Hindawi
Journal of Function Spaces
Volume 2021, Article ID 9948143, 12 pages
https://doi.org/10.1155/2021/9948143

Research Article

On the Porous-Elastic System with Thermoelasticity of Type III and Distributed Delay: Well-Posedness and Stability

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Received 7 March 2021; Revised 19 March 2021; Accepted 25 March 2021; Published 2 April 2021

Academic Editor: Ioan Rasa

The paper deals with a one-dimensional porous-elastic system with thermoelasticity of type III and distributed delay term. This model is dealing with dynamics of engineering structures and nonclassical problems of mathematical physics. We establish the well posedness of the system, and by the energy method combined with Lyapunov functions, we discuss the stability of system for both cases of equal and nonequal speeds of wave propagation.

1. Introduction

Let $\mathscr{H} = (0,1) \times (\tau_1, \tau_2) \times (0,\infty)$, $\tau_1, \tau_2 > 0$. For $(x, s, t) \in \mathscr{H}$, we consider the following porous-elastic system:

$$\begin{cases} \rho_1 u_{tt} = \mu u_{xx} + b\phi_x, \\ \rho_2 \phi_{tt} = \delta\phi_{xx} - bu_x - \xi\phi - \beta\theta_x - \mu_1\phi_t - \displaystyle\int_{\tau_1}^{\tau_2} |\mu_2(s)|\phi_t(x, t-s)ds, \\ \rho_3\theta_{tt} = l\theta_{xx} - \gamma\phi_{ttx} + k\theta_{txx}, \end{cases} \tag{1}$$

with the initial data

$$\begin{aligned} & u(x,0) = u_0(x), u_t(x,0) = u_1(x), \\ & \phi(x,0) = \phi_0(x), \phi_t(x,0) = \phi_1(x), \phi_t(x,-t) = f_0(x,t), \\ & \theta(x,0) = \theta_0(x), \theta_t(x,0) = \theta_1(x), x \in (0,1), \quad t > 0 \end{aligned} \tag{2}$$

and boundary conditions

$$u_x(0,t) = u_x(1,t) = \phi(0,t) = \phi(1,t) = \theta_x(0,t) = \theta_x(1,t) = 0, \quad t \geq 0. \tag{3}$$

Here, ϕ is the volume fraction of the solid elastic material, u is the longitudinal displacement, and θ is the difference in temperatures. The parameters $\rho_1, \rho_2, \rho_3, \mu, b, \delta, \xi, l, \gamma, \beta, k$ are positive constants with $\mu\xi > b^2$. The integral represents the distributed delay term withτ_1, τ_2which are time delays, μ_1 is positive constant, and μ_2 is an L^∞ function such that

(Hyp1) $\mu_2 : [\tau_1, \tau_2] \longrightarrow \mathbb{R}$ is a bounded function satisfying

$$\int_{\tau_1}^{\tau_2} |\mu_2(s)|ds < \mu_1. \tag{4}$$

This type of problem was mainly based on the following equations for one-dimensional theories of porous materials with temperature

$$\begin{cases} \rho_1 u_{tt} - T_x = 0, \\ \rho_2 \phi_{tt} - H_x - G = 0, \\ \rho_3 \theta_t + q_x + \gamma \phi_{tx} = 0, \end{cases} \tag{5}$$

where $(x,t) \in (0,L) \times (0,\infty)$.

According to Green and Naghdis theory, the constitutive equations of system (5) are given by

$$T = \mu u_x + b\phi, \tag{6}$$

$$G = -bu_x - \xi\phi - \mu_1 \phi_t - \int_{\tau_1}^{\tau_2} |\mu_2(s)| \phi_t(x, t-s)\, ds, \tag{7}$$

$$H = \delta\phi_x - \beta\theta, \tag{8}$$

$$q = -l\Phi_x - k\Phi_{tx}, \tag{9}$$

where $l, k > 0$ are the thermal conductivity and Φ is the thermal displacement whose time derivative is the empirical temperature θ, that is $\Phi_t = \theta$.

We substitute (9) in (5) with the condition $b \neq 0$, which results in

$$\begin{cases} \rho_1 u_{tt} = \mu u_{xx} + b\phi_x, \\ \rho_2 \phi_{tt} = \delta\phi_{xx} - bu_x - \xi\phi - \mu_1\phi_t - \int_{\tau_1}^{\tau_2} |\mu_2(s)| \phi_t(x, t-s)\, ds - \beta\theta_x, \\ \rho_3 \theta_t = l\Phi_{xx} - \gamma\phi_{tx} + k\Phi_{txx}. \end{cases} \tag{10}$$

By using $\Phi_t = \theta$ in the system (10), we find directly our system (1).

By using the multiplier techniques, the exponential decay results have been established. Next, in [1–3], the authors considered three types of thermoelastic theories based on an entropy equality instead of the usual entropy inequality (see [1–21] for more details).

According to the distributed delay, we mention, as a matter of course, the work by Nicaise and Pignotti in [16], where the authors studied the following system with distributed delay:

$$\begin{cases} u_{tt} - \Delta u = 0, \\ u = 0, \\ \dfrac{du}{dv}(t) + \int_{\tau_1}^{\tau_2} \mu(s) u_t(t-s)\, ds + \mu_0 u_t = 0, \\ u(.,0) = u_0, u_t(.,0) = u_1,\ u_t(x,-t) = f_0(x,t), \end{cases} \tag{11}$$

and proved the exponential stability result with condition

$$\int_{\tau_1}^{\tau_2} \mu(s)\, ds < \mu_0. \tag{12}$$

See for example [8, 22, 23]. Hao and Wei [24] considered the following problem:

$$\begin{cases} \rho_1 \phi_{tt} - K(\phi_x + \psi_x)_x = 0, \\ \rho_2 \psi_{tt} - b\psi_{xx} + K(\phi_x + \psi) + \beta\theta_{tx} + \mu_1 \psi_t + \mu_2 \psi_t(t-s) + f(\psi_t) = 0, \\ \rho_3 \theta_{tt} - \delta\theta_{xx} + \gamma\varphi_{ttx} - k\theta_{txx} = 0, \end{cases} \tag{13}$$

and obtained the well-posedness and stability of system.

There are many other works done by the authors in this context; our work differs from all of them, since we took the delay in the second equation to make the distributed delay in the rotation angle of the filament, which makes the contributions clear and important. In addition, we established the well-posedness of the system, and we obtain the exponential decay rate when $\delta/\rho_2 = \mu/\rho_1$ and the energy takes the algebraic rate for the case $\delta/\rho_2 \neq \mu/\rho_1$; these results are mainly stated in Theorem 8.

In order to show the dissipativity of systems (1)–(3), we introduce the new variables $\varphi = u_t$ and $\psi = \phi_t$. So, problems (1)–(3) take the form

$$\begin{cases} \rho_1 \varphi_{tt} = \mu\varphi_{xx} + b\psi_x, \\ \rho_2 \psi_{tt} = \delta\psi_{xx} - b\varphi_x - \xi\psi - \mu_1\psi_t - \int_{\tau_1}^{\tau_2} |\mu_2(s)| \psi_t(x, t-s)\, ds - \beta\theta_{tx}, \\ \rho_3 \theta_{tt} = l\theta_{xx} - \gamma\psi_{tx} + k\theta_{txx}, \end{cases} \tag{14}$$

with the initial data

$$\begin{aligned} &\varphi(x,0) = \varphi_0(x), \varphi_t(x,0) = \varphi_1(x), \psi(x,0) = \psi_0(x), \\ &\psi_t(x,0) = \psi_1(x), \theta(x,0) = \theta_0(x), \theta_t(x,0) = \theta_1(x), \\ &\psi_t(x,-t) = -f_0(x,t), \quad x \in (0,1) \end{aligned} \tag{15}$$

and boundary conditions

$$\varphi_x(0,t) = \varphi_x(1,t) = \psi(0,t) = \psi(1,t) = \theta_x(0,t) = \theta_x(1,t) = 0, \quad t \geq 0. \tag{16}$$

First, as in [16], taking the following new variable:

$$z(x, \rho, s, t) = \psi_t(x, t - s\rho), \tag{17}$$

then we obtain

$$\begin{cases} s z_t(x,\rho,s,t) + z_\rho(x,\rho,s,t) = 0, \\ z(x,0,s,t) = \psi_t(x,t). \end{cases} \tag{18}$$

Consequently, the problem was rewritten as

$$\begin{cases} \rho_1\varphi_{tt} = \mu\varphi_{xx} + b\psi_x, \\ \rho_2\psi_{tt} = \delta\psi_{xx} - b\varphi_x - \xi\psi - \mu_1\psi_t - \int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x,1,s,t)ds - \beta\theta_{tx}, \\ \rho_3\theta_{tt} = l\theta_{xx} - \gamma\psi_{tx} + k\theta_{txx}, \\ sz_t(x,\rho,s,t) + z_\rho(x,\rho,s,t) = 0, \end{cases} \quad (19)$$

where

$$(x,\rho,s,t) \in (0,1) \times \mathcal{H}, \quad (20)$$

with the boundary and the initial conditions

$$\varphi_x(0,t) = \varphi_x(1,t) = \psi(0,t) = \psi(1,t) = \theta_x(0,t) = \theta_x(1,t) = 0, \quad t \geq 0. \quad (21)$$

$$\varphi(x,0) = \varphi_0(x), \varphi_t(x,0) = \varphi_1(x), \psi(x,0) = \psi_0(x), \quad (22)$$

$$\psi_t(x,0) = \psi_1(x), \theta(x,0) = \theta_0(x), \theta_t(x,0) = \theta_1(x), \quad x \in (0,1), \quad (23)$$

$$z(x,\rho,s,0) = -f_0(x,\rho s) = h_0(x,\rho s), \quad x \in (0.1), \rho \in (0.1), s \in (0,\tau_2). \quad (24)$$

Meanwhile, from (19) and (24), it follows that

$$\frac{d^2}{dt^2}\int_0^1 \varphi(x,t)dx = 0. \quad (25)$$

So, by solving (25) and using (24), we get

$$\int_0^1 \varphi(x,t)dx = t\int_0^1 \varphi_1(x)dx + \int_0^1 \varphi_0(x)dx. \quad (26)$$

Consequently, if we let

$$\bar{\varphi}(x,t) = \varphi(x,t) - t\int_0^1 \varphi_1(x)dx - \int_0^1 \varphi_0(x)dx, \quad (27)$$

we get

$$\int_0^1 \bar{\varphi}(x,t)dx = 0, \quad \forall t \geq 0, \quad (28)$$

and from (19), we have

$$\frac{d^2}{dt^2}\int_0^1 \theta(x,t)dx = 0. \quad (29)$$

So, by solving (29) and using (24), we get

$$\int_0^1 \theta(x,t)dx = t\int_0^1 \theta_1(x)dx + \int_0^1 \theta_0(x)dx. \quad (30)$$

Consequently, if we let

$$\bar{\theta}(x,t) = \theta(x,t) - t\int_0^1 \theta_1(x)dx - \int_0^1 \theta_0(x)dx, \quad (31)$$

we get

$$\int_0^1 \bar{\theta}(x,t)dx = 0, \quad \forall t \geq 0. \quad (32)$$

Then, the Poincaré's inequality was used for $\bar{\varphi}$ and θ^- which are justified. A simple substitution shows that $(\bar{\varphi}, \psi, \bar{\theta})$ satisfies system (19) with initial data for $\bar{\varphi}$ and $\bar{\theta}$ given as

$$\begin{aligned} \bar{\varphi}_0(x) &= \varphi_0(x) - \int_0^1 \varphi_0(x)dx, \\ \bar{\varphi}_1(x) &= \varphi_1(x) - \int_0^1 \varphi_1(x)dx, \\ \bar{\theta}_0(x) &= \theta_0(x) - \int_0^1 \theta_0(x)dx, \\ \bar{\theta}_1(x) &= \theta_1(x) - \int_0^1 \theta_1(x)dx. \end{aligned} \quad (33)$$

Now, we use $\bar{\varphi}, \bar{\theta}$ instead of φ, θ and writing φ, θ for simplicity.

2. Well-Posedness

In this section, we give the existence and uniqueness result of the system (19)–(24) using the semigroup theory.

First, we introduce the vector function

$$U = (\varphi, \varphi_t, \psi, \psi_t, \theta, \theta_t, z)^T, \quad (34)$$

and the new dependent variables $u = \varphi_t, v = \psi_t, w = \theta_t$; then the system (19) can be written as follows:

$$\begin{cases} U_t = \mathcal{A}U, \\ U(0) = U_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, \theta_1, h_0)^T, \end{cases} \quad (35)$$

where $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} : \longrightarrow \mathcal{H}$ is the linear operator defined by

$$\mathcal{A}U = \begin{pmatrix} u \\ \frac{1}{\rho_1}[\mu\varphi_{xx} + b\psi_x] \\ v \\ \frac{1}{\rho_2}\left[\delta\psi_{xx} - b\varphi_x - \xi\psi - \beta w_x - \mu_1\psi_t - -\int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x,1,s,t)ds\right] \\ w \\ \frac{1}{\rho_3}[l\theta_{xx} - \gamma v_x + kw_{xx}] \\ -\frac{1}{s}z_\rho \end{pmatrix}, \quad (36)$$

and $\mathcal{H}$ is the energy space given by

$$\mathcal{H} = H_*^1 \times L_*^2(0,1) \times H_0^1 \times L^2(0,1) \times H_*^1 \times L^2(0,1) \times L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)), \quad (37)$$

where

$$L_*^2(0,1) = \left\{ \frac{\phi \in L^2(0,1)}{\int_0^1 \phi(x)dx} = 0 \right\},$$
$$H_*^1(0,1) = H^1(0,1) \cap L_*^2(0,1), \quad (38)$$
$$H_*^2(0,1) = \left\{ \frac{\phi \in H^2(0,1)}{\phi_x(1) = \phi_x(0)} = 0 \right\}.$$

For every

$$U = (\varphi, u, \psi, v, \theta, w, z)^T \in \mathcal{H},$$
$$\widehat{U} = (\varphi\wedge, u\wedge, \psi\wedge, v\wedge, \theta\wedge, w\wedge, z\wedge)^T \in \mathcal{H}, \quad (39)$$

we equip $\mathcal{H}$ with the inner product defined by

$$\begin{aligned} <U, \widehat{U}>_{\mathcal{H}} &= \gamma\rho_1 \int_0^1 u\widehat{u}dx + \gamma\rho_2 \int_0^1 v\widehat{v}dx + \gamma\xi \int_0^1 \psi\widehat{\psi}dx \\ &+ \beta\rho_3 \int_0^1 w\widehat{w}dx + \gamma\mu \int_0^1 \varphi_x\widehat{\varphi}_x dx + \gamma\delta \int_0^1 \psi_x\widehat{\psi}_x dx \\ &+ \gamma b \int_0^1 \left(\varphi_x\widehat{\psi} + \psi\widehat{\varphi}\right)dx + l\beta \int_0^1 \theta_x\widehat{\theta}_x dx \\ &+ \gamma \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu_2(s)| z\widehat{z} ds d\rho dx. \end{aligned} \quad (40)$$

The domain of $\mathcal{A}$ is given by

$$\mathcal{D}(\mathcal{A}) = \left\{ \begin{array}{l} U \in \mathcal{H}/\varphi, \theta \in H_*^2(0,1) \cap H_*^1(0,1), \psi \in H^2(0,1) \cap H_0^1(0,1) \\ u, w \in H_*^1(0,1), v \in H_0^1(0,1), z(x,0,s,t) = v \\ z, z_\rho \in L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)) \end{array} \right\} \quad (41)$$

Clearly, $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{H}$. Now, we can give the following existence result.

Theorem 1. *Let $U_0 \in \mathcal{H}$ and assume that (4) holds. Then, there exists a unique solution $U \in \mathcal{C}(\mathbb{R}_+, \mathcal{H})$ of problem (19).*
Moreover, if $U_0 \in \mathcal{D}(\mathcal{A})$, then

$$U \in \mathcal{C}(\mathbb{R}_+, \mathcal{D}(\mathcal{A})) \cap \mathcal{C}^1(\mathbb{R}_+, \mathcal{H}). \quad (42)$$

Proof. First, we prove that the operator $\mathcal{A}$ is dissipative. For any $U_0 \in \mathcal{D}(\mathcal{A})$ and by using (40), we have

$$\begin{aligned} <\mathcal{A}U, U>_{\mathcal{H}} &= -\gamma\mu_1 \int_0^1 v^2 dx - \gamma \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| v z(x,1,s,t) ds dx \\ &- \gamma \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z_\rho z ds d\rho dx - \beta k \int_0^1 w_x^2 dx. \end{aligned} \quad (43)$$

For the third term of the right-hand side of (43), we have

$$\begin{aligned} -\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z_\rho z ds d\rho dx &= -\frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} \int_0^1 |\mu_2(s)| \frac{d}{d\rho} z^2 d\rho ds dx \\ &= -\frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t) ds dx \\ &+ \frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,0,s,t) ds dx. \end{aligned} \quad (44)$$

By using Young's inequality, we get

$$\begin{aligned} -\int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| v z(x,1,s,t) ds dx &\leq \frac{1}{2} \left(\int_{\tau_1}^{\tau_2} |\mu_2(s)| ds \right) \int_0^1 v^2 dx \\ &+ \frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t) ds dx. \end{aligned} \quad (45)$$

Substituting (44) and (45) into (43), using the fact that $z(x,0,s,t) = v(x,t)$ and (4), we obtained

$$<\mathcal{A}U, U>_{\mathcal{H}} \leq -\gamma \left(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)| ds \right) \int_0^1 v^2 dx - \beta k \int_0^1 w_x^2 dx \leq 0. \quad (46)$$

Hence, the operator $\mathcal{A}$ is dissipative.

Next, we prove the operator $\mathcal{A}$ is maximal. It is sufficient to show that the operator $(Id - \mathcal{A})$ is surjective.

Indeed, for any $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7)^T \in \mathcal{H}$, we prove that there exists a unique $V = (\varphi, u, \psi, v, \theta, w, z) \in \mathcal{D}(\mathcal{A})$ such that

$$(Id - \mathcal{A})V = F. \quad (47)$$

That is

$$\begin{cases} \varphi - u = f_1, \\ \rho_1 u - \mu\varphi_{xx} - b\psi_x = \rho_1 f_2, \\ \psi - v = f_3, \\ \rho_2 v - \delta\psi_{xx} + b\varphi_x + \xi\psi + \beta w_x + \mu_1 v + \int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x,1,s,t) ds = \rho_2 f_4, \\ \theta - w = f_5, \\ \rho_3 w - l\theta_{xx} + \gamma v_x - k w_{xx} = \rho_3 f_6, \\ s z_t(x,\rho,s,t) + z_\rho(x,\rho,s,t) = s f_7, \end{cases} \quad (48)$$

We note that the last equation in (48) with $z(x,0,s,t) = v(x,t)$ has a unique solution given by

$$z(x,\rho,s,t) = e^{-\rho s} v + s e^{s\rho} \int_0^{\rho} e^{s\sigma} f_7(x,\sigma,s,t)\, d\sigma, \tag{49}$$

then

$$z(x,1,s,t) = e^{-s} v + s e^{s} \int_0^{1} e^{s\sigma} f_7(x,\sigma,s,t)\, d\sigma, \tag{50}$$

we have

$$u = \varphi - f_1, v = \psi - f_3, w = \theta - f_5. \tag{51}$$

Inserting (50) and (51) into (48), (48), and (48), we get

$$\begin{cases} \rho_1 \varphi - \mu \varphi_{xx} - b\psi_x = h_1, \\ \mu_4 \psi - \delta \psi_{xx} + b\varphi_x + \beta\theta_x = h_2, \\ rho_3 \theta - (l+k)\theta_{xx} + \gamma\psi_x = h_3, \end{cases} \tag{52}$$

where

$$\begin{cases} \mu_4 = \rho_2 + \xi + \mu_1 + \dfrac{4}{3}\gamma + \displaystyle\int_{\tau_1}^{\tau_2} |\mu_2(s)| e^{-s} ds, \\ h_1 = \rho_1 (f_1 + f_2), \\ h_2 = \rho_2 (f_3 + f_4) + \mu_1 - \displaystyle\int_{\tau_1}^{\tau_2} |\mu_2(s)| e^{-s} ds) f_3 ds - \int_{\tau_1}^{\tau_2} s |\mu_2(s)| e^{s} \int_0^1 e^{s\sigma} f_7(x,\sigma,s,t)\, d\sigma\, ds + \beta f_{5x} \\ h_3 = \rho_3 (f_5 + f_6) + \gamma f_{3x} - k f_{5xx}. \end{cases} \tag{53}$$

We multiply (52) by $\hat{\varphi}, \hat{\psi}, \hat{\theta}$, respectively, and integrate their sum over $(0,1)$ to get the following variational formulation:

$$B\left((\varphi,\psi,\theta), \left(\hat{\varphi},\hat{\psi},\hat{\theta}\right)\right) = \Gamma\left(\hat{\varphi},\hat{\psi},\hat{\theta}\right), \tag{54}$$

where

$$B : \left(H_*^1(0,1) \times H_0^1(0,1) \times H_*^1(0,1)\right)^2 \longrightarrow \mathbb{R} \tag{55}$$

is the bilinear form defined by

$$\begin{aligned} B\left((\varphi,\psi,\theta), \left(\hat{\varphi},\hat{\psi},\hat{\theta}\right)\right) = {} & \gamma\rho_1 \int_0^1 \varphi\hat{\varphi}\, dx + \gamma\mu \int_0^1 \varphi_x \hat{\varphi}_x\, dx \\ & + \gamma b \int_0^1 \left(\psi\hat{\varphi}_x + \varphi\hat{\psi}_x\right) dx \\ & + \gamma\mu_4 \int_0^1 \psi\hat{\psi}\, dx + \gamma\delta \int_0^1 \psi_x \hat{\psi}_x\, dx \\ & + \gamma\beta \int_0^1 \theta_x \hat{\psi}\, dx + \beta\gamma \int_0^1 \psi_x \hat{\theta}\, dx \\ & + \beta\rho_3 \int_0^1 \theta\hat{\theta}\, dx + \beta(l+k)^2 \int_0^1 \theta_x \hat{\theta}_x\, dx, \end{aligned}$$

$$\Gamma : \left(H_*^1(0,1) \times H_0^1(0,1) \times H_*^1(0,1)\right) \longrightarrow \mathbb{R} \tag{56}$$

is the linear functional given by

$$\Gamma\left(\hat{\varphi},\hat{\psi},\hat{\theta}\right) = \int_0^1 h_1 \hat{\varphi}\, dx + \int_0^1 h_2 \hat{\psi}\, dx + \int_0^1 h_3 \hat{\theta}\, dx. \tag{57}$$

Now, for $V = H_*^1(0,L) \times H_0^1(0,L) \times H_*^1(0,L)$, equipped with the norm

$$\|(\varphi,\psi,\theta)\|_V^2 = \|\varphi\|_2^2 + \|\varphi_x\|_2^2 + \|\psi\|_2^2 + \|\psi_x\|_2^2 + \|\theta\|_2^2 + \|\theta_x\|_2^2, \tag{58}$$

then, we have

$$\begin{aligned} B((\varphi,\psi,\theta),(\varphi,\psi,\theta)) = {} & \gamma\rho_1 \int_0^1 \varphi^2 dx + \gamma\mu \int_0^1 \varphi_x^2 dx \\ & + \gamma\mu_4 \int_0^1 \psi^2 dx + \gamma\delta \int_0^1 \psi_x^2 dx \\ & + \rho_3\beta \int_0^1 \theta^2 dx + \beta(l+k) \int_0^1 \theta_x^2 dx \\ & + 2\gamma b \int_0^1 \varphi_x \psi\, dx, \end{aligned} \tag{59}$$

we have

$$\begin{aligned} \mu\varphi_x^2 + \mu_4\psi^2 + 2b\varphi_x\psi = {} & \frac{1}{2}\left[\mu\left(\varphi_x + \frac{b}{\mu}\psi\right)^2 + \mu_4\left(\psi + \frac{b}{\mu_4}\varphi_x\right)^2 \right. \\ & \left. + \left(\mu - \frac{b^2}{\mu_4}\right)\varphi_x^2 + \left(\mu_4 - \frac{b^2}{\mu}\right)\psi^2\right] \\ > {} & \frac{1}{2}\left[\left(\mu - \frac{b^2}{\mu_4}\right)\varphi_x^2 + \left(\mu_4 - \frac{b^2}{\mu}\right)\psi^2\right], \end{aligned} \tag{60}$$

by assuming $\mu\xi - b^2 > 0$, we get

$$\mu - \frac{b^2}{\mu_4} > 0, \mu_4 - \frac{b^2}{\mu} > 0, \quad (61)$$

then, for some $M_0 > 0$,

$$B((\varphi, \psi, \theta), (\varphi, \psi, \theta)) \geq M_0 \|(\varphi, \psi, \theta)\|_V^2. \quad (62)$$

Thus, B is coercive. Consequently, using the Lax-Milgram theorem, we conclude that the existence of a unique solution $((\varphi, \psi, \theta))$ in V satisfies

$$\begin{aligned} u &= \varphi - f_1 \in H_*^1(0,1), \\ v &= \psi - f_3 \in H_0^1(0,1), \\ w &= \theta - f_5 \in H_*^1(0,1). \end{aligned} \quad (63)$$

Substituting φ, ψ, θ into (50) and (51), respectively, we have

$$\begin{aligned} & u, \theta \in H_*^1(0,1), \\ & \psi \in H_0^1(0,1), \\ & z, z_\rho \in L^2((0,1) \times (0,1) \times (\tau_1, \tau_2)). \end{aligned} \quad (64)$$

Let $\widehat{\varphi} \in H_0^1(0,1)$ and denote

$$\widehat{\widehat{\widehat{\varphi}}} = \widehat{\varphi}(x) - \int_0^1 \widehat{\varphi}(\xi) d\xi, \quad (65)$$

which gives us $\widehat{\widehat{\widehat{\varphi}}} \in H_*^1(0,1)$. Now, we replace $(\widehat{\varphi}, \widehat{\psi}, \widehat{\theta})$ by $(\widehat{\widehat{\widehat{\varphi}}}, 0, 0)$ in (54) to obtain

$$\gamma\rho_1 \int_0^1 \varphi \widehat{\widehat{\widehat{\varphi}}} dx + \gamma\mu \int_0^1 \varphi_x \widehat{\widehat{\widehat{\varphi}}}_x dx + \gamma b \int_0^1 \psi_x \widehat{\widehat{\widehat{\varphi}}} dx = \int_0^1 h_1 \widehat{\widehat{\widehat{\varphi}}} dx. \quad (66)$$

We get

$$\gamma\mu \int_0^1 \varphi_x \widehat{\widehat{\widehat{\varphi}}}_x dx = \int_0^1 (h_1 - \gamma\rho_1\varphi - \gamma b\psi_x) \widehat{\widehat{\widehat{\varphi}}} dx, \quad (67)$$

which yields

$$\gamma\mu\varphi_{xx} = \gamma\rho_1\varphi - \gamma b\psi_x - h_1 \in L^2(0,1). \quad (68)$$

Thus,

$$\varphi \in H^2(0,1). \quad (69)$$

Moreover, (52) also holds for any every $\widehat{\varphi} \in C^1([0,1])$. Then, by using integration by parts, we obtain

$$\gamma\mu \int_0^1 \varphi_x \widehat{\varphi}_x dx = \int_0^1 (h_1 - \gamma\rho_1\varphi - \gamma b\psi_x) \widehat{\varphi} dx. \quad (70)$$

Then, we get for any $\widehat{\varphi} \in C^1([0,1])$

$$\varphi_x(1)\widehat{\varphi}(1) - \varphi_x(0)\widehat{\varphi}(0) = 0. \quad (71)$$

Since $\widehat{\varphi}$ is arbitrary, we get that $\varphi_x(0) = \varphi_x(1) = 0$. Hence, $\varphi \in H_*^2(0,1)$. Using similar arguments as above, we can obtain

$$\begin{aligned} \psi &\in H^2(0,1) \cap H_0^1(0,1), \\ \theta &\in H_*^2(0,1). \end{aligned} \quad (72)$$

Finally, the application of regularity theory for the linear elliptic equations guarantees the existence of unique $U \in \mathcal{D}(\mathcal{A})$ such that (47) is satisfied.

Consequently, we conclude that $\mathcal{A}$ is a maximal dissipative operator. Hence, by Lumer-Philips theorem (see [25, 26]), we have the well-posedness result. This completes the proof.

3. Stability Results

We prepare the next lemmas (Lemmas 2–7) which will be useful to introduce the Lyapunov function in (104).

Lemma 2. *The energy functional E associated with our problem defined by*

$$\begin{aligned} E(t) = {} & \frac{\gamma}{2}\left\{\int_0^1 \left[\rho_1\varphi_t^2 + \mu\varphi_x^2 + \rho_2\psi_t^2 + \delta\psi_x^2 + \xi\psi^2 + 2b\varphi_x\psi\right] dx\right\} \\ & + \frac{\beta}{2}\left\{\int_0^1 \left[l\theta_x^2 + \rho_3\theta_t^2\right] dx\right\} \\ & + \frac{\gamma}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)| z^2(x,\rho,s,t) ds d\rho dx \end{aligned} \quad (73)$$

satisfies

$$E'(t) \leq -k\beta \int_0^1 \theta_{tx}^2 dx - \gamma\eta_0 \int_0^1 \psi_t^2 dx \leq 0, \quad (74)$$

where $\eta_0 = \mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)| ds \geq 0.$

Proof. Multiplying (19) by $\gamma\varphi_t$, (19) by $\gamma\psi_t$, and (19) by $\beta\theta_t$ then integration by parts over $(0,1)$, we get

$$\begin{aligned}&\frac{\gamma}{2}\frac{d}{dt}\int_0^1\left[\rho_1\varphi_t^2+\mu\varphi_x^2+\rho_2\psi_t^2+\delta\psi_x^2+\xi\psi^2+2b\varphi_x\psi\right]dx\\&+\gamma\mu_1\int_0^1\psi_t^2dx+\frac{\beta}{2}\frac{d}{dt}\int_0^1\left[l\theta_x^2+\rho_3\theta_t^2\right]dx\\&+\gamma\int_0^1\psi_t\int_{\tau_1}^{\tau_2}|\mu_2(s)|z(x,1,s,t)dsdx=0.\end{aligned}\tag{75}$$

Now, multiplying (19) by $z\,|\,\mu_2(s)\,|$ and integrating the result over $(0,1)\times(0,1)\times(\tau_1,\tau_2)$, we get

$$\begin{aligned}&\frac{d}{dt}\frac{\gamma}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}s|\mu_2(s)|z^2(x,\rho,s,t)dsd\rho dx\\&=-\gamma\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|zz_\rho(x,\rho,s,t)dsd\rho dx\\&=-\frac{\gamma}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|\frac{d}{d\rho}z^2(x,\rho,s,t)dsd\rho dx\\&=\frac{\gamma}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|\left(z^2(x,0,s,t)-z^2(x,1,s,t)\right)dsdx\\&=\frac{\gamma}{2}\left(\int_{\tau_1}^{\tau_2}|\,\mu_2(s)\,|\,ds\right)\int_0^1\psi_t^2dx-\frac{\gamma}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,s,t)dsdx.\end{aligned}\tag{76}$$

From (75) and (76), we get (73) and (74).

Now, using Young's inequality, (74) can be written as

$$E'(t)\le-k\beta\int_0^1\theta_{tx}^2dx-\gamma\left(\mu_1-\int_{\tau_1}^{\tau_2}|\,\mu_2(s)\,|\,ds\right)\int_0^1\psi_t^2dx.\tag{77}$$

Then, by (4), there exists a positive constant η_0 such that

$$E'(t)\le-k\beta\int_0^1\theta_{tx}^2dx-\gamma\eta_0\int_0^1\psi_t^2dx.\tag{78}$$

Thus, the functional E is nonincreasing.

Lemma 3. *The function*

$$F_1(t):=\rho_2\int_0^1\psi_t\psi dx+\frac{b\rho_1}{\mu}\int_0^1\psi\int_0^x\varphi_t(y)dydx+\frac{\mu_1}{2}\int_0^1\psi^2dx\tag{79}$$

satisfies

$$\begin{aligned}F_1'(t)\le&-\frac{\delta}{2}\int_0^1\psi_x^2dx-\mu_3\int_0^1\psi^2dx+\varepsilon_1\int_0^1\varphi_t^2dx\\&+c\left(1+\frac{1}{\varepsilon_1}\right)\int_0^1\psi_t^2dx+c\int_0^1\theta_{tx}^2dx\\&+c\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,s,t)dsdx,\end{aligned}\tag{80}$$

where $\mu_3=\xi-(b^2/\mu)>0$.

Proof. Direct computation, using integration by parts and Young's inequality, for $\varepsilon_1>0$, yields

$$\begin{aligned}F_1'(t)=&-\delta\int_0^1\psi_x^2dx-\left(\xi-\frac{b^2}{\mu}\right)\int_0^1\psi^2dx+\rho_2\int_0^1\psi_t^2dx\\&+\frac{b\rho_1}{\mu}\int_0^1\psi_t\int_0^x\varphi_t(y)dydx-\beta\int_0^1\psi\theta_{tx}dx\\&-\int_0^1\psi\int_{\tau_1}^{\tau_2}|\mu_2(s)|z(x,1,s,t)dsdx\le-\delta\int_0^1\psi_x^2dx\\&-\left(\xi-\frac{b^2}{\mu}\right)\int_0^1\psi^2dx+c\left(1+\frac{1}{\varepsilon_1}\right)\int_0^1\psi_t^2dx\\&+\varepsilon_1\int_0^1\left(\int_0^x\varphi_t(y)dy\right)^2dx-\beta\int_0^1\psi\theta_{tx}dx\\&-\int_0^1\psi\int_{\tau_1}^{\tau_2}|\mu_2(s)|z(x,1,s,t)dsdx.\end{aligned}\tag{81}$$

By Cauchy-Schwartz's inequality, it is clear that

$$\int_0^1\left(\int_0^x\varphi_t(y)dy\right)^2dx\le\int_0^1\left(\int_0^1\varphi_tdx\right)^2dx\le\int_0^1{\varphi_t}^2dx.\tag{82}$$

So, estimate (81) becomes

$$\begin{aligned}F_1'(t)\le&-\delta\int_0^1\psi_x^2dx-\left(\xi-\frac{b^2}{\mu}\right)\int_0^1\psi^2dx+c\left(1+\frac{1}{\varepsilon_1}\right)\int_0^1\psi_t^2dx\\&+\varepsilon_1\int_0^1{\varphi_t}^2dx-\beta\int_0^1\psi\theta_{tx}dx-\int_0^1\psi\int_{\tau_1}^{\tau_2}|\mu_2(s)|z(x,1,s,t)dsdx,\end{aligned}\tag{83}$$

where the Cauchy-Schwartz, Young, and Poincaré's inequalities have been used, for $\varepsilon_1>0$.

By the fact that $\mu\xi>b^2$, we get the desired result (80).

Lemma 4. *Assume that ((4)) holds. Then, the function*

$$F_2(t):=\int_0^1\psi_x\varphi_tdx+\int_0^1\psi_t\varphi_xdx\tag{84}$$

satisfies

$$F_2'(t) \le -\frac{b}{2\rho_2}\int_0^1 \varphi_x^2 dx + c\int_0^1 \psi_x^2 dx + c\int_0^1 \psi_t^2 + c\int_0^1 \theta_{tx}^2 + c\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t)dx + \left(\frac{\delta}{\rho_2} - \frac{\mu}{\rho_1}\right)\int_0^1 \varphi_x \psi_{xx} dx. \quad (85)$$

Proof. By differentiating F_2, then using (19), integration by parts gives

$$F_2'(t) = -\frac{b}{\rho_2}\int_0^1 \varphi_x^2 dx + \left(\frac{\delta}{\rho_2} - \frac{\mu}{\rho_1}\right)\int_0^1 \varphi_x\psi_{xx}dx + \frac{b}{\rho_1}\int_0^1 \psi_x^2 dx - \frac{\xi}{\rho_2}\int_0^1 \varphi_x \psi dx - \frac{\mu_1}{\rho_2}\int_0^1 \psi_t\varphi_x dx - \frac{\beta}{\rho_2}\int_0^1 \theta_{tx}\varphi_x dx - \frac{1}{\rho_2}\int_0^1 \varphi_x \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t) ds dx. \quad (86)$$

Thanks to Young, Cauchy-Schwartz, and Poincaré's inequalities to estimate terms in RHS of (86). For $\delta_1, \delta_2, \delta_3, \delta_4 > 0$, we have

$$-\frac{\xi}{\rho_2}\int_0^1 \varphi_x\psi dx \le \delta_1 \int_0^1 \varphi_x^2 dx + \frac{c}{4\delta_1}\int_0^1 \psi^2 dx, \quad (87)$$

$$-\frac{\mu_1}{\rho_2}\int_0^1 \psi_t\varphi_x dx \le \delta_2 \int_0^1 \varphi_x^2 dx + \frac{c}{4\delta_2}\int_0^1 \psi_t^2 dx, \quad (88)$$

$$-\frac{\beta}{\rho_2}\int_0^1 \theta_{tx}\varphi_x dx \le \delta_3 \int_0^1 \varphi_x^2 dx + \frac{c}{4\delta_3}\int_0^1 \theta_{tx}^2 dx, \quad (89)$$

$$-\frac{1}{\rho_2}\int_0^1 \varphi_x \int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x,1,s,t) ds dx \le \delta_4 \int_0^1 \varphi_x^2 dx + \frac{c}{4\delta_4}\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t) ds. \quad (90)$$

The replacement of (87)–(90) into (86) and setting$\delta_1 = \delta_2 = \delta_3 = \delta_4 = b/8\rho_2$helps to obtain (85).

Lemma 5. *The function*

$$F_3(t) := -\rho_1 \int_0^1 \varphi_t \varphi dx \quad (91)$$

satisfies

$$F_3'(t) \le -\rho_1 \int_0^1 \varphi_t^2 dx + \frac{3\mu}{2}\int_0^1 \varphi_x^2 dx + c\int_0^1 \psi_x^2 dx. \quad (92)$$

Proof. Direct computations give

$$F_3'(t) = -\rho_1 \int_0^1 \varphi_t^2 dx + \mu \int_0^1 \varphi_x^2 dx + b\int_0^1 \varphi_x \psi dx. \quad (93)$$

Estimate (92) easily follows by using Young's and Poincaré's inequalities

$$F_3'(t) \le -\rho_1 \int_0^1 \varphi_t^2 dx + \mu \int_0^1 \varphi_x^2 dx + \delta_5 \int_0^1 \varphi_x^2 dx + \frac{c}{4\delta_5}\int_0^1 \psi_x^2 dx, \quad (94)$$

setting $\delta_5 = \mu/2$ to obtain (92).

Lemma 6. *The function*

$$F_4(t) := -\rho_3 \int_0^1 \theta_t \theta dx \quad (95)$$

satisfies

$$F_4'(t) \le -\frac{l}{2}\int_0^1 \theta_x^2 dx + c\int_0^1 \psi_t^2 dx + c\int_0^1 \theta_{tx}^2 dx. \quad (96)$$

Proof. Direct computations give

$$F_4'(t) = -l\int_0^1 \theta_x^2 dx + \gamma\int_0^1 \theta_x \psi_t dx - k\int_0^1 \theta_x \theta_{tx} dx + \rho_3 \int_0^1 \theta_t^2 dx. \quad (97)$$

By using Young and Poincaré's inequalities, we get (96).

Lemma 7. *The function*

$$F_5(t) := \int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s e^{-s\rho} |\mu_2(s)| z^2(x,\rho,s,t) ds d\rho dx \quad (98)$$

satisfies

$$F_5'(t) \le -\eta_1 \int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)| z^2(x,\rho,s,t) ds d\rho dx + \mu_1 \int_0^1 \psi_t^2 dx - \eta_1 \int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t) ds dx, \quad (99)$$

where η_1 is a given positive constant.

Proof. By differentiating F_5 with respect to t and using the last equation in $(Hyp1)$, we have

$$F_5'(t) = -2\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} e^{-s\rho}|\mu_2(s)| z z_\rho(x,\rho,s,t) ds d\rho dx = -\frac{d}{d\rho}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s e^{-s\rho}|\mu_2(s)| z^2(x,\rho,s,t) ds d\rho dx - \int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)| \left[e^{-s} z^2(x,1,s,t) - z^2(x,0,s,t)\right] ds dx. \quad (100)$$

Using the fact that $z(x,0,s,t)=\psi_t(x,t-s)$ and $e^{-s}\leq e^{-s\rho}\leq 1$, for all $0<\rho<1$, we obtain

$$F_5'(t)=-\eta_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\rho,s,t)dsd\rho dx - \int_0^1\int_{\tau_1}^{\tau_2} e^{-s}|\mu_2(s)|z^2(x,1,s,t)dsdx + \int_{\tau_1}^{\tau_2}|\mu_2(s)|ds\int_0^1\psi_t^2dx. \quad (101)$$

We have $-e^{-s}\leq -e^{-\tau_2}\forall s\in[\tau_1,\tau_2]$. Set $\eta_1=e^{-\tau_2}$, and by (4), we get (99).

We state and prove the decay result in Theorem 8.

Theorem 8. *Let ((4)) hold. Then, there exist positive constants λ_1 and λ_2 such that the function ((73)) satisfies, for any $t>0$*

$$E(t)\leq \lambda_2 e^{-\lambda_1 t}, \quad \text{if } \frac{\delta}{\rho_2}=\frac{\mu}{\rho_1}, \quad (102)$$

$$E(t)\leq C(E_1(0)+E_2(0))t^{-1}, \quad \text{if } \frac{\delta}{\rho_2}\neq\frac{\mu}{\rho_1}. \quad (103)$$

Proof. We define a class of an appropriate Lyapunov function as

$$\mathscr{L}(t):=NE(t)+N_1F_1(t)+N_2F_2(t)+F_3(t)+F_4(t)+N_5F_5(t), \quad (104)$$

where N, N_1, N_2, and N_5 are positive constants to be selected later.

Differentiating (104) and by (74), (80), (85), (92), (96), and (99), we have

$$\begin{aligned}\mathscr{L}'(t)\leq &-\left[\frac{\delta N_1}{2}-cN_2-c\right]\int_0^1\psi_x^2dx-[\rho_1-N_1\varepsilon_1]\int_0^1\varphi_t^2dx\\ &-\left[\gamma\eta_0N-cN_1\left(1+\frac{1}{\varepsilon_1}\right)-N_2c-\mu_1N_5-c\right]\int_0^1\psi_t^2dx\\ &-\left[\frac{bN_2}{2\rho_2}-\frac{3\mu}{2}\right]\int_0^1\varphi_x^2dx-N_1\mu_3\int_0^1\psi^2dx\\ &-[N_5\eta_1-cN_1-cN_2]\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,s,t)dsdx\\ &-\frac{l}{2}\int_0^1\theta_x^2dx-N_5\eta_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}s|\mu_2(s)|z^2(x,\rho,s,t)dsd\rho dx\\ &-[Nk\beta-cN_1-cN_2-c]\int_0^1\theta_{tx}^2dx+N_2\left(\frac{\delta}{\rho_2}-\frac{\mu}{\rho_1}\right)\int_0^1\varphi_x\psi_{xx}dx.\end{aligned} \quad (105)$$

By setting $\varepsilon_1=\rho_1/2N_1$, we obtain

$$\begin{aligned}\mathscr{L}'(t)\leq &-\left[\frac{\delta N_1}{2}-cN_2-c\right]\int_0^1\psi_x^2dx-\frac{\rho_1}{2}\int_0^1\varphi_t^2dx\\ &-\left[\frac{bN_2}{2\rho_2}-\frac{3\mu}{2}\right]\int_0^1\varphi_x^2dx\\ &-[\gamma\eta_0N-cN_1(1+N_1)-cN_2-\mu_1N_5-c]\int_0^1\psi_t^2dx\\ &-[N_5\eta_1-cN_1-cN_2]\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,s,t)dsdx\\ &-N_1\mu_3\int_0^1\psi^2dx-[Nk\beta-cN_1-cN_2-c]\int_0^1\theta_{tx}^2dx\\ &-\frac{l}{2}\int_0^1\theta_x^2dx-N_5\eta_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}s|\mu_2(s)|z^2(x,\rho,s,t)dsd\rho dx\\ &+N_2\left(\frac{\delta}{\rho_2}-\frac{\mu}{\rho_1}\right)\int_0^1\varphi_x\psi_{xx}dx.\end{aligned} \quad (106)$$

Next, we carefully choose the constants, starting by N_2 to be large enough such that

$$\alpha_1=\frac{bN_2}{2J}-\frac{3\mu}{2}>0, \quad (107)$$

and N_1 so that

$$\alpha_2=\frac{\delta N_1}{2}-cN_2-c>0, \quad (108)$$

and N_5 large enough such that

$$\alpha_3=N_5\eta_1-cN_1-cN_2>0. \quad (109)$$

We arrive at

$$\begin{aligned}\mathscr{L}'(t)\leq &-\alpha_2\int_0^1\psi_x^2dx-\alpha_0\int_0^1\psi^2dx-\frac{\rho}{2}\int_0^1\varphi_t^2dx-\alpha_1\int_0^1\varphi_x^2dx\\ &-[\gamma\eta_0N-c]\int_0^1\psi_t^2dx-[k\beta N-c]\int_0^1\theta_{tx}^2dx-\frac{l}{2}\int_0^1\theta_x^2dx\end{aligned} \quad (110)$$

$$\begin{aligned}&-\alpha_3\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,s,t)dsdx+\alpha_5\int_0^1\varphi_x\psi_{xx}dx\\ &-\alpha_4\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}s|\mu_2(s)|z^2(x,\rho,s,t)dsd\rho dx,\end{aligned} \quad (111)$$

where $\alpha_0=\mu_3N_1=(\xi-(b^2/\mu))N_1$, $\alpha_4=N_5\eta_1$, $\alpha_5=N_2k_0=N_2((\delta/\rho_2)-(\mu/\rho_1))$.

Now, let us define the related function

$$\mathfrak{L}(t)=N_1F_1(t)+N_2F_2(t)+F_3(t)+F_4(t)+N_5F_5(t), \quad (112)$$

then

$$|\mathfrak{L}(t)| \le JN_1\int_0^1 |\psi\psi_t|dx + \frac{b\rho_1 N_1}{\mu}\int_0^1 \left|\psi\int_0^x \varphi_t(y)dy\right|dx + \frac{\mu_1 N_1}{2}\int_0^1 \psi^2 dx + N_2\int_0^1 |\psi_x\varphi_t + \varphi_x\psi_t|dx + \rho_1\int_0^1 |\varphi_t\varphi|dx + \rho_3\int_0^1 |\theta_t\theta|dx + N_5\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\rho}|\mu_2(s)|z^2(x,\rho,s,t)dsd\rho dx. \tag{113}$$

Thanks to Young, Cauchy-Schwartz, and Poincaré's inequalities, we get

$$|\mathfrak{L}(t)| \le c\int_0^1 \left(\varphi_t^2 + \psi_t^2 + \psi_x^2 + \varphi_x^2 + \psi^2 + \theta_t^2 + \theta_x^2\right)dx + c\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\rho,s,t)dsd\rho \le cE(t). \tag{114}$$

Then,

$$|\mathfrak{L}(t)| = |\mathscr{L}(t) - NE(t)| \le cE(t). \tag{115}$$

Thus,

$$(N-c)E(t) \le \mathscr{L}(t) \le (N+c)E(t). \tag{116}$$

One can now$N$large enough such that

$$N - c > 0, k\beta N - c > 0, N\gamma\eta_0 - c > 0. \tag{117}$$

We get

$$c_2 E(t) \le \mathscr{L}(t) \le c_3 E(t), \quad \forall t \ge 0, \tag{118}$$

and using (73), (110), and (116), and the fact that

$$\int_0^1 \theta_t^2 dx \le \int_0^1 \theta_{tx}^2 dx, \tag{119}$$

which gives

$$\mathscr{L}'(t) \le -k_1 E(t) + \alpha_5\int_0^1 \varphi_x\psi_{xx}dx, \quad \forall t \ge 0. \tag{120}$$

for some $k_1, c_2, c_3 > 0$.

Case 1. If $k_0 = (\delta/\rho_2) - (\mu/\rho_1) = 0$, in this case, ((120)) takes the form

$$\mathscr{L}'(t) \le -k_1 E(t), \quad \forall t \ge 0. \tag{121}$$

The combination of (118) and (121) gives

$$\mathscr{L}'(t) \le -\lambda_1\mathscr{L}(t), \quad \forall t \ge 0, \lambda_1 = \frac{k_1}{c_2}. \tag{122}$$

Finally, by integrating (122) and recalling (118), we obtain the first result of (103).

Case 2. If $k_0 = (\delta/\rho_2) - (\mu/\rho_1) \ne 0$, then

$$\begin{cases} k_0 < \dfrac{k_1\mu^2\gamma\delta}{2N_2(\rho_1 + b)}, & \text{if } k_0 > 0, \\ |k_0| < \dfrac{k_1\mu^2\gamma}{2N_2\rho_1}, & \text{if } k_0 < 0. \end{cases} \tag{123}$$

Let

$$E(t) = E(\varphi, \psi, \theta, z) = E_1(t), \tag{124}$$

be denoted by

$$E_2(t) = E(\varphi_t, \psi_t, \theta_t, z_t). \tag{125}$$

Then, we have

$$E_2'(t) \le -k\beta\int_0^1 \theta_{ttx}^2 dx - \gamma\eta_0\int_0^1 \psi_{tt}^2 dx. \tag{126}$$

The last term in (120), by using (19), and Young's inequality, and by setting $K = -\rho_1\alpha_5/\mu$, we have

$$\begin{aligned} \alpha_5\int_0^1 \varphi_x\psi_{xx}dx &= -\frac{\alpha_5\rho_1}{\mu}\int_0^1 \psi_x\varphi_{tt}dx + \frac{b\alpha_5}{\mu}\int_0^1 \psi_x^2 dx \\ &= -K\left(\frac{d}{dt}\left[\int_0^1 \psi\varphi_{xt}dx - \int_0^1 \psi_t\varphi_x dx\right]\right) - K\int_0^1 \varphi_x\psi_{tt}^2 dx + \frac{b\alpha_5}{\mu}\int_0^1 \psi_x^2 dx \\ &\le -K\left(\frac{d}{dt}\left[\int_0^1 \psi\varphi_{xt}dx - \int_0^1 \psi_t\varphi_x dx\right]\right) + \frac{b\alpha_5}{\mu}\int_0^1 \psi_x^2 dx + \frac{|K|}{4}\int_0^1 \psi_{tt}^2 dx + |K|\int_0^1 \varphi_x^2 dx. \end{aligned} \tag{127}$$

Let

$$\mathscr{N}(t) = \int_0^1 \psi\varphi_{xt}dx - \int_0^1 \psi_t\varphi_x dx, \tag{128}$$

then (120)

$$\mathscr{L}'(t) + K\mathscr{N}'(t) \leq -k_1 E_1'(t) + \frac{b\alpha_5}{\mu}\int_0^1 \psi_x^2 dx + \frac{|K|}{4}\int_0^1 \psi_{tt}^2 dx + |K|\int_0^1 \varphi_x^2 dx \leq -k_2 E_1'(t) + \frac{|K|}{4}\int_0^1 \psi_{tt}^2 dx, \tag{129}$$

where

$$k_2 = k_1 - \frac{2}{\mu\gamma}\left(|K| + \frac{b\alpha_5}{\delta}\right). \tag{130}$$

Let

$$G(t) = \mathscr{L}(t) + K\mathscr{N}(t) + N_3(E_1(t) + E_2(t)). \tag{131}$$

If $N_3 > \max\{C_0 \mid K|-c_1, |K|, |K|/4C\}$, indeed,

$$|\mathscr{N}(t)| = |\int_0^1 \psi\varphi_{xt} dx| + |\int_0^1 \psi_t \varphi_x dx| \leq \frac{1}{2}\int_0^1 \varphi_{tx}^2 dx + \frac{1}{2}\int_0^1 \psi_t^2 dx + \frac{1}{2}\int_0^1 \psi^2 dx + \frac{1}{2}\int_0^1 \varphi_x^2 dx \leq E_2(t) + C_0 E_1(t), \tag{132}$$

where $C_0 = \max\{2/\gamma\xi, 2/\gamma\mu, 2/\gamma\rho_2\}$. By (118), we obtain

$$G(t) \leq c_1 E_1(t) - |K|(E_2(t) + C_0 E_1(t)) + N_3(E_1(t) + E_2(t)) \leq (N_3 + c_1 - C_0 \mid K \mid) E_1(t) + (N_3 - |K \mid) E_2(t). \tag{133}$$

It is not hard to prove

$$m_1(E_1(t) + E_2(t)) \leq G(t) \leq m_2(E_1(t) + E_2(t)), \tag{134}$$

where $m_1, m_2 > 0$. By using (129) and (128), we obtain

$$G'(t) = \mathscr{L}'(t) + K\mathscr{N}'(t) + N_3\left(E_1'(t) + E_2'(t)\right) \leq -k_2 E_1(t) + \left(-CN_3 + \frac{|K|}{4}\right)\int_0^1 \psi_{tt}^2 dx. \tag{135}$$

Choosing N_3 such that

$$CN_3 - \frac{|K|}{4} > 0, \tag{136}$$

we have

$$G'(t) \leq -k_2 E_1(t). \tag{137}$$

Integrating (137), we get

$$\int_0^t E_1(y) dy \leq \frac{1}{k_2}(G(0) - G(1)) \leq \frac{1}{k_2} G(0) \leq \frac{m_2}{k_2}(E_1(0) + E_2(0)), \tag{138}$$

using the fact that

$$(tE_1(t))' = tE_1'(t) + E_1(t) \leq E_1(t). \tag{139}$$

We get that

$$tE_1(t) \leq \frac{m_2}{C_2}(E_1(0) + E_2(0)), \tag{140}$$

which is desired to be the second result of (103). This completes the proof.

4. Conclusion

This paper studied the asymptotic behavior of a one-dimensional thermoelastic system with distributed time delay; namely, an integral damping term on a time interval $[t-\tau_2, t-\tau_1]$ is taken into account. Beside the distributed delay term, a standard undelayed damping is included in the model $(-\mu_1\phi_t)$. We established the well-posedness of the system, and we proved stability estimates by means of appropriate Lyapunov functions. Exponential decay estimates are proved by nonclassical condition between the delay damping coefficient and the coefficient of the undelayed one which is satisfied. Several papers have been proposed for models including both undelayed and delayed damping of the same form, and exponential stability results have been obtained if the coefficient of the delay is smaller than the one of the undelayed term. This analysis has been extended to the case of a distributed delay in [16]. Also in this case, there are now a few literature, dealing with different PDE models, including thermoelastic systems. Typically, under the assumption (4), the system keeps the same properties, the one without delay but only with a standard frictional damping $c\phi_t$, for some coefficient c. Then, this paper introduced a considerable novelties different from those of [15].

Data Availability

No data were used to support the study.

Conflicts of Interest

This work does not have any conflicts of interest.

Acknowledgments

The third author extend their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through Research Group Project under Grant No. R.G.P-1/3/42.

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