Research Article

Numerical Analysis of Time-Fractional Diffusion Equations via a Novel Approach

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The aim of this paper is a new semianalytical technique called the variational iteration transform method for solving fractional-order diffusion equations. In the variational iteration technique, identifying of the Lagrange multiplier is an essential rule, and variational theory is commonly used for this purpose. The current technique has the edge over other methods as it does not need extra parameters and polynomials. The validity of the proposed method is verified by considering some numerical problems. The solution achieved has shown that the better accuracy of the proposed technique. This paper proposes a simpler method to calculate the multiplier using the Shehu transformation, making a valuable technique to researchers dealing with various linear and nonlinear problems.

1. Introduction

In the last decade, significant achievements have been made to applying and the theory of fractional differential equations (FDEs). These problems are increasingly implemented to model equations in research fields as diverse as mechanical schemes, dynamical schemes, chaos, continuous-time random walks, control, chaos synchronization, subdiffusive systems and anomalous diffusive, wave propagation phenomena and unification of diffusion, and so on. The benefits of the fractional-order scheme are that it allows more significant degrees of freedom in the problem. An integer-order differential operator (DO) is a local operator, while a fractional-order DO is a nonlocal operator, taking into account that a potential state depends not only on the current state but also on the past of all its previous instances. Fractional-order schemes have become famous for this valuable property. Another explanation for applying fractional-order derivatives is that they are naturally linked to memory structures that prevail in most physical and scientific structure models. The book by Spanier and Oldham [1] continued to play an

essential role in the improvement of the subject. A few other primary results connected to the solution of FDEs can be identified in the books of Ross and Miller [2], Podlubny [3], and Kilbas et al. [4].

Adolf Fick introduces the laws of diffusion of Fick in 1885. After that, the second law of Fick became identified as the diffusion equation. Diffusion is the mesh atom's movement from a high chemical potential or higher concentration field to a lower concentration or low chemical potential field. Investigators have investigated classical wave and diffusion equations to many physical schemes, such as classical diffusion, slow diffusion, diffusion-wave combination, and classical wave equation. Many diffusion equation implementations, such as phase transformation, electrochemistry, magnetic fields, filtration, microbiology, acoustics, astrophysics, and biochemical group structures. Diffusion is determined by the gradient of the potential energy of the diffusing form. The gradient is the difference in the value of a number, e.g., concentration, strain, or temperature, with differences in one or more variables is often differentiated. Researchers have been seeking to recognize and reduce manufacturing systems problems to

reach better productivity. In applied science schemes, there are various causes for entropy production. In heat engines, heat transfer, the primary source of entropy production is a mass transfer, the coupling between heat, entropy generation and chemical reaction, electrical conduction, as described in the seminal sequence of publications by Bejan et al. [5, 6]. Scholars have utilized different methods for the analysis of diffusion equations such as Chebyshev collocation technique [7], finite difference technique [8], finite volume element technique [9], variational iteration technique [10], two-step Adomian decomposition technique [11], finite volume technique [12], and Laplace transform [10].

In this paper, we implemented the variational iteration transform method to solve the fractional-order diffusion equations.

The fractional-order two-dimensional diffusion equation is given as

$$\frac{\partial^\delta \mu}{\partial \eta^\delta} = \frac{\partial^2 \mu}{\partial \zeta^2} + \frac{\partial^2 \mu}{\partial \psi^2}, 0 < \delta \leq 1, \eta \geq 0, \quad (1)$$

with initial condition

$$\mu(\zeta, \psi, 0) = g(\zeta, \psi). \quad (2)$$

The fractional-order three-dimensional diffusion equation

$$\frac{\partial^\delta \mu}{\partial \eta^\delta} = \frac{\partial^2 \mu}{\partial \zeta^2} + \frac{\partial^2 \mu}{\partial \psi^2} + \frac{\partial^2 \mu}{\partial \mathfrak{F}^2}, 0 < \delta \leq 1, \eta \geq 0, \quad (3)$$

with initial condition

$$\mu(\zeta, \psi, \mathfrak{F}, 0) = g(\zeta, \psi, \mathfrak{F}). \quad (4)$$

A Lagrange multiplier technique has been widely utilized to solve different types of nonlinear equations [13]. This occurs in mathematics and physics or certain related fields but has been established as a basic analytical method, i.e., a variational iteration method (VIM) to model differential equations [14]. The VIM was first recommended by He [15] and was implemented effectively to address the heat transformation problem [15–17]. Recently, several researchers have widely used this method to solve linear and nonlinear equations. The approach offers a consistent and efficient mechanism for a wide variety of applications in engineering and science. It is based on a specific Lagrange multiplier and has the merits of simplicity and easy implementation. Unlike conventional numerical approaches, VIM does not require linearization, discretization, or perturbation. The successive approximation provides quick convergence for the exact result [18–21]. The variational iteration method was introduced in 2010 using the modified Riemann-Liouville derivative [22]. Recently, a procedure combining in this sense Laplace transformation and VIM was proposed [23, 24], and Wu developed a modification via fractional calculus and Laplace transformation [25]. LVIM for solving nonlinear PDEs [26] and system of fractional PDEs [27].

2. Basic Definitions

2.1. Definition. The fractional-order Riemann-Liouville integral is defined as [28, 29]

$$I_0^\delta h(\eta) = \frac{1}{\Gamma(\delta)} \int_0^\eta (\eta - s)^{\delta-1} h(s) ds. \quad (5)$$

2.2. Definition. The fractional-order Caputo's derivative of $f(\eta)$ is given as [28, 29]

$$D_\eta^\delta f(\eta) = I_\eta^{J-\delta} f^j, J-1 < \delta < J, J \in \mathbb{N}; \quad (6)$$

$$\frac{d^J}{d\eta^J} h(\eta), \delta = J, J \in \mathbb{N}.$$

2.3. Definition. Shehu transformation is new and identical to other integral transformations defined for exponential order functions. In Set A, the function is defined by [30–32]

$$A = \{v(\eta): \exists, \rho_1, \rho_2 > 0, |v(\eta)| < M e^{\frac{|\eta|}{\rho_1}}, \text{ if } \eta \in [0, \infty). \quad (7)$$

The Shehu transform which is described as $S(\cdot)$ for a function $v(\eta)$ is defined as

$$S\{v(\eta)\} = V(s, u) = \int_0^\infty e^{-\frac{s\eta}{u}} v(\eta) d\eta, \eta > 0, s > 0. \quad (8)$$

The Shehu transform of a function $v(\eta)$ is $V(s, u)$: then, $v(\eta)$ is called the inverse of $V(s, u)$ which is given as

$$S^{-1}\{V(s, u)\} = v(\eta), \text{ for } \eta \geq 0, S^{-1} \text{ is inverse Shehu transform.} \quad (9)$$

2.4. Definition. Shehu transform for n th derivatives is given as [30–32]

$$S\{v^{(J)}(\eta)\} = \frac{s^J}{u^J} V(s, u) - \sum_{k=0}^{J-1} \left(\frac{s}{u}\right)^{J-k-1} v^{(k)}(0). \quad (10)$$

2.5. Definition. The fractional-order derivatives of Shehu transformation are defined as [30–32]

$$S\{v^{(\delta)}(\eta)\} = \frac{s^\delta}{u^\delta} V(s, u) - \sum_{k=0}^{J-1} \left(\frac{s}{u}\right)^{\delta-k-1} v^{(k)}(0), 0 < \beta \leq n. \quad (11)$$

2.6. Definition. The Mittag-Leffler function, $E_\delta(z)$ for $\delta > 0$, is given as

$$E_\delta(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\delta j + 1)}, \delta > 0, z \in \mathbb{C}. \quad (12)$$

3. The Methodology of VITM

This section introduces the general producer of VITM to solve time-fractional partial differential equation [23].

$$D_{\eta}^{\delta} \mu(\zeta, \eta) + \mathcal{M}(\zeta, \eta) + \mathcal{N}(\zeta, \eta) - \mathcal{K}(\zeta, \eta) = 0, 0 < \delta \leq 1, \quad (13)$$

with the initial sources

$$\mu(\zeta, 0) = g_1(\zeta), \quad (14)$$

where $D_{\eta}^{\delta} = \partial^{\delta} / (\partial \eta^{\delta})$ is the fractional-order Caputo derivative and δ , $\mathcal{M}$, and $\mathcal{N}$, are linear and nonlinear functions, respectively, and sources function $\mathcal{K}$.

The implementation of Shehu transformation to Eq. (13)

$$S[D_{\eta}^{\delta} \mu(\zeta, \eta)] + S[\mathcal{M}(\zeta, \eta) + \mathcal{N}(\zeta, \eta) - \mathcal{K}(\zeta, \eta)] = 0. \quad (15)$$

Using the differentiation property of Shehu transformation, we get

$$S[\mu(\zeta, \eta)] - \sum_{k=0}^{m-1} \frac{s^{\delta-k-1}}{u^{\delta-k}} \frac{\partial^k \mu(\zeta, \eta)}{\partial^k \eta} \Big|_{\eta=0} = -S[\mathcal{M}(\zeta, \eta) + \mathcal{N}(\zeta, \eta) - \mathcal{K}(\zeta, \eta)]. \quad (16)$$

The Lagrange multiplier of the iterative system as

$$S[\mu_{m+1}(\zeta, \eta)] = S[\mu_m(\zeta, \eta)] + \lambda(s) \left[\frac{s^{\delta}}{u^{\delta}} \mu_m(\zeta, \eta) - \sum_{k=0}^{m-1} \frac{s^{\delta-k-1}}{u^{\delta-k}} \frac{\partial^k \mu(\zeta, \eta)}{\partial^k \eta} \Big|_{\eta=0} - S[\mathcal{K}(\zeta, \eta)] - S[\mathcal{M}(\zeta, \eta) + \mathcal{N}(\zeta, \eta)] \right]. \quad (17)$$

A Lagrange multiplier as

$$\lambda(s) = -\frac{u^{\delta}}{s^{\delta}}. \quad (18)$$

Applying inverse Shehu transform S^{-1} , Eq. (17) can be defined as

$$\mu_{m+1}(\zeta, \eta) = \mu_m(\zeta, \eta) - S^{-1} \left[\frac{u^{\delta}}{s^{\delta}} \left[\sum_{k=0}^{m-1} \frac{s^{\delta-k-1}}{u^{\delta-k}} \frac{\partial^k \mu(\zeta, \eta)}{\partial^k \eta} \Big|_{\eta=0} - S[\mathcal{K}(\zeta, \eta)] - S[\mathcal{M}(\zeta, \eta) + \mathcal{N}(\zeta, \eta)] \right] \right], \quad (19)$$

the initial value can be described as

$$\mu_0(\zeta, \eta) = S^{-1} \left[\frac{u^{\delta}}{s^{\delta}} \left\{ \sum_{k=0}^{m-1} \frac{s^{\delta-k-1}}{u^{\delta-k}} \frac{\partial^k \mu(\zeta, \eta)}{\partial^k \eta} \Big|_{\eta=0} \right\} \right]. \quad (20)$$

4. Implementation of VITM

4.1. *Problem.* Consider the fractional-order diffusion equation [11]

$$\frac{\partial^{\delta} \mu}{\partial \eta^{\delta}} = \frac{\partial^2 \mu}{\partial \zeta^2} - \frac{\partial \mu}{\partial \zeta} + \mu \frac{\partial^2 \mu}{\partial \zeta^2} - \mu^2 + \mu, 0 < \alpha \leq 1, \quad (21)$$

with the initial condition

$$\mu(\zeta, 0) = e^{\zeta}. \quad (22)$$

Applying VITM on equation (21), we have

$$\begin{aligned} \mu_{m+1}(\zeta, \eta) &= S^{-1} \left[\frac{\mu_m(\zeta, \eta)}{s} \right] + S^{-1} \\ &\cdot \left[\lambda(s) S \left\{ \frac{\partial^2 \mu_m}{\partial \zeta^2} - \frac{\partial \mu_m}{\partial \zeta} + \mu_m \frac{\partial^2 \mu_m}{\partial \zeta^2} - \mu_m^2 + \mu_m \right\} \right], \end{aligned} \quad (23)$$

where the Lagrange multiplier is $\lambda(s)$

$$\begin{aligned} \lambda(s) &= -\frac{u^{\delta}}{s^{\delta}}, \\ \mu_{m+1}(\zeta, \eta) &= S^{-1} \left[\frac{\mu_m(\zeta, \eta)}{s} \right] - S^{-1} \\ &\cdot \left[\frac{u^{\delta}}{s^{\delta}} S \left\{ \frac{\partial^2 \mu_m}{\partial \zeta^2} - \frac{\partial \mu_m}{\partial \zeta} + \mu_m \frac{\partial^2 \mu_m}{\partial \zeta^2} - \mu_m^2 + \mu_m \right\} \right]. \end{aligned} \quad (24)$$

Now take,

$$\mu_0(\zeta, \eta) = e^{\zeta}, \quad (25)$$

consequently, we get

$$\begin{aligned} m &= 0, 1, 2, 3 \dots \\ \mu_1(\zeta, \eta) &= S^{-1} \left[\frac{\mu_0(\zeta, \eta)}{s} \right] - S^{-1} \left[\frac{u^{\delta}}{s^{\delta}} S \left\{ \frac{\partial^2 \mu_0}{\partial \zeta^2} - \frac{\partial \mu_0}{\partial \zeta} + \mu_0 \frac{\partial^2 \mu_0}{\partial \zeta^2} - \mu_0^2 + \mu_0 \right\} \right], \\ \mu_1(\zeta, \eta) &= e^{\zeta} + \frac{e^{\zeta} \eta^{\delta}}{\Gamma(\delta+1)}, \\ \mu_2(\zeta, \eta) &= S^{-1} \left[\frac{\mu_1(\zeta, \eta)}{s} \right] - S^{-1} \left[\frac{u^{\delta}}{s^{\delta}} S \left\{ \frac{\partial^2 \mu_1}{\partial \zeta^2} - \frac{\partial \mu_1}{\partial \zeta} + \mu_1 \frac{\partial^2 \mu_1}{\partial \zeta^2} - \mu_1^2 + \mu_1 \right\} \right], \\ \mu_2(\zeta, \eta) &= e^{\zeta} + \frac{e^{\zeta} \eta^{\delta}}{\Gamma(\delta+1)} + \frac{e^{\zeta} \eta^{2\delta}}{\Gamma(2\delta+1)}, \\ \mu_3(\zeta, \eta) &= S^{-1} \left[\frac{\mu_2(\zeta, \eta)}{s} \right] - S^{-1} \left[\frac{u^{\delta}}{s^{\delta}} S \left\{ \frac{\partial^2 \mu_2}{\partial \zeta^2} - \frac{\partial \mu_2}{\partial \zeta} + \mu_2 \frac{\partial^2 \mu_2}{\partial \zeta^2} - \mu_2^2 + \mu_2 \right\} \right], \\ \mu_3(\zeta, \eta) &= e^{\zeta} + \frac{e^{\zeta} \eta^{\delta}}{\Gamma(\delta+1)} + \frac{e^{\zeta} \eta^{2\delta}}{\Gamma(2\delta+1)} + \frac{e^{\zeta} \eta^{3\delta}}{\Gamma(3\delta+1)}. \end{aligned} \quad (26)$$

The approximate result of equation (21) can be achieved as

$$\mu(\zeta, \eta) = e^\zeta + \frac{e^\zeta \eta^\delta}{\Gamma(\delta+1)} + \frac{e^\zeta \eta^{2\delta}}{\Gamma(2\delta+1)} + \frac{e^\zeta \eta^{3\delta}}{\Gamma(3\delta+1)} + \cdots + \frac{e^\zeta \eta^{m\delta}}{\Gamma(m\delta+1)},$$

$$\mu(\zeta, \eta) = e^\zeta \sum_{m=0}^{\infty} \frac{(\eta^\delta)^m}{\Gamma(m\delta+1)} = e^\zeta E_\delta(\eta^\delta). \quad (27)$$

The exact result of equation (21)

$$\mu(\zeta, \eta) = e^{(\zeta+\eta)}. \quad (28)$$

In Figure 1, the exact and the VITM solutions of problem 1 at $\delta = 1$ are shown by subgraphs, respectively. From the given figures, it can be seen that both the VITM and exact results are in close contact with each other. Also, in Figure 2, the VITM results of problem 1 are investigated at different fractional-order $\delta = 0.8$ and 0.6 of 3D and 2D. It is analyzed that in Table 1, the time-fractional problem results are convergent to an integer order effect as time-fractional analysis to integer order.

4.2. Problem. Consider the two-dimensional fractional-order diffusion equation [11]

$$\frac{\partial^\delta \mu}{\partial \eta^\delta} = \frac{\partial^2 \mu}{\partial \zeta^2} + \frac{\partial^2 \mu}{\partial \psi^2} \quad 0 < \delta \leq 1, \quad (29)$$

with the initial condition

$$\mu(\zeta, \psi, 0) = (1 - \psi)e^\zeta. \quad (30)$$

Applying VITM on equation (38), we have

$$\mu_{m+1}(\zeta, \psi, \eta) = S^{-1} \left[\frac{\mu_m(\zeta, \psi, \eta)}{s} \right] + S^{-1} \left[\lambda(s) S \left\{ \frac{\partial^2 \mu_m}{\partial \zeta^2} + \frac{\partial^2 \mu_m}{\partial \psi^2} \right\} \right], \quad (31)$$

where the Lagrange multiplier $\lambda(s)$ is

$$\lambda(s) = -\frac{u^\delta}{s^\delta},$$

$$\mu_{m+1}(\zeta, \psi, \eta) = S^{-1} \left[\frac{\mu_m(\zeta, \psi, \eta)}{s} \right] - S^{-1} \left[\frac{u^\delta}{s^\delta} S \left\{ \frac{\partial^2 \mu_m}{\partial \zeta^2} + \frac{\partial^2 \mu_m}{\partial \psi^2} \right\} \right]. \quad (32)$$

Now take,

$$\mu_0(\zeta, \psi, \eta) = (1 - \psi)e^\zeta, \quad (33)$$

consequently, we get

$$m = 0, 1, 2, 3, \dots$$

$$\mu_1(\zeta, \psi, \eta) = S^{-1} \left[\frac{\mu_0(\zeta, \psi, \eta)}{s} \right] - S^{-1} \left[\frac{u^\delta}{s^\delta} S \left\{ \frac{\partial^2 \mu_0}{\partial \zeta^2} + \frac{\partial^2 \mu_0}{\partial \psi^2} \right\} \right],$$

$$\mu_1(\zeta, \psi, \eta) = (1 - \psi)e^\zeta + (1 - \psi)e^\zeta \frac{\eta^\delta}{\Gamma(\delta+1)},$$

$$\mu_2(\zeta, \psi, \eta) = S^{-1} \left[\frac{\mu_1(\zeta, \psi, \eta)}{s} \right] - S^{-1} \left[\frac{u^\delta}{s^\delta} S \left\{ \frac{\partial^2 \mu_1}{\partial \zeta^2} + \frac{\partial^2 \mu_1}{\partial \psi^2} \right\} \right],$$

$$\mu_2(\zeta, \psi, \eta) = (1 - \psi)e^\zeta + (1 - \psi)e^\zeta \frac{\eta^\delta}{\Gamma(\delta+1)} + (1 - \psi)e^\zeta \frac{\eta^{2\delta}}{\Gamma(2\delta+1)},$$

$$\mu_3(\zeta, \psi, \eta) = S^{-1} \left[\frac{\mu_2(\zeta, \psi, \eta)}{s} \right] - S^{-1} \left[\frac{u^\delta}{s^\delta} S \left\{ \frac{\partial^2 \mu_2}{\partial \zeta^2} + \frac{\partial^2 \mu_2}{\partial \psi^2} \right\} \right],$$

$$\mu_3(\zeta, \psi, \eta) = (1 - \psi)e^\zeta + (1 - \psi)e^\zeta \frac{\eta^\delta}{\Gamma(\delta+1)} + (1 - \psi)e^\zeta \frac{\eta^{2\delta}}{\Gamma(2\delta+1)} + (1 - \psi)e^\zeta \frac{\eta^{3\delta}}{\Gamma(3\delta+1)},$$

$$\vdots \quad (34)$$

The approximate result of equation (38) can be achieved as

$$\mu(\zeta, \psi, \eta) = (1 - \psi)e^\zeta + (1 - \psi)e^\zeta \frac{\eta^\delta}{\Gamma(\delta+1)} + (1 - \psi)e^\zeta \frac{\eta^{2\delta}}{\Gamma(2\delta+1)} + (1 - \psi)e^\zeta \frac{\eta^{3\delta}}{\Gamma(3\delta+1)} + \cdots + (1 - \psi)e^\zeta \frac{\eta^{m\delta}}{\Gamma(m\delta+1)}. \quad (35)$$

When $\alpha = 1$, the VITM solution is

$$\mu(\zeta, \psi, \eta) = (1 - \psi)e^\zeta \sum_{m=0}^{\infty} \frac{(\eta)^m}{m!}. \quad (36)$$

The exact solution in closed form is

$$\mu(\zeta, \psi, \eta) = (1 - \psi)e^{(\zeta+\eta)}. \quad (37)$$

In Figure 3, the exact and the VITM solutions of problem 2 at $\delta = 1$ are shown by subgraphs, respectively. From the given figures, it can be seen that both the VITM and exact results are in close contact with each other. Also, in Figure 4, the VITM results of problem 2 are investigated at different fractional-order $\delta = 0.8$ and 0.6 of 3D and 2D. It is analyzed that in Table 2, the time-fractional problem results are convergent to an integer order effect as time-fractional analysis to integer order.

4.3. Problem. Consider the two-dimensional fractional-order diffusion equation [11]

$$\frac{\partial^\delta \mu}{\partial \eta^\delta} = \frac{\partial^2 \mu}{\partial \zeta^2} + \frac{\partial^2 \mu}{\partial \psi^2} \quad 0 < \delta \leq 1, \quad (38)$$

with the initial condition

$$\mu(\zeta, \psi, 0) = e^{(\zeta+\psi)}. \quad (39)$$

FIGURE 1: Graph of exact and analytical results of Problem 3.1.

FIGURE 2: The different fractional-order graphs of Problem 3.1.

Applying VITM on equation (38), we have

$$\mu_{m+1}(\zeta, \psi, \eta) = S^{-1} \left[\frac{\mu_m(\zeta, \psi, \eta)}{s} \right] + S^{-1} \left[\lambda(s) S \left\{ \frac{\partial^2 \mu_m}{\partial \zeta^2} + \frac{\partial^2 \mu_m}{\partial \psi^2} \right\} \right], \quad (40)$$

where the Lagrange multiplier $\lambda(s)$ is

$$\lambda(s) = -\frac{u^\delta}{s^\delta},$$

$$\mu_{m+1}(\zeta, \psi, \eta) = S^{-1} \left[\frac{\mu_m(\zeta, \psi, \eta)}{s} \right] - S^{-1} \left[\frac{u^\delta}{s^\delta} S \left\{ \frac{\partial^2 \mu_m}{\partial \zeta^2} + \frac{\partial^2 \mu_m}{\partial \psi^2} \right\} \right]. \quad (41)$$

Now take,

$$\mu_0(\zeta, \psi, \eta) = e^{(\zeta+\psi)}, \quad (42)$$

consequently, we get

$$m = 0, 1, 2, 3 \dots$$

$$\mu_1(\zeta, \psi, \eta) = S^{-1} \left[\frac{\mu_0(\zeta, \psi, \eta)}{s} \right] - S^{-1} \left[\frac{u^\delta}{s^\delta} S \left\{ \frac{\partial^2 \mu_0}{\partial \zeta^2} + \frac{\partial^2 \mu_0}{\partial \psi^2} \right\} \right],$$

$$\mu_1(\zeta, \psi, \eta) = e^{(\zeta+\psi)} + 2e^{(\zeta+\psi)} \frac{\eta^\delta}{\Gamma(\delta+1)},$$

$$\mu_2(\zeta, \psi, \eta) = S^{-1} \left[\frac{\mu_1(\zeta, \psi, \eta)}{s} \right] - S^{-1} \left[\frac{u^\delta}{s^\delta} S \left\{ \frac{\partial^2 \mu_1}{\partial \zeta^2} + \frac{\partial^2 \mu_1}{\partial \psi^2} \right\} \right],$$

$$\mu_2(\zeta, \psi, \eta) = e^{(\zeta+\psi)} + 2e^{(\zeta+\psi)} \frac{\eta^\delta}{\Gamma(\delta+1)} + 4e^{(\zeta+\psi)} \frac{\eta^{2\delta}}{\Gamma(2\delta+1)},$$

$$\mu_3(\zeta, \psi, \eta) = S^{-1} \left[\frac{\mu_2(\zeta, \psi, \eta)}{s} \right] - S^{-1} \left[\frac{u^\delta}{s^\delta} S \left\{ \frac{\partial^2 \mu_2}{\partial \zeta^2} + \frac{\partial^2 \mu_2}{\partial \psi^2} \right\} \right],$$

$$\mu_3(\zeta, \psi, \eta) = e^{(\zeta+\psi)} + 2e^{(\zeta+\psi)} \frac{\eta^\delta}{\Gamma(\delta+1)} + 4e^{(\zeta+\psi)} \frac{\eta^{2\delta}}{\Gamma(2\delta+1)}$$

$$+ 16e^{(\zeta+\psi)} \frac{\eta^{3\delta}}{\Gamma(3\delta+1)},$$

$$\vdots \quad (43)$$

TABLE 1: VITM at fractional-order value $\delta = 0.8$ and absolute error $\delta = 1$ of example 1.

η	ζ	$\delta = 0.75$	$\delta = 1$	Exact	AE($\delta = 1$)
0.5	1	0.853687662	0.7165300518	0.7165306597	7.56E-08
	2	0.383587401	0.3331299364	0.3331301601	3.37E-08
	3	0.210647180	0.0930849163	0.0930849986	9.35E-09
	4	0.048026028	0.0411973531	0.0411973834	4.24E-09
	5	0.024621114	0.0222089854	0.0222089965	2.10E-09
1	1	1.389724464	0.9988367591	1.0000000000	9.35E-06
	2	0.580784320	0.4778488184	0.4778794412	4.42E-06
	3	0.283191872	0.2453240178	0.2453352832	2.24E-06
	4	0.074713729	0.0587829240	0.0587870683	5.45E-07
	5	0.034438971	0.0193141142	0.0193156388	2.32E-07

FIGURE 3: Graph of exact and approximate solutions of Problem 3.2.

The approximate result of equation (38) can be achieved as

$$\begin{aligned}
 \mu(\zeta, \psi, \eta) &= e^{(\zeta+\psi)} + 2e^{(\zeta+\psi)} \frac{\eta^\delta}{\Gamma(\delta+1)} + 4e^{(\zeta+\psi)} \frac{\eta^{2\delta}}{\Gamma(2\delta+1)} \\
 &\quad + 16e^{(\zeta+\psi)} \frac{\eta^{3\delta}}{\Gamma(3\delta+1)} + \dots + (2)^m e^{(\zeta+\psi)} \frac{\eta^{m\delta}}{\Gamma(m\delta+1)}, \\
 \mu(\zeta, \psi, \eta) &= e^{(\zeta+\psi)} \left(1 + \frac{2\eta^\delta}{\Gamma(\delta+1)} + \frac{(2\eta^\delta)^2}{\Gamma(2\delta+1)} + \frac{(2\eta^\delta)^3}{\Gamma(3\delta+1)} \right. \\
 &\quad \left. + \frac{(2\eta^\delta)^4}{\Gamma(4\delta+1)} + \dots + \frac{(2\eta^\delta)^m}{\Gamma(m\delta+1)} \right), \\
 \mu(\zeta, \psi, \eta) &= e^{(\zeta+\psi)} \sum_{m=0}^{\infty} \frac{(\eta^\delta)^m}{\Gamma(m\delta+1)} = (1-\psi)e^\zeta E_\delta(\eta^\delta). \quad (44)
 \end{aligned}$$

When $\delta = 1$, then the VITM solution is

$$\mu(\zeta, \psi, \eta) = e^{(\zeta+\psi)} \sum_{m=0}^{\infty} \frac{(\eta)^m}{m!}. \quad (45)$$

The exact solution in closed form is

$$\mu(\zeta, \psi, \eta) = e^{(\zeta+\psi+\eta)}. \quad (46)$$

In Figure 5, the exact and the VITM solutions of problem 3 at $\delta = 1$ are shown by subgraphs, respectively. From the given figures, it can be seen that both the VITM and exact results are in close contact with each other. Also, in Figure 6, the VITM results of problem 3 are investigated at different fractional-order $\delta = 0.8$ and 0.6 of 3D and 2D. It is analyzed that time-fractional problem results are convergent to an integer order effect as time-fractional analysis to integer order.

4.4. Problem. Consider the three-dimensional fractional-order diffusion equation [11]

$$\frac{\partial^\delta \mu}{\partial \eta^\delta} = \frac{\partial^2 \mu}{\partial \zeta^2} + \frac{\partial^2 \mu}{\partial \psi^2} + \frac{\partial^2 \mu}{\partial \mathfrak{S}^2}, \quad 0 < \delta \leq 1, \eta \geq 0, \quad (47)$$

FIGURE 4: The different fractional-order δ graphs of Problem 3.2.TABLE 2: VITM at fractional-order value $\delta = 0.8$ and absolute error $\delta = 1$ of example 2.

η	ζ	$\delta = 0.75$	$\delta = 1$	Exact	AE($\delta = 1$)
0.5	1	0.6873975264	0.4262266609	0.326227766	2.23E-07
	2	0.0687397526	0.0426226660	0.042622776	2.34E-08
	3	0.0067739752	0.0042622666	0.004262277	0.21E-09
	4	0.0006773975	0.0004362266	0.000436227	1.12E-10
	5	0.0000687397	0.0000446226	0.000044622	2.14E-10
1	1	2.7771190120	0.9984599728	1.000000000	4.56E-05
	2	0.3567119012	0.0998953372	0.100000000	5.89E-07
	3	0.0376711901	0.0098746437	0.010000000	7.12E-07
	4	0.0042671190	0.0009988544	0.001000000	7.89E-08
	5	0.0004567119	0.0000998459	0.000100000	7.22E-09

with the initial condition

$$\mu(\zeta, \psi, \mathfrak{F}, 0) = \sin \zeta \sin \psi \sin \mathfrak{F}. \quad (48)$$

Applying VITM on equation (47), we have

$$\begin{aligned} \mu_{m+1}(\zeta, \psi, \mathfrak{F}, \eta) = & S^{-1} \left[\frac{\mu_m(\zeta, \psi, \mathfrak{F}, \eta)}{s} \right] \\ & + S^{-1} \left[\lambda(s) S \left\{ \frac{\partial^2 \mu_m}{\partial \zeta^2} + \frac{\partial^2 \mu_m}{\partial \psi^2} + \frac{\partial^2 \mu_m}{\partial \mathfrak{F}^2} \right\} \right], \end{aligned} \quad (49)$$

where the Lagrange multiplier $\lambda(s)$ is

$$\begin{aligned} \lambda(s) = & -\frac{u^\delta}{s^\delta}, \\ \mu_{m+1}(\zeta, \psi, \mathfrak{F}, \eta) = & S^{-1} \left[\frac{\mu_m(\zeta, \psi, \mathfrak{F}, \eta)}{s} \right] \\ & - S^{-1} \left[\frac{u^\delta}{s^\delta} S \left\{ \frac{\partial^2 \mu_m}{\partial \zeta^2} + \frac{\partial^2 \mu_m}{\partial \psi^2} + \frac{\partial^2 \mu_m}{\partial \mathfrak{F}^2} \right\} \right]. \end{aligned} \quad (50)$$

Now take,

$$\mu_0(\zeta, \psi, \mathfrak{F}, \eta) = \sin \zeta \sin \psi \sin \mathfrak{F}, \quad (51)$$

FIGURE 5: Graph of exact and approximate solutions of Problem 3.3.

consequently, we get

$$m = 0, 1, 2, 3 \dots$$

$$\begin{aligned} \mu_1(\zeta, \psi, \mathfrak{F}, \eta) &= S^{-1} \left[\frac{\mu_0(\zeta, \psi, \mathfrak{F}, \eta)}{s} \right] \\ &\quad - S^{-1} \left[\frac{u^\delta}{s^\delta} S \left\{ \frac{\partial^2 \mu_0}{\partial \zeta^2} + \frac{\partial^2 \mu_0}{\partial \psi^2} + \frac{\partial^2 \mu_0}{\partial \mathfrak{F}^2} \right\} \right], \\ \mu_1(\zeta, \psi, \mathfrak{F}, \eta) &= \sin \zeta \sin \psi \sin \mathfrak{F} \left(1 - 3 \frac{\eta^\delta}{\Gamma(\delta+1)} \right), \\ \mu_2(\zeta, \psi, \mathfrak{F}, \eta) &= S^{-1} \left[\frac{\mu_1(\zeta, \psi, \mathfrak{F}, \eta)}{s} \right] \\ &\quad - S^{-1} \left[\frac{u^\delta}{s^\delta} S \left\{ \frac{\partial^2 \mu_1}{\partial \zeta^2} + \frac{\partial^2 \mu_1}{\partial \psi^2} + \frac{\partial^2 \mu_1}{\partial \mathfrak{F}^2} \right\} \right], \\ \mu_2(\zeta, \psi, \mathfrak{F}, \eta) &= \sin \zeta \sin \psi \sin \mathfrak{F} \left(1 - 3 \frac{\eta^\delta}{\Gamma(\delta+1)} \right. \\ &\quad \left. + (-3)^2 \frac{\eta^{2\delta}}{\Gamma(2\delta+1)} \right), \\ \mu_3(\zeta, \psi, \mathfrak{F}, \eta) &= S^{-1} \left[\frac{\mu_2(\zeta, \psi, \mathfrak{F}, \eta)}{s} \right] \\ &\quad - S^{-1} \left[\frac{u^\delta}{s^\delta} S \left\{ \frac{\partial^2 \mu_2}{\partial \zeta^2} + \frac{\partial^2 \mu_2}{\partial \psi^2} + \frac{\partial^2 \mu_2}{\partial \mathfrak{F}^2} \right\} \right], \\ \mu_3(\zeta, \psi, \mathfrak{F}, \eta) &= \sin \zeta \sin \psi \sin \mathfrak{F} \left(1 - 3 \frac{\eta^\delta}{\Gamma(\delta+1)} \right. \\ &\quad \left. + (-3)^2 \frac{\eta^{2\delta}}{\Gamma(2\delta+1)} + (-3)^3 \frac{\eta^{3\delta}}{\Gamma(3\delta+1)} \right), \\ &\quad \vdots \end{aligned} \quad (52)$$

The approximate result of equation (47) can be achieved as

$$\begin{aligned} u(\zeta, \psi, \mathfrak{F}, \eta) &= \sin \zeta \sin \psi \sin \mathfrak{F} \\ &\quad \cdot \left(1 - 3 \frac{\eta^\delta}{\Gamma(\delta+1)} + (-3)^2 \frac{\eta^{2\delta}}{\Gamma(2\delta+1)} \right. \\ &\quad \left. + (-3)^3 \frac{\eta^{3\delta}}{\Gamma(3\delta+1)} + \dots + (-3)^m \frac{\eta^{m\delta}}{\Gamma(m\delta+1)} \right). \end{aligned} \quad (53)$$

When $\delta = 1$, then the VITM solution in a closed form is

$$\begin{aligned} \mu(\zeta, \psi, \mathfrak{F}, \eta) &= \sin \zeta \sin \psi \sin \mathfrak{F} \\ &\quad \cdot \left(1 - 3\eta + \frac{(-3\eta)^2}{2!} + \frac{(-3\eta)^3}{3!} + \frac{(-3\eta)^4}{4!} + \dots \right). \end{aligned} \quad (54)$$

The exact solution in closed form is

$$\mu(\zeta, \psi, \mathfrak{F}, \eta) = e^{-3\eta} \sin \zeta \sin \psi \sin \mathfrak{F}. \quad (55)$$

In Figure 7, the exact and the VITM solutions of problem 4 at $\delta = 1$ are shown by subgraphs, respectively. From the given figures, it can be seen that both the VITM and exact results are in close contact with each other. Also, in Figure 8, the VITM results of problem 4 are investigated at different fractional-order $\delta = 0.8$ and 0.6 . It is analyzed that time-fractional problem results are convergent to an integer order effect as time-fractional analysis to integer order.

4.5. Example. Consider the fractional-order nonlinear convection-diffusion equation

$$\frac{\partial^\delta \mu}{\partial \eta^\delta} = \frac{\partial^2 \mu}{\partial \zeta^2} - \frac{\partial \mu}{\partial \zeta} + \frac{\partial}{\partial \eta} \left(\mu \frac{\partial^2 \mu}{\partial \zeta^2} \right) - 2\zeta, \quad 0 < \zeta \leq 1, \quad 0 < \delta \leq 1, \quad \eta > 0, \quad (56)$$

FIGURE 6: The different fractional-order graphs of δ Problem 3.3.

FIGURE 7: Graph of exact and approximate solutions of Problem 3.4.

with the boundary conditions

$$\mu(0, \eta) = 2\eta, \mu(1, \eta) = 1 + 2\eta, \quad (57)$$

and initial condition

$$\mu(\zeta, 0) = \zeta^2, \quad (58)$$

Applying VITM on equation (56), we have

$$\begin{aligned} \mu_{m+1}(\zeta, \eta) = & S^{-1} \left[\frac{\mu_m(\zeta, \eta)}{s} \right] \\ & + S^{-1} \left[\lambda(s) S \left\{ \frac{\partial^2 \mu_m}{\partial \zeta^2} - \frac{\partial \mu_m}{\partial \zeta} + \frac{\partial}{\partial \eta} \left(\mu_m \frac{\partial^2 \mu_m}{\partial \zeta^2} \right) - 2\zeta \right\} \right], \end{aligned} \quad (59)$$

where the Lagrange multiplier is $\lambda(s)$.

$$\lambda(s) = -\frac{u^\delta}{s^\delta},$$

$$\begin{aligned} \mu_{m+1}(\zeta, \eta) = & S^{-1} \left[\frac{\mu_m(\zeta, \eta)}{s} \right] \\ & - S^{-1} \left[\frac{u^\delta}{s^\delta} S \left\{ \frac{\partial^2 \mu_m}{\partial \zeta^2} - \frac{\partial \mu_m}{\partial \zeta} + \frac{\partial}{\partial \eta} \left(\mu_m \frac{\partial^2 \mu_m}{\partial \zeta^2} \right) - 2\zeta \right\} \right]. \end{aligned} \quad (60)$$

Now take,

$$\mu_0(\zeta, \eta) = \zeta^2, \quad (61)$$

FIGURE 8: The fractional-order of $\delta = 0.8$ and 0.6 of Problem 3.4.

consequently, we get

$$\begin{aligned}
 m &= 0, 1, 2, 3 \dots \\
 \mu_1(\zeta, \eta) &= S^{-1} \left[\frac{\mu_0(\zeta, \eta)}{s} \right] \\
 &\quad - S^{-1} \left[\frac{u^\delta}{s^\delta} S \left\{ \frac{\partial^2 \mu_0}{\partial \zeta^2} - \frac{\partial \mu_0}{\partial \zeta} + \frac{\partial}{\partial \eta} \left(\mu_0 \frac{\partial^2 \mu_0}{\partial \zeta^2} \right) - 2\zeta \right\} \right], \\
 \mu_1(\zeta, \eta) &= \zeta^2 + (2 - 4\zeta) \frac{\eta^\delta}{\Gamma(\delta + 1)}, \\
 \mu_2(\zeta, \eta) &= S^{-1} \left[\frac{\mu_1(\zeta, \eta)}{s} \right] \\
 &\quad - S^{-1} \left[\frac{u^\delta}{s^\delta} S \left\{ \frac{\partial^2 \mu_1}{\partial \zeta^2} - \frac{\partial \mu_1}{\partial \zeta} + \frac{\partial}{\partial \eta} \left(\mu_1 \frac{\partial^2 \mu_1}{\partial \zeta^2} \right) - 2\zeta \right\} \right], \\
 \mu_2(\zeta, \eta) &= \zeta^2 + (2 - 4\zeta) \frac{\eta^\delta}{\Gamma(\delta + 1)} + 4 \frac{\eta^{2\delta}}{\Gamma(2\delta + 1)} \\
 &\quad - 4\zeta(3\zeta - 1) \frac{\eta^{2\delta-1}}{\Gamma(2\delta)}, \\
 \mu_3(\zeta, \eta) &= S^{-1} \left[\frac{\mu_2(\zeta, \eta)}{s} \right] \\
 &\quad - S^{-1} \left[\frac{u^\delta}{s^\delta} S \left\{ \frac{\partial^2 \mu_2}{\partial \zeta^2} - \frac{\partial \mu_2}{\partial \zeta} + \frac{\partial}{\partial \eta} \left(\mu_2 \frac{\partial^2 \mu_2}{\partial \zeta^2} \right) - 2\zeta \right\} \right], \\
 \mu_3(\zeta, \eta) &= \zeta^2 + (2 - 4\zeta) \frac{\eta^\delta}{\Gamma(\delta + 1)} + 4 \frac{\eta^{2\delta}}{\Gamma(2\delta + 1)} \\
 &\quad - 4\zeta(3\zeta - 1) \frac{\eta^{2\delta-1}}{\Gamma(2\delta)} - 24 \frac{\eta^{3\delta-1}}{\Gamma(3\delta)} - 4(6\zeta + 1) \frac{\eta^{3\delta-2}}{\Gamma(3\delta)} \\
 &\quad - 4\zeta^2(6\zeta - 1) \frac{\eta^{3\delta-2}}{\Gamma(3\delta - 1)} - 8(1 - 2\zeta) \frac{\Gamma(2\delta + 1)\eta^{3\delta-1}}{\Gamma(3\delta)(\Gamma(\delta + 1))^2} \\
 &\quad + 8\zeta \frac{\eta^{3\delta-1}}{\Gamma(3\delta)} + 8\zeta^2(1 - 3\zeta) \frac{\eta^{3\delta-2}}{\Gamma(3\delta - 1)}. \tag{62}
 \end{aligned}$$

The approximate result of equation (56) can be achieved as

$$\begin{aligned}
 \mu(\zeta, \eta) &= \zeta^2 + (2 - 4\zeta) \frac{\eta^\delta}{\Gamma(\delta + 1)} + 4 \frac{\eta^{2\delta}}{\Gamma(2\delta + 1)} \\
 &\quad - 4\zeta(3\zeta - 1) \frac{\eta^{2\delta-1}}{\Gamma(2\delta)} - 24 \frac{\eta^{3\delta-1}}{\Gamma(3\delta)} \\
 &\quad - 4(6\zeta + 1) \frac{\eta^{3\delta-2}}{\Gamma(3\delta)} - 4\zeta^2(6\zeta - 1) \frac{\eta^{3\delta-2}}{\Gamma(3\delta - 1)} \tag{63} \\
 &\quad - 8(1 - 2\zeta) \frac{\Gamma(2\delta + 1)\eta^{3\delta-1}}{\Gamma(3\delta)(\Gamma(\delta + 1))^2} + 8\zeta \frac{\eta^{3\delta-1}}{\Gamma(3\delta)} \\
 &\quad + 8\zeta^2(1 - 3\zeta) \frac{\eta^{3\delta-2}}{\Gamma(3\delta - 1)} + \dots
 \end{aligned}$$

The exact result of equation (56) is

$$\mu(\zeta, \eta) = \zeta^2 + 2\eta. \tag{64}$$

5. Conclusion

In this article, an extended variational iteration transform method is implemented to achieve the analytical result of time-fractional diffusion equations. The suggested method is an effective and simple tool to solve fractional-order partial differential equations, because it applies the Lagrange multiplier directly to solve fractional-order partial differential equations. In conclusion, the current technique has the small number of calculations and straightforward implementation and therefore can be applied to other fractional-order partial differential equation, which frequently arises in applied science.

Abbreviations

VITM: Variational iteration transform method
 ST: Shehu transform
 DE: Diffusion equation
 FC: Fractional calculus
 PDEs: Partial differential equations
 ADM: Adomian decomposition method.

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Nehad Ali Shah and Jae Dong Chung contributed equally to this work and are first coauthors.

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