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Research Article

Fixed Point Results for Rational Orbitally (Θ, δ_b)-Contractions with an Application

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The purpose of this paper is to define a rational orbitally (Θ, δ_b)-contraction and prove some new results in the context of b-metric spaces. Our results extend, generalize, and unify some known results in the literature. As application of our main result, we investigate the solution of Fredholm integral inclusion. We also provide an example to substantiate the advantage and usefulness of obtained results.

1. Introduction

The fixed point theory is a very essential tool for nonlinear analysis of solvability of nonlinear integral equations and others. A suitable selection of a generalized and extended metric space allows to get nontrivial conditions guaranteeing the existence of solutions for a considered equation. Therefore, it is necessary to flourish the fixed point theory in various generalization of metric spaces. One of the famous extensions of metric space is the notion of b-metric space which has been given by Bakhtin [1] in 1989. It was properly defined by Czerwik [2] with the aspect of relaxing triangle inequality in metric spaces in 1993 and proved famous Banach Contraction Principle in this generalized metric space. Khamsi and Hussain [3] discussed the topology of b-metric space and established fixed point results for KKM mappings in metric type spaces. Van An et al. [4] proved the Stone-type theorem on b-metric spaces and obtained a sufficient condition for a b-metric space to be metrizable. On the other, Czerwik [5, 6] introduced set-valued mappings in b-metric spaces and generalized Nadler's fixed point theorem. In 2012, Aydi et al. [7, 8] gave fixed point and common fixed point theorems for set-valued quasicontraction mappings and set-valued weak ϕ-contraction mappings in the setting of b-metric spaces, respectively. Many authors followed the concept of b-metric space and established impressive results [9–19].

In 2012, Jleli and Samet [20] introduced a new type of contraction named as Θ-contraction and obtained a fixed point result to generalize the celebrated Banach Contraction Principle in Branciari metric spaces. Ali et al. [21] defined multivalued Suzuki-type θ-contractions and obtained some generalized fixed point results. Afterwards, Jleli et al. [22] established a new fixed point theorem for Θ-contraction in the setting of Branciari metric spaces and extended the main result of Jleli and Samet [20]. Recently, Alamri et al. [23] adapted Jleli's approach to the b-metric space and obtained some generalized fixed point results. For more details in the direction of Θ-contractions, we refer the reader to [21–30].

In this paper, we define the notion of the rational (Θ, δ_b)-contraction in b-metric spaces and explore the existence of solutions for certain integral problems of Fredholm type as applications of our main results. We obtain our results by using fixed point theorems for multivalued mappings, under new contractive conditions, in the setting of complete b-metric spaces. Evidently, the given results generalized some notable results of the literature to b-metric spaces.

2. Preliminaries

In this section, we give some fundamental notations, definitions, and lemmas which will be used throughout the paper. Throughout this paper, we denote $\mathbb{N}$ the set of positive integers and $\mathbb{R}^+$ the set of all nonnegative real numbers.

Czerwik [2] gave the notion of the b-metric space as follows.

Definition 1 (see [2]). Let $\mathscr{M}$ be a nonempty set $\mathscr{M}$ and $s \geq 1$. A function σ_b: $\mathscr{M} \times \mathscr{M} \longrightarrow \mathbb{R}_0^+$ is called a b-metric if these assertions hold: for all $\rho, \varpi, \omega \in \mathscr{M}$,

$(B1)\sigma_b(\rho, \varpi) = 0 \Leftrightarrow \rho = \varpi$

$(B2)\sigma_b(\rho, \varpi) = \sigma_b(\varpi, \rho)$

$(B3)\sigma_b(\rho, \omega) \leq s(\sigma_b(\rho, \varpi) + \sigma_b(\varpi, \omega))$

The triple $(\mathscr{M}, \sigma_b, s)$ is called a b -metric space.

Now, we give an elementary example of a b-metric space, but it is not a metric space as follows.

Example 2 (see [2]). Let $\mathscr{M} = \mathbb{R}$ and $\sigma_b : \mathscr{M} \times \mathscr{M} \longrightarrow \mathbb{R}$ be a mapping defined by

$$\sigma_b(\rho, \varpi) = |\rho - \varpi|^2, \quad \forall \rho, \varpi \in \mathscr{M}, s = 2. \tag{1}$$

Then, $(\mathscr{M}, \sigma_b, s)$ is a b-metric space which is not a metric space.

A brief but short history for multivalued mappings defined in $(\mathscr{M}, \sigma_b, s)$ is given in this way.

Let $P_{cb}(\mathscr{M})$ be the family of all bounded and closed subsets of $\mathscr{M}$. For any $E_1, E_2, E_3 \in P_b(\mathscr{M})$, we define

$$\begin{aligned} \sigma_b(E_1, E_2) &= \inf \{\sigma_b(\rho, \varpi) : \rho \in E_1, \varpi \in E_2\}, \\ \delta_b(E_1, E_2) &= \sup \{\sigma_b(\rho, \varpi) : \rho \in E_1, \varpi \in E_2\}, \end{aligned} \tag{2}$$

with

$$\sigma_b(\rho, E_3) = \sigma_b(\{\rho\}, E_3) = \inf \{\sigma_b(\rho, \omega) : \rho \in E_1, \omega \in E_3\}. \tag{3}$$

Here, we provide some useful properties of δ_b and σ_b (see [2, 5, 6]):

(1) If $E_1 = \{\rho\}$ and $E_2 = \{\varpi\}$, then $\sigma_b(E_1, E_2) = \delta_b(E_1, E_2) = \sigma_b(\rho, \varpi)$

(2) $\sigma_b(E_1, E_2) \leq \delta_b(E_1, E_2)$

(3) $\sigma_b(\rho, E_2) \leq \sigma_b(\rho, \varpi)$ for any $\varpi \in E_2$

(4) $\delta_b(E_1, E_3) \leq s[\delta_b(E_1, E_2) + \delta_b(E_2, E_3)]$

(5) $\delta_b(E_1, E_2) = 0 \Leftrightarrow E_1 = E_2 = \{\rho\}$

Moreover, we will always suppose that

(6) the function σ_b is continuous in its variables

Now, we present the concepts of orbit and orbital continuity of a mapping in the setting of $(\mathscr{M}, \sigma_b, s)$ which are generalization and extensions of the same notions for metric spaces given in [31, 32].

Definition 3. Let $\mathscr{J}, \mathscr{J}_{1,}\mathscr{J}_2 : (\mathscr{M}, \sigma_b, s) \longrightarrow P_b(\mathscr{M})$.

(1) An *orbit* $O(\rho_0, \mathscr{J})$ of $\mathscr{J}$ at ρ_0 is any sequence $\{\rho_n\}$ such that $\rho_n \in \mathscr{J}\rho_{n-1}$ for each $n \in \mathbb{N}$

(2) If, for any $\rho_0 \in \mathscr{M}$, there exists $\{\rho_n\}$ in $\mathscr{M}$ such that $\rho_{2n+1} \in \mathscr{J}_2\rho_{2n}$ and $\rho_{2n+2} \in \mathscr{J}_1\rho_{2n+1}$ for each $n \in \mathbb{N} \cup \{0\}$, then $O(\rho_0, \mathscr{J}_1, \mathscr{J}_2) = \{\rho_n\}$ for each $n \in \mathbb{N}$ is said to be an *orbit* of $(\mathscr{J}_1, \mathscr{J}_2)$ at ρ_0

(3) $(\mathscr{M}, \sigma_b, s)$ is said to be $(\mathscr{J}_1, \mathscr{J}_2)$ *-orbitally complete* if any Cauchy subsequence $\{\rho_{n_i}\}$ of $O(\rho_0, \mathscr{J}_1, \mathscr{J}_2)$ converges in $\mathscr{M}$. Specifically, for $\mathscr{J}_1 = \mathscr{J}_2 = \mathscr{J}$, $\mathscr{M}$ is said to be $\mathscr{J}$ *-orbitally complete*

(4) $\mathscr{J}$ is said to be *orbitally continuous* at $\rho_0 \in \mathscr{M}$ if, for any $\{\rho_n\} \subset O(\rho_0, \mathscr{J})$ for each $n \in \mathbb{N} \cup \{0\}$ and $\rho^* \in \mathscr{M}$,$\sigma_b(\rho_n, \rho^*) \longrightarrow 0$ as $n \longrightarrow \infty$ gives that $\delta_b(\mathscr{J}\rho_n, \mathscr{J}\rho^*) \longrightarrow 0$ as $n \longrightarrow \infty$

(5) A graph $G(\mathscr{J})$ of $\mathscr{J}$ is constructed as follows:

$$G(\mathscr{J}) = \{(\rho, \varpi) : \rho \in \mathscr{M}, \varpi \in \mathscr{J}\rho\} \tag{4}$$

We need the following property of $G(\mathscr{J})$ in our proof:

(G_p) $G(\mathscr{J})$ is said to be $\mathscr{J}$ *-orbitally closed* if, for any sequence $\{\rho_n\}$ in $\mathscr{M}$, we get $(\rho^*, \rho^*) \in G(\mathscr{J})$ whenever $(\rho_n, \rho_{n+1}) \in G(\mathscr{J})$ and $\lim_{n\longrightarrow\infty}\rho_n = \rho^*$

In 2012, Jleli and Samet [20] gave the notion of Θ-contractions and proved a contemporary fixed point theorem for these contractions in generalized metric spaces. Motivate by Jleli and Samet [20], Alamri et al. [23] present the following definition.

Definition 4 (see [23]). Let $\Omega_s(s \geq 1)$ denote the family of all mappings $\Theta : \mathbb{R}^+ \longrightarrow (1,\infty)$ which satisfy these conditions:

$(\Theta_1)\Theta$ is nondecreasing

(Θ_2)For any $\{\rho_n\} \subseteq \mathbb{R}^+$, $\lim_{n\longrightarrow\infty}\Theta(\rho_n) = 1 \Leftrightarrow \lim_{n\longrightarrow\infty}(\rho_n) = 0$

(Θ_3)There exist $h \in (0, 1)$ and $l \in (0,\infty]$ such that $\lim_{\rho\longrightarrow 0^+}\Theta(\rho) - 1/\rho^h = l$

(Θ_4)For $\{\rho_n\} \subseteq \mathbb{R}^+$ such that $\Theta(s\rho_n) \leq \Theta(\rho_{n-1})]^k$ for each $n \in \mathbb{N}$ and some $0<k<1$, then $\Theta(s_n^n\rho) \leq \Theta(s^{n-1}\rho_{n-1})]^k$ for each $n \in \mathbb{N}$

They provided the following example.

Example 5. Let $\Theta : \mathbb{R}^+ \longrightarrow (1,\infty)$ be a mapping given by $\theta(\varsigma) = e^{\sqrt{\varsigma e^{\varsigma}}}$. Clearly, Θ satisfies the conditions (Θ_1)-(Θ_4). Here, we show only the condition (Θ_4). Suppose that, for each $n \in \mathbb{N}$ and some $0 < k \in <1$, we have $\theta(s\rho_n) \leq \theta(\rho_{n-1})]^k$. This implies that

$$e^{\sqrt{s\rho_n e^{s\rho_n}}} \leq \left[e^{\sqrt{\rho_{n-1}e^{\rho_{n-1}}}}\right]^k, \tag{5}$$

i.e.,

$$\sqrt{s\rho_n e^{s\rho_n}} \le k\sqrt{\rho_{n-1} e^{\rho_{n-1}}}. \quad (6)$$

This implies that

$$\sqrt{s\rho_n e^{s\rho_n - \rho_{n-1}}} \le k\sqrt{\rho_{n-1}}. \quad (7)$$

It follows that $\theta(s\rho_n) \le \theta(\rho_{n-1})]^k \le \theta(\rho_{n-1})$ and θ is non-decreasing, and so, $s\rho_n \le \rho_{n-1}$ and $s\rho_n - \rho_{n-1} \le 0$ implies $e^{s^{n-1}(s\rho_n - \rho_{n-1})} \le e^{s\rho_n - \rho_{n-1}}$. Therefore, (7) implies that

$$\begin{aligned}\sqrt{s\rho_n e^{s^{n-1}(s\rho_n - \rho_{n-1})}} \le k\sqrt{\rho_{n-1}} &\Rightarrow \sqrt{\frac{s\rho_n e^{s^n\rho_n}}{e^{s^{n-1}\rho_{n-1}}}} \le k\sqrt{\rho_{n-1}} \Rightarrow \sqrt{s\rho_n e^{s^n\rho_n}} \\ &\le k\sqrt{\rho_{n-1} e^{s^{n-1}\rho_{n-1}}} \Rightarrow \sqrt{s^n\rho_n e^{s^n\rho_n}} \\ &\le k\sqrt{s^{n-1}\rho_{n-1} e^{s^{n-1}\rho_{n-1}}} \Rightarrow e^{\sqrt{s^n\rho_n e^{s^n\rho_n}}} \\ &\le e^{k\sqrt{s^{n-1}\rho_{n-1}e^{s^{n-1}\rho_{n-1}}}} \Rightarrow \theta(s^n\rho_n) \le \left[\theta\left(s^{n-1}\rho_{n-1}\right)\right]^k,\end{aligned} \quad (8)$$

and hence, the condition (Θ_4) holds.

3. Main Results

In this way, we define the notion of rational (Θ, δ_b)-contraction.

Definition 6. Let $(\mathscr{M}, \sigma_b)$ be a b -metric space. A mapping $\mathscr{J} : \mathscr{M} \longrightarrow P_b(\mathscr{M})$ is called a rational (Θ, δ_b)-contraction if $\exists \Theta \in \Omega_s$, $0 < k < 1$ and $0 \le L$ such that

$$\Theta(s\delta_b(\mathscr{J}\rho, \mathscr{J}\varpi)) \le [\Theta(m_1(\rho, \varpi) + Lm_2(\rho, \varpi))]^k, \quad (9)$$

$\forall \rho, \varpi \in \mathscr{M}$ with $\min\{\delta_b(\mathscr{J}\rho, \mathscr{J}\varpi), \sigma_b(\rho, \varpi)\} > 0$, where

$$m_1(\rho,\varpi) = \max\left\{\begin{array}{c}\sigma_b(\rho,\varpi), \sigma_b(\rho,\mathscr{J}\rho), \sigma_b(\varpi,\mathscr{J}\varpi), \dfrac{\sigma_b(\rho,\mathscr{J}\varpi)+\sigma_b(\varpi,\mathscr{J}\rho)}{2s}, \\ \dfrac{\sigma_b(\varpi,\mathscr{J}\varpi)[1+\sigma_b(\rho,\mathscr{J}\rho)]}{s[1+\sigma_b(\rho,\varpi)]}, \dfrac{\sigma_b(\varpi,\mathscr{J}\rho)[1+\sigma_b(\rho,\mathscr{J}\rho)]}{s[1+\sigma_b(\rho,\varpi)]}\end{array}\right\},$$

$$m_2(\rho, \varpi) = \min\{\sigma_b(\rho, \mathscr{J}\rho), \sigma_b(\varpi, \mathscr{J}\varpi), \sigma_b(\rho, \mathscr{J}\varpi), \sigma_b(\varpi, \mathscr{J}\rho)\}. \quad (10)$$

If (9) holds for all $\rho, \varpi \in O(\bar{\rho_0}, \mathscr{J})$ for some $\rho_0 \in \mathscr{M}$, then $\mathscr{J}$ is called a rational orbitally (Θ, δ_b)-contraction.

Theorem 7. *Let $(\mathscr{M}, \sigma_b)$ be a b -metric space such that σ_b is a continuous functional and $\mathscr{J} : \mathscr{M} \longrightarrow P_b(\mathscr{M})$ be a rational orbitally (Θ, δ_b)-contraction. If the following conditions hold:*

(a) $(\mathscr{M}, \sigma_b, s)$ is $\mathscr{J}$-orbitally complete for some $\rho_0 \in \mathscr{M}$

(b) Θ is continuous and $\mathscr{J}\rho$ is closed, $\forall \rho \in O(\bar{\rho_0}, \mathscr{J})$ or the property (Gp) holds, then there exists $\rho^* \in \mathscr{M}$ such that $\rho^* \in \mathscr{J}\rho^*$

Proof. For any $\rho_0 \in \mathscr{M}$, we generate a sequence $\{\rho_n\}$ in $\mathscr{M}$ as $\rho_{n+1} \in \mathscr{J}\rho_n$ for each $n \ge 0$.

If there exists $n_0 \in \mathbb{N} \cup \{0\}$ for which $\rho_{n_0} = \rho_{n_0+1}$, then ρ_{n_0} is a fixed point of $\mathscr{J}$, and so, the proof is completed. Thus, assume that, for each $n \in \mathbb{N} \cup \{0\}, \rho_n \ne \rho_{n+1}$. So, we have $\sigma_b(\rho_{n+1}, \rho_{n+2}) > 0$ and $\delta_b(\mathscr{J}\rho_n, \mathscr{J}\rho_{n+1}) > 0$ for each $n \ge 0$. Then, it follows from (9) with $\rho = \rho_n$ and $\varpi = \rho_{n+1}$ that

$$\begin{aligned}\Theta(s\sigma_b(\rho_{n+1}, \rho_{n+2})) &\le \Theta(s\delta_b(\mathscr{J}\rho_n, \mathscr{J}\rho_{n+1})) \\ &\le [\Theta(m_1(\rho_n, \rho_{n+1}) + Lm_2(\rho_n, \rho_{n+1}))]^k,\end{aligned} \quad (11)$$

where

$$\begin{aligned}m_1(\rho_n,\rho_{n+1}) &= \max\left\{\begin{array}{c}\sigma_b(\rho_n,\rho_{n+1}), \sigma_b(\rho_n,\mathscr{J}\rho_n), \sigma_b(\rho_{n+1},\mathscr{J}\rho_{n+1}), \dfrac{\sigma_b(\rho_n,\mathscr{J}\rho_{n+1})+\sigma_b(\rho_{n+1},\mathscr{J}\rho_n)}{2s}, \\ \dfrac{\sigma_b(\rho_{n+1},\mathscr{J}\rho_{n+1})[1+\sigma_b(\rho_n,\mathscr{J}\rho_n)]}{s[1+\sigma_b(\rho_n,\rho_{n+1})]}, \dfrac{\sigma_b(\rho_{n+1},\mathscr{J}\rho_n)[1+\sigma_b(\rho_n,\mathscr{J}\rho_n)]}{s[1+\sigma_b(\rho_n,\rho_{n+1})]}\end{array}\right\} \\ &\le \max\left\{\sigma_b(\rho_n,\rho_{n+1}), \sigma_b(\rho_n,\rho_{n+1}), \sigma_b(\rho_{n+1},\rho_{n+2}), \frac{1}{2s}\sigma_b(\rho_n,\rho_{n+2}), \frac{1}{s}\sigma_b(\rho_{n+1},\rho_{n+2})\right\} \\ &= \max\left\{\sigma_b(\rho_n,\rho_{n+1}), \sigma_b(\rho_{n+1},\rho_{n+2}), \frac{1}{2s}\sigma_b(\rho_n,\rho_{n+2})\right\},\end{aligned}$$

$$m_2(\rho_n, \rho_{n+1}) = \min\{\sigma_b(\rho_n, \mathscr{J}\rho_n), \sigma_b(\rho_{n+1}, \mathscr{J}\rho_{n+1}), \sigma_b(\rho_n, \mathscr{J}\rho_{n+1}), \sigma_b(\rho_{n+1}, \mathscr{J}\rho_n)\} = 0. \quad (12)$$

Since

$$\frac{1}{2s}\sigma_b(\rho_n, \rho_{n+2}) \le \max\{\sigma_b(\rho_n, \rho_{n+1}), \sigma_b(\rho_{n+1}, \rho_{n+2})\}, \quad (13)$$

it follows from (11) that

$$\Theta(s\sigma_b(\rho_{n+1}, \rho_{n+2})) \le [\Theta(\max\{\sigma_b(\rho_n, \rho_{n+1}), \sigma_b(\rho_{n+1}, \rho_{n+2})\})]^k. \quad (14)$$

Assume that $\sigma_b(\rho_n, \rho_{n+1}) \le \sigma_b(\rho_{n+1}, \rho_{n+2})$ for some $n \in \mathbb{N}$. Then, from (14), we get

$$\Theta(s\sigma_b(\rho_{n+1}, \rho_{n+2})) \le [\Theta(\sigma_b(\rho_{n+1}, \rho_{n+2}))]^k, \quad (15)$$

which is a contradiction with (Θ_1). Hence, we have

$$\max\{\sigma_b(\rho_n, \rho_{n+1}), \sigma_b(\rho_{n+1}, \rho_{n+2})\} = \sigma_b(\rho_n, \rho_{n+1}), \quad (16)$$

and consequently,

$$\Theta(s\sigma_b(\rho_{n+1}, \rho_{n+2})) \le [\Theta(\sigma_b(\rho_n, \rho_{n+1}))]^k, \quad \forall n \in \mathbb{N} \cup \{0\}. \quad (17)$$

It follows from (17) and (Θ_4) that

$$\Theta(s^n\sigma_b(\rho_n, \rho_{n+1})) \le \left[\Theta\left(s^{n-1}\sigma_b(\rho_{n-1}, \rho_n)\right)\right]^k, \quad \forall n \in \mathbb{N} \cup \{0\}. \quad (18)$$

Let us represent $\lambda_n = \sigma_b(\rho_n, \rho_{n+1})$ for $n \in \mathbb{N} \cup \{0\}$. Then, $\lambda_n > 0$ for all $n \in \mathbb{N} \cup \{0\}$. Thus, we have

$$\begin{aligned}\Theta(s^n\lambda_n) &\le \left[\Theta\left(s^{n-1}\lambda_{n-1}\right)\right]^k \le \left[\Theta\left(s^{n-2}\lambda_{n-2}\right)\right]^{k^2} \\ &\le \cdots \le [\Theta(\lambda_0)]^{k^n}, \quad \forall n \in \mathbb{N} \cup \{0\}.\end{aligned} \quad (19)$$

Taking $n \longrightarrow \infty$ in (19), we get

$$\lim_{n\longrightarrow\infty} \Theta(s^n\lambda_n) = 1, \quad (20)$$

which implies that

$$\lim_{n\longrightarrow\infty} s^n\lambda_n = 0, \quad (21)$$

by (Θ_2). By (Θ_3), $\exists h \in (0, 1)$ and $\tau \in (0,\infty]$ such that

$$\lim_{n\longrightarrow\infty} \frac{\Theta(s^n\lambda_n) - 1}{(s^n\lambda_n)^h} = \tau. \quad (22)$$

Assume that $\tau < \infty$. For this case, let $q_2 = \tau/2 > 0$. So, $\exists$ $n_1 \in \mathbb{N}$ such that

$$\left|\frac{\Theta(s^n\lambda_n) - 1}{(s^n\lambda_n)^h} - \tau\right| \le q_2, \quad \forall n > n_1, \quad (23)$$

which implies that

$$\frac{\Theta(s^n\lambda_n) - 1}{(s^n\lambda_n)^h} \ge \tau - q_2 = \frac{\tau}{2} = q_2, \quad \forall n > n_1. \quad (24)$$

Then, we have

$$n(s^n\lambda_n)^h \le q_1 n[\Theta(s^n\lambda_n) - 1], \quad \forall n > n_1, \quad (25)$$

where $q_1 = 1/q_2$.

Now, assume that $\tau = \infty$. Let $q_2 > 0$. So, $\exists n_1 \in \mathbb{N}$ such that

$$q_2 \le \frac{\Theta(s^n\lambda_n) - 1}{(s^n\lambda_n)^h}, \quad \forall n > n_1, \quad (26)$$

which implies that

$$n(s^n\lambda_n)^h \le q_1 n[\Theta(s^n\lambda_n) - 1], \quad \forall n > n_1, \quad (27)$$

where $q_1 = 1/q_2$. Thus, in all cases, there exist $q_1 > 0$ and $n_1 \in \mathbb{N}$ such that

$$n(s^n\lambda_n)^h \le q_1 n[\Theta(s^n\lambda_n) - 1], \quad \forall n > n_1. \quad (28)$$

Hence, by (19) and (28), we get

$$n(s^n\lambda_n)^h \le q_1 n\left([\Theta(\lambda_0)]^{k^n} - 1\right). \quad (29)$$

Taking $n \longrightarrow \infty$ in (29), we have

$$\lim_{n\longrightarrow\infty} n(s^n\lambda_n)^h = 0, \quad (30)$$

and hence, $\lim_{n\longrightarrow\infty} n^{1/h}s^n\lambda_n = 0$, which yields that $\sum_{n=1}^{\infty} s^n\lambda_n$ is convergent. Thus, $\{\rho_n\}$ is a Cauchy sequence in $O(\rho_0, \mathcal{J})$. Since $\mathcal{M}$ is $\mathcal{J}$-orbitally complete, there exists $\rho^* \in \mathcal{M}$ such that

$$\rho_n \longrightarrow \rho^* \text{ as } n \longrightarrow \infty. \quad (31)$$

Suppose that $\mathcal{J}\rho^*$ is closed. We notice that if $\exists\{n_k\} \subset \mathbb{N}$ such that $\rho_{n_k} \in \mathcal{J}\rho^*$ for each $k \in \mathbb{N}$. Since $\mathcal{J}\rho^*$ is closed and $\lim_{k\longrightarrow\infty}\rho_{n_k} = \rho^*$, we conclude that $\rho^* \in \mathcal{J}\rho^*$, and so, the proof is finished. Hence, we suppose that there exists $n_0 \in \mathbb{N}$ so that $\rho_n \notin \mathcal{J}\rho^*$ for each $n \in \mathbb{N}$ with $n \ge n_0$. This implies that $\delta_b(\mathcal{J}\rho_n, \mathcal{J}\rho^*) > 0$ for each $n \ge n_0$. Then, it follows from (9) with $\rho = \rho_n$ and $\varpi = \rho^*$ that

$$\begin{aligned}\Theta(s\sigma_b(\rho_{n+1}, \mathcal{J}\rho^*)) &= \Theta(s\delta_b(\mathcal{J}\rho_n, \mathcal{J}\rho^*)) \\ &\le [\Theta(m_1(\rho_n, \rho^*) + Lm_2(\rho_n, \rho^*))]^k,\end{aligned} \quad (32)$$

where

$$
\begin{aligned}
m_1(\rho_n,\rho^*) &= \max\left\{\begin{array}{c}\sigma_b(\rho_n,\rho^*),\sigma_b(\rho_n,\mathcal{J}\rho_n),\sigma_b(\rho^*,\mathcal{J}\rho^*),\dfrac{\sigma_b(\rho_n,\mathcal{J}\rho^*)+\sigma_b(\rho^*,\mathcal{J}\rho_n)}{2s},\\ \dfrac{\sigma_b(\rho^*,\mathcal{J}\rho^*)[1+\sigma_b(\rho_n,\mathcal{J}\rho_n)]}{s[1+\sigma_b(\rho_n,\rho^*)]},\dfrac{\sigma_b(\rho^*,\mathcal{J}\rho_n)[1+\sigma_b(\rho_n,\mathcal{J}\rho_n)]}{s[1+\sigma_b(\rho_n,\rho^*)]}\end{array}\right\}\\
&\le \max\left\{\begin{array}{c}\sigma_b(\rho_n,\rho^*),\sigma_b(\rho_n,\rho_{n+1}),\sigma_b(\rho^*,\mathcal{J}\rho^*),\dfrac{\sigma_b(\rho_n,\mathcal{J}\rho^*)+\sigma_b(\rho^*,\rho_{n+1})}{2s},\\ \dfrac{\sigma_b(\rho^*,\mathcal{J}\rho^*)[1+\sigma_b(\rho_n,\rho_{n+1})]}{s[1+\sigma_b(\rho_n,\rho^*)]},\dfrac{\sigma_b(\rho^*,\rho_{n+1})[1+\sigma_b(\rho_n,\rho_{n+1})]}{s[1+\sigma_b(\rho_n,\rho^*)]}\end{array}\right\}\\
&\longrightarrow \sigma_b(\rho^*,\mathcal{J}\rho^*) \text{ as } n\longrightarrow\infty,\\
m_2(\rho_n,\rho^*) &= \min\{\sigma_b(\rho_n,\mathcal{J}\rho_n),\sigma_b(\rho^*,\mathcal{J}\rho^*),\sigma_b(\rho_n,\mathcal{J}\rho^*),\sigma_b(\rho^*,\mathcal{J}\rho_n)\}\\
&\le \min\{\sigma_b(\rho_n,\rho_{n+1}),\sigma_b(\rho^*,\mathcal{J}\rho^*),\sigma_b(\rho_n,\mathcal{J}\rho^*),\sigma_b(\rho^*,\rho_{n+1})\}\\
&\longrightarrow 0 \text{ as } n\longrightarrow\infty.
\end{aligned}\tag{33}
$$

Using the continuity of Θ and σ_b, so applying the limit of (32) as $n\longrightarrow\infty$, we get

$$\Theta(s\sigma_b(\rho^*,\mathcal{J}\rho^*))\le[\Theta(\sigma_b(\rho^*,\mathcal{J}\rho^*)]^k,\tag{34}$$

which is impossible from $0<k\in<1$ and $s\ge 1$. By the condition (Θ_1), we get $\sigma_b(\rho^*,\mathcal{J}\rho^*)=0$. Since $\mathcal{J}\rho^*$ is closed, thus we get $\rho^*\in\mathcal{J}\rho^*$. Assume that $G(\mathcal{J})$ is $\mathcal{J}$-orbitally closed. Since $(\rho_n,\rho_{n+1})\in G(\mathcal{J})$ and $\lim_{n\longrightarrow\infty}\rho_n=\rho^*\forall n\in\mathbb{N}\cup\{0\}$, we get $(\rho^*,\rho^*)\in G(\mathcal{J})$. Hence, $\rho^*\in\mathcal{J}\rho^*$. This completes the proof. □

If $\Theta(\varsigma)=e^{\sqrt{\varsigma}}$ for any $\varsigma>0$ in Theorem 7, we get following result.

Corollary 8. *Let $(\mathcal{M},\sigma_b)$ be a b-metric space such that σ_b is a continuous functional and $\mathcal{J}:(\mathcal{M},\sigma_b)\longrightarrow P_b(\mathcal{M})$ be a mapping satisfying the following condition: for some $0<k<1$,$\rho_0\in\mathcal{M}$ and $L\ge 0$,*

$$s\delta_b(\mathcal{J}\rho,\mathcal{J}\varpi)\le k(m_1(\rho,\varpi)+Lm_2(\rho,\varpi))\tag{35}$$

for all $\rho,\varpi\in\mathcal{M}$ with $\min\{\delta_b(\mathcal{J}\rho,\mathcal{J}\varpi),\sigma_b(\rho,\varpi)\}>0$, where

$$
\begin{aligned}
m_1(\rho,\varpi) &= \max\left\{\begin{array}{c}\sigma_b(\rho,\varpi),\sigma_b(\rho,\mathcal{J}\rho),\sigma_b(\varpi,\mathcal{J}\varpi),\dfrac{\sigma_b(\rho,\mathcal{J}\varpi)+\sigma_b(\varpi,\mathcal{J}\rho)}{2s},\\ \dfrac{\sigma_b(\varpi,\mathcal{J}\varpi)[1+\sigma_b(\rho,\mathcal{J}\rho)]}{s[1+\sigma_b(\rho,\varpi)]},\dfrac{\sigma_b(\varpi,\mathcal{J}\rho)[1+\sigma_b(\rho,\mathcal{J}\rho)]}{s[1+\sigma_b(\rho,\varpi)]}\end{array}\right\},\\
m_2(\rho,\varpi) &= \min\{\sigma_b(\rho,\mathcal{J}\rho),\sigma_b(\varpi,\mathcal{J}\varpi),\sigma_b(\rho,\mathcal{J}\varpi),\sigma_b(\varpi,\mathcal{J}\rho)\}
\end{aligned}\tag{36}
$$

for all $\rho,\varpi\in O(\bar{\rho_0},\mathcal{J})$. Assume that $(\mathcal{M},\sigma_b,s)$ is $\mathcal{J}$-orbitally complete for some $\rho_0\in\mathcal{M}$. If $\mathcal{J}\rho$ is closed, for all $\rho\in O(\bar{\rho_0},\mathcal{J})$ or the property (Gp) holds, then $\exists\rho^*\in\mathcal{M}$ such that $\rho^*\in\mathcal{J}\rho^*$.

Example 9. Let $\mathcal{M}=[0,1]$. The b-metric is defined by

$$\sigma_b(\rho,\varpi)=(\rho-\varpi)^2,\tag{37}$$

with coefficient $s=2$. Define $\mathcal{J}:\mathcal{M}\longrightarrow P_{cb}(\mathcal{M})$ given by

$$\mathcal{J}\rho=\begin{cases}\left\{\dfrac{1}{3}\right\}, & 0\le\rho<1,\\ \left[0,\dfrac{1}{4}\right], & \rho=1.\end{cases}\tag{38}$$

If $\rho,\varpi\in[0,1)$, then $\delta_b(\mathcal{J}\rho,\mathcal{J}\varpi)=0$. Let $\rho\in[0,1)$ and $\varpi=1$. Then, $\mathcal{J}\rho=\{1/3\}$,$\mathcal{J}\varpi=[0,1/4]$, and $\delta_b(\mathcal{J}\rho,\mathcal{J}\varpi)=1/12$,

$$
\begin{aligned}
m_1(\rho,\varpi) &= \max\left\{\begin{array}{c}(1-\rho)^2,\left(\dfrac{1}{3}-\rho\right)^2,\left(\dfrac{3}{4}\right)^2,\dfrac{1}{4}\left[\sigma_b(\rho,\mathcal{J}\varpi)+\left(\dfrac{2}{3}\right)^2\right],\\ \dfrac{(3/4)^2\left[1+(1/3-\rho)^2\right]}{2\left[1+(1-\rho)^2\right]},\dfrac{(2/3)^2\left[1+(1/3-\rho)^2\right]}{2\left[1+(1-\rho)^2\right]}\end{array}\right\}\ge\frac{9}{16},\\
m_2(\rho,\varpi) &= \min\left\{\left(\frac{1}{3}-\rho\right)^2,\left(\frac{3}{4}\right)^2,\sigma_b(\rho,\mathcal{J}\varpi),\left(\frac{2}{3}\right)^2\right\}\ge 0.
\end{aligned}\tag{39}
$$

Take $k=2/9$, $\Theta(\varsigma)=e^{\sqrt{\varsigma}}$ and $0\le L$. Then,

$$
\begin{aligned}
\Theta(s\delta_b(\mathcal{J}\rho,\mathcal{J}\varpi)) &= \Theta\left(2\cdot\frac{1}{12}\right)=e^{\sqrt{1/6}}<e^{2/3}=e^{8/9\cdot 3/4}\\
&= [\Theta(m_1(\rho,\varpi)+Lm_2(\rho,\varpi))]^k.
\end{aligned}\tag{40}
$$

Thus, all conditions of Theorem 7 are satisfied and $\rho^*=1/3$ is the required point.

The family Ω_s contains a wide set of functions; that is, if we take

$$\Theta(\rho) = 2 - \frac{2}{\pi}\arctan\left(\frac{1}{\rho^{\beta}}\right), \tag{41}$$

where $\rho > 0$ and $0 < \beta < 1$, then we can obtain the following corollary from Theorem 7.

Corollary 10. *Let $(\mathcal{M}, \sigma_b)$ be a b-metric space such that σ_b is a continuous functional. Assume that $\mathcal{J} : \mathcal{M} \longrightarrow P_b(\mathcal{M})$ be a mapping such that for some $0 < k, \beta < 1, \rho_0 \in \mathcal{M}$ and $0 \le L$,*

$$\begin{aligned} &2 - \frac{2}{\pi}\arctan\left(\frac{1}{(s\delta_b(\mathcal{J}\rho, \mathcal{J}\varpi))^{\beta}}\right) \\ &\quad \le \left[2 - \frac{2}{\pi}\arctan\left(\frac{1}{(m_1(\rho,\varpi) + Lm_2(\rho,\varpi))^{\beta}}\right)\right]^k, \end{aligned} \tag{42}$$

$\forall \rho, \varpi \in \mathcal{M}$ with $\min\{\delta_b(\mathcal{J}\rho, \mathcal{J}\varpi), \sigma_b(\rho,\varpi)\} > 0$, where

$$m_1(\rho,\varpi) = \max\left\{\begin{array}{c} \sigma_b(\rho,\varpi), \sigma_b(\rho,\mathcal{J}\rho), \sigma_b(\varpi,\mathcal{J}\varpi), \dfrac{\sigma_b(\rho,\mathcal{J}\varpi) + \sigma_b(\varpi,\mathcal{J}\rho)}{2s}, \\ \dfrac{\sigma_b(\varpi,\mathcal{J}\varpi)[1+\sigma_b(\rho,\mathcal{J}\rho)]}{s[1+\sigma_b(\rho,\varpi)]}, \dfrac{\sigma_b(\varpi,\mathcal{J}\rho)[1+\sigma_b(\rho,\mathcal{J}\rho)]}{s[1+\sigma_b(\rho,\varpi)]} \end{array}\right\},$$

$$m_2(\rho,\varpi) = \min\{\sigma_b(\rho,\mathcal{J}\rho), \sigma_b(\varpi,\mathcal{J}\varpi), \sigma_b(\rho,\mathcal{J}\varpi), \sigma_b(\varpi,\mathcal{J}\rho)\} \tag{43}$$

for all $\rho, \varpi \in O(\overline{\rho_0}, \mathcal{J})$. Suppose that $(\mathcal{M}, \sigma_b, s)$ is $\mathcal{J}$-orbitally complete for some $\rho_0 \in \mathcal{M}$. If $\mathcal{J}\rho$ is closed for all $\rho \in O(\overline{\rho_0}, \mathcal{J})$ or the property (Gp) holds, then there exists $\rho^* \in \mathcal{M}$ such that $\rho^* \in \mathcal{J}\rho^*$.

If we replace self-mapping $\mathcal{J} : \mathcal{M} \longrightarrow \mathcal{M}$ on the place of multivalued mapping in Theorem 7, we can get following results as consequences.

Corollary 11. *Let $(\mathcal{M}, \sigma_b)$ be a b-metric space such that σ_b is a continuous functional and $\mathcal{J} : \mathcal{M} \longrightarrow \mathcal{M}$ be a mapping such that $\mathcal{M}$ is $\mathcal{J}$-orbitally complete at some ρ_0. Assume that there exist $\Theta \in \Omega_s$, $k \in (0,1)$ and $L \ge 0$ such that*

$$\Theta(s\sigma_b(\mathcal{J}\rho, \mathcal{J}\varpi)) \le \left[\Theta(m_1'(\rho,\varpi) + Lm_2'(\rho,\varpi))\right]^k \tag{44}$$

for all $\rho, \varpi \in \mathcal{M}$ with $\min\{\sigma_b(\mathcal{J}\rho, \mathcal{J}\varpi), \sigma_b(\rho,\varpi)\} > 0$, where

$$m_1'(\rho,\varpi) = \max\left\{\begin{array}{c} \sigma_b(\rho,\varpi), \sigma_b(\rho,\mathcal{J}\rho), \sigma_b(\varpi,\mathcal{J}\varpi), \dfrac{\sigma_b(\rho,\mathcal{J}\varpi) + \sigma_b(\varpi,\mathcal{J}\rho)}{2s}, \\ \dfrac{\sigma_b(\varpi,\mathcal{J}\varpi)[1+\sigma_b(\rho,\mathcal{J}\rho)]}{s[1+\sigma_b(\rho,\varpi)]}, \dfrac{\sigma_b(\varpi,\mathcal{J}\rho)[1+\sigma_b(\rho,\mathcal{J}\rho)]}{s[1+\sigma_b(\rho,\varpi)]} \end{array}\right\},$$

$$m_2'(\rho,\varpi) = \min\{\sigma_b(\rho,\mathcal{J}\rho), \sigma_b(\varpi,\mathcal{J}\varpi), \sigma_b(\rho,\mathcal{J}\varpi), \sigma_b(\varpi,\mathcal{J}\rho)\}. \tag{45}$$

If Θ is continuous, then there exists $\rho^* \in \mathcal{M}$ such that $\rho^* = \mathcal{J}\rho^*$.

4. Applications

In this section, we solve the Fredholm integral inclusion:

$$\rho(\varsigma) \in g(\varsigma) + \int_a^b K(\varsigma, \zeta, \rho(\zeta))\sigma\zeta, \quad \forall \varsigma \in [a,b], \tag{46}$$

given in the start of this paper.

Let σ_b on $C[a,b]$ be given by

$$\sigma_b(\rho,\varpi) = \left(\max_{\varsigma\in[a,b]} |\rho(\varsigma) - \varpi(\varsigma)|\right)^p = \max_{\varsigma\in[a,b]} |\rho(\varsigma) - \varpi(\varsigma)|^p, \quad \text{with } p \ge 1. \tag{47}$$

Then, $(C[a,b], \sigma_b, 2^{p-1})$ is a complete b-metric space. Assume that for $\varsigma, \zeta \in [a,b]$, these conditions hold:

(a) $\forall \rho \in C[a,b], K_\rho(\varsigma,\zeta) = K(\varsigma,\zeta,\rho(\zeta))$ is continuous

(b) $\exists Y : [a,b] \times [a,b] \longrightarrow [0,+\infty)$ such that

$$\begin{aligned} &|k_\varphi(\varsigma,\zeta) - k_\psi(\varsigma,\zeta)|^p \le Y(\varsigma,\zeta) \\ &\cdot\left[\begin{array}{c} \max\left\{\begin{array}{c} \sigma_b(\varphi(\zeta),\psi(\zeta)), \sigma_b(\varphi(\zeta), K(\varsigma,\zeta,\varphi(\zeta))), \\ \sigma_b(\psi(\zeta), K(\varsigma,\zeta,\psi(\zeta))), \\ \dfrac{\sigma_b(\varphi(\zeta), K(\varsigma,\zeta,\psi(\zeta))) + \sigma_b(\psi(\zeta), K(\varsigma,\zeta,\varphi(\zeta)))}{2^p}, \\ \dfrac{\sigma_b(\psi(\zeta), K(\varsigma,\zeta,\psi(\zeta)))[1+\sigma_b(\varphi(\zeta), K(\varsigma,\zeta,\varphi(\zeta)))]}{2^{p-1}[1+\sigma_b(\varphi(\zeta),\psi(\zeta))]}, \\ \dfrac{\sigma_b(\psi(\zeta), K(\varsigma,\zeta,\varphi(\zeta)))[1+\sigma_b(\varphi(\zeta), K(\varsigma,\zeta,\varphi(\zeta)))]}{2^{p-1}[1+\sigma_b(\varphi(\zeta),\psi(\zeta))]} \end{array}\right\} + \\ \lambda\min\{\sigma_b(\varphi(\zeta), K(\varsigma,\zeta,\varphi(\zeta))), \sigma_b(\psi(\zeta), K(\varsigma,\zeta,\psi(\zeta))), \\ \sigma_b(\varphi(\zeta), K(\varsigma,\zeta,\psi(\zeta))), \sigma_b(\psi(\zeta), K(\varsigma,\zeta,\varphi(\zeta)))\} \end{array}\right] \end{aligned} \tag{48}$$

$\forall \varsigma, \zeta \in [a,b], \varphi, \psi \in C[a,b]$ and $k_\varphi(\varsigma,\zeta) \in K_\varphi(\varsigma,\zeta), k_\psi(\varsigma,\zeta) \in K_\psi(\varsigma,\zeta)$, where $p > 1$ and $\lambda \ge 0$

(c) $\exists 0 < k < 1$ such that

$$\sup_{\varsigma\in[a,b]} \int_a^b Y(\varsigma,\zeta)\sigma\zeta \le \frac{k}{2^{p-1}} \tag{49}$$

Theorem 12. *Under the conditions (a)–(c), the integral inclusion given in (46) has a solution in $C[a,b]$.*

Proof. Let $\mathcal{M} = C[a,b]$. Define b-metric σ_b as in (47) and a multivalued mapping $\mathcal{J} : \mathcal{M} \longrightarrow P_{cb}(\mathcal{M})$ by

$$\mathcal{J}\rho = \left\{\varpi \in \mathcal{M} : \varpi(\varsigma) \in g(\varsigma) + \int_a^b K(\varsigma,\zeta,\rho(\zeta))\sigma\zeta, \varsigma \in [a,b]\right\}. \tag{50}$$

Let $\rho \in \mathcal{M}$. For a mapping $K_\rho(\varsigma,\zeta)$: $[a,b] \times [a,b] \longrightarrow P_{cb}(\mathbb{R})$, there exists $k_\rho(\varsigma,\zeta)$: $[a,b] \times [a,b] \longrightarrow \mathbb{R}$ such that

$$k_\rho(\varsigma,\zeta) \in K_\rho(\varsigma,\zeta), \tag{51}$$

$\forall\varsigma,\zeta\in[a,b]$. Then,

$$g(\varsigma)+\int_a^b k_\rho(\varsigma,\zeta)\sigma\zeta\in\mathcal{J}\rho. \tag{52}$$

Hence, $\mathcal{J}\rho\neq\varnothing$. Since $K_\rho(\varsigma,\zeta)$ is continuous and $g\in C([a,b])$, the ranges of both functions are bounded. This shows that $\mathcal{J}\rho$ is also bounded.

Now, we show that (9) holds for any $\mathcal{J}$ on $\mathcal{M}$ with some $0<k<1$, $0\le L$ and $\Theta\in\Omega_\zeta$, i.e.,

$$\begin{aligned}&\Theta(\zeta\delta_b(\mathcal{J}\rho_1,\mathcal{J}\rho_2))\\ &\le\left[\Theta\left[\begin{matrix}\max\left\{\begin{matrix}\sigma_b(\rho_1,\rho_2),\sigma_b(\rho_1,\mathcal{J}\rho_1),\sigma_b(\rho_2,\mathcal{J}\rho_2),\\ \dfrac{\sigma_b(\rho_1,\mathcal{J}\rho_2)+\sigma_b(\rho_2,\mathcal{J}\rho_1)}{2^p},\\ \dfrac{\sigma_b(\rho_2,\mathcal{J}\rho_2)[1+\sigma_b(\rho_1,\mathcal{J}\rho_1)]}{2^{p-1}[1+\sigma_b(\rho_1,\rho_2)]},\\ \dfrac{\sigma_b(\rho_2,\mathcal{J}\rho_1)[1+\sigma_b(\rho_1,\mathcal{J}\rho_1)]}{2^{p-1}[1+\sigma_b(\rho_1,\rho_2)]}\end{matrix}\right\}\\ +L\min\{\sigma_b(\rho_1,\mathcal{J}\rho_1),\sigma_b(\rho_2,\mathcal{J}\rho_2),\\ \sigma_b(\rho_1,\mathcal{J}\rho_2),\sigma_b(\rho_2,\mathcal{J}\rho_1)\}\end{matrix}\right]\right]^k\end{aligned} \tag{53}$$

for all $\rho_1,\rho_2\in\mathcal{M}$. Let $\varpi_1\in\mathcal{J}\rho_1$ be such that

$$\varpi_1(\varsigma)\in g(\varsigma)+\int_a^b K(\varsigma,\zeta,\rho_1(\zeta))\sigma\zeta,\quad\forall\varsigma\in[a,b]. \tag{54}$$

It follows that $\exists k_{\rho_1}(\varsigma,\zeta)\in K_{\rho_1}(\varsigma,\zeta)=K(\varsigma,\zeta,\rho_1(\zeta))$ such that

$$\varpi_1(\varsigma)=g(\varsigma)+\int_a^b k_{\rho_1}(\varsigma,\zeta)\sigma\zeta,\quad\forall\varsigma,\zeta\in[a,b]. \tag{55}$$

Now, for all $\rho_1,\rho_2\in\mathcal{M}$, it follows from (b) that

$$\begin{aligned}&\left|k_{\rho_1}(\varsigma,\zeta)-k_{\rho_2}(\varsigma,\zeta)\right|^p\le Y(\varsigma,\zeta)\\ &\cdot\left[\begin{matrix}\max\left\{\begin{matrix}\sigma_b(\rho_1(\zeta),\rho_2(\zeta)),\sigma_b(\rho_1(\zeta),K(\varsigma,\zeta,\rho_1(\zeta))),\\ \sigma_b(\rho_2(\zeta),K(\varsigma,\zeta,\rho_2(\zeta))),\\ \dfrac{\sigma_b(\rho_1(\zeta),K(\varsigma,\zeta,\rho_2(\zeta)))+\sigma_b(\rho_2(\zeta),K(\varsigma,\zeta,\rho_1(\zeta)))}{2^p}\\ \dfrac{\sigma_b(\rho_2(\zeta),K(\varsigma,\zeta,\rho_2(\zeta)))[1+\sigma_b(\rho_1(\zeta),K(\varsigma,\zeta,\rho_1(\zeta)))]}{2^{p-1}[1+\sigma_b(\rho_1(\zeta),\rho_2(\zeta))]},\\ \dfrac{\sigma_b(\rho_2(\zeta),K(\varsigma,\zeta,\rho_1(\zeta)))[1+\sigma_b(\rho_1(\zeta),K(\varsigma,\zeta,\rho_1(\zeta)))]}{2^{p-1}[1+\sigma_b(\rho_1(\zeta),\rho_2(\zeta))]}\end{matrix}\right\}\\ +L\min\{\sigma_b(\rho_1(\zeta),K(\varsigma,\zeta,\rho_1(\zeta))),\\ \sigma_b(\rho_2(\zeta),K(\varsigma,\zeta,\rho_2(\zeta))),\\ \sigma_b(\rho_1(\zeta),K(\varsigma,\zeta,\rho_2(\zeta))),\\ \sigma_b(\rho_2(\rho),K(\varsigma,\zeta,\rho_1(\zeta)))\}\end{matrix}\right]\end{aligned} \tag{56}$$

It follows that $\exists\omega(\varsigma,\zeta)\in K_{\rho_2}(\varsigma,\zeta)$ such that

$$\begin{aligned}&\left|k_{\rho_1}(\varsigma,\zeta)-\omega(\varsigma,\zeta)\right|^p\le Y(\varsigma,\zeta)\\ &\cdot\left[\begin{matrix}\max\left\{\begin{matrix}\sigma_b(\rho_1(\zeta),\rho_2(\zeta)),\sigma_b(\rho_1(\zeta),K(\varsigma,\zeta,\rho_1(\zeta))),\\ \sigma_b(\rho_2(\zeta),\omega(\varsigma,\zeta)),\\ \dfrac{\sigma_b(\rho_1(\zeta),\omega(\varsigma,\zeta))+\sigma_b(\rho_2(\zeta),K(\varsigma,\zeta,\rho_1(\zeta)))}{2^p},\\ \dfrac{\sigma_b(\rho_2(\zeta),\omega(\varsigma,\zeta))[1+\sigma_b(\rho_1(\zeta),K(\varsigma,\zeta,\rho_1(\zeta)))]}{2^{p-1}[1+\sigma_b(\rho_1(\zeta),\rho_2(\zeta))]},\\ \dfrac{\sigma_b(\rho_2(\zeta),K(\varsigma,\zeta,\rho_1(\zeta)))[1+\sigma_b(\rho_1(\zeta),K(\varsigma,\zeta,\rho_1(\zeta)))]}{2^{p-1}[1+\sigma_b(\rho_1(\zeta),\rho_2(\zeta))]}\end{matrix}\right\}\\ +L\min\{\sigma_b(\rho_1(\zeta),K(\varsigma,\zeta,\rho_1(\zeta))),\sigma_b(\rho_2(\zeta),\omega(\varsigma,\zeta)),\\ \sigma_b(\rho_1(\zeta),\omega(\varsigma,\zeta)),\sigma_b(\rho_2(\rho),K(\varsigma,\zeta,\rho_1(\zeta)))\}\end{matrix}\right]\\ &=R(\varsigma,\zeta),\forall\varsigma,\zeta\in[a,b].\end{aligned} \tag{57}$$

Now, we show the function $U(\varsigma,\zeta)\colon[a,b]\times[a,b]\longrightarrow P_{cb}(\mathbb{R})$ by

$$U(\varsigma,\zeta)=K_{\rho_2}(\varsigma,\zeta)\bigcap\left\{\varphi\in\mathbb{R}:\sigma_b\left(k_{\rho_1}(\varsigma,\zeta),\varphi\right)\le R(\varsigma,\zeta)\right\}. \tag{58}$$

Thus, by (a), U is lower semicontinuous; this implies that there exists $k_{\rho_2}(\varsigma,\zeta)\colon[a,b]\times[a,b]\longrightarrow\mathbb{R}$ such that $k_{\rho_2}(\varsigma,\zeta)\in U(\varsigma,\zeta),\forall\varsigma,\zeta\in[a,b]$. Then, $\varpi_2(\varsigma)=g(\varsigma)+\int_a^b k_{\rho_1}(\varsigma,\zeta)\sigma\zeta$ satisfies

$$\varpi_1(\varsigma)\in g(\varsigma)+\int_a^b K(\varsigma,\zeta,\rho_1(\zeta))\sigma\zeta,\quad\forall\varsigma\in[a,b], \tag{59}$$

which implies that $\varpi_2\in\mathcal{J}\rho_2$ and

$$\begin{aligned}&\sigma_b(\varpi_1,\varpi_2)\le\max_{\varsigma\in[a,b]}\int_a^b\left|k_{\rho_1}(\varsigma,\zeta)-k_{\rho_2}(\varsigma,\zeta)\right|^p\sigma\zeta\le\max_{\varsigma\in[a,b]}\int_a^b Y(\varsigma,\zeta)\\ &\cdot\left[\begin{matrix}\max\left\{\begin{matrix}\sigma_b(\rho_1(\zeta),\rho_2(\zeta)),\sigma_b(\rho_1(\zeta),K(\varsigma,\zeta,\rho_1(\zeta))),\\ \sigma_b(\rho_2(\zeta),K(\varsigma,\zeta,\rho_2(\zeta))),\\ \dfrac{\sigma_b(\rho_1(\zeta),K(\varsigma,\zeta,\rho_2(\zeta)))+\sigma_b(\rho_2(\zeta),K(\varsigma,\zeta,\rho_1(\zeta)))}{2^p},\\ \dfrac{\sigma_b(\rho_2(\zeta),K(\varsigma,\zeta,\rho_2(\zeta)))[1+\sigma_b(\rho_1(\zeta),K(\varsigma,\zeta,\rho_1(\zeta)))]}{2^{p-1}[1+\sigma_b(\rho_1(\zeta),\rho_2(\zeta))]},\\ \dfrac{\sigma_b(\rho_2(\zeta),K(\varsigma,\zeta,\rho_1(\zeta)))[1+\sigma_b(\rho_1(\zeta),K(\varsigma,\zeta,\rho_1(\zeta)))]}{2^{p-1}[1+\sigma_b(\rho_1(\zeta),\rho_2(\zeta))]}\end{matrix}\right\}\\ +L\min\{\sigma_b(\rho_1(\zeta),K(\varsigma,\zeta,\rho_1(\zeta))),\\ \sigma_b(\rho_2(\zeta),K(\varsigma,\zeta,\rho_2(\zeta))),\\ \sigma_b(\rho_1(\zeta),K(\varsigma,\zeta,\rho_2(\zeta))),\\ \sigma_b(\rho_2(\rho),K(\varsigma,\zeta,\rho_1(\zeta)))\}\end{matrix}\right]\sigma\zeta\le\frac{k}{2^{p-1}}\\ &\cdot\left[\begin{matrix}\max\left\{\begin{matrix}\sigma_b(\rho_1,\rho_2),\sigma_b(\rho_1,\mathcal{J}\rho_1),\sigma_b(\rho_2,\mathcal{J}\rho_2),\dfrac{\sigma_b(\rho_1,\mathcal{J}\rho_2)+\sigma_b(\rho_2,\mathcal{J}\rho_1)}{2^p},\\ \dfrac{\sigma_b(\rho_2,\mathcal{J}\rho_2)[1+\sigma_b(\rho_1,\mathcal{J}\rho_1)]}{\zeta[1+\sigma_b(\rho_1,\rho_2)]},\dfrac{\sigma_b(\rho_2,\mathcal{J}\rho_1)[1+\sigma_b(\rho_1,\mathcal{J}\rho_1)]}{\zeta[1+\sigma_b(\rho_1,\rho_2)]}\end{matrix}\right\}\\ +L\min\{\sigma_b(\rho_1,\mathcal{J}\rho_1),\sigma_b(\rho_2,\mathcal{J}\rho_2),\sigma_b(\rho_1,\mathcal{J}\rho_2),\sigma_b(\rho_2\mathcal{J}\rho_1)\}\end{matrix}\right],\end{aligned} \tag{60}$$

$\forall \varsigma, \zeta \in [a, b]$. Thus, we get

$$\delta_b(\mathcal{J}\rho_1, \mathcal{J}\rho_2) \le \frac{k}{2^{p-1}} \left[\begin{array}{c} \max \left\{ \begin{array}{c} \sigma_b(\rho_1, \rho_2), \sigma_b(\rho_1, \mathcal{J}\rho_1), \sigma_b(\rho_2, \mathcal{J}\rho_2), \\ \dfrac{\sigma_b(\rho_1, \mathcal{J}\rho_2) + \sigma_b(\rho_2, \mathcal{J}\rho_1)}{2^p}, \\ \dfrac{\sigma_b(\rho_2, \mathcal{J}\rho_2)[1 + \sigma_b(\rho_1, \mathcal{J}\rho_1)]}{2^{p-1}[1 + \sigma_b(\rho_1, \rho_2)]}, \\ \dfrac{\sigma_b(\rho_2, \mathcal{J}\rho_1)[1 + \sigma_b(\rho_1, \mathcal{J}\rho_1)]}{2^{p-1}[1 + \sigma_b(\rho_1, \rho_2)]} \end{array} \right\} \\ +L \min \{\sigma_b(\rho_1, \mathcal{J}\rho_1), \sigma_b(\rho_2, \mathcal{J}\rho_2), \sigma_b(\rho_1, \mathcal{J}\rho_2), \\ \sigma_b(\rho_2, \mathcal{J}\rho_1)\} \end{array} \right],$$

$$\zeta\delta_b(\mathcal{J}\rho_1, \mathcal{J}\rho_2) \le k \left[\begin{array}{c} \max \left\{ \begin{array}{c} \sigma_b(\rho_1, \rho_2), \sigma_b(\rho_1, \mathcal{J}\rho_1), \sigma_b(\rho_2, \mathcal{J}\rho_2), \\ \dfrac{\sigma_b(\rho_1, \mathcal{J}\rho_2) + \sigma_b(\rho_2, \mathcal{J}\rho_1)}{2^p}, \\ \dfrac{\sigma_b(\rho_2, \mathcal{J}\rho_2)[1 + \sigma_b(\rho_1, \mathcal{J}\rho_1)]}{2^{p-1}[1 + \sigma_b(\rho_1, \rho_2)]}, \\ \dfrac{\sigma_b(\rho_2, \mathcal{J}\rho_1)[1 + \sigma_b(\rho_1, \mathcal{J}\rho_1)]}{2^{p-1}[1 + \sigma_b(\rho_1, \rho_2)]} \end{array} \right\} \\ +L \min \{\sigma_b(\rho_1, \mathcal{J}\rho_1), \sigma_b(\rho_2, \mathcal{J}\rho_2), \\ \sigma_b(\rho_1, \mathcal{J}\rho_2), \sigma_b(\rho_2, \mathcal{J}\rho_1)\} \end{array} \right]. \tag{61}$$

Taking the exponential on both sides, we have

$$e^{\zeta\delta_b(\mathcal{J}\rho_1, \mathcal{J}\rho_2)}\delta_b(\mathcal{J}\rho_1, \mathcal{J}\rho_2) \le ek \left[\begin{array}{c} \max \left\{ \begin{array}{c} \sigma_b(\rho_1, \rho_2), \sigma_b(\rho_1, \mathcal{J}\rho_1), \sigma_b(\rho_2, \mathcal{J}\rho_2), \\ \dfrac{\sigma_b(\rho_1, \mathcal{J}\rho_2) + \sigma_b(\rho_2, \mathcal{J}\rho_1)}{2^p}, \\ \dfrac{\sigma_b(\rho_2, \mathcal{J}\rho_2)[1 + \sigma_b(\rho_1, \mathcal{J}\rho_1)]}{2^{p-1}[1 + \sigma_b(\rho_1, \rho_2)]}, \\ \dfrac{\sigma_b(\rho_2, \mathcal{J}\rho_1)[1 + \sigma_b(\rho_1, \mathcal{J}\rho_1)]}{2^{p-1}[1 + \sigma_b(\rho_1, \rho_2)]} \end{array} \right\} \\ +L \min \{\sigma_b(\rho_1, \mathcal{J}\rho_1), \sigma_b(\rho_2, \mathcal{J}\rho_2), \sigma_b(\rho_1, \mathcal{J}\rho_2), \sigma_b(\rho_2, \mathcal{J}\rho_1)\} \end{array} \right]. \tag{62}$$

Taking the function $\Theta \in \Omega_s$, it follows that (18) is fulfilled. Therefore, by Theorem 7, we show that the integral inclusion (46) has a solution. This completes the proof. ☐

5. Conclusion

In this paper, we have defined a rational (Θ, δ_b)-contraction and established some fixed point results in b-metric spaces. In this way, we generalized some known results of literature. We also discussed the solution of Fredholm integral inclusion as application of our obtained result. We expect that the results given in this article will make new directions for those who are working in the theory of fixed points.

In this direction, our future work will pivot on studying the fixed points of fuzzy mappings and L-fuzzy mappings for Θ-contractions in b-metric spaces, with fractional differential inclusion problems as applications.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final paper.

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