Research Article

Blow-Up for a Stochastic Viscoelastic Lamé Equation with Logarithmic Nonlinearity

Amina Benramdane,¹ Nadia Mezouar,² Mohammed Sulaiman Alqawba,³
Salah Mahmoud Boulaaras^{3,4} and Bahri Belkacem Cherif^{3,5}

¹Mascara University, Faculty of Exact Sciences, Mascara 29000, Algeria

²Mascara University, Faculty of Economics Sciences, Mascara 29000, Algeria

³Department of Mathematics, College of Sciences and Arts, Qassim University, Ar Rass, Saudi Arabia

⁴Laboratory of Fundamental and Applied Mathematics of Oran (LMFAO), University of Oran 1, Ahmed Benbella, Algeria

⁵Preparatory Institute for Engineering Studies in Sfax, Tunisia

Correspondence should be addressed to Bahri Belkacem Cherif; bahi1968@yahoo.com

Received 7 March 2021; Accepted 2 April 2021; Published 19 April 2021

Academic Editor: Chuanjun Chen

Copyright © 2021 Amina Benramdane et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In this paper, we consider an initial boundary value problem of stochastic viscoelastic wave equation with nonlinear damping and logarithmic nonlinear source terms. We proved a blow-up result for the solution with decreasing kernel.

1. Introduction

In recent years, stochastic partial differential equations in a separable Hilbert space have been studied by many authors, and various results on the existence, uniqueness, stability,

blow-up, and other quantitative and qualitative properties of solutions have been established.

In this work, we consider the following problem of stochastic wave equation:

$$\begin{cases} u_{tt} - \mu \Delta u - (\lambda + \mu) \nabla(\operatorname{div} u) + \int_0^t h(t-s) \Delta u(s) ds + |u_t|^{q-2} u_t = |u|^{p-2} \ln |u|^k + \varepsilon \sigma(x, t) W_t(x, t) \text{ in } \mathcal{D} \times]0, +\infty[, \\ u(x, t) = 0 \text{ on } \partial \mathcal{D} \times [0, +\infty[, \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x) \text{ in } \bar{\mathcal{D}}, \end{cases} \quad (1)$$

where $\mathcal{D}$ is a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}^*$, with a smooth boundary $\partial \mathcal{D}$; μ, λ are the Lamé constants which satisfy $\mu > 0$, $\lambda + \mu \geq 0$; h is a positive function, $p > q \geq 2$; the constant k is a small nonnegative real number; and $L^2(\mathcal{D})$ is the set of square integrable function on $\mathcal{D}$ equipped with the inner product $\langle \cdot, \cdot \rangle$ and its norm $\|\cdot\|_2$.

$W(x, t)$ is an infinite dimensional Wiener process, $\sigma(x, t)$ is $L^2(\mathcal{D})$ valued progressively measurable, and ε is a positive constant which measures the strength of noise.

It is common to observe a wave motion as a physical phenomenon which is mathematically modeled by a partial differential equation of hyperbolic type. Much has been

written about such equations regarding their widespread applications to engineering and sciences. However, for more realistic models, the random fluctuation had been taken into consideration which led to introduced stochastic wave equation in 1960's. Several examples of linear stochastic wave propagation and applications can be found in [1]. Mueller [2] was the first who investigate the existence of explosive solutions for some stochastic wave equation. Motivated by Mueller [2], Chow [3] was interested by knowing how does a random perturbation affect the solution behavior for a wave equation with a polynomial nonlinearity. He was concerned with the existence of local and global solutions of the stochastic equation:

$$\begin{cases} u_{tt} = \Delta u + f(u) + \sigma(u)W_t(x, t) \text{ in } x \in \mathbb{R}^d, & t > 0, \\ u(x, 0) = g(x), u_t(x, 0) = h(x), \end{cases} \quad (2)$$

where the initial data g and h are given functions and the nonlinear terms $f(u)$ and $\sigma(u)$ are assumed to be polynomials in u . Four years later, he [4] established an energy inequality and the exponential bound for a linear stochastic

equation and gave the existence theorem for a unique global solution for the randomly perturbed wave equation:

$$\begin{cases} u_{tt} + 2\alpha u_t - A(x, \partial x)u(x, t) = f(x, t) + \sigma(x, t)W_t(x, t) \text{ in } x \in \mathbb{R}^d, & t > 0, \\ u(x, 0) = u_0(x), u_t(x, 0) = v_0(x). \end{cases} \quad (3)$$

In 2009, Chow [5] studied the problem of explosive solutions for a class of nonlinear stochastic wave equation in a domain $\mathcal{D} \subset \mathbb{R}^d$ for $d \geq 3$,

$$\begin{cases} u_{tt} = (c^2 \Delta - \alpha)u + f(u) + \sigma(u, x, t)W_t(x, t) \text{ in } x \in \mathcal{D}, & t > 0, \\ u(x, 0) = g(x), u_t(x, 0) = h(x). \end{cases} \quad (4)$$

We can mention some other works such as Cheng et al. [6] who studied the existence of a global solution and blow-up solutions for the nonlinear stochastic viscoelastic wave equation with nonlinear damping and source terms:

$$\begin{cases} u_{tt} - \Delta u + \int_0^t h(t-s)\Delta u(s)ds + |u_t|^{q-2}u_t = u|u|^{p-2} + \varepsilon \sigma(x, t)W_t(x, t) \text{ in } \mathcal{D} \times]0, +\infty[, \\ u(x, t) = 0 \text{ on } \partial\mathcal{D} \times [0, +\infty[, \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x) \text{ in } \bar{\mathcal{D}}. \end{cases} \quad (5)$$

The authors proved that finite time blow-up with non-negative probability is explosive or it is explosive in energy sense for $p > q$.

Moreover, Kim et al. [7] considered the stochastic quasi-linear viscoelastic wave equation with nonlinear damping and source terms:

$$\begin{cases} |u_t|^p u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t h(t-s)\Delta u(s)ds + |u_t|^{q-2}u_t = u|u|^{p-2} + \varepsilon \sigma(x, t)W_t(x, t) \text{ in } \mathcal{D} \times]0, +\infty[, \\ u(x, t) = 0 \text{ on } \partial\mathcal{D} \times [0, +\infty[, \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x) \text{ in } \bar{\mathcal{D}}. \end{cases} \quad (6)$$

They showed the existence of a global solution and blow-up in finite time.

Recently, Yang et al. [8] treated the following stochastic nonlinear viscoelastic wave equation:

$$\begin{cases} |u_t|^p u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t h(t-s)\Delta u(s)ds = \sigma(x, t)W_t(x, t) \text{ in } \mathcal{D} \times]0, +\infty[, \\ u(x, t) = 0 \text{ on } \partial\mathcal{D} \times [0, +\infty[, \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x) \text{ in } \bar{\mathcal{D}}. \end{cases} \quad (7)$$

They established the existence of global solution and asymptotic stability of the solution by using some properties of the convex function.

However, it was noticed that the logarithmic nonlinearity appears naturally in many branches of physics such as nuclear physics, optics, and geophysics (see [9, 10]). These specific applications in physics and other fields attract a lot of mathematical scientists to work with such problems. In the deterministic case, Al-Gharabli [11] investigated the stability of the solution of a viscoelastic plate equation with a logarithmic nonlinearity source term for the following problem:

$$\begin{cases} u_{tt} + \Delta^2 u + u + \int_0^t h(t-s)\Delta u^2(s)ds = u \ln |u|^k \text{ in } \mathcal{D} \times]0, +\infty[, \\ u = \frac{\partial u}{\partial \nu} = 0 \text{ in } \partial\mathcal{D} \times]0, +\infty[, \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x) \text{ in } \bar{\mathcal{D}}, \end{cases} \quad (8)$$

where $\mathcal{D} \subseteq \mathbb{R}^2$ is a bounded domain with a smooth boundary $\partial\mathcal{D}$. The vector ν is the unit outer normal to $\partial\mathcal{D}$, and h is the nondecreasing nonnegative function.

Mezouar et al. [12] treated a more general problem where they considered the following nonlinear viscoelastic Kirchhoff equation with a time-varying delay term:

$$\begin{cases} |u_t|^l u_{tt} - M(\|\nabla u\|^2) \Delta u - \Delta u_{tt} + \int_0^t h(t-s)\Delta u(s)ds + \mu_1 g_1(u_t(x, t)) + \mu_2 g_2(u_t(x, t - \tau(t))) = ku \ln |u| \text{ in } \mathcal{D} \times]0, +\infty[, \\ u(x, t) = 0 \text{ on } \partial\mathcal{D} \times [0, +\infty[, \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x) \text{ in } \mathcal{D}, \\ u_t(x, t - \tau(0)) = f_0(x, t - \tau(0)) \text{ in } \mathcal{D} \times]0, \tau(0)[. \end{cases} \quad (9)$$

The paper is organized as follows: in Section 2, we introduce some basic definitions, necessary assumptions, and lemmas that are helpful in proving our main result. Section 3 is devoted to show the blow-up of the solution of our problem.

2. Preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space for which a filtration $\{\mathcal{F}_t, t \geq 0\}$ of increasing sub σ -fields $\mathcal{F}_t$ is given and $W(x, t)$ be a continuous Wiener random field in this space with a mean zero and the covariance operator Q satisfying

$$\text{Tr}(Q) = \sum_{i \geq 1} \lambda_i < \infty. \quad (10)$$

$W(x, t)$ is defined by

$$W(x, t) = \sum_{j=1}^{\infty} \sqrt{\lambda_j} \beta_j(t) e_j(t), \quad j \in \mathbb{N}^*, t \geq 0, \quad (11)$$

where $\beta_j(t)$ is a sequence of real-valued standard Brownian motions mutually independent on the probability space $(\Omega, \mathcal{F}, P)$, λ_j are the eigenvalues of Q , and e_j are the corresponding eigenvectors. That is,

$$Qe_j = \lambda_j e_j. \quad (12)$$

Note $E(\cdot)$ stands for expectation with respect to probability measure P . Let $\mathcal{H}$ be the set of $L_2^0 = L^2(Q^{1/2}V, V)$ -valued processes with the norm

$$\|\phi(t)\|_{\mathcal{H}}^2 = E \int_0^t \|\phi(s)\|_{L_2^0}^2 ds = E \int_0^t \text{Tr}(\phi(s)Q\phi^*(s))ds < \infty, \quad (13)$$

where $\phi^*(s)$ denotes the adjoint operator of $\phi(s)$ and $V = H_0^1(\mathcal{D})$ which is equivalent to $H^1(\mathcal{D})$. For any process $\phi(s) \in \mathcal{H}$, we can define the stochastic integral with respect to the Q -Wiener process as $\int_0^t \phi(s)dW(s)$ which is a martingale. For more details about the infinite dimension Wiener process and stochastic integral, we refer to Da Prato and Zabczyk (pp. 90-96, [13]).

To state and prove our result, we need some assumptions.

A1. Assume that $h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a C^1 nonincreasing function satisfying

$$h(0) > 0, \mu - \int_0^{\infty} h(s)ds = l > 0, \quad (14)$$

and there exist two nonnegative constants c_1 and c_2 such that

$$-c_1 h(t) \leq h'(t) \leq -c_2 h(t), \quad t \geq 0. \quad (15)$$

A2.

$$\int_0^\infty h(s)ds < \mu \frac{(p-2)p}{(p-1)^2}. \quad (16)$$

A3. $p > q \geq 2$ and

$$\begin{cases} 2 < p \leq \frac{2(n-1)}{n-2}, & \text{if } n \geq 3, \\ 2 < k \leq +\infty, & \text{if } n = 1, 2. \end{cases} \quad (17)$$

The following theorem states the existence and uniqueness of a local solution of our problem; the proof can be established by combining the proof given in [6, 12].

Theorem 1. Assume that (A1) and (A3) hold. If $(u_0, u_1) \in H_0^1(\mathcal{D}) \times L^2(\mathcal{D})$ and $E \int_0^t \|\sigma(t)\|_2^2 dt < \infty$, then there exists a solution in which holds (1) on the interval $[0, T]$ in the sense of distributions over $(0, T) \times \mathcal{D}$ for almost all w a test function such that

$$(u, u_t) \in L^2(\Omega; L^\infty([0, T]; (H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})) \times H_0^1(\mathcal{D}))) \cap L^2(\Omega; C([0, T]; H_0^1(\mathcal{D}) \times L^2(\mathcal{D}))). \quad (18)$$

We define the energy associated to the solution of system (1) by

$$\begin{aligned} e(t) = & \frac{1}{2} \|u_t\|_2^2 + \frac{1}{2} \left(\mu - \int_0^t h(s)ds \right) \|\nabla u\|_2^2 \\ & + \frac{\lambda + \mu}{2} \|\operatorname{div} u\|_2^2 + \frac{1}{2} (h \circ \nabla u)(t)^k dx, \\ & + \frac{k}{p^2} \|u\|_p^p - \frac{1}{p} \int_{\mathcal{D}} |u|^p \ln |u| \end{aligned} \quad (19)$$

where

$$(h \circ \nabla u)(t) = \int_0^t h(t-s) \|\nabla u(\cdot, s) - \nabla u(\cdot, t)\|^2 ds. \quad (20)$$

We rewrite (1) as an equivalent Itô's system

$$\begin{cases} du = v dt, \\ dv = \left[\mu \Delta u + (\lambda + \mu) \nabla(\operatorname{div} u) - \int_0^t h(t-s) \Delta u(s) ds - |v|^{q-2} v + u|u|^{p-2} \ln |u|^k \right] dt + \varepsilon \sigma(x, t) dW_t(x, t) \text{ in } \mathcal{D} \times]0, +\infty[, \\ u(x, t) = 0 \text{ on } \partial \mathcal{D} \times [0, +\infty[, \\ u(x, 0) = u_0(x), v(x, 0) = u_1(x) \text{ in } \bar{\mathcal{D}}, \end{cases} \quad (21)$$

which can be written as the integral equation

$$\begin{cases} u(t) = u_0 + \int_0^t v(s) ds, \\ v(t) = v(0) + \int_0^t \left[\mu \Delta u + (\lambda + \mu) \nabla(\operatorname{div} u) - \int_0^t h(s-r) \Delta u(r) dr - |v|^{q-2} v + u|u|^{p-2} \ln |u|^k \right] ds + \int_0^t \varepsilon \sigma(x, s) dW_s(x, t) \text{ in } \mathcal{D} \times]0, +\infty[, \\ u(x, t) = 0 \text{ on } \partial \mathcal{D} \times [0, +\infty[, \\ u(x, 0) = u_0(x), v(x, 0) = u_1(x) \text{ in } \bar{\mathcal{D}}. \end{cases} \quad (22)$$

Lemma 2 [14] (Sobolev-Poincaré's inequality). Let m be a number with

$$2 \leq m \leq +\infty (n = 1, 2) \quad (23)$$

or

$$\frac{2 \leq m \leq 2n}{(n-2)(n \geq 3)}. \quad (24)$$

Then there exists a constant $C_s = C_s(\mathcal{D}, m)$ such that

$$\|u\|_m \leq C_s \|\nabla u\|_2, \quad \text{for } u \in H_0^1(\mathcal{D}). \quad (25)$$

Lemma 3 [15]. For $h, \varphi \in C^1([0, +\infty[, IR)$, we have

$$\begin{aligned} \int_{\mathcal{D}} h * \varphi \varphi_t dx &= -\frac{1}{2} h(t) \|\varphi(t)\|_2^2 + \frac{1}{2} (h' \circ \varphi)(t) \\ &\quad - \frac{1}{2} \frac{d}{dt} \left[(h \circ \varphi)(t) - \left(\int_0^t h(s) ds \right) \|\varphi\|^2 \right]. \end{aligned} \quad (26)$$

Lemma 4. Let (u, v) be a solution of the problem (21) with the initial data $(u_0, v_0) \in H_0^1(\mathcal{D}) \times L^2(\mathcal{D})$, $E \int_0^t \|\sigma(s)\|_2^2 ds < \infty$. Then, the energy functional defined by (19) satisfies

$$\begin{aligned} e(t) &= e(0) - \int_0^t \|v\|_q^q ds - \frac{1}{2} \int_0^t h(s) \|\nabla u(s)\|_2^2 ds \\ &\quad + \frac{1}{2} \int_0^t (h' \circ \nabla u)(s) ds + \int_0^t \langle v(s), \varepsilon \sigma(x, s) dW_s \rangle \\ &\quad + \frac{\varepsilon^2}{2} \sum_{j=1}^{\infty} \int_0^t \int_{\mathcal{D}} \lambda_j e_j^2(x) \sigma^2(x, s) dx ds. \end{aligned} \quad (27)$$

Proof. We can apply the Itô's formula to (21) for each $x \in \mathcal{D}$ after integrating the above equation over $\mathcal{D}$ to get

$$\begin{aligned} \|v(t)\|_2^2 &= \|v(0)\|_2^2 + 2 \int_{\mathcal{D}} \int_0^t v(s) [\mu \Delta u + (\lambda + \mu) \nabla(\operatorname{div} u) \\ &\quad - \int_0^s h(s-\tau) \Delta u(\tau) d\tau - |v|^{q-2} v + u|u|^{p-2} \ln |u|^k] ds dx \\ &\quad + 2 \int_0^t \langle v(s), \varepsilon \sigma(x, s) dW_s \rangle \\ &\quad + \varepsilon^2 \sum_{j=1}^{\infty} \int_0^t \int_{\mathcal{D}} \lambda_j e_j^2(x) \sigma^2(x, s) dx ds. \end{aligned} \quad (28)$$

By using integration by parts, we get

$$\begin{aligned} \mu \int_{\mathcal{D}} \int_0^t \Delta u v(s) dx ds &= -\mu \int_{\mathcal{D}} \int_0^t \nabla u \nabla v ds dx \\ &= -\frac{\mu}{2} (\|\nabla u(t)\|_2^2 - \|\nabla u(0)\|_2^2), \end{aligned} \quad (29)$$

$$\begin{aligned} \int_{\mathcal{D}} \int_0^t (\lambda + \mu) \nabla(\operatorname{div} u(s)) v(s) ds dx \\ &= -(\lambda + \mu) \int_{\mathcal{D}} \int_0^t \operatorname{div} u(s) \operatorname{div} v(s) ds dx \\ &= -\frac{\lambda + \mu}{2} [\|\operatorname{div} u(t)\|_2^2 - \|\operatorname{div} u(0)\|_2^2]. \end{aligned} \quad (30)$$

By applying Lemma 3, we have

$$\begin{aligned} \int_0^t \int_{\mathcal{D}} \int_0^s h(s-\tau) \Delta u(\tau) v(s) d\tau dx ds \\ &= -\int_0^t \int_{\mathcal{D}} \int_0^s h(s-\tau) \nabla u(\tau) \nabla v(s) d\tau dx ds \\ &= \int_0^t \left(\frac{1}{2} h(s) \|\nabla u(s)\|_2^2 - \frac{1}{2} (h' \circ \nabla u)(s) \right. \\ &\quad \left. + \frac{1}{2} \frac{d}{ds} \left[(h \circ \nabla u)(s) - \int_0^s h(\tau) d\tau \|\nabla u(s)\|_2^2 \right] \right) ds. \end{aligned} \quad (31)$$

We have

$$\begin{aligned} \int_{\mathcal{D}} \int_0^t u|u|^{p-2} \ln |u|^k u_s ds dx \\ &= \int_0^t \int_{\mathcal{D}} \frac{1}{p} \frac{d}{ds} (|u(s)|^p) \ln |u|^k dx ds \\ &= \int_0^t \left\{ \int_{\mathcal{D}} \left\{ \frac{1}{p} \frac{d}{ds} (|u(s)|^p \ln |u|^k) \right. \right. \\ &\quad \left. \left. - \frac{1}{p} |u(s)|^p \frac{d}{ds} (\ln |u|^k) \right\} dx \right\} ds \\ &= \int_{\mathcal{D}} \left(\frac{1}{p} (|u|^p \ln |u|^k) \right) dx - \frac{k}{p^2} \|u\|_p^p. \end{aligned} \quad (32)$$

By replacing (29)–(32) in (28) and multiplying equation (28) by 1/2, we arrive at (27).

3. Blow-Up

We prove our main result for $p > q$; we purpose

$$E \int_0^{\infty} \int_{\mathcal{D}} \sigma^2(x, t) dx dt < \infty, \quad (33)$$

$$G(t) = \frac{\varepsilon^2}{2} \sum_{j=1}^{\infty} E \int_0^t \int_{\mathcal{D}} \lambda_j e_j^2(x) \sigma^2(x, s) dx ds, \quad (34)$$

$$\begin{aligned} G(\infty) &= \frac{\varepsilon^2}{2} \sum_{j=1}^{\infty} E \int_0^{\infty} \int_{\mathcal{D}} \lambda_j e_j^2(x) \sigma^2(x, s) dx ds \\ &\leq \frac{\varepsilon^2}{2} \operatorname{Tr}(Q) c_0^2 E \int_0^{\infty} \int_{\mathcal{D}} \sigma^2(x, s) dx ds := E_1 < \infty, \end{aligned} \quad (35)$$

where

$$\operatorname{Tr}(Q) = \sum_{j=1}^{\infty} \lambda_j < \infty \text{ and } c_0 = \sup_{j \geq 1} \|e_j\|_{\infty} < \infty. \quad (36)$$

Lemma 5. Let (u, v) be a solution of system (21) with initial data $(u_0, v_0) \in H_0^1(\mathcal{D}) \times L^2(\mathcal{D})$. Then, we have

$$\begin{aligned} \frac{d}{dt} Ee(t) &= -E\|v(t)\|_q^q - \frac{1}{2}h(t)E\|\nabla u(t)\|_2^2 + \frac{1}{2}E(h' \circ \nabla u)(t) \\ &\quad + \frac{\varepsilon^2}{2} \sum_{j=1}^{\infty} E \int_{\mathcal{D}} \lambda_j e_j^2(x) \sigma^2(x, t) dx, \end{aligned} \quad (37)$$

$$\begin{aligned} E\langle u(t), v(t) \rangle &= E\langle u_0, u_1 \rangle - \mu \int_0^t E\|\nabla u(s)\|_2^2 ds \\ &\quad - (\lambda + \mu) \int_0^t E\|\operatorname{div} u(s)\|_2^2 ds \\ &\quad + E \int_0^t \int_0^s h(s-r) \langle \nabla u(r), \nabla u(s) \rangle dr ds \\ &\quad - E \int_0^t \langle u(s), |v(s)|^{q-2} v(s) \rangle ds \\ &\quad + E \int_0^t \left\langle u(s), u(s) |u(s)|^{p-2} \ln |u(s)|^k \right\rangle ds \\ &\quad + E \int_0^t \|v(s)\|_2^2 ds. \end{aligned} \quad (38)$$

Proof. Using the Itô's formula and by following the same way as our discussions in Lemma 4 with taking the expectations, we obtain (37).

We multiply the second equation in (22) by u and integrate the result over $\mathcal{D}$, and we take expectation; we obtain (38).

We set $H(t) = G(t) - Ee(t)$. As h is a positive decreasing function so

$$\begin{aligned} H'(t) &= G'(t) - \frac{d}{dt} Ee(t) = E\|v\|_q^q + \frac{1}{2}h(t)E\|\nabla u(t)\|_2^2 \\ &\quad - \frac{1}{2}E(h' \circ \nabla u)(t) \geq E\|v\|_q^q. \end{aligned} \quad (39)$$

Consequently,

$$H'(t) \geq 0. \quad (40)$$

Lemma 6. Let (u, v) be a solution of system (21). Assume that (A1) holds. Then, there exists a positive constant C such that

$$\begin{aligned} E\|u(t)\|_{p+1}^s &\leq C \left(G(t) - H(t) - \frac{1}{2}E\|v\|_2^2 \right. \\ &\quad + \frac{1}{p}E \int_{\mathcal{D}} |u|^p \ln |u|^k dx - \frac{1}{2}E(h \circ \nabla u)(t) \\ &\quad \left. - \frac{\lambda + \mu}{2}E\|\operatorname{div} u\|_2^2 + E\|u\|_{p+1}^{p+1} \right), \end{aligned} \quad (41)$$

where $2 \leq s \leq p+1$.

Proof.

$$\begin{aligned} G(t) - H(t) &- \frac{1}{2}E\|v\|_2^2 + \frac{1}{p}E \int_{\mathcal{D}} |u|^p \ln |u|^k dx \\ &- \frac{1}{2}E(h \circ \nabla u)(t) - \frac{\lambda + \mu}{2}E\|\operatorname{div} u\|_2^2 + E\|u\|_{p+1}^{p+1} \\ &= Ee(t) - \frac{1}{2}E\|v\|_2^2 + \frac{1}{p}E \int_{\mathcal{D}} |u|^p \ln |u|^k dx \\ &\quad - \frac{1}{2}E(h \circ \nabla u)(t) + E\|u\|_{p+1}^{p+1} - \frac{\lambda + \mu}{2}\|\operatorname{div} u\|_2^2 \\ &= \frac{1}{2}E\|u_t\|_2^2 + \frac{1}{2}E \left(\mu - \int_0^t h(s) ds \right) \|\nabla u\|_2^2 \\ &\quad + \frac{\lambda + \mu}{2}E\|\operatorname{div} u\|_2^2 + \frac{1}{2}E(h \circ \nabla u)(t) + \frac{k}{p^2}E\|u\|_p^p \\ &\quad - \frac{1}{p}E \int_{\mathcal{D}} |u|^p \ln |u|^k dx - \frac{1}{2}E\|v\|_2^2 \\ &\quad + \frac{1}{p}E \int_{\mathcal{D}} |u|^p \ln |u|^k dx - \frac{1}{2}E(h \circ \nabla u)(t) \\ &\quad - \frac{\lambda + \mu}{2}E\|\operatorname{div} u\|_2^2 + E\|u\|_{p+1}^{p+1} \\ &= \frac{1}{2} \left(\mu - \int_0^t h(s) ds \right) E\|\nabla u\|_2^2 + \frac{k}{p^2}E\|u\|_p^p + E\|u\|_{p+1}^{p+1} \\ &\geq \frac{1}{2}IE\|\nabla u\|_2^2 + E\|u\|_{p+1}^{p+1}. \end{aligned} \quad (42)$$

The last inequality is getting from (A1).

Case 7. If $\|u\|_{p+1} \leq 1$, then $\|u\|_{p+1}^s \leq \|u\|_{p+1}^2$.

By applying Lemma 2, we obtain $\|u\|_{p+1}^s \leq c\|\nabla u\|_2^2$, then

$$\frac{1}{2}IE\|\nabla u\|_2^2 + \frac{k}{p^2}E\|u\|_{p+1}^{p+1} \geq \frac{1}{2}IE\|u\|_{p+1}^s + \frac{k}{p^2}E\|u\|_{p+1}^{p+1} \geq E\|u\|_{p+1}^s. \quad (43)$$

Case 8. If $\|u\|_{p+1} \geq 1$, then $\|u\|_{p+1}^{p+1} \geq \|u\|_{p+1}^s$.

Hence,

$$\frac{1}{2}IE\|\nabla u\|_2^2 + E\|u\|_{p+1}^{p+1} \geq \frac{1}{2}IE\|\nabla u\|_2^2 + E\|u\|_{p+1}^s \geq E\|u\|_{p+1}^s. \quad (44)$$

Consequently, we obtain (41).

We are ready to state and prove our main result for $p > q$. For this purpose, we define

$$L(t) := H^{1-\alpha}(t) + \delta E\langle u, v \rangle, \quad (45)$$

where

$$0 < \alpha < \min \left\{ \frac{p-1}{2(p+1)}, \frac{p+1-q}{(p+1)q} \right\} \quad (46)$$

and δ is a very small constant determined later.

Theorem 9. Assume (A1) and (A2) hold. Let (u, v) be a solution of system (21) with initial data $(u_0, v_0) \in H_0^1(\mathcal{D}) \times L^2(\mathcal{D})$ satisfying

$$Ee(0) \leq -(1 + \beta)E_1, \quad (47)$$

where β is a nonnegative constant and E_1 is given in (35). If $p > q$, then there exists a positive time $T_0 \in [0, T]$ such that

$$\lim_{t \rightarrow T_0} E(e(t)) = +\infty, \quad (48)$$

where

$$T_0 = \frac{1 - \alpha}{\alpha K L^{\alpha/1-\alpha}(0)}, \quad (49)$$

$$L(0) = H^{1-\alpha}(0) + \delta E\langle u_0, u_1 \rangle > 0,$$

and K is given later.

Proof. Let

$$L(t) = H^{1-\alpha}(t) + \delta E\langle u, v \rangle. \quad (50)$$

A direct differentiation of $L(t)$ gives

$$\begin{aligned} L'(t) &= (1 - \alpha)H^{-\alpha}(t)H'(t) + \delta[-\mu E\|\nabla u(t)\|_2^2 \\ &\quad - (\lambda + \mu)E\|\operatorname{div} u\|_2^2 + E\int_0^t h(t-r)\langle \nabla u(r), \nabla u(t) \rangle dr \\ &\quad - E\langle u(t), |v(t)|^{q-2}v(t) \rangle \\ &\quad + E\langle u(t), u(t)|u(t)|^{p-2} \ln |u(t)|^k \rangle + E\|v(t)\|_2^2] \\ &= (1 - \alpha)H^{-\alpha}(t)H'(t) + \delta[-\mu E\|\nabla u(t)\|_2^2 \\ &\quad - (\lambda + \mu)E\|\operatorname{div} u\|_2^2 + E\int_0^t h(t-r)\langle \nabla u(r), \nabla u(t) \rangle dr \\ &\quad - E\langle u(t), |v(t)|^{q-2}v(t) \rangle + E\langle u(t), u(t)|u(t)|^{p-2} \ln |u(t)|^k \rangle \\ &\quad + E\|v(t)\|_2^2] + \delta p[H(t) - G(t) + Ee(t)]. \end{aligned} \quad (51)$$

Recalling (39) and (19), (51) leads to

$$\begin{aligned} L'(t) &\geq (1 - \alpha)H^{-\alpha}(t)E\|v\|_q^q + \delta p(H(t) - G(t)) \\ &\quad + \delta\left(\frac{\mu p}{2} - \mu\right)E\|\nabla u(t)\|_2^2 + \delta\left(\frac{p}{2} + 1\right)E\|v\|_2^2 \\ &\quad - \delta E\langle u(t), |v(t)|^{q-2}v(t) \rangle - \frac{\delta p}{2}E\int_0^t h(s)ds\|\nabla u\|_2^2 \\ &\quad + \delta E\int_0^t h(t-r)\langle \nabla u(r), \nabla u(t) \rangle dr \\ &\quad + (\lambda + \mu)\delta\left(\frac{p}{2} - 1\right)E\|\operatorname{div} u\|_2^2 + \frac{\delta k}{p}E\|u\|_p^p \\ &\quad + \frac{\delta p}{2}E(h \circ \nabla u)(t). \end{aligned} \quad (52)$$

By using Young's and Hölder's inequalities, we get

$$\begin{aligned} &E\int_0^t h(t-r)\langle \nabla u(r), \nabla u(t) \rangle dr \\ &= E\int_0^t h(t-r)\langle \nabla u(r) - \nabla u(t), \nabla u(t) \rangle dr + E\int_0^t h(s)ds\|\nabla u(t)\|_2^2 \\ &\geq -\frac{p}{2}E(h \circ \nabla u)(t) - \frac{1}{2p}E\int_0^t h(s)ds\|\nabla u(t)\|_2^2 \\ &\quad + E\int_0^t h(s)ds\|\nabla u(t)\|_2^2. \end{aligned} \quad (53)$$

Hence,

$$\begin{aligned} L'(t) &\geq (1 - \alpha)H^{-\alpha}(t)E\|v\|_q^q + \delta p(H(t) - G(t)) \\ &\quad + \delta\left(\frac{\mu p}{2} - \mu\right)E\|\nabla u(t)\|_2^2 + \delta\left(\frac{p}{2} + 1\right)E\|v\|_2^2 \\ &\quad - \delta E\langle u(t), |v(t)|^{q-2}v(t) \rangle \\ &\quad + \delta\left(1 - \frac{1}{2p} - \frac{p}{2}\right)E\int_0^t h(s)ds\|\nabla u(t)\|_2^2 \\ &\quad - \frac{\delta p}{2}E(h \circ \nabla u)(t) + \frac{\delta p}{2}E(h \circ \nabla u)(t) \\ &\quad + (\lambda + \mu)\delta\left(\frac{p}{2} - 1\right)E\|\operatorname{div} u\|_2^2 + \frac{\delta k}{p}E\|u\|_p^p \\ &\geq (1 - \alpha)H^{-\alpha}(t)E\|v\|_q^q + \delta p(H(t) - G(t)) \\ &\quad + \delta\left(\frac{\mu p}{2} - \mu\right)E\|\nabla u(t)\|_2^2 + \delta\left(\frac{p}{2} + 1\right)E\|v\|_2^2 \\ &\quad - \delta E\langle u(t), |v(t)|^{q-2}v(t) \rangle \\ &\quad + \delta\left(1 - \frac{1}{2p} - \frac{p}{2}\right)E\int_0^t h(s)ds\|\nabla u(t)\|_2^2 \\ &\quad + (\lambda + \mu)\delta\left(\frac{p}{2} - 1\right)E\|\operatorname{div} u\|_2^2 + \frac{\delta k}{p}E\|u\|_p^p. \end{aligned} \quad (54)$$

As $q < p + 1$, then $E\|u(t)\|_q^q \leq cE\|u(t)\|_{p+1}^{p+1}$ so by using Young's and Hölder's inequality; we obtain

$$\begin{aligned} E\langle u(t), |v(t)|^{q-2}v(t) \rangle &\leq \left(E\|v(t)\|_q^q\right)^{q-1/q} \left(E\|u(t)\|_q^q\right)^{1/q} \\ &\leq c\left(E\|v(t)\|_q^q\right)^{q-1/q} \left(E\|u(t)\|_{p+1}^{p+1}\right)^{1/p+1} \\ &\leq c\left(E\|v(t)\|_q^q\right)^{q-1/q} \left(E\|u(t)\|_{p+1}^{p+1}\right)^{(1/p+1)-(1/q)} \left(E\|u(t)\|_{p+1}^{p+1}\right)^{(1/q)} \\ &\leq c\left(\frac{q-1}{q}\xi\left(E\|v(t)\|_q^q\right) + \frac{\xi^{1-q}}{q}\left(E\|u(t)\|_{p+1}^{p+1}\right)\right) \\ &\quad \times \left(E\|u(t)\|_{p+1}^{p+1}\right)^{(1/p+1)-(1/q)}, \end{aligned} \quad (55)$$

where ξ and c are constants.

We consider the following partition of $\mathcal{D}$:

$$\mathcal{D}_1 = \{x \in \mathcal{D} : |u| > 1\}, \mathcal{D}_2 = \{x \in \mathcal{D} : |u| \leq 1\}. \quad (56)$$

We have

$$\begin{aligned}
 E \int_{\mathcal{D}} |u|^p \ln |u|^k dx &= E \int_{\mathcal{D}_1} |u|^p \ln |u|^k dx + E \int_{\mathcal{D}_2} |u|^p \ln |u|^k dx \\
 &\leq E \int_{\mathcal{D}_1} |u|^p \ln |u|^k dx \\
 &\leq E \int_{\mathcal{D}_1} k |u|^{p+1} dx \\
 &\leq k E \|u\|_{p+1}^{p+1}.
 \end{aligned} \tag{57}$$

By (40), (47), and $-Ee(0) = H(0)$, we have

$$\begin{aligned}
 (1 + \beta)G(t) &< (1 + \beta)E_1 \leq H(0) \leq H(t) \\
 &\leq G(t) + \frac{1}{p} E \int_{\mathcal{D}} |u|^p \ln |u|^k dx.
 \end{aligned} \tag{58}$$

Therefore,

$$G(t) \leq \frac{1}{1 + \beta} H(t). \tag{59}$$

From (57), (58), and (59), we get

$$k E \|u(t)\|_{p+1}^{p+1} \geq E \int_{\mathcal{D}_1} k |u|^{p+1} dx \geq p(H(t) - G(t)) \geq p \frac{\beta}{1 + \beta} H(t). \tag{60}$$

As H is increasing positive nonnegative function and by recalling (46), we get

$$\begin{aligned}
 &\left(E \|u(t)\|_{p+1}^{p+1} \right)^{(1/p+1)-(1/q)} \\
 &\leq \left(p \frac{\beta}{k(1 + \beta)} \right)^{(1/p+1)-(1/q)} H^{(1/p+1)-(1/q)}(t) \\
 &\leq \left(p \frac{\beta}{k(1 + \beta)} \right)^{(1/p+1)-(1/q)} H^{-\alpha}(t) \\
 &\leq \left(p \frac{\beta}{k(1 + \beta)} \right)^{(1/p+1)-(1/q)} H^{-\alpha}(0).
 \end{aligned} \tag{61}$$

Taking into account (61) in (55), we find

$$\begin{aligned}
 &E \langle u(t), |v(t)|^{q-2} v(t) \rangle \\
 &\leq \left(c \left(\frac{p\beta}{k(1 + \beta)} \right)^{(1/p+1)-(1/q)} \right) \frac{q-1}{q} \xi \left(E \|v(t)\|_q^q \right) H^{-\alpha}(t) \\
 &\quad + \left(c \left(\frac{p\beta}{k(1 + \beta)} \right)^{(1/p+1)-(1/q)} \right) \frac{\xi^{1-q}}{q} \left(E \|u(t)\|_{p+1}^{p+1} \right) H^{-\alpha}(0).
 \end{aligned} \tag{62}$$

Substituting (62) into (54), we get

$$\begin{aligned}
 L'(t) &\geq (1 - \alpha) H^{-\alpha}(t) E \|v\|_q^q + \delta p(H(t) - G(t)) \\
 &\quad + \delta \mu \left(\frac{p}{2} - 1 \right) E \|\nabla u(t)\|_2^2 + \delta \left(\frac{p}{2} + 1 \right) E \|v\|_2^2 \\
 &\quad + \delta \left(1 - \frac{1}{2p} - \frac{p}{2} \right) E \int_0^t h(s) ds \|\nabla u(t)\|_2^2 \\
 &\quad + (\lambda + \mu) \delta \left(\frac{p}{2} - 1 \right) E \|\operatorname{div} u\|_2^2 \\
 &\quad - \delta \frac{a_1(q-1)}{q} \xi \left(E \|v\|_q^q \right) H^{-\alpha}(t) \\
 &\quad - \delta \frac{a_1}{q} \xi^{1-q} \left(E \|u\|_{p+1}^{p+1} \right) H^{-\alpha}(0) + \frac{\delta k}{p} E \|u\|_p^p,
 \end{aligned} \tag{63}$$

where $a_1 = c(p\beta/(k(1 + \beta)))^{(1/p+1)-(1/q)}$.

Using Lemma 6, we arrive at

$$\begin{aligned}
 L'(t) &\geq \left(1 - \alpha - \delta \frac{a_1(q-1)}{q} \xi \right) H^{-\alpha}(t) E \|v\|_q^q + \delta p(H(t) \\
 &\quad - G(t)) + \delta \mu \left(\frac{p}{2} - 1 \right) E \|\nabla u(t)\|_2^2 + \delta \left(\frac{p}{2} + 1 \right) E \|v\|_2^2 \\
 &\quad + \delta \left(1 - \frac{1}{2p} - \frac{p}{2} \right) E \int_0^t h(s) ds \|\nabla u(t)\|_2^2 \\
 &\quad + (\lambda + \mu) \delta \left(\frac{p}{2} - 1 \right) \|\operatorname{div} u\|_2^2 - \delta \frac{a_1}{q} \xi^{1-q} H^{-\alpha}(0) C \\
 &\quad \cdot \left(G(t) - H(t) - \frac{1}{2} E \|v\|_2^2 + \frac{1}{p} E \int_{\mathcal{D}} |u|^p \ln |u|^k dx \right. \\
 &\quad \left. + E \|u\|_{p+1}^{p+1} - \frac{1}{2} E(h \circ \nabla u)(t) - \frac{\lambda + \mu}{2} E \|\operatorname{div} u\|_2^2 \right) \\
 &\quad + \frac{\delta k}{p} E \|u\|_p^p.
 \end{aligned} \tag{64}$$

Once ξ is fixed, we pick δ small enough so that

$$1 - \alpha - \delta \frac{a_1(q-1)}{q} \xi \geq 0, \tag{65}$$

It implies that

$$\begin{aligned}
 L'(t) &\geq \delta \left(p + a_2 \xi^{1-q} \right) (H(t) - G(t)) \\
 &\quad + \delta \left(\frac{p}{2} + 1 + a_2 \frac{1}{2} \xi^{1-q} \right) E \|v\|_2^2 \\
 &\quad - \delta a_2 \xi^{1-q} \frac{1}{p} E \int_{\mathcal{D}} |u|^p \ln |u|^k dx + \delta (\lambda + \mu) \\
 &\quad \cdot \left(\xi^{1-q} \frac{a_2}{2} + \left(\frac{p}{2} - 1 \right) \right) E \|\operatorname{div} u\|_2^2 \\
 &\quad + \delta a_2 \xi^{1-q} \frac{1}{2} E(h \circ \nabla u)(t) + \delta a_3 E \|\nabla u(t)\|_2^2 \\
 &\quad + \frac{\delta k}{p} E \|u\|_p^p - \delta a_2 \xi^{1-q} E \|u\|_{p+1}^{p+1}.
 \end{aligned} \tag{66}$$

where $a_2 = C(a_1/q)H^{-\alpha}(0)$ and $a_3 = \mu((p/2) - 1) + (1 - (1/2p) - (p/2)) \int_0^\infty h(s)ds$ which is positive from (A2).

From (A1), (19), and Lemma 2, we have

$$\begin{aligned} H(t) - G(t) &\geq -\frac{1}{2}E\|v\|_2^2 - \left(\frac{\mu}{2} + 2C_s\right)E\|\nabla u\|_2^2 \\ &\quad - \frac{\lambda + \mu}{2}E\|\operatorname{div} u\|_2^2 - \frac{1}{2}E(h \circ \nabla u)(t) \\ &\quad - \frac{k}{p^2}E\|u\|_p^p + \frac{1}{p}E \int_{\mathcal{D}} |u|^p \ln |u|^k dx + E\|u\|_{p+1}^{p+1}. \end{aligned} \quad (67)$$

Now we add and subtract $\delta a_4(H(t) - G(t))$ in (66), and using (67), we find

$$\begin{aligned} L'(t) &\geq \delta(p - a_4 + a_2\xi^{1-q})(H(t) - G(t)) \\ &\quad + \delta\left(\frac{p}{2} + 1 - \frac{a_4}{2} + a_2\frac{1}{2}\xi^{1-q}\right)E\|v\|_2^2 \\ &\quad + \delta(\lambda + \mu)\left(\xi^{1-q}\frac{a_2}{2} + \left(\frac{p}{2} - 1\right) - \frac{a_4}{2}\right)E\|\operatorname{div} u\|_2^2 \\ &\quad + \delta\left(a_2\xi^{1-q}\frac{1}{2} - a_4\right)E(h \circ \nabla u)(t) \\ &\quad + \delta\left(a_3 - a_4\left(\frac{\mu + 4C_s}{2}\right)\right)E\|\nabla u(t)\|_2^2 \\ &\quad + \frac{\delta k}{p}\left(1 - \frac{a_4}{p}\right)E\|u\|_p^p + \frac{\delta}{p}\left(a_4 - a_2\xi^{1-q}\right)E \int_{\mathcal{D}} |u|^p \ln |u|^k dx \\ &\quad + \delta\left(a_4 - a_2\xi^{1-q}\right)E\|u\|_{p+1}^{p+1}, \end{aligned} \quad (68)$$

where $a_4 = \min \{a_2\xi^{1-q}, (2a_3/(\mu + 4C_s))\} > 0$.

Using (60), we obtain

$$\begin{aligned} L'(t) &\geq \delta p \frac{\beta}{1 + \beta} H(t) + \delta\left(\frac{p}{2} + 1 - \frac{a_4}{2} + a_2\frac{1}{2}\xi^{1-q}\right)E\|v\|_2^2 \\ &\quad + \delta(\lambda + \mu)\left(\xi^{1-q}\frac{a_2}{2} + \left(\frac{p}{2} - 1\right) - \frac{a_4}{2}\right)E\|\operatorname{div} u\|_2^2 \\ &\quad + \frac{\delta}{2}\left(a_2\xi^{1-q} - a_4\right)E(h \circ \nabla u)(t) \\ &\quad + \delta\left(a_3 - a_4\left(\frac{\mu + 4C_s}{2}\right)\right)E\|\nabla u(t)\|_2^2 \\ &\quad + \frac{\delta k}{p}\left(1 - \frac{a_4}{p}\right)E\|u\|_p^p + \delta\left(a_4 - a_2\xi^{1-q}\right)E\|u\|_{p+1}^{p+1} \\ &\geq \gamma(H(t) + E\|v\|_2^2 + E\|\operatorname{div} u\|_2^2 + E(h \circ \nabla u)(t) \\ &\quad + E\|\nabla u(t)\|_2^2 + E\|u\|_p^p + E\|u\|_{p+1}^{p+1}) \geq 0, \end{aligned} \quad (69)$$

where $\gamma > 0$ is the minimum of the coefficients of $H(t)$, $E\|v\|_2^2$, $E\|\operatorname{div} u\|_2^2$, $E(h \circ \nabla u)(t)$, $E\|\nabla u(t)\|_2^2$, and $E\|u\|_p^p$ in (69).

Consequently,

$$L(t) \geq L(0) > 0, \quad \forall t > 0. \quad (70)$$

Next, we have

$$\begin{aligned} (L(t))^{1/1-\alpha} &= (H^{1-\alpha}(t) + \delta E\langle u, v \rangle)^{1/1-\alpha} \\ &\leq 2^{1/1-\alpha} \left(H(t) + \delta^{1/1-\alpha} \left| E \int_{\mathcal{D}} uv dx \right|^{1/1-\alpha} \right). \end{aligned} \quad (71)$$

Therefore, by using Hölder's and Young's inequalities, we obtain

$$\begin{aligned} \left| E \int_{\mathcal{D}} uv dx \right|^{1/1-\alpha} &\leq \left(c \left(E\|u\|_{p+1}^2 \right)^{1/2} \left(E\|v\|_2^2 \right)^{1/2} \right)^{1/1-\alpha} \\ &\leq c \left(E\|u\|_{p+1}^2 \right)^{1/(2(1-\alpha))} \left(E\|v\|_2^2 \right)^{1/(2(1-\alpha))} \\ &\leq c \left[\frac{\left(E\|u\|_{p+1}^2 \right)^{\eta/(2(1-\alpha))}}{\eta} + \frac{\left(E\|v\|_2^2 \right)^{\zeta/(2(1-\alpha))}}{\zeta} \right], \end{aligned} \quad (72)$$

with $(1/\eta) + (1/\zeta) = 1$.

We choose $\eta = 2(1-\alpha)$, $\zeta = (2(1-\alpha))/(1-2\alpha)$, and we use (46), so (72) becomes

$$\begin{aligned} \left| E \int_{\mathcal{D}} uv dx \right|^{1/1-\alpha} &\leq c \left[(1-2\alpha)E\|u\|_{p+1}^{2/1-2\alpha} + E\|v\|_2^2 \right] \\ &\leq c \left[E\|u\|_{p+1}^{2/1-2\alpha} + E\|v\|_2^2 \right]. \end{aligned} \quad (73)$$

By applying Lemma 6 with $s = 2/1 - 2\alpha$ and recalling (19), we obtain

$$\begin{aligned} \left| E \int_{\mathcal{D}} uv dx \right|^{1/1-\alpha} &\leq c \left[G(t) - H(t) - \frac{1}{2}E\|v\|_2^2 \right. \\ &\quad + \frac{1}{p}E \int_{\mathcal{D}} |u|^p \ln |u|^k dx - \frac{1}{2}E(h \circ \nabla u)(t) \\ &\quad \left. - \frac{\lambda + \mu}{2}E\|\operatorname{div} u\|_2^2 + E\|u\|_{p+1}^{p+1} + E\|v\|_2^2 \right] \\ &\leq c \left[\frac{1}{2}E\|v\|_2^2 + \frac{1}{2} \left(\mu - \int_0^t h(s)ds \right) E\|\nabla u\|_2^2 \right. \\ &\quad + \frac{\lambda + \mu}{2}E\|\operatorname{div} u\|_2^2 + \frac{k}{p^2}E\|u\|_p^p + \frac{1}{2}E(h \circ \nabla u)(t) \\ &\quad - \frac{1}{p}E \int_{\mathcal{D}} |u|^p \ln |u|^k dx - \frac{1}{2}E\|v\|_2^2 + \frac{1}{p}E \int_{\mathcal{D}} |u|^p \ln |u|^k dx \\ &\quad \left. - \frac{1}{2}E(h \circ \nabla u)(t) - \frac{\lambda + \mu}{2}E\|\operatorname{div} u\|_2^2 + E\|v\|_2^2 + E\|u\|_{p+1}^{p+1} \right] \\ &\leq c \left[E\|v\|_2^2 + \frac{1}{2}\mu E\|\nabla u\|_2^2 + \frac{k}{p^2}E\|u\|_p^p + \frac{\lambda + \mu}{2}E\|\operatorname{div} u\|_2^2 \right. \\ &\quad \left. + \frac{1}{2}E(h \circ \nabla u)(t) + E\|u\|_{p+1}^{p+1} \right]. \end{aligned} \quad (74)$$

Hence,

$$\begin{aligned} (L(t))^{1/1-\alpha} &\leq 2^{1/1-\alpha} \left(H(t) + \delta^{1/1-\alpha} c \left[E\|v\|_2^2 + \frac{1}{2} \mu E\|\nabla u\|_2^2 \right. \right. \\ &\quad \left. \left. + \frac{\lambda + \mu}{2} E\|\operatorname{div} u\|_2^2 + \frac{k}{p^2} E\|u\|_p^p + \frac{1}{2} E(ho\nabla u)(t) + E\|u\|_{p+1}^{p+1} \right] \right) \\ &\leq \tilde{C} \left[H(t) + E\|v\|_2^2 + E\|\nabla u\|_2^2 + E\|\operatorname{div} u\|_2^2 + E\|u\|_p^p \right. \\ &\quad \left. + E(ho\nabla u)(t) + E\|u\|_{p+1}^{p+1} \right], \end{aligned} \quad (75)$$

where $\tilde{C} = 2^{1/1-\alpha} \max \{1, c\delta^{1/1-\alpha}, c\delta^{1/1-\alpha}((\lambda + \mu)/2), c\delta^{1/1-\alpha}(k/p^2)\}$.

According to (69) and (75), we have

$$(L(t))^{1/1-\alpha} \leq \frac{\tilde{C}}{\gamma} L'(t) \leq \tilde{K} L'(t). \quad (76)$$

In a direct integration of (76), we get

$$(L(t))^{1/1-\alpha} \geq \frac{1}{(L(0))^{1/1-\alpha} - (\tilde{K}\alpha t/(1-\alpha))}. \quad (77)$$

Therefore, $L(t)$ blows up in time $T \leq T_0 = (1-\alpha)/(\alpha\tilde{K}L^{\alpha/1-\alpha}(0))$, and the proof is completed.

Data Availability

No data were used to support the study.

Conflicts of Interest

This work does not have any conflicts of interest.

References

- [1] P. L. Chow, W. Kohler, and G. Papanicolaou, *Multiple Scattering and Waves in Random Media*, North-Holland Publishing Company, North-Holland, Amsterdam, 1981.
- [2] C. Mueller, "Long time existence for the wave equation with a noise term," *The Annals of Probability*, vol. 25, no. 1, pp. 133–151, 1997.
- [3] P. L. Chow, "Stochastic wave equations with polynomial nonlinearity," *The Annals of Applied Probability*, vol. 12, no. 1, pp. 361–381, 2002.
- [4] P. L. Chow, "Asymptotics of solutions to semilinear stochastic wave equations," *The Annals of Applied Probability*, vol. 16, no. 2, pp. 757–789, 2006.
- [5] P. L. Chow, "Nonlinear stochastic wave equations: blow-up of second moments in L2-norm," *Annals of Applied Probability*, vol. 19, no. 6, pp. 2039–2046, 2009.
- [6] S. Cheng, Y. Guo, and Y. Tang, "Stochastic viscoelastic wave equations with nonlinear damping and source terms," *Journal of Applied Mathematics*, vol. 2014, 15 pages, 2014.
- [7] S. Kim, J. Y. Park, and Y. H. Kang, "Stochastic quasilinear viscoelastic wave equation with nonlinear damping and source terms," *Boundary Value Problems*, vol. 2018, no. 1, Article ID 14, 2018.
- [8] H. Yang, S. Fang, F. Liang, and M. Li, "A general stability result for second order stochastic quasilinear evolution equations with memory," *Boundary Value Problems*, vol. 2020, no. 1, Article ID 62, 2020.
- [9] K. Enqvist and J. McDonald, "Q-balls and baryogenesis in the MSSM," *Physics Letters B*, vol. 425, no. 3–4, pp. 309–321, 1998.
- [10] P. Górka, "Logarithmic Klein-Gordon equation," *Acta Physica Polonica B*, vol. 40, no. 1, pp. 59–66, 2009.
- [11] M. M. Al-Gharabli, "New general decay results for a viscoelastic plate equation with a logarithmic nonlinearity," *Boundary Value Problems*, vol. 2019, no. 1, article 194, 2019.
- [12] N. Mezouar, S. Boulaaras, and A. Allahem, "Global existence of solutions for the viscoelastic Kirchhoff equation with logarithmic source terms," *Complexity*, vol. 2020, Article ID 7105387, 25 pages, 2020.
- [13] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, 1992.
- [14] R. A. Adams, "Sobolev spaces, academic press," *Pure and Applied Mathematics*, vol. 65, 1978.
- [15] J. Y. Park and J. R. Kang, "Global existence and uniform decay for a nonlinear viscoelastic equation with damping," *Acta Applicandae Mathematicae*, vol. 110, no. 3, pp. 1393–1406, 2010.