v(s) \rangle dr ds
- E \int_0^t (u(s), |v(s)|^{p-2} v(s)) ds
+ E \int_0^t \langle u(s), u(s)|u(s)|^{p-2} \ln |u(s)|^k \rangle ds
+ E \int_0^t \|\nabla v(s)\|_2^2 ds.
\]

\[
(38)
\]

**Proof.** Using the Itô’s formula and by following the same way as our discussions in Lemma 4 with taking the expectations, we obtain (37).

We multiply the second equation in (22) by \(u\) and integrate the result over \(\Omega\), and we take expectation; we obtain (38).

We set \(H(t) = G(t) - Ee(t)\). As \(h\) is a positive decreasing function so

\[
H'(t) = G'(t) - \frac{d}{dt} E(t) = E\|v\|_q^q + \frac{1}{2} h(t) E\|\nabla u(t)\|_2^2
- \frac{1}{2} E \left( h' \nabla u \right)(t) \geq E\|v\|_q^q.
\]

Consequently,

\[
H'(t) \geq 0.
\]

**Lemma 6.** Let \((u, v)\) be a solution of system (21). Assume that \((A1)\) holds. Then, there exists a positive constant \(C\) such that

\[
E\|u(t)\|_{p+1}^p \leq C \left( G(t) - H(t) - \frac{1}{2} E\|v\|_q^q \right)
+ \frac{1}{p} \int_{\Omega} |u|^p \ln |u|^k dx - \frac{1}{2} E(h \nabla u)(t)
- \frac{\lambda + \mu}{2} E\|\text{div } u\|_2^2 + E\|u\|_{p+1}^{p+1},
\]

\[
(41)
\]

where \(2 \leq s \leq p + 1\).
**Theorem 9.** Assume (A1) and (A2) hold. Let \((u, v)\) be a solution of system (21) with initial data \((u_0, v_0) \in H_0^1(\mathcal{D}) \times L^2(\mathcal{D})\) satisfying

\[
E(0) \leq -(1 + \beta)E_1, 
\]

where \(\beta\) is a nonnegative constant and \(E_1\) is given in (35). If \(p > q\), then there exists a positive time \(T_0 \in [0, T]\) such that

\[
\lim_{t \to T_0} E(v(t)) = +\infty, 
\]

where

\[
T_0 = \frac{1 - \alpha}{aKL^{\alpha-\beta}(0)}, 
\]

\(L(0) = H^{1-\alpha}(0) + \delta E(u_0, u_1) > 0,\)

and \(K\) is given later.

**Proof.**

A direct differentiation of \(L(t)\) gives

\[
L'(t) = (1 - \alpha)H^{-\alpha}(t)H'(t) + \delta[-\mu E\|\nabla u(t)\|_2^2 \\
- (\lambda + \mu)E\|\nabla u(t)\|_2 + E\|u(t)\|_2^2 + E\int_0^t \|h(r)\|_2 \|\nabla u(r)\|_2 dr \\
- E(u(t), v(t)\|^{q+2}v(t)\| \\
+ \delta E(v(t)\|^{q+2}v(t)\|) + E(u(t), u(t)\|^{p+2}ln |u(t)|^4) + E\|v(t)\|_2^2 \\
+ E\|v(t)\|_2^2 + \delta|H(t) - G(t) + Ee(t)|] 
\]

Recalling (39) and (19), (51) leads to

\[
L'(t) \geq (1 - \alpha)H^{-\alpha}(t)E\|v\|_2^2 + \delta E(H(t) - G(t)) \\
\quad + \delta\left(\frac{HP}{2} - \mu\right)E\|v\|_2^2 + \delta\left(\frac{P}{2} + 1\right)E\|v\|_2^2 \\
- \delta E(u(t), v(t)\|^{q+2}v(t)\|) - \frac{\delta p}{2} E\int_0^t \|h(s)\|_2 \|\nabla u(s)\|_2^2 \\
+ \delta E(v(t)\|^{q+2}v(t)\|) - \frac{\delta p}{2} E\int_0^t \|h(r)\|_2 \|\nabla u(r)\|_2 dr \\
+ (\lambda + \mu)\delta\left(\frac{P}{2} - 1\right)E\|\nabla u\|_p^2 + \frac{\delta k}{p} E\|u\|_p^2 \\
+ \frac{\delta p}{2} E(h \circ \nabla u)(t). 
\]

By using Young’s and H"older’s inequalities, we get

\[
\begin{align*}
E\int_0^t h(t-r)\langle \nabla v, \nabla u(t)\rangle dr \\
= E\int_0^t h(t-r)\langle \nabla v - \nabla u(t), \nabla u(t)\rangle dr + E\int_0^t h(s)ds\|\nabla u(t)\|_2^2 \\
\geq \frac{-P}{2} E(h \circ \nabla u)(t) - \frac{1}{2P} E\int_0^t h(s)ds\|\nabla u(t)\|_2^2 \\
+ E\int_0^t h(s)ds\|\nabla u(t)\|_2^2 
\end{align*}
\]

Hence,

\[
L'(t) \geq (1 - \alpha)H^{-\alpha}(t)E\|v\|_2^2 + \delta E(H(t) - G(t)) \\
\quad + \delta\left(\frac{HP}{2} - \mu\right)E\|v\|_2^2 + \delta\left(\frac{P}{2} + 1\right)E\|v\|_2^2 \\
- \delta E(u(t), v(t)\|^{q+2}v(t)\|) - \frac{\delta p}{2} E\int_0^t \|h(s)\|_2 \|\nabla u(s)\|_2^2 \\
+ (\lambda + \mu)\delta\left(\frac{P}{2} - 1\right)E\|\nabla u\|_p^2 + \frac{\delta k}{p} E\|u\|_p^2,
\]

As \(q < p + 1\), then \(E\|u(t)\|_p^2 \leq cE\|u(t)\|_p^{p+1}\) so by using Young’s and H"older’s inequality, we obtain

\[
E\langle u(t), v(t)\|^{q+2}v(t)\| \leq \left(\frac{E\|v(t)\|_2^2}{q}\right)^{q-1} \left(\frac{E\|u(t)\|_p^2}{q+1}\right)^{1/q} \\
\leq c\left(\frac{E\|v(t)\|_2^2}{q}\right)^{q-1} \left(\frac{E\|u(t)\|_p^2}{q+1}\right)^{1/q} \\
\leq c\left(\frac{q-1}{q+1}\right)^{1/q} \left(\frac{E\|v(t)\|_2^2}{q}\right)^{q-1} \left(\frac{E\|u(t)\|_p^2}{q+1}\right)^{1/q} \left(\frac{E\|u(t)\|_p^{p+1}}{E\|u(t)\|_p^{p+1}}\right)^{(1/q)} \\
\leq c\left(\frac{q-1}{q+1}\right)^{1/q} \left(\frac{E\|v(t)\|_2^2}{q}\right)^{q-1} \left(\frac{E\|u(t)\|_p^2}{q+1}\right)^{1/q} \left(\frac{E\|u(t)\|_p^{p+1}}{E\|u(t)\|_p^{p+1}}\right)^{(1/q)} \\
\times \left(\frac{E\|u(t)\|_p^{p+1}}{E\|u(t)\|_p^{p+1}}\right)^{(1/q)}.
\]
We have

\[ E \int_{\mathcal{D}} |u|^p \ln |u|^k \, dx = E \int_{\mathcal{D}_1} |u|^p \ln |u|^k \, dx + E \int_{\mathcal{D}_2} |u|^p \ln |u|^k \, dx \]
\[ \leq E \int_{\mathcal{D}_1} |u|^p \ln |u|^k \, dx \]
\[ \leq E \int_{\mathcal{D}_1} k|u|^{p+1} \, dx \]
\[ \leq kE \|u\|_{p+1}^{p+1}. \]

(57)

By (40), (47), and \(-Ec(0) = H(0)\), we have

\[ (1 + \beta)G(t) < (1 + \beta)E_1 \leq H(0) \leq H(t) \]
\[ \leq G(t) + \frac{1}{p} E \int_{\mathcal{D}} |u|^p \ln |u|^k \, dx. \]

(58)

Therefore,

\[ G(t) \leq \frac{1}{1 + \beta} H(t). \]

(59)

From (57), (58), and (59), we get

\[ kE \|u(t)\|_{p+1}^{p+1} \geq E \int_{\mathcal{D}_1} k|u|^{p+1} \, dx \geq p(H(t) - G(t)) \geq p \frac{\beta}{1 + \beta} H(t). \]

(60)

As \(H\) is increasing positive nonnegative function and by recalling (46), we get

\[ \left( E \|u(t)\|_{p+1}^{p+1} \right)^{(1/p+1)-(1/q)} \]
\[ \leq \left( \frac{p \beta}{k(1 + \beta)} \right)^{(1/p+1)-(1/q)} H^{(1/p+1)-(1/q)}(t) \]
\[ \leq \left( \frac{p \beta}{k(1 + \beta)} \right)^{(1/p+1)-(1/q)} H^{-\alpha}(t) \]
\[ \leq \left( \frac{p \beta}{k(1 + \beta)} \right)^{(1/p+1)-(1/q)} H^{-\alpha}(0). \]

(61)

Taking into account (61) in (55), we find

\[ E(u(t), |v(t)|^{q-2}v(t)) \]
\[ \leq \left( \frac{c \left( p \beta \right)^{(1/p+1)-(1/q)} q-1 q \xi (E\|v(t)\|_q^q) H^{-\alpha}(t) \right) \]
\[ + \left( \frac{c \left( p \beta \right)^{(1/p+1)-(1/q)} q-1 q \xi (E\|u(t)\|_{p+1}^{p+1}) H^{-\alpha}(0) \right. \]

(62)

Substituting (62) into (54), we get

\[ L'(t) \geq (1 - \alpha) H^{-\alpha}(t) E\|v\|_q^q + \delta p(H(t) - G(t)) \]
\[ + \delta \left( \frac{p}{2} - 1 \right) E\|v\|_q^q + \delta \left( \frac{p}{2} + 1 \right) E\|v\|_q^q \]
\[ + \delta \left( 1 - \frac{1}{2p} - \frac{p}{2} \right) E \int_0^t h(s)ds\|v\|_q^q \]
\[ + (\lambda + \mu) \delta \left( \frac{p}{2} - 1 \right) E \|\text{div } u\|_2^2 \]
\[ - \delta \frac{a_2(q-1) \xi (E\|v\|_q^q) H^{-\alpha}(t)}{q} \]
\[ - \delta \frac{a_2(1-q) \xi (E\|u\|_{p+1}^{p+1}) H^{-\alpha}(0) + \frac{\delta k}{p} E\|u\|_p^p,} \]

(63)

where \(a_2 = c(p\beta/(k(1 + \beta)))^{(1/p+1)-(1/q)}\).

Using Lemma 6, we arrive at

\[ L'(t) \geq \left( 1 - \alpha - \delta \frac{a_2(q-1) \xi}{q} \right) H^{-\alpha}(t) E\|v\|_q^q + \delta p(H(t) - G(t)) \]
\[ - G(t) + \delta \left( \frac{p}{2} - 1 \right) E\|v\|_q^q + \delta \left( \frac{p}{2} + 1 \right) E\|v\|_q^q \]
\[ + \delta \left( 1 - \frac{1}{2p} - \frac{p}{2} \right) E \int_0^t h(s)ds\|v\|_q^q \]
\[ + (\lambda + \mu) \delta \left( \frac{p}{2} - 1 \right) E\|\text{div } u\|_2^2 - \delta \frac{a_2(1-q) \xi \|v\|_q^q H^{-\alpha}(0) + C} {q} \]
\[ \cdot \left( G(t) - H(t) - \frac{1}{2} E\|v\|_q^q + \frac{1}{p} E \int_{\mathcal{D}} |u|^p \ln |u|^k \, dx \right. \]
\[ + E\|u\|_{p+1}^{p+1} - \frac{1}{2} E(h o \text{div } u)(t) - \frac{\lambda + \mu}{2} E\|\text{div } u\|_2^2 \]
\[ + \frac{\delta k}{p} E\|u\|_p^p. \]

(64)

Once \(\xi\) is fixed, we pick \(\delta\) small enough so that

\[ 1 - \alpha - \delta \frac{a_2(q-1) \xi}{q} \xi \geq 0. \]

(65)

It implies that

\[ L'(t) \geq \delta \left( p + a_2 \xi^{1-q} \right) \frac{(H(t) - G(t))}{H^{-\alpha}(t)} \]
\[ + \delta \left( \frac{p}{2} + 1 + a_2 \right) \frac{1}{2} \xi^{1-q} ) E\|v\|_q^q \]
\[ - \delta a_2 \xi^{1-q} \frac{1}{p} E \int_{\mathcal{D}} |u|^p \ln |u|^k \, dx + (\lambda + \mu) \]
\[ \cdot \left( \xi^{1-q} a_2 + \frac{1}{2} \right) E\|\text{div } u\|_2^2 \]
\[ + \delta a_2 \xi^{1-q} \frac{1}{2} E(h o \text{div } u)(t) + \delta a_2 E\|v\|_q^q \]
\[ + \frac{\delta k}{p} E\|u\|_p^p - \delta a_2 \xi^{1-q} E\|u\|_{p+1}^{p+1}. \]
where \( a_2 = C(a_1/q)H^a(0) \) and \( a_3 = \mu((p/2) - 1) + (1 - (1/p))H(0) \), which is positive from (A2).

From (A1), (19), and Lemma 2, we have

\[
H(t) - G(t) \geq -\frac{1}{2} E\| v \|_2^2 - \left( \frac{\mu}{2} + 2G_\epsilon \right) E\| \nabla u \|_2^2 \\
- \frac{\lambda + \mu}{2} E\| \nabla u \|_2^2 + \left( \frac{\mu}{2} - 1 \right) E(\chi_\nu(t)) \\
- \frac{k}{p^2} E\| u \|_p^2 + \frac{1}{p} \int_\mathcal{O} |u|^p \ln |u|^k dx + E\| u \|_{p+1}^{p+1}.
\]

Now we add and subtract \( \delta a_4 (H(t) - G(t)) \) in (66), and using (67), we find

\[
L'(t) \geq \delta \left( p - a_4 + a_2 \xi^{1-q} \right) (H(t) - G(t)) \\
+ \delta \left( \frac{p}{2} + 1 - a_2 + a_2 \xi^{1-q} \right) E\| v \|_2^2 \\
+ \delta (\lambda + \mu) \xi^{1-q} \frac{a_2}{2} - \left( \frac{p}{2} - 1 \right) - a_4 \right) E\| \nabla u \|_2^2 \\
+ \delta \left( a_2 \xi^{1-q} - a_4 \right) E(\chi_\nu(t)) \\
+ \delta \left( a_3 - a_4 (\mu + 4C_\nu) \right) E\| \nabla u \|_2^2 \\
+ \frac{\delta k}{p} \left( 1 - a_2 \right) E\| u \|_p^2 + \frac{1}{p} \left( a_4 - a_2 \xi^{1-q} \right) E\| \nabla u \|_2^2 \\
+ \delta \left( a_4 - a_2 \xi^{1-q} \right) E\| u \|_{p+1}^{p+1}.
\]

Next, we have

\[
(L(t))^{1/1-a} = (H^{1-a}(t) + \delta E(u, v))^{1/1-a} \\
\leq 2^{1/1-a} \left( H(t) + \delta E(u, v) \right)^{1/1-a} \left( \int_\mathcal{O} uv dx \right)^{1/1-a}.
\]

Therefore, by using Hölder’s and Young’s inequalities, we obtain

\[
\left( \int_\mathcal{O} uv dx \right)^{1/1-a} \leq c \left( E\| u \|_{p+1}^2 \right)^{1/2} \left( E\| v \|_2^2 \right)^{1/2} \left( \frac{\eta}{1} + \left( \frac{E\| u \|_{p+1}^2}{\xi} \right)^{\eta(2(1-a))} \right).
\]

with \((1/n) + (1/\xi) = 1\).

We choose \( = 2(1 - a), \eta = (2(1 - a)) \), and we use (46), so (72) becomes

\[
\left( \int_\mathcal{O} uv dx \right)^{1/1-a} \leq c \left( (1 - 2a) E\| u \|_{p+1}^{2/1-2a} + E\| v \|_2^2 \right) \\
\leq c \left( E\| u \|_{p+1}^{2/1-2a} + E\| v \|_2^2 \right).
\]

By applying Lemma 6 with \( s = 2/1 - 2a \) and recalling (19), we obtain

\[
\left( \int_\mathcal{O} uv dx \right)^{1/1-a} \leq c \left[ G(t) - H(t) - \frac{1}{2} E\| u \|_2^2 \\
+ \frac{1}{p} \int_\mathcal{O} |u|^p \ln |u|^k dx - \frac{1}{2} E(\chi_\nu(t)) \\
- \frac{\lambda + \mu}{2} E\| \nabla u \|_2^2 + E\| u \|_{p+1}^{p+1} + E\| v \|_2^2 \right] \\
\leq c \left( \frac{1}{2} E\| v \|_2^2 \right) + \left( \frac{1}{2} \left( \mu - \int_0^t h(s) ds \right) E\| u \|_2^2 \\
+ \frac{\lambda + \mu}{2} E\| \nabla u \|_2^2 + \frac{k}{p^2} E\| u \|_p^2 + \frac{1}{2} E(\chi_\nu(t)) \\
- \frac{1}{p} \int_\mathcal{O} |u|^p \ln |u|^k dx - \frac{1}{2} E\| v \|_2^2 \right) + \frac{1}{p} \int_\mathcal{O} |u|^p \ln |u|^k dx \\
- \frac{1}{2} E(\chi_\nu(t)) - \frac{\lambda + \mu}{2} E\| \nabla u \|_2^2 + E\| u \|_{p+1}^{p+1} \right] \\
\leq c \left[ E\| u \|_2^2 + \frac{1}{2} \mu E\| u \|_2^2 + \frac{k}{p^2} E\| u \|_p^2 + \frac{\lambda + \mu}{2} E\| \nabla u \|_2^2 \\
+ \frac{1}{2} E(\chi_\nu(t)) + E\| u \|_{p+1}^{p+1} \right].
\]

where \( \gamma > 0 \) is the minimum of the coefficients of \( H(t), E\| u \|_2^2, E\| \nabla u \|_2^2, E(\chi_\nu(t)), E\| \nabla u \|_2^2, \) and \( E\| u \|_{p+1}^2 \).
where \( C = 2^{1/1-\alpha} \max \{1, c\delta^{1/1-\alpha}, c\delta^{1/1-\alpha}(\lambda + \mu)/2, c\delta^{1/1-\alpha} (k/p^2) \} \).

According to (69) and (75), we have

\[
(L(t))^{1/1-\alpha} \leq \frac{\tilde{C}}{p} L'(t) \leq \tilde{K} L'(t).
\]

(76)

In a direct integration of (76), we get

\[
(L(t))^{1/1-\alpha} \geq \frac{1}{(L(0))^{1/1-\alpha} - (\tilde{K} aT/(1-\alpha))}.
\]

(77)

Therefore, \( L(t) \) blows up in time \( T \leq T_0 = (1-\alpha)/(\alpha KL^{(1-\alpha)}(0)) \), and the proof is completed.

**Data Availability**

No data were used to support the study.

**Conflicts of Interest**

This work does not have any conflicts of interest.

**References**


