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Research Article

Commutators of Multilinear Calderón-Zygmund Operator on Weighted Herz-Morrey Spaces with Variable Exponents

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In this paper, we acquire the boundedness of commutators generated by multilinear Calderón-Zygmund operator and BMO functions on products of weighted Herz-Morrey spaces with variable exponents.

1. Introduction

The space of all Schwartz functions on $\mathbb{R}^n$ was denoted by $\mathcal{S}(\mathbb{R}^n)$, and the space of all tempered distributions on $\mathbb{R}^n$ was denoted by $\mathcal{S}'(\mathbb{R}^n)$. The space of compactly supported bounded functions denoted by $L_C^\infty(\mathbb{R}^n)$, and the support set of function f was denoted by supp (f). On the m-fold of the Schwartz function space $\mathcal{S}(\mathbb{R}^n)$, we also set T as an m-linear operator originally defined and $m \geq 2$, and its value belongs to $\mathcal{S}'(\mathbb{R}^n)$:

$$T : \underbrace{\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \times \cdots \times \mathcal{S}(\mathbb{R}^n)}_{m} \longrightarrow \mathcal{S}'(\mathbb{R}^n). \quad (1)$$

We say that T is an m-linear Calderón-Zygmund operator, if for some $p_1, \cdots, p_m \in [1, \infty)$, it extends to a bounded multilinear operator from $L^{p_1} \times L^{p_2} \times \cdots \times L^{p_m}$ to L^p with $1/p_1 + 1/p_2 + \cdots + 1/p_m = 1/p$, and for $f_1, f_2, \cdots, f_m \in L_C^\infty(\mathbb{R}^n)$, $x \notin \cap_{j=1}^m$ supp (f_j)

$$T(f_1, f_2, \cdots, f_m)(x) \coloneqq \int_{(\mathbb{R}^n)^m} K(x, y_1, y_2, \cdots, y_m) \prod_{i=1}^m f_i(y_i) dy_1 dy_2 \cdots dy_m, \quad (2)$$

where kernel K is a function in $(\mathbb{R}^n)^{m+1}$ away from the diagonal $x = y_1 = y_2 = \cdots = y_m$ and there exist positive constants ε, A satisfies the following:

$$|K(x, y_1, \cdots, y_m)| \leq A \left(\sum_{i=1}^m |x - y_i| \right)^{-mn},$$
$$\left| K(x, y_1, \cdots, y_m) - K\left(x', y_1, \cdots, y_m\right) \right| \leq \frac{A |x - x'|^\varepsilon}{\left(\sum_{i=1}^m |x - y_i|\right)^{mn+\varepsilon}}, \quad (3)$$

whenever $|x - x'| \leq 1/2 \max \{|x - y_1|, |x - y_2|, \cdots, |x - y_m|\}$, and for all $1 \leq i \leq m$,

$$|K(x, y_1, \cdots, y_i, \cdots, y_m)| - \left| K\left(x, y_1, \cdots, y_i', \cdots, y_m\right) \right| \leq \frac{A |y_i - y_i'|^\varepsilon}{\left(\sum_{i=1}^m |x - y_i|\right)^{mn+\varepsilon}}, \quad (4)$$

where $|y_i - y_i'| \leq 1/2 \max \{|x - y_1|, |x - y_2|, \cdots, |x - y_m|\}$.

If $b \in L^1_{loc}(\mathbb{R}^n)$, set

$$\|b\|_* \coloneqq \sup_B \frac{1}{|B|} \int_B |b(x) - b_B| dx, \quad (5)$$

where $b_B = (1/|B|) \int_B b(y) dy$ and the supremum is taken over all $B \subset \mathbb{R}^n$, and what follows $|B|$ is the Lebesgue measure of measurable set B in $\mathbb{R}^n$. A function b is called bounded mean oscillation if $\|b\|_* < \infty$. Denote by $BMO(\mathbb{R}^n)$ the set of all bounded mean oscillation functions on $\mathbb{R}^n$.

Although our method suits any multilinear operator, only the bilinear Calderón-Zygmund operator will be considered here for the sake of simplicity. Specifically, we will discuss the commutator of a bilinear Calderón-Zygmund operator T, BMO functions b_1 and b_2, and suitable functions f_1 and f_2,

$$\begin{aligned}[b_1, b_2, T](f_1, f_2)(x) = b_1(x)b_2(x)T(f_1, f_2)(x) \\ - b_1(x)T(f_1, b_2f_2)(x) \\ - b_2(x)T(b_1f_1, f_2)(x) \\ + T(b_1f_1, b_2f_2).\end{aligned} \tag{6}$$

Many analyses of linear commutators have been extended to other fields, such as weighted space, homogeneous space, multiparameter, and multilinear settings. Huang and Xu [1] obtained boundedness of multilinear singular integrals and their commutators from products of variable exponent Lebesgue spaces to variable exponent Lebesgue spaces. Huet al. [2] proved the boundedness of commutators generated by fractional integrals and BMO on generalized Herz spaces with general Muckenhoupt weights. Tang et al. [3] obtained the boundedness of a commutator generated by the multilinear Calderón-Zygmund operator and BMO functions in Herz-Morrey spaces with variable exponents. Chen et al. [4] studied multiple weighted norm inequalities for maximal vector-valued multilinear singular operator and maximal commutators. Wang et al. [5] proved the boundedness for a class of multisublinear singular integral operators on the product central Morrey spaces with variable exponents.

Motivated by the mentioned works, we will consider the boundedness of commutators generated by multilinear Calderón-Zygmund operator and BMO functions on products of weighted Herz-Morrey spaces with variable exponents.

2. Notations and Main Result

In this section, we recall some notations and definitions; then, we describe our results. Assume $p(\cdot)$ be a measurable function on $\mathbb{R}^n$ and take values in $[1, \infty)$, the Lebesgue space with variable exopnent $L^{p(\cdot)}(\mathbb{R}^n)$ is acquired by

$$L^{p(\cdot)}(\mathbb{R}^n) := \left\{ f \text{ is measurable} : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx < \infty \text{for some } \lambda > 0 \right\}. \tag{7}$$

The norm is defined by

$$\|f\|_{L^{p(\cdot)}} := \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \le 1 \right\}. \tag{8}$$

On a Banach function space, the Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ is equipped with the norm $\|f\|_{L^{p(\cdot)}}$ The space $L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n)$ is defined by

$$L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n) := \left\{ f : f_{\chi_K} \in L^{p(\cdot)}(\mathbb{R}^n) \text{ for all compact subsets } K \subset \mathbb{R}^n \right\}, \tag{9}$$

where and what follows, χ_A denotes the characteristic function of a measurable set $A \subset \mathbb{R}^n$.

Let $p(\cdot)$: $\mathbb{R}^n \longrightarrow (0,\infty)$, we denote

$$p_- := \text{ess} \inf_{x\in\mathbb{R}^n} p(x), p_+ := \text{ess} \sup_{x\in\mathbb{R}^n} p(x). \tag{10}$$

The set $\mathscr{P}(\mathbb{R}^n)$ consists of all $p(\cdot)$ satisfying $p_- > 1$ and $p_+ < \infty$; $\mathscr{P}_0(\mathbb{R}^n)$ consists of all $p(\cdot)$ satisfying $p_- > 0$ and $p_+ < \infty$. $L^{p(\cdot)}$ can be equally defined as above for $p(\cdot) \in \mathscr{P}_0(\mathbb{R}^n)$. $q'(\cdot)$ is the conjugate exponent of $p(\cdot)$, defined pointwise by $1/p(\cdot) + 1/p'(\cdot) = 1$.

Let $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ and w be a weight which is a nonnegative measurable function on $\mathbb{R}^n$. Then, the weighted variable exponent Lebesgue space $L^{p(\cdot)}(w)$ is the set of all complex-valued measurable function f such that $fw \in L^{p(\cdot)}$. The space $L^{p(\cdot)}(w)$ is a Banach space equipped with the norm

$$\|f\|_{L^{p(\cdot)}(w)} := \|fw\|_{L^{p(\cdot)}}. \tag{11}$$

Let $f \in L^1_{loc}(\mathbb{R}^n)$. Then, the standard Hardy-Littlewood maximal function of f is defined by

$$Mf(x) := \sup_{B\ni x} \frac{1}{|B|} \int_B |f(y)|dy, \forall x \in \mathbb{R}^n, \tag{12}$$

where the supremum is taken over all balls containing x in $\mathbb{R}^n$. Generally speaking, on weighted variable Lebesgue spaces, the Hardy-Littlewood maximal operator is not bounded. But if it meets certain conditions, it will be established. Namely, let $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ and meet the following global log-Hölder continuous and $w \in A_{p(\cdot)}$ such that M is bounded on $L^{p(\cdot)}(\mathrm{w})$, see [6].

Definition 1. Assume $\alpha(\cdot)$ be a real-valued measurable function on $\mathbb{R}^n$.

(i) We say that $\alpha(\cdot)$ satisfies the local log-Hölder continuity condition if there exists a constant C_1 such that

$$|\alpha(x) - \alpha(y)| \le \frac{C_1}{\log(e + (1/|x - y|))}, x, y\hat{G} \in \mathbb{R}^n, |x - y| < \frac{1}{2} \tag{13}$$

(ii) We say that $\alpha(\cdot)$ satisfies the log-Hölder continuous at the origin if there exists a constant C_2 such that

$$|\alpha(x)-\alpha(0)| \le \frac{C_2}{\log\left(e+(1/|x|)\right)}, \forall x \in \mathbb{R}^n \tag{14}$$

Denote by $\mathscr{P}_0^{\log}(\mathbb{R}^n)$ the set of all log-Hölder continuous functions at the origin.

(iii) We say that $\alpha(\cdot)$ satisfies the log-Hölder continuous at the infinity if there exists $\alpha_\infty \in \mathbb{R}$ and a constant C_3 such that

$$|\alpha(x)-\alpha_\infty| \le \frac{C_3}{\log\left(e+|x|\right)}, \forall x \in \mathbb{R}^n \tag{15}$$

Denote by $\mathscr{P}_\infty^{\log}(\mathbb{R}^n)$ the set of all log-Hölder continuous functions at infinity.

(iv) We say that $\alpha(\cdot)$ satisfies the global log-Hölder continuous if $\alpha(\cdot)$ is both log-Hölder continuous and locally log-Hölder continuous at infinity. We denote by $\mathscr{P}^{\log}(\mathbb{R}^n)$ the set of all global log-Hölder continuous functions

Definition 2. Given $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ and a positive measurable function w, we say that $w \in A_{p(\cdot)}$ if there exists a positive constant C for all balls B in $\mathbb{R}^n$ such that

$$\frac{1}{|B|}\|w\chi_B\|_{L^{p(\cdot)}}\left\|w^{-1}\chi_B\right\|_{L^{p'(\cdot)}} \le C. \tag{16}$$

Remark 3. In [7], Cruz-Uribe et al. obtained that if $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ and $w \in A_{p(\cdot)}$, then $w^{-1} \in A_{p'(\cdot)}$.

The Muckenhoupt A_p class with constant exponent $p \in (1,\infty)$ was firstly proposed by Muckenhoupt in [8]. The variable Muckenhoupt $A_{p(\cdot)}$ was considered in [7, 9–12].

Lemma 4 (see [7, Theorem 1.5]). *If $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n) \cap \mathscr{P}(\mathbb{R}^n)$ and $w \in A_{p(\cdot)}$, then there is a positive constant C such that for each $f \in L^{p(\cdot)}(w)$,*

$$\|(Mf)w\|_{L^{p(\cdot)}} \le C\|fw\|_{L^{p(\cdot)}}. \tag{17}$$

Next, we define the weighted Herz-Morrey space with variable exponents, and we use the following concepts. Let $k \in \mathbb{Z}$, we define

$$\begin{aligned} B_k &:= \left\{x \in \mathbb{R}^n : |x| \le 2^k\right\}, \\ D_k &:= B_k \setminus B_{k-1}, \\ \chi_k &:= \chi_{D_k}, \\ \tilde{\chi}_m &= \chi_m,\ m \ge 1, \\ \tilde{\chi}_0 &= \chi_{B_0}. \end{aligned} \tag{18}$$

Definition 5. Let $q(\cdot), p(\cdot) \in \mathscr{P}_0(\mathbb{R}^n)$, $\lambda \in [0,\infty)$. Let $\alpha(\cdot)$ be a bounded real-valued measurable function on $\mathbb{R}^n$. The nonhomogeneous weighted Herz-Morrey space $MK_{q(\cdot),\lambda}^{\alpha(\cdot),p(\cdot)}(w)$ and homogeneous weighted Herz-Morrey space $M\dot{K}_{q(\cdot),\lambda}^{\alpha(\cdot),p(\cdot)}(w)$ are defined, respectively, by

$$\begin{aligned} MK_{q(\cdot),\lambda}^{\alpha(\cdot),p(\cdot)}(w) &:= \left\{f \in L_{\mathrm{loc}}^{p(\cdot)}(\mathbb{R}^n, w) \colon \|f\|_{MK_{q(\cdot),\lambda}^{\alpha(\cdot),p(\cdot)}(w)} < \infty\right\}, \\ M\dot{K}_{q(\cdot),\lambda}^{\alpha(\cdot),p(\cdot)}(w) &:= \left\{f \in L_{\mathrm{loc}}^{p(\cdot)}(\mathbb{R}^n \setminus \{0\}, w) \colon \|f\|_{M\dot{K}_{q(\cdot),\lambda}^{\alpha(\cdot),p(\cdot)}(w)} < \infty\right\}, \end{aligned} \tag{19}$$

and where

$$\begin{aligned} \|f\|_{M\dot{K}_{q(\cdot),\lambda}^{\alpha(\cdot),p(\cdot)}(w)} &:= \sup_{L \in \mathbb{Z}} 2^{-L\lambda}\left\|\left(2^{\alpha(\cdot)k} f\chi_k\right)_{k \le L}\right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}(w)\right)}, \\ \|f\|_{MK_{q(\cdot),\lambda}^{\alpha(\cdot),p(\cdot)}(w)} &:= \sup_{L \in \mathbb{N}_0} 2^{-L\lambda}\left\|\left(2^{\alpha(\cdot)k} f\tilde{\chi}_k\right)_{k=0}^{L}\right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}(w)\right)}. \end{aligned} \tag{20}$$

Let B and C be two real numbers. If there exists a constant $K > 0$ such that $B \le KC$, we denote $B \lesssim C$. If $B \lesssim C$ and $C \lesssim B$, we denote $B \approx C$.

Proposition 6 (see [13, Proposition 1]). *Let $p(\cdot)$, $q(\cdot) \in \mathscr{P}_0(\mathbb{R}^n)$, w be a weight, $\lambda \in [0,\infty)$, and $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$.*

(i) If $\alpha(\cdot), q(\cdot) \in \mathscr{P}_0^{\log}(\mathbb{R}^n) \cap \mathscr{P}_\infty^{\log}(\mathbb{R}^n)$, then for all $f \in L_{loc}^{p(\cdot)}(\mathbb{R}^n \setminus \{0\}, w)$

$$\begin{aligned} \|f\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q(\cdot)}(w)} \approx \max\Bigg\{ & \sup_{L \lesssim 0, L \in \mathbb{Z}} 2^{-L\lambda}\left\|\left(2^{k\alpha(0)} f\chi_k\right)_{k \le L}\right\|_{\ell^{q_0}\left(L^{p(\cdot)}(w)\right)}, \sup_{L>0, L \in \mathbb{Z}} \\ & \cdot \left[2^{-L\lambda}\left\|\left(2^{k\alpha(0)} f\chi_k\right)_{k<0}\right\|_{\ell^{q_0}\left(L^{p(\cdot)}(w)\right)} \right. \\ & \left. + 2^{-L\lambda}\left\|\left(2^{k\alpha_\infty} f\chi_k\right)_{k=0}^{L}\right\|_{\ell^{q_\infty}\left(L^{p(\cdot)}(w)\right)}\right]\Bigg\}, \end{aligned} \tag{21}$$

where and hereafter, $q_0 := q(0)$

(ii) If $\alpha(\cdot), q(\cdot) \in \mathscr{P}_\infty^{\log}(\mathbb{R}^n)$, then

$$MK_{p(\cdot),\lambda}^{\alpha(\cdot),q(\cdot)}(w) = MK_{p(\cdot),\lambda}^{\alpha_\infty,q_\infty}(w) \tag{22}$$

Lemma 7 has been proved by Noi and Izuki in [14, 15].

Lemma 7. *If $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n) \cap \mathscr{P}(\mathbb{R}^n)$ and $w \in A_{p(\cdot)}$, then there exist constants δ_1, $\delta_2 \in (0,1)$ and $C > 0$ such that for all balls B in $\mathbb{R}^n$ and all measurable subsets $S \subset B$,*

$$\begin{aligned} \frac{\|\chi_S\|_{L^{p(\cdot)}(w)}}{\|\chi_B\|_{L^{p(\cdot)}(w)}} &\leq C\left(\frac{|S|}{|B|}\right)^{\delta_1}, \\ \frac{\|\chi_S\|_{L^{p'(\cdot)}(w^{-1})}}{\|\chi_B\|_{L^{p'(\cdot)}(w^{-1})}} &\leq C\left(\frac{|S|}{|B|}\right)^{\delta_2}. \end{aligned} \tag{23}$$

The following is the main result.

Theorem 8. *Let T be a bilinear Calderón-Zygmund operator and let b_1 and b_2 be BMO functions. Given $p_1(\cdot)$, $p_2(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n) \cap \mathscr{P}(\mathbb{R}^n)$ satisfying $1/p(x) = (1/p_1(x)) + (1/p_2(x))$ for $x \in \mathbb{R}^n$. Let w_1,w_2 be weights, $w = w_1 w_2$, $w_i \in A_{p_i(\cdot)}$, $i = 1, 2$. Assume that $\alpha(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathscr{P}_0^{\log}(\mathbb{R}^n) \cap \mathscr{P}_\infty^{\log}(\mathbb{R}^n)$, $\alpha(0) = \alpha_1(0) + \alpha_2(0)$, $\alpha_\infty = \alpha_{1\infty} + \alpha_{2\infty}$, $q(\cdot) \in \mathscr{P}_0^{\log}(\mathbb{R}^n) \cap \mathscr{P}_\infty^{\log}(\mathbb{R}^n)$,$1/q(0) = 1/q_1(0) + 1/q_2(0)$, $1/q_\infty = 1/q_{1\infty} + 1/q_{2\infty}$, $\lambda = \lambda_1 + \lambda_2$, $0 \leq \lambda_i < \infty$, δ_{i1}, $\delta_{i2} \in (0,1)$ are the constants in Lemma 7 for exponents $p_i(\cdot)$ and weights w_i, $i = 1, 2$. If $\lambda_i - n\delta_{i1} < \alpha_{i\infty}, \alpha_i(0) < n\delta_{i2}$,$i = 1, 2$, then*

$$\|[b_1, b_2, T](f_1, f_2)\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q(\cdot)}(w)} \lesssim \|f_1\|_{M\dot{K}_{p_1(\cdot),\lambda_1}^{\alpha_1(\cdot),q_1(\cdot)}(w_1)} \|f_2\|_{M\dot{K}_{p_2(\cdot),\lambda_2}^{\alpha_2(\cdot),q_2(\cdot)}(w_2)}. \tag{24}$$

3. Proof of Theorem 8

Before we prove Theorem 8, we need to introduce some lemmas.

Lemma 9 (see [1, Theorem 2.3]). *Let $p(\cdot)$, $p_1(\cdot)$, $p_2(\cdot) \in \mathscr{P}_0(\mathbb{R}^n)$ such that $1/p(x) = 1/p_1(x) + 1/p_2(x)$ for $x \in \mathbb{R}^n$. Then, there exists a constant C_{p,p_1} independent of functions f and g such that*

$$\|fg\|_{L^{p(\cdot)}} \leq C_{p,p_1} \|f\|_{L^{p_1(\cdot)}} \|g\|_{L^{p_2(\cdot)}}, \tag{25}$$

holds for every $f \in L^{p_1(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p_2(\cdot)}(\mathbb{R}^n)$.

If $p \in \mathscr{P}(\mathbb{R}^n)$, $w \in A_{p(\cdot)}$ with $w = w_1 w_2$, then by the Hölder inequality, we have

$$\|fg\|_{L^{p(\cdot)}(w)} \leq C_{p,p_1} \|f\|_{L^{p_1(\cdot)}(w_1)} \|g\|_{L^{p_2(\cdot)}(w_2)}. \tag{26}$$

Lemma 10 (see [16, Corollary 3.11]). *Let $b \in BMO(\mathbb{R}^n)$, $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n) \cap \mathscr{P}(\mathbb{R}^n)$, $w \in A_{p(\cdot)}$, $t \in [1,\infty)$, and $k, i \in \mathbb{N}$ such that $k > i$, then one has*

$$\left\| \left| b - b_{B_i} \right|^t \chi_{B_k} \right\|_{L^{p(\cdot)}(w)} \leq C(k-i)^t \|b\|_*^t \left\| \chi_{B_k} \right\|_{L^{p(\cdot)}(w)}. \tag{27}$$

Lemma 11 (see [17, Theorem 2.6]). *If $0 < p < \infty$ and $\delta > 0$, then there exists a positive constant C such that*

$$\left(\sum_{l=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} 2^{-|k-l|\delta} a_k \right)^p \right)^{1/p} \leq C \left(\sum_{l=-\infty}^{\infty} a_l^p \right)^{1/p}, \tag{28}$$

for nonnegative sequences $\{a_l\}_{l=-\infty}^{\infty}$.

Lemma 12 (see [10, Theorems 2.23 and 2.24]). *Assume that for some p_0, $p_0 \in (1,\infty)$, and every $w_0 \in A_\infty$,*

$$\int_{\mathbb{R}^n} f(x)^{p_0} w_0(x) dx \leq C \int_{\mathbb{R}^n} g(x)^{p_0} w_0(x) dx, (f, g) \in \mathscr{F}, \tag{29}$$

where $\mathscr{F}$ is a pair of nonnegative functions. Given $p(\cdot) \in \mathscr{P}_0(\mathbb{R}^n)$, assume that there exists $s \leq p_-$ such that $w^s \in A_{p(\cdot)/s}$ and M is bounded on $L^{(p(\cdot)/s)'}(w^{-s})(\mathbb{R}^n)$. Then,

$$\|f\|_{L^{p(\cdot)}(w)} \leq C \|g\|_{L^{p(\cdot)}(w)}, (f, g) \in \mathscr{F}. \tag{30}$$

Lemma 13. *Let $p_1(\cdot)$, $p_2(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, $1 < (p_i)_- \leq (p_i)_+ < \infty$, and $p_i(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n) \cap \mathscr{P}(\mathbb{R}^n)$ satisfying $1/p(x) = 1/p_1(x) + 1/p_2(x)$ for $x \in \mathbb{R}^n$, $i = 1, 2$. Let b_1 and b_2 be BMO functions. Let $w_1 \in A_{p_1(\cdot)}$, $w_2 \in A_{p_2(\cdot)}$, and $w = w_1 w_2$. If T is a bilinear Calderón-Zygmund operator, then*

$$\|[b_1, b_2, T](f_1, f_2)\|_{L^{p(\cdot)}(w)} \lesssim \|f_1\|_{L^{p_1(\cdot)}(w_1)} \|f_2\|_{L^{p_2(\cdot)}(w_2)}. \tag{31}$$

Proof. We assume that b_1 and b_2 are bounded functions. Let f_1 and f_2 be bounded functions with compact support. By the same argument in the proof of [1, Theorem 2.2], we have

$$\begin{aligned} &\int_{\mathbb{R}^n} |[b_1, b_2, T](f_1, f_2)(x)|^{p_0} w_0(x) dx \\ &\quad \lesssim \|b_1\|_*^{p_0} \|b_2\|_*^{p_0} \int_{\mathbb{R}^n} (M_1 f_1(x) M_2 f_2(x))^{p_0} w_0(x) dx. \end{aligned} \tag{32}$$

By Lemma 12, we have

$$\|[b_1, b_2, T](f_1, f_2)\|_{L^{p(\cdot)}(w)} \lesssim \|b_1\|_* \|b_2\|_* \|M_1 f_1 M_2 f_2\|_{L^{p(\cdot)}(w)}. \tag{33}$$

By Lemmas 9 and 4, we have

$$\|[b_1, b_2, T](f_1, f_2)\|_{L^{p(\cdot)}(w)} \lesssim \|b_1\|_* \|b_2\|_* \|f_1\|_{L^{p_1(\cdot)}(w_1)} \|f_2\|_{L^{p_2(\cdot)}(w_2)}. \tag{34}$$

□

Proof of Theorem 8. Assume f_1 and f_2 are bounded functions with compact support and write

$$f_i = \sum_{l=-\infty}^{\infty} f_i \chi_l =: \sum_{l=-\infty}^{\infty} f_{il},\ i = 1, 2. \tag{35}$$

By Proposition 6, we have

$$\begin{aligned}
&\|[b_1, b_2, T](f_1, f_2)\|_{M\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot),\lambda}(w)} \\
&\approx \max\left\{ \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda} \left\| \left(2^{k\alpha(0)}[b_1, b_2, T](f_1, f_2)\chi_k\right)_{k\le L} \right\|_{l^{q_0}(L^{p(\cdot)}(w))}, \sup_{L>0, L\in\mathbb{Z}} \right. \\
&\quad \times \left[2^{-L\lambda} \left\| \left(2^{k\alpha(0)}[b_1, b_2, T](f_1, f_2)\chi_k\right)_{k<0} \right\|_{l^{q_0}(L^{p(\cdot)}(w))} \right. \\
&\quad \left.\left. + 2^{-L\lambda} \left\| \left(2^{k\alpha_\infty}[b_1, b_2, T](f_1, f_2)\chi_k\right)_{k=0}^{L} \right\|_{l^{q_\infty}(L^{p(\cdot)}(w))} \right] \right\} := \max\{E, H\},
\end{aligned} \tag{36}$$

where

$$\begin{aligned}
E &:= \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda} \left\| \left(2^{k\alpha(0)}[b_1, b_2, T](f_1, f_2)\chi_k\right)_{k\le L} \right\|_{l^{q_0}(L^{p(\cdot)}(w))}, \\
H &:= \sup_{L\in\mathbb{N}} \{F + G\}, \\
F &:= 2^{-L\lambda} \left\| \left(2^{k\alpha(0)}[b_1, b_2, T](f_1, f_2)\chi_k\right)_{k<0} \right\|_{l^{q_0}(L^{p(\cdot)}(w))}, \\
G &:= 2^{-L\lambda} \left\| \left(2^{k\alpha_\infty}[b_1, b_2, T](f_1, f_2)\chi_k\right)_{k=0}^{L} \right\|_{l^{q_\infty}(L^{p(\cdot)}(w))}.
\end{aligned} \tag{37}$$

Since the estimates of E and F are essentially analogical, we only need to obtain E and G bounded in the Herz-Morrey space with variable exponents. It is easy to see that

$$E \le C\sum_{i=i}^{9} E_i,\ G \le C\sum_{i=i}^{9} G_i, \tag{38}$$

where

$$\begin{aligned}
E_1 &:= \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda} \left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)} \left\| \sum_{l=-\infty}^{k-2} \sum_{j=-\infty}^{k-2} [b_1, b_2, T]\left(f_{1l}, f_{2j}\right)\chi_k \right\|_{L^{p(\cdot)}(w)}^{q(0)} \right)^{1/q(0)}, \\
E_2 &:= \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda} \left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)} \left\| \sum_{l=-\infty}^{k-2} \sum_{j=k-1}^{k+1} [b_1, b_2, T]\left(f_{1l}, f_{2j}\right)\chi_k \right\|_{L^{p(\cdot)}(w)}^{q(0)} \right)^{1/q(0)}, \\
E_3 &:= \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda} \left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)} \left\| \sum_{l=-\infty}^{k-2} \sum_{j=k+2}^{\infty} [b_1, b_2, T]\left(f_{1l}, f_{2j}\right)\chi_k \right\|_{L^{p(\cdot)}(w)}^{q(0)} \right)^{1/q(0)}, \\
E_4 &:= \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda} \left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)} \left\| \sum_{l=k-1}^{k+1} \sum_{j=-\infty}^{k-2} [b_1, b_2, T]\left(f_{1l}, f_{2j}\right)\chi_k \right\|_{L^{p(\cdot)}(w)}^{q(0)} \right)^{1/q(0)}, \\
E_5 &:= \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda} \left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)} \left\| \sum_{l=k-1}^{k+1} \sum_{j=k-1}^{k+1} [b_1, b_2, T]\left(f_{1l}, f_{2j}\right)\chi_k \right\|_{L^{p(\cdot)}(w)}^{q(0)} \right)^{1/q(0)}, \\
E_6 &:= \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda} \left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)} \left\| \sum_{l=k-1}^{k+1} \sum_{j=k+2}^{\infty} [b_1, b_2, T]\left(f_{1l}, f_{2j}\right)\chi_k \right\|_{L^{p(\cdot)}(w)}^{q(0)} \right)^{1/q(0)}, \\
E_7 &:= \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda} \left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)} \left\| \sum_{l=k+2}^{\infty} \sum_{j=-\infty}^{k-2} [b_1, b_2, T]\left(f_{1l}, f_{2j}\right)\chi_k \right\|_{L^{p(\cdot)}(w)}^{q(0)} \right)^{1/q(0)}, \\
E_8 &:= \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda} \left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)} \left\| \sum_{l=k+2}^{\infty} \sum_{j=k-1}^{k+1} [b_1, b_2, T]\left(f_{1l}, f_{2j}\right)\chi_k \right\|_{L^{p(\cdot)}(w)}^{q(0)} \right)^{1/q(0)}, \\
E_9 &:= \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda} \left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)} \left\| \sum_{l=k+2}^{\infty} \sum_{j=k+2}^{\infty} [b_1, b_2, T]\left(f_{1l}, f_{2j}\right)\chi_k \right\|_{L^{p(\cdot)}(w)}^{q(0)} \right)^{1/q(0)}, \\
G_1 &:= 2^{-L\lambda} \left(\sum_{k=0}^{L} 2^{k\alpha_\infty q_\infty} \left\| \sum_{l=-\infty}^{k-2} \sum_{j=-\infty}^{k-2} [b_1, b_2, T]\left(f_{1l}, f_{2j}\right)\chi_k \right\|_{L^{p(\cdot)}(w)}^{q_\infty} \right)^{1/q_\infty}, \\
G_2 &:= 2^{-L\lambda} \left(\sum_{k=0}^{L} 2^{k\alpha_\infty q_\infty} \left\| \sum_{l=-\infty}^{k-2} \sum_{j=k-1}^{k+1} [b_1, b_2, T]\left(f_{1l}, f_{2j}\right)\chi_k \right\|_{L^{p(\cdot)}(w)}^{q_\infty} \right)^{1/q_\infty}, \\
G_3 &:= 2^{-L\lambda} \left(\sum_{k=0}^{L} 2^{k\alpha_\infty q_\infty} \left\| \sum_{l=-\infty}^{k-2} \sum_{j=k+2}^{\infty} [b_1, b_2, T]\left(f_{1l}, f_{2j}\right)\chi_k \right\|_{L^{p(\cdot)}(w)}^{q_\infty} \right)^{1/q_\infty}, \\
G_4 &:= 2^{-L\lambda} \left(\sum_{k=0}^{L} 2^{k\alpha_\infty q_\infty} \left\| \sum_{l=k-1}^{k+1} \sum_{j=-\infty}^{k-2} [b_1, b_2, T]\left(f_{1l}, f_{2j}\right)\chi_k \right\|_{L^{p(\cdot)}(w)}^{q_\infty} \right)^{1/q_\infty}, \\
G_5 &:= 2^{-L\lambda} \left(\sum_{k=0}^{L} 2^{k\alpha_\infty q_\infty} \left\| \sum_{l=k-1}^{k+1} \sum_{j=k-1}^{k+1} [b_1, b_2, T]\left(f_{1l}, f_{2j}\right)\chi_k \right\|_{L^{p(\cdot)}(w)}^{q_\infty} \right)^{1/q_\infty}, \\
G_6 &:= 2^{-L\lambda} \left(\sum_{k=0}^{L} 2^{k\alpha_\infty q_\infty} \left\| \sum_{l=k-1}^{k+1} \sum_{j=k+2}^{\infty} [b_1, b_2, T]\left(f_{1l}, f_{2j}\right)\chi_k \right\|_{L^{p(\cdot)}(w)}^{q_\infty} \right)^{1/q_\infty}, \\
G_7 &:= 2^{-L\lambda} \left(\sum_{k=0}^{L} 2^{k\alpha_\infty q_\infty} \left\| \sum_{l=k+2}^{\infty} \sum_{j=-\infty}^{k-2} [b_1, b_2, T]\left(f_{1l}, f_{2j}\right)\chi_k \right\|_{L^{p(\cdot)}(w)}^{q_\infty} \right)^{1/q_\infty}, \\
G_8 &:= 2^{-L\lambda} \left(\sum_{k=0}^{L} 2^{k\alpha_\infty q_\infty} \left\| \sum_{l=k+2}^{\infty} \sum_{j=k-1}^{k+1} [b_1, b_2, T]\left(f_{1l}, f_{2j}\right)\chi_k \right\|_{L^{p(\cdot)}(w)}^{q_\infty} \right)^{1/q_\infty}, \\
G_9 &:= 2^{-L\lambda} \left(\sum_{k=0}^{L} 2^{k\alpha_\infty q_\infty} \left\| \sum_{l=k+2}^{\infty} \sum_{j=k+2}^{\infty} [b_1, b_2, T]\left(f_{1l}, f_{2j}\right)\chi_k \right\|_{L^{p(\cdot)}(w)}^{q_\infty} \right)^{1/q_\infty}.
\end{aligned} \tag{39}$$

We shall use the following estimates. If $l \le k-1$, then pass Hölder's inequality, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} |b_i(x) - b_i(y_i)| f_{il} dy_i \\
&\quad \lesssim \int_{\mathbb{R}^n} \left| b_i(x) - (b_i)_{B_l} \right| f_{il} dy_i + \int_{\mathbb{R}^n} \left| b_i(y_i) - (b_i)_{B_l} \right| f_{il} dy_i \\
&\quad \lesssim \left| b_i(x) - (b_i)_{B_l} \right| \|f_{il}\chi_l\|_{L^{p_i(\cdot)}(w_i)} \|\chi_l\|_{L^{p_i'(\cdot)}(w_i^{-1})} \\
&\qquad + \|f_{il}\chi_l\|_{L^{p_i(\cdot)}(w_i)} \left\| \left(b_i - (b_i)_{B_l}\right)\chi_l \right\|_{L^{p_i'(\cdot)}(w_i^{-1})}.
\end{aligned} \tag{40}$$

By Lemmas 7 and 10, Hölder's inequality, and Definition 2, we acquire that

$$\begin{aligned}
&\left\|2^{-kn}\int_{\mathbb{R}^n}|b_i(x)-b_i(y_i)|f_{il}dy_i\chi_k\right\|_{L^{p_i(\cdot)}(w_i)}\\
&\quad\lesssim 2^{-kn}\left\|\left(b_i-(b_i)_{B_l}\right)\cdot\chi_k\right\|_{L^{p(\cdot)}(w)}\|f_{il}\chi_l\|_{L^{p_i(\cdot)}(w_i)}\|\chi_l\|_{L^{p_i'(\cdot)}(w_i^{-1})}\\
&\qquad+2^{-kn}\|f_{il}\chi_l\|_{L^{p_i(\cdot)}(w_i)}\left\|\left(b_i-(b_i)_{B_l}\right)\chi_l\right\|_{L^{p_i'(\cdot)}(w_i^{-1})}\|\chi_k\|_{L^{p(\cdot)}(w)}\\
&\quad\lesssim 2^{-kn}\|b_i\|_*\|f_{il}\chi_l\|_{L^{p_i(\cdot)}(w_i)}\left\|\chi_{B_l}\right\|_{L^{p_i'(\cdot)}(w_i^{-1})}\left\|\chi_{B_k}\right\|_{L^{p(\cdot)}(w)}\\
&\qquad+2^{-kn}\|b_i\|_*(k-l)\|f_{il}\chi_l\|_{L^{p_i(\cdot)}(w_i)}\left\|\chi_{B_l}\right\|_{L^{p_i'(\cdot)}(w_i^{-1})}\left\|\chi_{B_k}\right\|_{L^{p(\cdot)}(w)}\\
&\quad\lesssim (k-l)2^{-kn}\|b_i\|_*\|f_{il}\chi_l\|_{L^{p_i(\cdot)}(w_i)}\left\|\chi_{B_l}\right\|_{L^{p_i'(\cdot)}(w_i^{-1})}\left\|\chi_{B_k}\right\|_{L^{p(\cdot)}(w)}\\
&\quad\lesssim (k-l)2^{-kn}\|b_i\|_*\left\|\chi_{B_k}\right\|_{L^{p_i(\cdot)}(w_i)}\|f_{il}w_i\chi_l\|_{L^{p_i(\cdot)}}\|\chi_l w_i^{-1}\|_{L^{p_i'(\cdot)}}\\
&\quad\lesssim (k-l)2^{-kn}\|b_i\|_*|B_k|\left\|\chi_{B_k}\right\|^{-1}_{L^{p_i'(\cdot)}(w_i^{-1})}\left\|\chi_{B_l}\right\|_{L^{p_i'(\cdot)}(w_i^{-1})}\|f_{il}\chi_l\|_{L^{p_i(\cdot)}(w_i)}\\
&\quad\lesssim (k-l)2^{(l-k)n\delta_{i2}}\|b_i\|_*\|f_{il}\|_{L^{p_i(\cdot)}(w_i)}.
\end{aligned}\tag{41}$$

If $l=k$, then

$$\begin{aligned}
&\left\|2^{-kn}\int_{\mathbb{R}^n}|b_i(x)-b_i(y_i)|f_{il}dy_i\chi_k\right\|_{L^{p_i(\cdot)}(w_i)}\\
&\quad\lesssim 2^{-kn}\left\|\left(b_i-(b_i)_{B_k}\right)\cdot\chi_k\right\|_{L^{p(\cdot)}(w)}\|f_{il}\chi_l\|_{L^{p_i(\cdot)}(w_i)}\|\chi_l\|_{L^{p_i'(\cdot)}(w_i^{-1})}\\
&\qquad+2^{-kn}\|f_{il}\chi_l\|_{L^{p_i(\cdot)}(w_i)}\left\|\left(b_i-(b_i)_{B_l}\right)\chi_l\right\|_{L^{p_i'(\cdot)}(w_i^{-1})}\|\chi_k\|_{L^{p(\cdot)}(w)}\\
&\quad\lesssim 2^{-kn}\|b_i\|_*\left\|\chi_{B_k}\right\|_{L^{p_i(\cdot)}(w_i)}\|f_{il}w_i\chi_l\|_{L^{p_i(\cdot)}}\|\chi_l w_i^{-1}\|_{L^{p_i'(\cdot)}}\\
&\quad\lesssim 2^{-kn}\|b_i\|_*\left\|\chi_{B_k}\right\|_{L^{p_i(\cdot)}(w_i)}\left\|\chi_{B_l}\right\|_{L^{p_i'(\cdot)}(w_i^{-1})}\|f_{il}\chi_l\|_{L^{p_i(\cdot)}(w_i)}\\
&\quad\lesssim \|b_i\|_*\|f_{il}\|_{L^{p_i(\cdot)}(w_i)}.
\end{aligned}\tag{42}$$

If $l\geq k+1$, then pass Hölder's inequality, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n}|b_i(x)-b_i(y_i)|f_{il}dy_i\\
&\quad\lesssim\int_{\mathbb{R}^n}\left|b_i(x)-(b_i)_{B_l}\right|f_{il}dy_i+\int_{\mathbb{R}^n}\left|b_i(y_i)-(b_i)_{B_l}\right|f_{il}dy_i\\
&\quad\lesssim\left|b_i(x)-(b_i)_{B_k}\right|\|f_{il}\chi_l\|_{L^{p_i(\cdot)}(w_i)}\|\chi_l\|_{L^{p_i'(\cdot)}(w_i^{-1})}\\
&\qquad+\|f_{il}\chi_l\|_{L^{p_i(\cdot)}(w_i)}\left\|\left(b_i-(b_i)_{B_k}\right)\chi_l\right\|_{L^{p_i'(\cdot)}(w_i^{-1})}.
\end{aligned}\tag{43}$$

By Lemmas 7 and 10, Hölder's inequality, and Definition 2, we acquire that

$$\begin{aligned}
&\left\|2^{-kn}\int_{\mathbb{R}^n}|b_i(x)-b_i(y_i)|f_{il}dy_i\chi_k\right\|_{L^{p_i(\cdot)}(w_i)}\\
&\quad\lesssim 2^{-kn}\left\|\left(b_i-(b_i)_{B_k}\right)\cdot\chi_k\right\|_{L^{p(\cdot)}(w)}\|f_{il}\chi_l\|_{L^{p_i(\cdot)}(w_i)}\|\chi_l\|_{L^{p_i'(\cdot)}(w_i^{-1})}\\
&\qquad+2^{-kn}\|f_{il}\chi_l\|_{L^{p_i(\cdot)}(w_i)}\left\|\left(b_i-(b_i)_{B_k}\right)\chi_l\right\|_{L^{p_i'(\cdot)}(w_i^{-1})}\|\chi_k\|_{L^{p(\cdot)}(w)}\\
&\quad\lesssim 2^{-kn}\|b_i\|_*\|f_{il}\chi_l\|_{L^{p_i(\cdot)}(w_i)}\left\|\chi_{B_l}\right\|_{L^{p_i'(\cdot)}(w_i^{-1})}\left\|\chi_{B_k}\right\|_{L^{p(\cdot)}(w)}\\
&\qquad+2^{-kn}\|b_i\|_*(l-k)\|f_{il}\chi_l\|_{L^{p_i(\cdot)}(w_i)}\left\|\chi_{B_l}\right\|_{L^{p_i'(\cdot)}(w_i^{-1})}\left\|\chi_{B_k}\right\|_{L^{p(\cdot)}(w)}\\
&\quad\lesssim 2^{-kn}(l-k)\|b_i\|_*\|f_{il}\chi_l\|_{L^{p_1(\cdot)}(w_1)}\left\|\chi_{B_l}\right\|_{L^{p_1'(\cdot)}(w_1^{-1})}\left\|\chi_{B_k}\right\|_{L^{p(\cdot)}(w)}\\
&\quad\lesssim 2^{-kn}(l-k)\|b_i\|_*\left\|\chi_{B_k}\right\|_{L^{p_i(\cdot)}(w_i)}\|f_{il}w_i\chi_l\|_{L^{p_i(\cdot)}}\|\chi_l w_i^{-1}\|_{L^{p_i'(\cdot)}}\\
&\quad\lesssim 2^{-kn}(l-k)\|b_i\|_*\left\|\chi_{B_k}\right\|_{L^{p_i(\cdot)}(w_i)}\left\|\chi_{B_l}\right\|_{L^{p_i(\cdot)}(w_i)}\left\|\chi_{B_l}\right\|^{-1}_{L^{p_i(\cdot)}(w_i)}\\
&\qquad\times\left\|\chi_{B_l}\right\|_{L^{p_i'(\cdot)}(w_i^{-1})}\|f_{il}\chi_l\|_{L^{p_i(\cdot)}(w_i)}\\
&\quad\lesssim 2^{(l-k)n(1-\delta_{1i})}(l-k)\|b_i\|_*\|f_{il}\|_{L^{p_i(\cdot)}(w_i)}.
\end{aligned}\tag{44}$$

By the interchange of f_1 and f_2, we see that the estimates of E_2, E_3, and E_6 are similar to E_4, E_7, and E_8, respectively. Thus, we only to estimate E_1, E_2, E_3, E_5, E_6, and E_9.

To estimate E_1, due to $l, j\leq k-2$, we infer that for $i=1,2$,

$$\begin{aligned}
|x-y_i|&\geq|x|-|y_i|>2^{k-1}-2^{\min\{l,j\}}\\
&\geq 2^{k-2},\ x\in D_k,\ y_1\in D_l,\ y_2\in D_j.
\end{aligned}\tag{45}$$

Therefore, for $x\in D_k$, we have

$$|K(x,y_1,y_2)|\leq C(|x-y_1|+|x-y_2|)^{-2n}\leq C2^{-2kn}.\tag{46}$$

Thus, $\forall x\in D_k$ and $l,j\leq k-2$, we have

$$\begin{aligned}
&[b_1,b_2,T]\left(f_{1l},f_{2j}\right)(x)\\
&\quad\leq\int_{\mathbb{R}^{2n}}\prod_{i=1}^{2}|b_i(x)-b_i(y_i)||K(x,y_1,y_2)||f_{1l}(y_1)|\left|f_{2j}(y_2)\right|dy_1dy_2\\
&\quad\leq 2^{-2kn}\int_{\mathbb{R}^{2n}}\prod_{i=1}^{2}|b_i(x)-b_i(y_i)|f_{1l}(y_1)|\left|f_{2j}(y_2)\right|dy_1dy_2.
\end{aligned}\tag{47}$$

Thus, according to Hölder's inequality, we have

$$\begin{aligned}&\left\|\sum_{l=-\infty}^{k-2}\sum_{j=-\infty}^{k-2}[b_1,b_2,T]\left(f_{1l},f_{2j}\right)\chi_k\right\|_{L^{p(\cdot)}(w)}\\&\lesssim\left\|\sum_{l=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}|b_1(x)-b_1(y_1)||f_{1l}(y_1)|dy_1\right.\\&\quad\times\left.\sum_{j=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}|b_2(x)-b_2(y_1)|\left|f_{2j}(y_2)\right|dy_2\chi_k\right\|_{L^{p(\cdot)}(w)}\\&\lesssim\left\|\sum_{l=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}|b_1(x)-b_1(y_1)||f_{1l}(y_1)|dy_1\chi_k\right\|_{L^{p_1(\cdot)}(w_1)}\\&\quad\times\left\|\sum_{j=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}|b_2(x)-b_2(y_2)|\left|f_{2j}(y_2)\right|dy_2\chi_k\right\|_{L^{p_2(\cdot)}(w_2)}.\end{aligned}\tag{48}$$

Since $1/q(0)=1/q_1(0)+1/q_2(0)$, $\lambda=\lambda_1+\lambda_2$, according to Hölder's inequality, we will have

$$\begin{aligned}E_1&\lesssim\sup_{L\le0,L\in\mathbb{Z}}2^{-L\lambda}\left(\sum_{k=-\infty}^{L}2^{k\alpha(0)q(0)}\left\|\sum_{l=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}|b_1(x)-b_1(y_1)||f_{1l}(y_1)|dy_1\chi_k\right\|_{L^{p_1(\cdot)}(w_1)}^{q(0)}\right.\\&\quad\times\left.\left\|\sum_{j=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}|b_2(x)-b_2(y_2)|\left|f_{2j}(y_2)\right|dy_2\chi_k\right\|_{L^{p_2(\cdot)}(w_2)}^{q(0)}\right)^{1/q(0)}\\&\lesssim\sup_{L\le0,L\in\mathbb{Z}}2^{-L\lambda_1}\\&\quad\times\left(\sum_{k=-\infty}^{L}2^{k\alpha_1(0)q_1(0)}\left\|\sum_{l=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}|b_1(x)-b_1(y_1)||f_{1l}(y_1)|dy_1\chi_k\right\|_{L^{p_1(\cdot)}(w_1)}^{q_1(0)}\right)^{1/q_1(0)}\\&\quad\times\sup_{L\le0,L\in\mathbb{Z}}2^{-L\lambda_2}\\&\quad\times\left(\sum_{k=-\infty}^{L}2^{k\alpha_2(0)q_2(0)}\left\|\sum_{j=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}|b_2(x)-b_2(y_2)|\left|f_{2j}(y_2)\right|dy_2\chi_k\right\|_{L^{p_2(\cdot)}(w_2)}^{q_2(0)}\right)^{1/q_2(0)}\\&\coloneqq E_{1,1}\times E_{1,2}.\end{aligned}\tag{49}$$

where

$$\begin{aligned}E_{1,i}&\coloneqq\sup_{L\le0,L\in\mathbb{Z}}2^{-L\lambda_i}\\&\quad\times\left\{\sum_{k=-\infty}^{L}2^{k\alpha_i(0)q_i(0)}\left\|\sum_{l=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}|b_i(x)-b_i(y_i)||f_{il}(y_i)|dy_i\chi_k\right\|_{L^{p_i(\cdot)}(w_i)}^{q_i(0)}\right\}^{1/q_i(0)}.\end{aligned}\tag{50}$$

Since $n\delta_{i2}-\alpha_i(0)>0$, by (41) and Lemma 11, we acquire that

$$\begin{aligned}E_{1,i}&\lesssim\sup_{L\le0,L\in\mathbb{Z}}2^{-L\lambda_i}\left\{\sum_{k=-\infty}^{L}2^{k\alpha_i(0)q_i(0)}\|b_i\|_*\left(\sum_{l=-\infty}^{k-2}2^{(l-k)n\delta_{i2}}\|f_{il}\|_{L^{p_i(\cdot)}(w_i)}\right)^{q_i(0)}\right\}^{1/q_i(0)}\\&=\|b_i\|_*\sup_{L\le0,L\in\mathbb{Z}}2^{-L\lambda_i}\times\left\{\sum_{k=-\infty}^{L}\left(\sum_{l=-\infty}^{k-2}2^{l\alpha_i(0)}\|f_{il}\|_{L^{p_i(\cdot)}(w_i)}2^{(l-k)(n\delta_{i2}-\alpha_i(0))}\right)^{q_i(0)}\right\}^{1/q_i(0)}\\&\lesssim\|b_i\|_*\sup_{L\le0,L\in\mathbb{Z}}2^{-L\lambda_i}\left(\sum_{l=-\infty}^{L-2}2^{l\alpha_i(0)q_i(0)}\|f_{il}\|_{L^{p_i(\cdot)}(w_i)}^{q_i(0)}\right)^{1/q_i(0)}\\&\lesssim\|b_i\|_*\|f_i\|_{M\dot{K}^{\alpha_i(\cdot),q_i(\cdot)}_{p_i(\cdot),\lambda_i}(w_i)}.\end{aligned}\tag{51}$$

Thus, we obtain that

$$E_1\lesssim\|f_1\|_{M\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot),\lambda_1}(w_1)}\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)}.\tag{52}$$

To estimate E_2, due to $l\le k-2, k-1\le j\le k+1$ for $i=1,2$, then, we have

$$|x-y_1|\ge|x|-|y_1|\ge2^{k-2},x\in D_k,y_1\in D_l.\tag{53}$$

So, according to Hölder's inequality, we have

$$\begin{aligned}&\left\|\sum_{l=-\infty}^{k-2}\sum_{j=k-1}^{k+1}[b_1,b_2,T]\left(f_{1l},f_{2j}\right)\chi_k\right\|_{L^{p(\cdot)}(w)}\\&\lesssim\left\|\sum_{l=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}|b_1(x)-b_1(y_1)||f_{1l}(y_1)|dy_1\right.\\&\quad\times\left.\sum_{j=k-1}^{k+1}2^{-kn}\int_{\mathbb{R}^n}|b_2(x)-b_2(y_2)|\left|f_{2j}(y_2)\right|dy_2\chi_k\right\|_{L^{p(\cdot)}(w)}\\&\lesssim\left\|\sum_{l=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}|b_1(x)-b_1(y_1)||f_{1l}(y_1)|dy_1\chi_k\right\|_{L^{p_1(\cdot)}(w_1)}\\&\quad\times\left\|\sum_{j=k-1}^{k+1}2^{-kn}\int_{\mathbb{R}^n}|b_2(x)-b_2(y_2)|\left|f_{2j}(y_2)\right|dy_2\chi_k\right\|_{L^{p_2(\cdot)}(w_2)}.\end{aligned}\tag{54}$$

Since $1/q(0)=1/q_1(0)+1/q_2(0)$, $\lambda=\lambda_1+\lambda_2$, by Hölder's inequality, we acquire that

$$\begin{aligned}E_2&\lesssim\sup_{L\le0,L\in\mathbb{Z}}2^{-L\lambda}\left(\sum_{k=-\infty}^{L}2^{k\alpha(0)q(0)}\left\|\sum_{l=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}|b_1(x)-b_1(y_1)||f_{1l}(y_1)|dy_1\chi_k\right\|_{L^{p_1(\cdot)}(w_1)}^{q(0)}\right.\\&\quad\times\left.\left\|\sum_{j=k-1}^{k+1}2^{-kn}\int_{\mathbb{R}^n}|b_2(x)-b_2(y_2)|\left|f_{2j}(y_2)\right|dy_2\chi_k\right\|_{L^{p_2(\cdot)}(w_2)}^{q(0)}\right)^{1/q(0)}\\&\lesssim\sup_{L\le0,L\in\mathbb{Z}}2^{-L\lambda_1}\left(\sum_{k=-\infty}^{L}2^{k\alpha_1(0)q_1(0)}\right.\\&\quad\times\left.\left\|\sum_{l=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}|b_1(x)-b_1(y_1)||f_{1l}(y_1)|dy_1\chi_k\right\|_{L^{p_1(\cdot)}(w_1)}^{q_1(0)}\right)^{1/q_1(0)}\\&\quad\times\sup_{L\le0,L\in\mathbb{Z}}2^{-L\lambda_2}\left(\sum_{k=-\infty}^{L}2^{k\alpha_2(0)q_2(0)}\right.\\&\quad\times\left.\left\|\sum_{j=k-1}^{k+1}2^{-kn}\int_{\mathbb{R}^n}|b_2(x)-b_2(y_2)|\left|f_{2j}(y_2)\right|dy_2\chi_k\right\|_{L^{p_2(\cdot)}(w_2)}^{q_2(0)}\right)^{1/q_2(0)}\\&\coloneqq E_{2,1}\times E_{2,2}.\end{aligned}\tag{55}$$

It is obviously that

$$E_{2,1}=E_{1,1}\lesssim\|b_1\|_*\|f_1\|_{M\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot),\lambda_1}(w_1)}.\tag{56}$$

Now we estimate $E_{2,2}$. Taking (41), (42), and (44) together, we have

$$E_{2,2} \lesssim \|b_2\|_* \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_2}\left(\sum_{k=-\infty}^{L} 2^{k\alpha_2(0)q_2(0)} \sum_{j=k-1}^{k+1} 2^{(j-k)nq_2(0)} \left\|f_{2j}\chi_j\right\|_{L^{p_2(\cdot)}(w_2)}^{q_2(0)}\right)^{1/q_2(0)}$$
$$\lesssim \|b_2\|_* \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_2}\left(\sum_{k=-\infty}^{L} 2^{k\alpha_2(0)q_2(0)} \left\|f_{2j}\chi_k\right\|_{L^{p_2(\cdot)}(w_2)}^{q_2(0)}\right)^{1/q_2(0)}$$
$$\lesssim \|b_2\|_* \|f_2\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)}, \tag{57}$$

where we used $2^{-n\delta_{22}} < 1$ and $2^{(j-k)n(1-\delta_{12})} < 2^{(j-k)n} < 2^{2n}$, $j \in \{k-1, k, k+1\}$ for (41) and (44), respectively. Therefore, we acquire that

$$E_2 \lesssim \|b_1\|_* \|b_2\|_* \|f_1\|_{M\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot),\lambda_1}(w_1)} \|f_2\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)}. \tag{58}$$

To estimate E_3, since $l \le k-2, j \ge k+2$, then, we have

$$|x-y_1| \ge |x| - |y_1| \ge 2^{k-2}, |x-y_2| \ge |y_2| - |x| > 2^{j-2}, x \in D_k, y_1 \in D_l, y_2 \in D_j. \tag{59}$$

Therefore, $\forall x \in D_k, l \le k-2, j \ge k+2$, we get

$$\left|[b_1, b_2, T]\left(f_{1l}, f_{2j}\right)(x)\right| \lesssim \int_{\mathbb{R}^{2n}} \prod_{i=1}^{2} |b_i(x) - b_i(y_i)| |K(x, y_1, y_2)| |f_{1l}(y_1)| \left|f_{2j}(y_2)\right| dy_1 dy_2$$
$$\lesssim 2^{-kn} 2^{-jn} \int_{\mathbb{R}^{2n}} \prod_{i=1}^{2} |b_i(x) - b_i(y_i)| f_{1l}(y_1)| \left|f_{2j}(y_2)\right| dy_1 dy_2. \tag{60}$$

So, according to Hölder's inequality, we acquire that

$$\left\|\sum_{l=-\infty}^{k-2} \sum_{j=k+2}^{\infty} [b_1, b_2, T]\left(f_{1l}, f_{2j}\right)\chi_k\right\|_{L^{p(\cdot)}(w)}$$
$$\lesssim \left\|\sum_{l=-\infty}^{k-2} 2^{-kn} \int_{\mathbb{R}^n} |b_1(x) - b_1(y_1)| |f_{1l}(y_1)| dy_1 \times \sum_{j=k+2}^{\infty} 2^{-jn} \int_{\mathbb{R}^n} |b_2(x) - b_2(y_2)| \left|f_{2j}(y_2)\right| dy_2 \chi_k\right\|_{L^{p(\cdot)}(w)}$$
$$\lesssim \left\|\sum_{l=-\infty}^{k-2} 2^{-kn} \int_{\mathbb{R}^n} |b_1(x) - b_1(y_1)| |f_{1l}(y_1)| dy_1 \chi_k\right\|_{L^{p_1(\cdot)}(w_1)} \times \left\|\sum_{j=k+2}^{\infty} 2^{-jn} \int_{\mathbb{R}^n} |b_2(x) - b_2(y_2)| \left|f_{2j}(y_2)\right| dy_2 \chi_k\right\|_{L^{p_2(\cdot)}(w_2)}. \tag{61}$$

Since $1/q(0) = 1/q_1(0) + 1/q_2(0), \lambda = \lambda_1 + \lambda_2$, according to Hölder's inequality, we acquire that

$$E_3 \lesssim \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)} \left\|\sum_{l=-\infty}^{k-2} 2^{-kn} \int_{\mathbb{R}^n} |b_1(x) - b_1(y_1)| |f_{1l}(y_1)| dy_1 \chi_k\right\|_{L^{p_1(\cdot)}(w_1)}^{q(0)} \times \left\|\sum_{j=k+2}^{\infty} 2^{-jn} \int_{\mathbb{R}^n} |b_2(x) - b_2(y_2)| \left|f_{2j}(y_2)\right| dy_2 \chi_k\right\|_{L^{p_2(\cdot)}(w_2)}^{q(0)}\right)^{1/q(0)}$$
$$\le \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_1}\left(\sum_{k=-\infty}^{L} 2^{k\alpha_1(0)q_1(0)} \times \left\|\sum_{l=-\infty}^{k-2} 2^{-kn} \int_{\mathbb{R}^n} |b_1(x) - b_1(y_1)| |f_{1l}(y_1)| dy_1 \chi_k\right\|_{L^{p_1(\cdot)}(w_1)}^{q_1(0)}\right)^{1/q_1(0)}$$
$$\times \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_2}\left(\sum_{k=-\infty}^{L} 2^{k\alpha_2(0)q_2(0)} \times \left\|\sum_{j=k+2}^{\infty} 2^{-jn} \int_{\mathbb{R}^n} |b_2(x) - b_2(y_2)| \left|f_{2j}(y_2)\right| dy_2 \chi_k\right\|_{L^{p_2(\cdot)}(w_2)}^{q_2(0)}\right)^{1/q_2(0)}$$
$$:= E_{3,1} \times E_{3,2}. \tag{62}$$

It is obviously that

$$E_{3,1} = E_{1,1} \lesssim \|f_1\|_{M\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot),\lambda_1}(w_1)}. \tag{63}$$

By (44), we acquire that

$$E_{3,2} \lesssim \|b_2\|_* \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_2} \times \left(\sum_{k=-\infty}^{L} 2^{k\alpha_2(0)q_2(0)} \left(\sum_{j=k+2}^{\infty} 2^{(k-j)n\delta_{21}} \left\|f_{2j}\chi_j\right\|_{L^{p_2(\cdot)}(w_2)}\right)^{q_2(0)}\right)^{1/q_2(0)}$$
$$\lesssim \|b_2\|_* \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_2} \times \left(\sum_{k=-\infty}^{L} \left(\sum_{j=k+2}^{L} 2^{j\alpha_2(0)} \left\|f_{2j}\chi_j\right\|_{L^{p_2(\cdot)}(w_2)} 2^{(k-j)(n\delta_{21}+\alpha_2(0))}\right)^{q_2(0)}\right)^{1/q_2(0)}$$
$$+ \|b_2\|_* \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_2} \times \left(\sum_{k=-\infty}^{L} \left(2^{k\alpha_2(0)} \sum_{j=L+1}^{0} \left\|f_{2j}\chi_j\right\|_{L^{p_2(\cdot)}(w_2)} 2^{(k-j)n\delta_{21}}\right)^{q_2(0)}\right)^{1/q_2(0)}$$
$$+ \|b_2\|_* \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_2} \times \left(\sum_{k=-\infty}^{L} \left(2^{k\alpha_2(0)} \sum_{j=1}^{\infty} \left\|f_{2j}\chi_j\right\|_{L^{p_2(\cdot)}(w_2)} 2^{(k-j)n\delta_{21}}\right)^{q_2(0)}\right)^{1/q_2(0)}$$
$$:= I_1 + I_2 + I_3. \tag{64}$$

We consider I_1. By Lemma 11, we have

$$I_1 \lesssim \|b_2\|_* \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_2} \times \left(\sum_{k=-\infty}^{L} \left(\sum_{j=k+2}^{L} 2^{j\alpha_2(0)} \left\|f_{2j}\chi_j\right\|_{L^{p_2(\cdot)}(w_2)} 2^{(k-j)(n\delta_{21}+\alpha_2(0))}\right)^{q_2(0)}\right)^{1/q_2(0)}$$
$$\lesssim \|b_2\|_* \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_2}\left(\sum_{j=-\infty}^{L+2} 2^{j\alpha_2(0)q_2(0)} \left\|f_{2j}\chi_j\right\|_{L^{p_2(\cdot)}(w_2)}^{q_2(0)}\right)^{1/q_2(0)}$$
$$\lesssim \|b_2\|_* \|f_2\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)}, \tag{65}$$

where $2^{-|k-j|(n\delta_{21}+\alpha_2(0))} = 2^{-|k-j|\alpha_2}$ for $\alpha_2 = n\delta_{21} + \alpha_2(0) > 0$.

We consider I_2. Since $n\delta_{21}+\alpha_2(0)-\lambda_2>0$, we obtain that

$$\begin{aligned}
I_2 &\lesssim \|b_2\|_* \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_2}\left(\sum_{k=-\infty}^{L}\left(2^{k(n\delta_{21}+\alpha_2(0))}\sum_{j=L+1}^{0}2^{j\alpha_2(0)}\left\|f_{2j}\chi_j\right\|_{L^{p_2(\cdot)}(w_2)}\right.\right.\\
&\quad \left.\left.\times 2^{-j(n\delta_{21}+\alpha_2(0))}\right)^{q_2(0)}\right)^{1/q_2(0)}\\
&\lesssim \|b_2\|_* \sup_{L\le 0, L\in\mathbb{Z}}\sup_{j\le 0} 2^{-j\lambda_2}2^{j\alpha_2(0)}\left\|f_{2j}\chi_j\right\|_{L^{p_2(\cdot)}(w_2)}\\
&\quad \times 2^{-L\lambda_2}\left(\sum_{k=-\infty}^{L}\left(2^{k(n\delta_{21}+\alpha_2(0))}\sum_{j=L+1}^{0}2^{-j(n\delta_{21}+\alpha_2(0)-\lambda_2)}\right)^{q_2(0)}\right)^{1/q_2(0)}\\
&\lesssim \|b_2\|_*\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)} \sup_{L\le 0, L\in\mathbb{Z}} 2^{L(-n\delta_{21}-\alpha_2(0))}\\
&\quad \cdot\left(\sum_{k=-\infty}^{L}2^{k(n\delta_{21}+\alpha_2(0))q_2(0)}\right)^{1/q_2(0)}\\
&\lesssim \|b_2\|_*\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)}.
\end{aligned} \tag{66}$$

We consider I_3. Since $n\delta_{21}+\alpha_{2\infty}-\lambda_2>0$ and $n\delta_{21}+\alpha_2(0)-\lambda_2>0$, we obtain

$$\begin{aligned}
I_3 &\lesssim \|b_2\|_* \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_2}\left(\sum_{k=-\infty}^{L}\left(2^{k(n\delta_{21}+\alpha_2(0))}\right.\right.\\
&\quad \left.\left.\times\sum_{j=1}^{\infty}2^{j\alpha_{2\infty}}\left\|f_{2j}\chi_j\right\|_{L^{p_2(\cdot)}(w_2)}2^{-j(n\delta_{21}+\alpha_{2\infty})}\right)^{q_2(0)}\right)^{1/q_2(0)}\\
&\lesssim \|b_2\|_* \sup_{L\le 0, L\in\mathbb{Z}}\sup_{j\ge 1} 2^{-j\lambda_2}2^{j\alpha_{2\infty}}\left\|f_{2j}\chi_j\right\|_{L^{p_2(\cdot)}(w_2)}\\
&\quad \times 2^{-L\lambda_2}\left(\sum_{k=-\infty}^{L}\left(2^{k(n\delta_{21}+\alpha_2(0))}\sum_{j=1}^{\infty}2^{-j(n\delta_{21}+\alpha_{2\infty}-\lambda_2)}\right)^{q_2(0)}\right)^{1/q_2(0)}\\
&\lesssim \|b_2\|_*\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)} \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_2}\left(\sum_{k=-\infty}^{L}2^{k(n\delta_{21}+\alpha_2(0))q_2(0)}\right)^{1/q_2(0)}\\
&\lesssim \|b_2\|_*\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)} \sup_{L\le 0, L\in\mathbb{Z}} 2^{L(-\lambda_2+n\delta_{21}+\alpha_2(0))}\\
&\lesssim \|b_2\|_*\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)}.
\end{aligned} \tag{67}$$

Thus, we have

$$E_3 \lesssim \|b_1\|_*\|b_2\|_*\|f_1\|_{M\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot),\lambda_1}(w_1)}\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)}. \tag{68}$$

To estimate E_5, according to Lemma 13 and Hölder's inequality, we have

$$\begin{aligned}
E_5 &\lesssim \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=-\infty}^{L}2^{k\alpha(0)q(0)}\left\|\sum_{l=k-1}^{k+1}\sum_{j=k-1}^{k+1}[b_1,b_2,T]\left(f_{1l},f_{2j}\right)\chi_k\right\|^{q(0)}_{L^{p(\cdot)}(w)}\right)^{1/q(0)}\\
&\lesssim \|b_1\|_*\|b_2\|_* \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=-\infty}^{L}2^{k\alpha(0)q(0)}\left(\|f_1\|_{L^{p_1(\cdot)}(w_1)}\|f_2\|_{L^{p_2(\cdot)}(w_2)}\right)^{q(0)}\right)^{1/q(0)}\\
&\lesssim \|b_1\|_*\|b_2\|_* \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_1}\left(\sum_{k=-\infty}^{L}2^{k\alpha_1(0)q_1(0)}\|f_1\|^{q_1(0)}_{L^{p_1(\cdot)}(w_1)}\right)^{1/q_1(0)}\\
&\quad \times 2^{-L\lambda_2}\left(\sum_{k=-\infty}^{L}2^{k\alpha_2(0)q_2(0)}\|f_2\|^{q_2(0)}_{L^{p_2(\cdot)}(w_2)}\right)^{1/q_2(0)}\\
&\lesssim \|b_1\|_*\|b_2\|_*\|f_1\|_{M\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot),\lambda_1}(w_1)}\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)}.
\end{aligned} \tag{69}$$

To estimate E_6, due to $k-1\le l\le k+1$ and $j\ge k+2$, then we have

$$|x-y_1|>2^{k-2}, |x-y_2|>2^{j-2}, x\in D_k, y_1\in D_l, y_2\in D_j. \tag{70}$$

Thus, $\forall x\in D_k, k-1\le l\le k+1$ and $j\ge k+2$, we obtain that

$$\begin{aligned}
&\left|[b_1,b_2,T]\left(f_{1l},f_{2j}\right)(x)\right|\\
&\quad\lesssim \int_{\mathbb{R}^{2n}}\prod_{i=1}^{2}|b_i(x)-b_i(y_i)||K(x,y_1,y_2)||f_{1l}(y_1)|\left|f_{2j}(y_2)\right|dy_1dy_2\\
&\quad\lesssim 2^{-kn}2^{-jn}\int_{\mathbb{R}^{2n}}\prod_{i=1}^{2}|b_i(x)-b_i(y_i)||f_{1l}(y_1)|\left|f_{2j}(y_2)\right|dy_1dy_2.
\end{aligned} \tag{71}$$

So, according to Hölder's inequality, we acquire that

$$\begin{aligned}
&\left\|\sum_{l=k-1}^{k+1}\sum_{j=k+2}^{\infty}[b_1,b_2,T]\left(f_{1l},f_{2j}\right)\chi_k\right\|_{L^{p(\cdot)}(w)}\\
&\quad\lesssim \left\|\sum_{l=k-1}^{k+1}2^{-kn}\int_{\mathbb{R}^n}|b_1(x)-b_1(y_1)||f_{1l}(y_1)|dy_1\right.\\
&\qquad \left.\times\sum_{j=k+2}^{\infty}2^{-jn}\int_{\mathbb{R}^n}|b_2(x)-b_2(y_2)|\left|f_{2j}(y_2)\right|dy_2\chi_k\right\|_{L^{p(\cdot)}(w)}\\
&\quad\lesssim \left\|\sum_{l=k-1}^{k+1}2^{-kn}\int_{\mathbb{R}^n}|b_1(x)-b_1(y_1)||f_{1l}(y_1)|dy_1\chi_k\right\|_{L^{p_1(\cdot)}(w_1)}\\
&\qquad \times\left\|\sum_{j=k+2}^{\infty}2^{-jn}\int_{\mathbb{R}^n}|b_2(x)-b_2(y_2)|\left|f_{2j}(y_2)\right|dy_2\chi_k\right\|_{L^{p_2(\cdot)}(w_2)}.
\end{aligned} \tag{72}$$

Since $1/q(0)=1/q_1(0)+1/q_2(0), \lambda=\lambda_1+\lambda_2$, according to Hölder's inequality, we have

$$
\begin{aligned}
E_6 \lesssim & \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda} \left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)} \left\| \sum_{l=k-1}^{k+1} 2^{-kn} \int_{\mathbb{R}^n} |b_1(x)-b_1(y_1)||f_{1l}(y_1)|dy_1\chi_k \right\|_{L^{p_1(\cdot)}(w_1)}^{q(0)} \right. \\
& \left. \times \left\| \sum_{j=k+2}^{\infty} 2^{-jn} \int_{\mathbb{R}^n} |b_2(x)-b_2(y_2)|\left|f_{2j}(y_2)\right| dy_2\chi_k \right\|_{L^{p_2(\cdot)}(w_2)}^{q(0)} \right)^{1/q(0)} \\
\lesssim & \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_1} \left(\sum_{k=-\infty}^{L} 2^{k\alpha_1(0)q_1(0)} \right. \\
& \left. \times \left\| \sum_{l=k-1}^{k+1} 2^{-kn} \int_{\mathbb{R}^n} |b_1(x)-b_1(y_1)||f_{1l}(y_1)|dy_1\chi_k \right\|_{L^{p_1(\cdot)}(w_1)}^{q_1(0)} \right)^{1/q_1(0)} \\
& \times \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_2} \left(\sum_{k=-\infty}^{L} 2^{k\alpha_2(0)q_2(0)} \right. \\
& \left. \times \left\| \sum_{j=k+2}^{\infty} 2^{-jn} \int_{\mathbb{R}^n} \left|f_{2j}(y_2)\right| dy_2\chi_k \right\|_{L^{p_2(\cdot)}(w_2)}^{q_2(0)} \right)^{1/q_2(0)} \\
\coloneqq & E_{6,1} \times E_{6,2}.
\end{aligned} \tag{73}
$$

By the interchange of f_1 and f_2, we acquire that the estimate of $E_{6,1}$ is analogical to the estimate of $E_{2,2}$ and $E_{6,2} = E_{3,2}$.

To estimate E_9, since $l,j \ge k+2$, then. we get

$$
|x - y_i| > 2^{k-2}, x \in D_k, y_1 \in D_l, y_2 \in D_j. \tag{74}
$$

Therefore, $\forall x \in D_k, l,j \ge k+2$, we have

$$
\begin{aligned}
& \left|[b_1, b_2, T]\left(f_{1l}, f_{2j}\right)(x)\right| \\
& \lesssim \int_{\mathbb{R}^{2n}} \prod_{i=1}^{2} |b_i(x)-b_i(y_i)||K(x,y_1,y_2)||f_{1l}(y_1)|\left|f_{2j}(y_2)\right| dy_1 dy_2 \\
& \lesssim 2^{-\ln} 2^{-jn} \int_{\mathbb{R}^{2n}} \prod_{i=1}^{2} |b_i(x)-b_i(y_i)||f_{1l}(y_1)|\left|f_{2j}(y_2)\right| dy_1 dy_2.
\end{aligned} \tag{75}
$$

So, according to Hölder's inequality, we obtain that

$$
\begin{aligned}
& \left\| \sum_{l=k+2}^{\infty} \sum_{j=k+2}^{\infty} [b_1, b_2, T]\left(f_{1l}, f_{2j}\right)\chi_k \right\|_{L^{p(\cdot)}(w)} \\
& \lesssim \left\| \sum_{l=k+2}^{\infty} 2^{-\ln} \int_{\mathbb{R}^n} |b_1(x)-b_1(y_1)||f_{1l}(y_1)|dy_1 \right. \\
& \qquad \left. \times \sum_{j=k+2}^{\infty} 2^{-jn} \int_{\mathbb{R}^n} |b_2(x)-b_2(y_2)|\left|f_{2j}(y_2)\right| dy_2\chi_k \right\|_{L^{p(\cdot)}(w)} \\
& \lesssim \left\| \sum_{l=k+2}^{\infty} 2^{-\ln} \int_{\mathbb{R}^n} |b_1(x)-b_1(y_1)||f_{1l}(y_1)|dy_1\chi_k \right\|_{L^{p_1(\cdot)}(w_1)} \\
& \qquad \times \left\| \sum_{j=k+2}^{\infty} 2^{-jn} \int_{\mathbb{R}^n} |b_2(x)-b_2(y_2)|\left|f_{2j}(y_2)\right| dy_2\chi_k \right\|_{L^{p_2(\cdot)}(w_2)}.
\end{aligned} \tag{76}
$$

Since $1/q(0) = 1/q_1(0) + 1/q_2(0)$, $\lambda = \lambda_1 + \lambda_2$, according to Hölder's inequality, we have

$$
\begin{aligned}
E_9 \lesssim & \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda} \left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)} \left\| \sum_{l=k+2}^{\infty} 2^{-\ln} \int_{\mathbb{R}^n} |b_1(x)-b_1(y_1)||f_{1l}(y_1)|dy_1\chi_k \right\|_{L^{p_1(\cdot)}(w_1)}^{q(0)} \right. \\
& \left. \times \left\| \sum_{j=k+2}^{\infty} 2^{-jn} \int_{\mathbb{R}^n} |b_2(x)-b_2(y_2)|\left|f_{2j}(y_2)\right| dy_2\chi_k \right\|_{L^{p_2(\cdot)}(w_2)}^{q(0)} \right)^{1/q(0)} \\
\lesssim & \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_1} \left(\sum_{k=-\infty}^{L} 2^{k\alpha_1(0)q_1(0)} \right. \\
& \left. \times \left\| \sum_{l=k+2}^{\infty} 2^{-\ln} \int_{\mathbb{R}^n} |b_1(x)-b_1(y_1)||f_{1l}(y_1)|dy_1\chi_k \right\|_{L^{p_1(\cdot)}(w_1)}^{q_1(0)} \right)^{1/q_1(0)} \\
& \times \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_2} \left(\sum_{k=-\infty}^{L} 2^{k\alpha_2(0)q_2(0)} \right. \\
& \left. \times \left\| \sum_{j=k+2}^{\infty} 2^{-jn} \int_{\mathbb{R}^n} |b_2(x)-b_2(y_2)|\left|f_{2j}(y_2)\right| dy_2\chi_k \right\|_{L^{p_2(\cdot)}(w_2)}^{q_2(0)} \right)^{1/q_2(0)} \\
\coloneqq & E_{9,1} \times E_{9,2}.
\end{aligned} \tag{77}
$$

clearly, the estimate $E_{9,i}$ is analogical to the estimated $E_{3,2}$ for $i = 1, 2$.

Combining all the estimates of E_i, $i = 1, 2, \cdots, 9$, we obtain that

$$
E \lesssim \|b_1\|_* \|b_2\|_* \|f_1\|_{M\dot{K}^{\alpha_1(\cdot), q_1(\cdot)}_{p_1(\cdot), \lambda_1}(w_1)} \|f_2\|_{M\dot{K}^{\alpha_2(\cdot), q_2(\cdot)}_{p_2(\cdot), \lambda_2}(w_2)}. \tag{78}
$$

In order to continue, we need further preparation. If $l < 0$, since Proposition 6, we obtain that

$$
\begin{aligned}
& \|f_{il}\chi_l\|_{L^{p_i(\cdot)}(w_i)} \\
& \quad = 2^{-l\alpha_i(0)} \left(2^{l\alpha_i(0)q_i(0)} \|f_{il}\chi_l\|_{L^{p_i(\cdot)}(w_i)}^{q_i(0)} \right)^{1/q_i(0)} \\
& \quad \lesssim 2^{-l\alpha_i(0)} \left(\sum_{t=-\infty}^{l} 2^{t\alpha_i(0)q_i(0)} \|f_{it}\chi_t\|_{L^{p_i(\cdot)}(w_i)}^{q_i(0)} \right)^{1/q_i(0)} \\
& \quad \lesssim 2^{l(\lambda_i - \alpha_i(0))} \left(2^{-l\lambda_i} \left(\sum_{t=-\infty}^{l} \left\| 2^{t\alpha_i(0)} f_{it}\chi_t \right\|_{L^{p_i(\cdot)}(w_i)}^{q_i(0)} \right) \right)^{1/q_i(0)} \\
& \quad \lesssim 2^{l(\lambda_i - \alpha_i(0))} \|f_i\|_{M\dot{K}^{\alpha_i(\cdot), q_i(\cdot)}_{p_i(\cdot), \lambda_i}(w_i)}.
\end{aligned} \tag{79}
$$

conclusively, we estimate G. according to the interchange of f_1 and f_2, we see that the estimates of G_2, G_3, and G_6 are similar to G_4, G_7, and G_8, respectively. Thus, it was only necessary to estimate G_1,G_2,G_3,G_5,G_6, and G_9.

To estimate G_1, due to $l,j \le k-2$, $1/q_{\infty} = 1/q_{1\infty} + 1/q_{2\infty}$, $\lambda = \lambda_1 + \lambda_2$, by (48) and Hölder's inequality, we obtain that

$$\begin{aligned} G_1 &\lesssim 2^{-L\lambda} \left(\sum_{k=0}^{L} 2^{k\alpha_{\infty} q_{\infty}} \left\| \sum_{l=-\infty}^{k-2} 2^{-kn} \int_{\mathbb{R}^n} |b_1(x) - b_1(y_1)| |f_{1l}(y_1)| dy_1 \chi_k \right\|_{L^{p_1(\cdot)}(w_1)}^{q_{\infty}} \right. \\ &\quad \times \left. \left\| \sum_{j=-\infty}^{k-2} 2^{-kn} \int_{\mathbb{R}^n} |b_2(x) - b_2(y_2)| \left| f_{2j}(y_2) \right| dy_2 \chi_k \right\|_{L^{p_2(\cdot)}(w_2)}^{q_{\infty}} \right)^{1/q_{\infty}} \\ &\lesssim 2^{-L\lambda_1} \left(\sum_{k=0}^{L} 2^{k\alpha_{1\infty} q_{1\infty}} \left\| \sum_{l=-\infty}^{k-2} 2^{-kn} \int_{\mathbb{R}^n} |b_1(x) - b_1(y_1)| |f_{1l}(y_1)| dy_1 \chi_k \right\|_{L^{p_1(\cdot)}(w_1)}^{q_{1\infty}} \right)^{1/q_{1\infty}} \\ &\quad \times 2^{-L\lambda_2} \left(\sum_{k=0}^{L} 2^{k\alpha_{2\infty} q_{2\infty}} \left\| \sum_{j=-\infty}^{k-2} 2^{-kn} \int_{\mathbb{R}^n} |b_2(x) - b_2(y_2)| \left| f_{2j}(y_2) \right| dy_2 \chi_k \right\|_{L^{p_2(\cdot)}(w_2)}^{q_{2\infty}} \right)^{1/q_{2\infty}} \\ &:= G_{1,1} \times G_{1,2}. \end{aligned} \tag{80}$$

where

$$G_{1,i} := 2^{-L\lambda_i} \left(\sum_{k=0}^{L} 2^{k\alpha_{i\infty} q_{i\infty}} \left\| \sum_{l=-\infty}^{k-2} 2^{-kn} \int_{\mathbb{R}^n} |b_i(x) - b_i(y_i)| |f_{il}(y_i)| dy_i \chi_k \right\|_{L^{p_i(\cdot)}(w_i)}^{q_{i\infty}} \right)^{1/q_{i\infty}}. \tag{81}$$

We consider $G_{1,i}$. By (41), we obtain that

$$\begin{aligned} G_{1,i} &\lesssim \|b_i\|_* 2^{-L\lambda_i} \left\{ \sum_{k=0}^{L} 2^{k\alpha_{i\infty} q_{i\infty}} \left(\sum_{l=-\infty}^{-1} \|f_{il}\chi_l\|_{L^{p_i(\cdot)}(w_i)} 2^{(l-k)n\delta_{i2}} \right. \right. \\ &\quad + \left. \left. \sum_{l=0}^{k} \|f_{il}\chi_l\|_{L^{p_i(\cdot)}(w_i)} 2^{(l-k)n\delta_{i2}} \right)^{q_{i\infty}} \right\}^{1/q_{i\infty}} \\ &\lesssim \|b_i\|_* 2^{-L\lambda_i} \left\{ \sum_{k=0}^{L} 2^{k\alpha_{i\infty} q_{i\infty}} \left(\sum_{l=-\infty}^{-1} \|f_{il}\chi_l\|_{L^{p_i(\cdot)}(w_i)} 2^{(l-k)n\delta_{i2}} \right)^{q_{i\infty}} \right\}^{1/q_{i\infty}} \\ &\quad + \|b_i\|_* 2^{-L\lambda_i} \left\{ \sum_{k=0}^{L} 2^{k\alpha_{i\infty} q_{i\infty}} \left(\sum_{l=0}^{k} \|f_{il}\chi_l\|_{L^{p_i(\cdot)}(w_i)} 2^{(l-k)n\delta_{i2}} \right)^{q_{i\infty}} \right\}^{1/q_{i\infty}} \\ &:= I_4 + I_5. \end{aligned} \tag{82}$$

If $q_{i\infty} \ge 1$, since $n\delta_{i2} - \alpha_{i\infty} > 0$ and $n\delta_{i2} - \alpha_i(0) > 0$, then by the Minkowski inequality and (79), we have

$$\begin{aligned} I_4 &= \|b_i\|_* 2^{-L\lambda_i} \left\{ \sum_{k=0}^{L} 2^{k\alpha_{i\infty} q_{i\infty}} \left(\sum_{l=-\infty}^{-1} \|f_{il}\chi_l\|_{L^{p_i(\cdot)}(w_i)} 2^{(l-k)n\delta_{i2}} \right)^{q_{i\infty}} \right\}^{1/q_{i\infty}} \\ &\lesssim \|b_i\|_* 2^{-L\lambda_i} \sum_{l=-\infty}^{-1} \|f_{il}\chi_l\|_{L^{p_i(\cdot)}(w_i)} \left\{ \sum_{k=0}^{L} \left(2^{k\alpha_{i\infty}} 2^{(l-k)n\delta_{i2}} \right)^{q_{i\infty}} \right\}^{1/q_{i\infty}} \\ &\lesssim \|b_i\|_* 2^{-L\lambda_i} \sum_{l=-\infty}^{-1} 2^{\ln \delta_{i2}} \|f_{il}\chi_l\|_{L^{p_i(\cdot)}(w_i)} \left\{ \sum_{k=0}^{L} 2^{-k(n\delta_{i2} - \alpha_{1\infty}) q_{i\infty}} \right\}^{1/q_{i\infty}} \\ &\lesssim \|b_i\|_* \|f_i\|_{M\dot{K}_{p_i(\cdot),\lambda_i}^{\alpha_i(\cdot),q_i(\cdot)}(w_i)} 2^{-L\lambda_i} \sum_{l=-\infty}^{-1} 2^{l(n\delta_{i2} + \lambda_i - \alpha_i(0))} \\ &\lesssim \|b_i\|_* \|f_i\|_{M\dot{K}_{p_i(\cdot),\lambda_i}^{\alpha_i(\cdot),q_i(\cdot)}(w_i)}, \end{aligned} \tag{83}$$

If $q_{i\infty} < 1$, since $n\delta_{i2} - \alpha_{i\infty} > 0$ and $n\delta_{i2} - \alpha_i(0) > 0$, then by (79), we have

$$\begin{aligned} I_4 &\lesssim \|b_i\|_* 2^{-L\lambda_i} \left(\sum_{k=0}^{L} 2^{k\alpha_{i\infty} q_{i\infty}} \sum_{l=-\infty}^{-1} \|f_{il}\chi_l\|_{L^{p_i(\cdot)}(w_1)}^{q_{i\infty}} 2^{(l-k)n\delta_{i2} q_{i\infty}} \right)^{1/q_{i\infty}} \\ &= \|b_i\|_* 2^{-L\lambda_i} \left(\sum_{l=-\infty}^{-1} \|f_{il}\chi_l\|_{L^{p_i(\cdot)}(w_i)}^{q_{i\infty}} 2^{\ln \delta_{i2} q_{i\infty}} \sum_{k=0}^{L} 2^{k\alpha_{i\infty} q_{i\infty}} 2^{-kn\delta_{i2} q_{1\infty}} \right)^{1/q_{i\infty}} \\ &= \|b_i\|_* 2^{-L\lambda_i} \left(\sum_{l=-\infty}^{-1} \|f_{il}\chi_l\|_{L^{p_i(\cdot)}(w_i)}^{q_{i\infty}} 2^{\ln \delta_{i2} q_{i\infty}} \sum_{k=0}^{L} 2^{-k(n\delta_{i2} - \alpha_{i\infty}) q_{i\infty}} \right)^{1/q_{i\infty}} \\ &\lesssim \|b_i\|_* 2^{-L\lambda_i} \left(\sum_{l=-\infty}^{-1} \|f_{il}\chi_l\|_{L^{p_i(\cdot)}(w_i)}^{q_{i\infty}} 2^{\ln \delta_{i2} q_{i\infty}} \right)^{1/q_{i\infty}} \\ &\lesssim \|b_i\|_* \|f_i\|_{M\dot{K}_{p_i(\cdot),\lambda_i}^{\alpha_i(\cdot),q_i(\cdot)}} 2^{-L\lambda_i} \left(\sum_{l=-\infty}^{-1} 2^{l(n\delta_{i2} + \lambda_i - \alpha_i(0)) q_{i\infty}} \right)^{1/q_{i\infty}} \\ &\lesssim \|b_i\|_* \|f_i\|_{M\dot{K}_{p_i(\cdot),\lambda_i}^{\alpha_i(\cdot),q_i(\cdot)}}, \end{aligned} \tag{84}$$

We consider I_5. Since $n\delta_{i2} - \alpha_{i\infty} > 0$, then by Lemma 11, we have

$$\begin{aligned} I_5 &= \|b_i\|_* 2^{-L\lambda_i} \left\{ \sum_{k=0}^{L} 2^{k\alpha_{i\infty} q_{i\infty}} \left(\sum_{l=0}^{k} \|f_{il}\chi_l\|_{L^{p_i(\cdot)}(w_i)} 2^{(l-k)n\delta_{i2}} \right)^{q_{i\infty}} \right\}^{1/q_{i\infty}} \\ &= \|b_i\|_* 2^{-L\lambda_i} \left\{ \sum_{k=0}^{L} \left(\sum_{l=0}^{k} 2^{l\alpha_{i\infty}} \|f_{il}\chi_l\|_{L^{p_i(\cdot)}(w_i)} 2^{(l-k)(n\delta_{i2} - \alpha_{i\infty})} \right)^{q_{i\infty}} \right\}^{1/q_{i\infty}} \\ &\lesssim \|b_i\|_* 2^{-L\lambda_i} \left(\sum_{l=0}^{k} 2^{l\alpha_{i\infty} q_{i\infty}} \|f_{il}\chi_l\|_{L^{p_i(\cdot)}(w_i)}^{q_{i\infty}} \right)^{1/q_{i\infty}} \\ &\lesssim \|b_i\|_* \|f_i\|_{M\dot{K}_{p_i(\cdot),\lambda_i}^{\alpha_i(\cdot),q_i(\cdot)}(w_i)}, \end{aligned} \tag{85}$$

where we wrote $2^{-|k-l|(n\delta_{i2} - \alpha_{i\infty})} \lesssim 2^{-|k-l|\alpha_i}$ for $\alpha_i = n\delta_{i2} - \alpha_{i\infty}$.

Thus, we get

$$G_1 \lesssim \|b_i\|_* \|b_2\|_* \|f_1\|_{M\dot{K}_{p_1(\cdot),\lambda_1}^{\alpha_1(\cdot),q_1(\cdot)}(w_1)} \|f_2\|_{M\dot{K}_{p_2(\cdot),\lambda_2}^{\alpha_2(\cdot),q_2(\cdot)}(w_2)}. \tag{86}$$

To estimate G_2, due to $l \le k-2$, $k-1 \le j \le k+1$, $1/q_{\infty} = 1/q_{1\infty} + 1/q_{2\infty}$, $\lambda = \lambda_1 + \lambda_2$, by (54) and Hölder's inequality, we obtain that

$$\begin{aligned} G_2 &\lesssim 2^{-L\lambda} \left(\sum_{k=0}^{L} 2^{k\alpha_{\infty} q_{\infty}} \left\| \sum_{l=-\infty}^{k-2} 2^{-kn} \int_{\mathbb{R}^n} |b_1(x) - b_1(y_1)| |f_{1l}(y_1)| dy_1 \chi_k \right\|_{L^{p_1(\cdot)}(w_1)}^{q_{\infty}} \right. \\ &\quad \times \left. \left\| \sum_{j=k-1}^{k+1} 2^{-kn} \int_{\mathbb{R}^n} |b_2(x) - b_2(y_2)| \left| f_{2j}(y_2) \right| dy_2 \chi_k \right\|_{L^{p_2(\cdot)}(w_2)}^{q_{\infty}} \right)^{1/q_{\infty}} \\ &\lesssim 2^{-L\lambda_1} \left(\sum_{k=0}^{L} 2^{k\alpha_{1\infty} q_{1\infty}} \left\| \sum_{l=-\infty}^{k-2} 2^{-kn} \int_{\mathbb{R}^n} |b_1(x) - b_1(y_1)| |f_{1l}(y_1)| dy_1 \chi_k \right\|_{L^{p_1(\cdot)}(w_1)}^{q_{1\infty}} \right)^{1/q_{1\infty}} \\ &\quad \times 2^{-L\lambda_2} \left(\sum_{k=0}^{L} 2^{k\alpha_{2\infty} q_{2\infty}} \left\| \sum_{j=k-1}^{k+1} 2^{-kn} \int_{\mathbb{R}^n} |b_2(x) - b_2(y_2)| \left| f_{2j}(y_2) \right| dy_2 \chi_k \right\|_{L^{p_2(\cdot)}(w_2)}^{q_{2\infty}} \right)^{1/q_{2\infty}} \\ &:= G_{2,1} \times G_{2,2}. \end{aligned} \tag{87}$$

It is obviously that

$$G_{2,1} = G_{1,1} \lesssim \|b_1\|_* \|f_1\|_{M\dot{K}_{p_1(\cdot),\lambda_1}^{\alpha_1(\cdot),q_1(\cdot)}(w_1)}. \tag{88}$$

Now, we estimate $G_{2,2}$. Combing (41), (42), and (44), we have

$$G_{2,2} \lesssim \|b_2\|_* 2^{-L\lambda_2} \left(\sum_{k=0}^{L} 2^{k\alpha_{2\infty} q_{2\infty}} \sum_{j=k-1}^{k+1} 2^{(j-k)nq_{2\infty}} \left\| f_{2j}\chi_j \right\|_{L^{p_2(\cdot)}(w_2)}^{q_{2\infty}} \right)^{1/q_{2\infty}} \lesssim \|b_2\|_* 2^{-L\lambda_2} \left(\sum_{k=0}^{L} 2^{k\alpha_{2\infty} q_{2\infty}} \|f_2\chi_k\|_{L^{p_2(\cdot)}(w_2)}^{q_{2\infty}} \right)^{1/q_{2\infty}} \lesssim \|b_2\|_* \|f_2\|_{M\dot{K}_{p_2(\cdot),\lambda_2}^{\alpha_2(\cdot),q_2(\cdot)}(w_2)}, \tag{89}$$

where we used $2^{-n\delta_{22}} < 1$ and $2^{(j-k)n(1-\delta_{12})} < 2^{(j-k)n}$ for (41) and (44), respectively.

Thus, we acquire that

$$G_2 \lesssim \|b_1\|_* \|b_2\|_* \|f_1\|_{M\dot{K}_{p_1(\cdot),\lambda_1}^{\alpha_1(\cdot),q_1(\cdot)}(w_1)} \|f_2\|_{M\dot{K}_{p_2(\cdot),\lambda_2}^{\alpha_2(\cdot),q_2(\cdot)}(w_2)}. \tag{90}$$

To estimate G_3, since $l \le k-2, j \ge k+2$, $1/q_\infty = 1/q_{1\infty} + 1/q_{2\infty}, \lambda = \lambda_1 + \lambda_2$, using (61) and Hölder's inequality, we obtain that

$$\begin{aligned} G_3 &\lesssim 2^{-L\lambda} \left(\sum_{k=0}^{L} 2^{k\alpha_\infty q_\infty} \left\| \sum_{l=-\infty}^{k-2} 2^{-kn} \int_{\mathbb{R}^n} |b_1(x) - b_1(y_1)| |f_{1l}(y_1)| dy_1 \chi_k \right\|_{L^{p_1(\cdot)}(w_1)}^{q_\infty} \right. \\ &\quad \left. \times \left\| \sum_{j=k+2}^{\infty} 2^{-jn} \int_{\mathbb{R}^n} |b_2(x) - b_2(y_2)| \left| f_{2j}(y_2) \right| dy_2 \chi_k \right\|_{L^{p_2(\cdot)}(w_2)}^{q_\infty} \right)^{1/q_\infty} \\ &\lesssim 2^{-L\lambda_1} \left(\sum_{k=0}^{L} 2^{k\alpha_{1\infty} q_{1\infty}} \left\| \sum_{l=-\infty}^{k-2} 2^{-kn} \int_{\mathbb{R}^n} |b_1(x) - b_1(y_1)| |f_{1l}(y_1)| dy_1 \chi_k \right\|_{L^{p_1(\cdot)}(w_1)}^{q_{1\infty}} \right)^{1/q_{1\infty}} \\ &\quad \times 2^{-L\lambda_2} \left(\sum_{k=0}^{L} 2^{k\alpha_{2\infty} q_{2\infty}} \left\| \sum_{j=k+2}^{\infty} 2^{-jn} \int_{\mathbb{R}^n} |b_2(x) - b_2(y_2)| \left| f_{2j}(y_2) \right| dy_2 \chi_k \right\|_{L^{p_2(\cdot)}(w_2)}^{q_{2\infty}} \right)^{1/q_{2\infty}} \\ &\coloneqq G_{3,1} \times G_{3,2}. \end{aligned} \tag{91}$$

It is obviously that

$$G_{3,1} = G_{1,1} \lesssim \|b_1\|_* \|f_1\|_{M\dot{K}_{p_1(\cdot),\lambda_1}^{\alpha_1(\cdot),q_1(\cdot)}(w_1)}. \tag{92}$$

Since $n\delta_{12} + \alpha_{2\infty} > 0$, by (44) and Lemma 11, we obtain that

$$\begin{aligned} G_{3,2} &\lesssim \|b_2\|_* 2^{-L\lambda_2} \left(\sum_{k=0}^{L} 2^{k\alpha_{2\infty} q_{2\infty}} \left(\sum_{j=k+2}^{\infty} 2^{(k-j)n\delta_{21}} \left\| f_{2j}\chi_j \right\|_{L^{p_2(\cdot)}(w_2)} \right)^{q_{2\infty}} \right)^{1/q_{2\infty}} \\ &\lesssim \|b_2\|_* 2^{-L\lambda_2} \left(\sum_{k=0}^{L} \left(\sum_{j=k+2}^{L+2} 2^{j\alpha_{2\infty}} \left\| f_{2j}\chi_j \right\|_{L^{p_2(\cdot)}(w_2)} 2^{(k-j)(n\delta_{21}+\alpha_{2\infty})} \right)^{q_{2\infty}} \right)^{1/q_{2\infty}} \\ &\quad + \|b_2\|_* 2^{-L\lambda_2} \left(\sum_{k=0}^{L} \left(2^{k\alpha_{2\infty}} \sum_{j=L+3}^{\infty} \left\| f_{2j}\chi_j \right\|_{L^{p_2(\cdot)}(w_2)} 2^{(k-j)n\delta_{21}} \right)^{q_{2\infty}} \right)^{1/q_{2\infty}} \\ &\coloneqq I_6 + I_7. \end{aligned} \tag{93}$$

We consider I_6. By Lemma 11, we obtain that

$$\begin{aligned} I_6 &\lesssim \|b_2\|_* 2^{-L\lambda_2} \left(\sum_{k=0}^{L} \left(\sum_{j=k+2}^{L+2} 2^{j\alpha_{2\infty}} \left\| f_{2j}\chi_j \right\|_{L^{p_2(\cdot)}(w_2)} 2^{(k-j)(n\delta_{21}+\alpha_{2\infty})} \right)^{q_{2\infty}} \right)^{1/q_{2\infty}} \\ &\lesssim \|b_2\|_* 2^{-L\lambda_2} \left(\sum_{j=0}^{L+2} 2^{j\alpha_{2\infty} q_{2\infty}} \left\| f_{2j}\chi_j \right\|_{L^{p_2(\cdot)}(w_2)}^{q_{2\infty}} \right)^{1/q_{2\infty}} \\ &\lesssim \|b_2\|_* \|f_2\|_{M\dot{K}_{p_2(\cdot),\lambda_2}^{\alpha_2(\cdot),q_2(\cdot)}(w_2)}, \end{aligned} \tag{94}$$

where wrote $2^{-|k-j|(n\delta_{21}+\alpha_{2\infty})} = 2^{-|k-j|\zeta_2}$ for $\zeta_2 = n\delta_{21} + \alpha_{2\infty} > 0$.

We consider I_7. Since $n\delta_{21} + \alpha_{2\infty} - \lambda_2 > 0$, we have

$$\begin{aligned} I_7 &\lesssim \|b_2\|_* \\ &\quad \cdot 2^{-L\lambda_2} \left(\sum_{k=0}^{L} \left(2^{k(n\delta_{21}+\alpha_{2\infty})} \sum_{j=L+3}^{\infty} 2^{j\alpha_{2\infty}} \left\| f_2\chi_j \right\|_{L^{p_2\binom{\ddot{A}}{n}}(w_2)} 2^{-j(n\delta_{21}+\alpha_{2\infty})} \right)^{q_{2\infty}} \right)^{1/q_{2\infty}} \\ &\lesssim \|b_2\|_* \sup_{j\ge1} 2^{-j\lambda_2} 2^{j\alpha_{2\infty}} \left\| f_{2j}\chi_j \right\|_{L^{p_2\binom{\ddot{A}}{n}}(w_2)} \\ &\quad \times 2^{-L\lambda_2} \left(\sum_{k=0}^{L} \left(2^{k(n\delta_{21}+\alpha_{2\infty})} \sum_{j=L+3}^{\infty} 2^{-j(n\delta_{21}+\alpha_{2\infty}-\lambda_2)} \right)^{q_{2\infty}} \right)^{1/q_{2\infty}} \\ &\lesssim \|b_2\|_* \|f_2\|_{MK_{p_2\binom{\ddot{A}}{n},\lambda_2}^{\alpha_2\binom{\ddot{A}}{n},q_2\binom{\ddot{A}}{n}}(w_2)} \\ &\quad \cdot 2^{-L\lambda_2} \left(\sum_{k=0}^{L} 2^{k(n\delta_{21}+\alpha_{2\infty})q_{2\infty}} 2^{-L(n\delta_{21}+\alpha_{2\infty}-\lambda_2)} \right)^{1/q_{2\infty}} \\ &\lesssim \|b_2\|_* \|f_2\|_{MK_{p_2\binom{\ddot{A}}{n},\lambda_2}^{\alpha_2\binom{\ddot{A}}{n},q_2\binom{\ddot{A}}{n}}(w_2)} \\ &\quad \cdot 2^{-L\lambda_2} 2^{(n\delta_{21}+\alpha_{2\infty})L} 2^{-L(n\delta_{21}+\alpha_{2\infty}-\lambda_2)} \\ &\lesssim \|b_2\|_* \|f_2\|_{MK_{p_2\binom{\ddot{A}}{n},\lambda_2}^{\alpha_2\binom{\ddot{A}}{n},q_2\binom{\ddot{A}}{n}}(w_2)} \end{aligned} \tag{95}$$

Thus, we get

$$G_3 \lesssim \|b_1\|_* \|b_2\|_* \|f_1\|_{M\dot{K}_{p_1(\cdot),\lambda_1}^{\alpha_1(\cdot),q_1(\cdot)}(w_1)} \|f_2\|_{M\dot{K}_{p_2(\cdot),\lambda_2}^{\alpha_2(\cdot),q_2(\cdot)}(w_2)}. \tag{96}$$

To estimate G_5, according to Lemma 13 and Hölder's inequality, we have

$$\begin{aligned} G_5 &\lesssim 2^{-L\lambda} \left(\sum_{k=0}^{L} 2^{k\alpha_\infty q_\infty} \left\| \sum_{l=k-1}^{k+1} \sum_{j=k-1}^{k+1} [b_1, b_2, T] \left(f_{1l}, f_{2j} \right) \chi_k \right\|_{L^{p(\cdot)}(w)}^{q_\infty} \right)^{1/q_\infty} \\ &\lesssim 2^{-L\lambda} \left(\sum_{k=0}^{L} 2^{k\alpha_\infty q_\infty} \left(\|b_1\|_* \|b_2\|_* \|f_1\|_{L^{p_1(\cdot)}(w_1)} \|f_2\|_{L^{p_2(\cdot)}(w_2)} \right)^{q_\infty} \right)^{1/q_\infty} \\ &\lesssim \|b_1\|_* 2^{-L\lambda_1} \left(\sum_{k=0}^{L} 2^{k\alpha_{1\infty} q_{1\infty}} \|f_1\|_{L^{p_1(\cdot)}(w_1)}^{q_{1\infty}} \right)^{1/q_{1\infty}} \\ &\quad \times \|b_2\|_* 2^{-L\lambda_2} \left(\sum_{k=0}^{L} 2^{k\alpha_{2\infty} q_{2\infty}} \|f_2\|_{L^{p_2(\cdot)}(w_2)}^{q_{2\infty}} \right)^{1/q_{2\infty}} \\ &\lesssim \|b_1\|_* \|b_2\|_* \|f_1\|_{M\dot{K}_{p_1(\cdot),\lambda_1}^{\alpha_1(\cdot),q_1(\cdot)}(w_1)} \|f_2\|_{M\dot{K}_{p_2(\cdot),\lambda_2}^{\alpha_2(\cdot),q_2(\cdot)}(w_2)}. \end{aligned} \tag{97}$$

To estimate G_6, due to $k-1 \le l \le k+1$ and $j \ge k+2$, $1/q_\infty = 1/q_{1\infty} + 1/q_{2\infty}$,$\lambda = \lambda_1 + \lambda_2$, using (72) and Hölder's inequality, we obtain that

$$\begin{aligned} G_6 &\le 2^{-L\lambda} \left(\sum_{k=0}^{L} 2^{k\alpha_\infty q_\infty} \left\| \sum_{l=k-1}^{k+1} 2^{-kn} \int_{\mathbb{R}^n} |b_1(x) - b_1(y_1)||f_{1l}(y_1)| dy_1 \chi_k \right\|_{L^{p_1(\cdot)}(w_1)}^{q_\infty} \right. \\ &\quad \left. \times \left\| \sum_{j=k+2}^{\infty} 2^{-jn} \int_{\mathbb{R}^n} |b_2(x) - b_2(y_2)| \left| f_{2j}(y_2) \right| dy_2 \chi_k \right\|_{L^{p_2(\cdot)}(w_2)}^{q_\infty} \right)^{1/q_\infty} \\ &\le 2^{-L\lambda_1} \left(\sum_{k=0}^{L} 2^{k\alpha_{1\infty} q_{1\infty}} \left\| \sum_{l=k-1}^{k+1} 2^{-kn} \int_{\mathbb{R}^n} |b_1(x) - b_1(y_1)||f_{1l}(y_1)| dy_1 \chi_k \right\|_{L^{p_1(\cdot)}(w_1)}^{q_{1\infty}} \right)^{1/q_{1\infty}} \\ &\quad \times 2^{-L\lambda_2} \left(\sum_{k=0}^{L} 2^{k\alpha_{2\infty} q_{2\infty}} \left\| \sum_{j=k+2}^{\infty} 2^{-jn} \int_{\mathbb{R}^n} |b_2(x) - b_2(y_2)| \left| f_{2j}(y_2) \right| dy_2 \chi_k \right\|_{L^{p_2(\cdot)}(w_2)}^{q_{2\infty}} \right)^{1/q_{2\infty}} \\ &\coloneqq G_{6,1} \times G_{6,2}. \end{aligned} \tag{98}$$

Since the symmetry of f_1 and f_2, we can know that the estimate $G_{6,1}$ is analogical to the estimated $G_{1,2}$ and $G_{6,2} = G_{3,2}$.

To estimate G_9, due to $l,j \ge k+2$, $1/q_\infty = 1/q_{1\infty} + 1/q_{2\infty}$, $\lambda = \lambda_1 + \lambda_2$, using (76) and Hölder's inequality, we obtain that

$$\begin{aligned} G_9 &\le 2^{-L\lambda} \left(\sum_{k=0}^{L} 2^{k\alpha_\infty q_\infty} \left\| \sum_{l=k+2}^{\infty} 2^{-\ln} \int_{\mathbb{R}^n} |b_1(x) - b_1(y_1)||f_{1l}(y_1)| dy_1 \chi_k \right\|_{L^{p_1(\cdot)}(w_1)}^{q_\infty} \right. \\ &\quad \left. \times \left\| \sum_{j=k+2}^{\infty} 2^{-jn} \int_{\mathbb{R}^n} |b_2(x) - b_2(y_2)| \left| f_{2j}(y_2) \right| dy_2 \chi_k \right\|_{L^{p_2(\cdot)}(w_2)}^{q_\infty} \right)^{1/q_\infty} \\ &\le 2^{-L\lambda_1} \left(\sum_{k=0}^{L} 2^{k\alpha_{1\infty} q_{1\infty}} \left\| \sum_{l=k+2}^{\infty} 2^{-\ln} \int_{\mathbb{R}^n} |b_1(x) - b_1(y_1)||f_{1l}(y_1)| dy_1 \chi_k \right\|_{L^{p_1(\cdot)}(w_1)}^{q_{1\infty}} \right)^{1/q_{1\infty}} \\ &\quad \times 2^{-L\lambda_2} \left(\sum_{k=0}^{L} 2^{k\alpha_{2\infty} q_{2\infty}} \left\| \sum_{j=k+2}^{\infty} 2^{-jn} \int_{\mathbb{R}^n} |b_2(x) - b_2(y_2)| \left| f_{2j}(y_2) \right| dy_2 \chi_k \right\|_{L^{p_2(\cdot)}(w_2)}^{q_{2\infty}} \right)^{1/q_{2\infty}} \\ &\coloneqq G_{9,1} \times G_{9,2}. \end{aligned} \tag{99}$$

clearly, the estimate $G_{9,i}$ is analogical to the estimated $G_{3,2}$ for $i = 1, 2$.

Combining all the estimates ofr G_i together, $i = 1, 2, \cdots, 9$, we obtain that

$$G \lesssim \|f_1\|_{M\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot),\lambda_1}(w_1)} \|f_2\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)}. \tag{100}$$

Combining the above estimates for E, F, and G, the proof of Theorem 8 is completed.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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