Research Article

Hyers-Ulam Stability of Functional Equation Deriving from Quadratic Mapping in Non-Archimedean (n, β) -Normed Spaces

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In this work, we have to introduce a generalized quadratic functional equation and derive its solution. The main objective of this work is to investigate the Hyers-Ulam stability of quadratic functional equation in non-Archimedean (n, β) -normed spaces.

1. Introduction

The theory of functional equation is one of the most interesting topics in the field of Mathematics. It deals with the search of functions which satisfies a given equation. A functional equation is like a regular algebraic equation; though instead of unknown elements in some set, we are interested in finding a function satisfying our equation.

The concept of stability of a functional equation arises when one replaces a functional equation by an inequality which acts as a perturbation of the equation. The first stability problem concerning group homomorphisms was raised by Ulam [1] in 1940 and affirmatively solved by Hyers [2]. Aoki was the first author who has generalized the theorem of Hyers (see [3]).

Moreover, Gavruta [4], Rassias [5], and Bourgin [6] have considered the stability problem with unbounded Cauchy difference (see also [7]). On the other hand, Rassias [8–13] considered the Cauchy difference controlled by a product of different powers of norm. This stability phenomenon is called the Ulam-Gavruta-Rassias stability (see also [14]).

The Hyers-Ulam stability issue for the quadratic functional equation was cleared by Skof [15]. In [16], Czerwik demonstrated the Hyers-Ulam-Rassias stability of the qua-

dratic functional equation. Afterward, Jung [17] has summed up the outcomes gotten by Skof and Czerwik.

The Hyers-Ulam stability problem for the quadratic functional equation was solved by Skof [15]. In [16], Czerwik proved the Hyers-Ulam-Rassias stability of quadratic functional equation. Later, Jung [17] has generalized the results obtained by Skof and Czerwik.

The first work on the Hyers-Ulam stability of functional equations in complete non-Archimedean normed spaces (some particular cases were considered earlier; see [18] for details) is [19]. After it, a lot of articles (see, for instance, [20] and the references given there) on the stability of other equations in such spaces were published. In [21], the stability of the additive Cauchy equation in non-Archimedean fuzzy normed spaces under the strongest t -norm TM Rassias has been established.

In 1897, Hensel [22] has introduced a normed space which does not have the Archimedean property. It turned out that non-Archimedean spaces have many nice applications [23–26].

Initially, Liu [27] introduced the notions of (n, β) -normed space and non-Archimedean (n, β) -normed space. Then, they investigated Hyers-Ulam stability of the Cauchy functional equation and the Jensen functional equation in non-Archimedean (n, β) -normed spaces and that of the pexiderized Cauchy functional equation in (n, β) -normed

spaces. During the most recent many years, a few stability issues of functional equations have been researched by various mathematicians (see [28–40]).

The objective of this work is to introduce that a generalized quadratic functional equation is

$$\sum_{1 \leq i < j \leq l} \phi(v_i + v_j) + \sum_{1 \leq i < j \leq l} \phi(v_i - v_j) = 2(l-1) \sum_{1 \leq i \leq l} \phi(v_i), \quad (1)$$

where $l \geq 2$ and examine the Hyers-Ulam stability of the above mentioned equation in non-Archimedean (n, β) -normed spaces.

2. Preliminaries

Now, we recall some notions and results which will be used.

Throughout this paper, let $\mathbb{N}$ denote the set of nonnegative integers and $i, t, p, n \in \mathbb{N}$, and let $n \geq 2$ be fixed.

Definition 1 (see [27]). Let E be a linear space over $\mathbb{R}$ with $\dim E \geq n$, $n \in \mathbb{N}$, and $0 < \beta \leq 1$, let a mapping $\|\cdot, \dots, \cdot\|_\beta : E^n \rightarrow \mathbb{R}$ satisfies the following conditions:

- (i) $\|m_1, \dots, m_n\|_\beta = 0$ if and only if $m_1, \dots, m_n$ are linearly dependent
- (ii) $\|m_1, \dots, m_n\|_\beta$ is invariant under permutations of $m_1, \dots, m_n$
- (iii) $\|\lambda m_1, \dots, m_n\|_\beta = |\lambda|^\beta \|m_1, \dots, m_n\|_\beta$
- (iv) $\|m_1, \dots, m_{n-1}, u + v\|_\beta \leq \|m_1, \dots, m_{n-1}, u\|_\beta + \|m_1, \dots, m_{n-1}, v\|_\beta$

for all $m_1, \dots, m_n \in E$, and $\lambda \in \mathbb{R}$.

Then, $\|\cdot, \dots, \cdot\|_\beta$ is called as (n, β) -norm on E , and $(E, \|\cdot, \dots, \cdot\|_\beta)$ is called a linear (n, β) -normed space or (n, β) -normed space.

We remark that the notion of a linear (n, β) -normed space is a summed up of a linear n -normed space ($\beta = 1$) and of a β -normed space ($n = 1$).

Definition 2 (see [27]). Let E be a real vector space with $\dim E \geq n$ over a scalar field K with a non-Archimedean nontrivial valuation $|\cdot|$, where $n \in \mathbb{Z}^+$ and a constant β with $0 < \beta \leq 1$. A real-valued function $\|\cdot, \dots, \cdot\|_\beta : E^n \rightarrow \mathbb{R}$ is called an (n, β) -norm on E if the upcoming conditions true:

- (a) $\|m_1, \dots, m_n\|_\beta = 0$ if and only if $m_1, \dots, m_n$ are linearly dependent
- (b) $\|m_1, \dots, m_n\|_\beta$ is invariant under permutations of $m_1, \dots, m_n$
- (c) $\|\lambda m_1, \dots, m_n\|_\beta = |\lambda|^\beta \|m_1, \dots, m_n\|_\beta$
- (d) $\|m_0 + m_1, \dots, m_n\|_\beta \leq \max \{ \|m_0, m_2, \dots, m_n\|_\beta, \|m_1, m_2, \dots, m_n\|_\beta \}$

for all $\lambda \in K$ and $m_0, m_1, \dots, m_n \in E$. Then, $(E, \|\cdot, \dots, \cdot\|_\beta)$ is called as non-Archimedean (n, β) -normed space.

Example 3. Let p be a prime number. For any nonzero rational number $x = (a/b)p^r$ such that a and b are coprime to the prime number p , define the p -adic absolute value $\|x\|_p := p^{-r}$. Then, $\|\cdot\|_p$ is a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to $\|\cdot\|_p$ is denoted by $\mathbb{Q}_p$ and is called the p -adic number field.

Note that if $p > 3$; then, $\|2^n\| = 1$ in for each integer n .

Remark 4 (see [27]). A sequence $\{v_m\}$ in a non-Archimedean (n, β) -normed space E is a Cauchy sequence if and only if $\{v_{m+1} - v_m\}$ converges to zero.

Lemma 5 (see [27]). For a convergent sequence $\{v_p\}$ in a linear (n, β) -normed space E ,

$$\lim_{p \rightarrow \infty} \|v_p, w_1, \dots, w_{n-1}\|_\beta = \left\| \lim_{p \rightarrow \infty} v_p, w_1, \dots, w_{n-1} \right\|_\beta, \quad (2)$$

for all $w_1, \dots, w_{n-1} \in E$.

Lemma 6 (see [27]). Let $(E, \|\cdot, \dots, \cdot\|_\beta)$ be a linear (n, β) -normed space, $n \geq 2$, $0 < \beta \leq 1$. If $v_1 \in E$ and $\|v_1, w_1, \dots, w_{n-1}\|_\beta = 0$ for all $w_1, \dots, w_{n-1} \in E$, then $v_1 = 0$.

3. General Solution

Here, the authors discussed the general solution of the equation (1). Consider E and F are real vector spaces.

Theorem 7. If $\phi : E \rightarrow F$ be a mapping satisfies the functional equation (1) for all $v_1, v_2, \dots, v_l \in E$, then the mapping $\phi : E \rightarrow F$ is quadratic, that is ϕ satisfies the equality

$$\phi(v_1 + v_2) + \phi(v_1 - v_2) = 2\phi(v_1) + 2\phi(v_2), \quad (3)$$

for all $v_1, v_2 \in E$.

Proof. Suppose that the mapping $\phi : E \rightarrow F$ satisfies the functional equation (1). Replacing $(v_1, v_2, \dots, v_l)$ by $(0, 0, \dots, 0)$ in (1), we get $\phi(0) = 0$. Now, setting $(v_1, v_2, \dots, v_l)$ by $(v, 0, \dots, 0)$ in (1), we get $\phi(-v) = \phi(v)$ for all $v \in E$. Therefore, the function ϕ is an even function. Substituting $(v_1, v_2, \dots, v_l)$ by $(v, v, 0, \dots, 0)$ and $(v, v, v, 0, \dots, 0)$ in (1), we obtain $\phi(2v) = 2^2 \phi(v)$, $\phi(3v) = 3^2 \phi(v)$, and so on, for every $v \in E$. In general, for any nonnegative integer $n > 0$, we get $\phi(nv) = n^2 \phi(v)$ for all $v \in E$. Next, replacing $(v_1, v_2, \dots, v_l)$ by $(v_1, v_2, 0, \dots, 0)$ in (1), we get our desired outcome.

4. Stability Results

Here, we consider $|2| \neq 1$ and examine the Hyers-Ulam stability of the functional equation (1).

Let us assume E and F are non-Archimedean β_1 -normed space and complete non-Archimedean (n, β) -normed space, respectively, where $n \geq 2$, $0 < \beta, \beta_1 \leq 1$.

Define a mapping $\Delta\phi : E \longrightarrow F$ by

$$\Delta\phi(v_1, v_2, \dots, v_l) = \sum_{1 \leq i < j \leq l} \phi(v_i + v_j) + \sum_{1 \leq i < j \leq l} \phi(v_i - v_j) - 2(l-1) \sum_{1 \leq i \leq l} \phi(v_i), \quad (4)$$

for all $v_1, v_2, \dots, v_l \in E$.

Theorem 8. Let $\tau \in [0, \infty)$ and $s \in (0, \infty)$ with $s\beta_1 > \beta$, and let $\chi : F \times F \times \dots \times F \longrightarrow [0, \infty)$ be a function. Suppose that $\phi : \underbrace{E \times \dots \times E}_{n-1} \longrightarrow F$ be a mapping satisfies

$$\|\Delta\phi(v_1, v_2, \dots, v_l), w_1, \dots, w_{n-1}\|_\beta \leq \tau \sum_{1 \leq j \leq l} \|v_j\|_{\beta_1}^s \chi(m_1, \dots, m_{n-1}), \quad (5)$$

for all $v_1, v_2, \dots, v_l$ and $m_1, \dots, m_{n-1} \in F$. Then, there exists a unique quadratic mapping $Q_2 : E \longrightarrow F$ satisfies

$$\|\phi(v) - Q_2(v), w_1, \dots, w_{n-1}\|_\beta \leq 2\tau \left| 2^{-2\beta} \right| \|v\|_{\beta_1}^s \chi(m_1, \dots, m_{n-1}), \quad (6)$$

for all $v \in E, m_1, \dots, m_{n-1} \in F$.

Proof. Replacing $(v_1, v_2, \dots, v_l)$ by $(v, v, 0, \dots, 0)$ in (5), we obtain

$$\|\phi(2v) - 2^2\phi(v), w_1, \dots, w_{n-1}\|_\beta \leq 2\tau \|v\|_{\beta_1}^s \chi(m_1, \dots, m_{n-1}), \quad (7)$$

for all $v \in E, m_1, \dots, m_{n-1} \in F$. Dividing both sides by $|2^{2\beta}|$, we have

$$\left\| \frac{\phi(2v)}{2^2} - \phi(v), w_1, \dots, w_{n-1} \right\|_\beta \leq 2 \left| 2^{-2\beta} \right| \tau \|v\|_{\beta_1}^s \chi(m_1, \dots, m_{n-1}), \quad (8)$$

for all $v \in E, m_1, \dots, m_{n-1} \in F$. Replacing v by $2^p v$ in (8), we attain

$$\begin{aligned} & \left\| \frac{\phi(2^{p+1}v)}{2^{2(p+1)}} - \frac{\phi(2^p v)}{2^{2p}}, m_1, \dots, m_{n-1} \right\|_\beta \\ & \leq 2 \left| 2^{-(p+1)\beta} \right| \tau \|2^p v\|_{\beta_1}^s \chi(m_1, \dots, m_{n-1}) \\ & \leq 2 \left| 2^{-2\beta} \right| \left| 2^{s\beta_1 - 2\beta} \right|^p \tau \|v\|_{\beta_1}^s \chi(m_1, \dots, m_{n-1}). \end{aligned} \quad (9)$$

As $s\beta_1 > \beta$ and $|2| \neq 1$, we obtain that

$$\lim_{p \rightarrow \infty} \left\| \frac{\phi(2^{p+1}v)}{2^{2(p+1)}} - \frac{\phi(2^p v)}{2^{2p}}, m_1, \dots, m_{n-1} \right\|_\beta = 0, \quad (10)$$

for all $v \in E, m_1, \dots, m_{n-1} \in F$. From Remark 4, we conclude that the sequence $\{\phi(2^p v)/2^{2p}\}$ is a Cauchy sequence in F . Since F is complete space, we can define $Q_2 : E \longrightarrow F$ by

$$Q_2(v) = \lim_{p \rightarrow \infty} \frac{\phi(2^p v)}{2^{2p}}, \quad (11)$$

for all $v \in E$. Next, our aim is to prove that the function Q_2 is quadratic. From (5), (11) and Lemma 5 that

$$\begin{aligned} & \|\Delta Q_2(v_1, v_2, \dots, v_l), m_1, \dots, m_{n-1}\|_\beta \\ & = \lim_{p \rightarrow \infty} \left| 2^{-2p\beta} \right| \|\Delta\phi(2^p v_1, 2^p v_2, \dots, 2^p v_l), m_1, \dots, m_{n-1}\|_\beta \\ & \leq \lim_{p \rightarrow \infty} \left| 2^{-2p\beta} \right| \tau \sum_{1 \leq j \leq l} \|2^p v_j\|_{\beta_1}^s \chi(m_1, \dots, m_{n-1}) \\ & \leq \lim_{p \rightarrow \infty} \tau \left| 2^{s\beta_1 - 2\beta} \right|^p \sum_{1 \leq j \leq l} \|v_j\|_{\beta_1}^s \chi(m_1, \dots, m_{n-1}), \end{aligned} \quad (12)$$

for all $v_1, v_2, \dots, v_l \in E$ and $m_1, \dots, m_{n-1} \in F$. As $s\beta_1 > \beta$ and $|2| \neq 1$, we obtain

$$\|\Delta Q_2(v_1, v_2, \dots, v_l), m_1, \dots, m_{n-1}\|_\beta = 0, \quad (13)$$

for all $v_1, v_2, \dots, v_l \in E, m_1, \dots, m_{n-1} \in F$. By Lemma 6, we have

$$\Delta Q_2(v_1, v_2, \dots, v_l) = 0, \quad (14)$$

for all $v_1, v_2, \dots, v_l \in E$. Therefore, the function Q_2 is quadratic. Switching v through $2v$ in (7) and divide by $|2^{2\beta}|$, we attain

$$\left\| \frac{\phi(2^2 v)}{2^4} - \frac{\phi(2v)}{2^2}, m_1, \dots, m_{n-1} \right\|_\beta \leq 2\tau \left| 2^{-4\beta} \right| \|2v\|_{\beta_1}^s \chi(m_1, \dots, m_{n-1}). \quad (15)$$

Thus, by (7) and (15), we reach

$$\begin{aligned} & \left\| \phi(v) - \frac{\phi(2^2 v)}{2^4}, m_1, \dots, m_{n-1} \right\|_\beta \\ & \leq \max \left\{ \left\| \frac{\phi(2v)}{2^2} - \phi(v), m_1, \dots, m_{n-1} \right\|_\beta, \left\| \frac{\phi(2^2 v)}{2^4} - \frac{\phi(2v)}{2^2}, m_1, \dots, m_{n-1} \right\|_\beta \right\} \\ & \leq \max \left\{ 2\tau \left| 2^{-2\beta} \right| \|v\|_{\beta_1}^s \chi(m_1, \dots, m_{n-1}), 2\tau \left| 2^{-4\beta} \right| \|2v\|_{\beta_1}^s \chi(m_1, \dots, m_{n-1}) \right\}. \end{aligned} \quad (16)$$

As $s\beta_1 > \beta$ and $|2| \neq 1$, we obtain

$$\left\| \phi(v) - \frac{\phi(2^2 v)}{2^4}, m_1, \dots, m_{n-1} \right\|_\beta \leq |2^{-2\beta}| 2\tau \|v\|_{\beta_1}^s \chi(m_1, \dots, m_{n-1}), \quad (17)$$

for all $v \in E, m_1, \dots, m_{n-1} \in F$. By induction on p , we can conclude that

$$\left\| \phi(v) - \frac{\phi(2^p v)}{2^{2p}}, m_1, \dots, m_{n-1} \right\|_\beta \leq 2 |2^{-2\beta}| \tau \|v\|_{\beta_1}^s \chi(m_1, \dots, m_{n-1}), \quad (18)$$

for all $p \in \mathbb{N}, v \in E$, and $m_1, \dots, m_{n-1} \in F$. Replacing v by $2v$ in (18) and dividing both sides by $|2^{2\beta}|$, we have

$$\left\| \frac{\phi(2v)}{2^2} - \frac{\phi(2^{p+1}v)}{2^{2(p+1)}}, m_1, \dots, m_{n-1} \right\|_\beta \leq 2 |2^{-4\beta}| \tau \|2v\|_{\beta_1}^s \chi(m_1, \dots, m_{n-1}), \quad (19)$$

for all $v \in E, m_1, \dots, m_{n-1} \in F$, and $p \in \mathbb{N}$. It follows from (7) and (19) that

$$\left\| \phi(v) - \frac{\phi(2^{p+1}v)}{2^{2(p+1)}}, m_1, \dots, m_{n-1} \right\|_\beta \leq 2 |2^{-2\beta}| \tau \|v\|_{\beta_1}^s \chi(m_1, \dots, m_{n-1}), \quad (20)$$

for all $v \in E, m_1, \dots, m_{n-1} \in F$, and $p \in \mathbb{N}$. Passing the limit as p tends to ∞ in inequality (18), we can get (6). Next, we want to prove that the function Q_2 is unique. Let Q'_2 be an another quadratic mapping which satisfies (6),

$$\begin{aligned} & \|Q_2(v) - Q'_2(v), m_1, \dots, m_{n-1}\|_\beta \\ &= |2^{-2p\beta}| \|Q_2(2^p v) - Q'_2(2^p v), m_1, \dots, m_{n-1}\|_\beta \\ &\leq |2^{-2p\beta}| \max \left\{ \|Q_2(2^p v) - \phi(2^p v), m_1, \dots, m_{n-1}\|_\beta, \|\phi(2^p v) - Q'_2(2^p v), m_1, \dots, m_{n-1}\|_\beta \right\} \\ &\leq 2 |2^{-2p\beta}| |2^{-2\beta}| \tau \|2^p v\|_{\beta_1}^s \chi(m_1, \dots, m_{n-1}) \\ &\leq 2\tau |2^{s\beta_1 - 2\beta}| |2^{-2\beta}| \|v\|_{\beta_1}^s \chi(m_1, \dots, m_{n-1}). \end{aligned} \quad (21)$$

Taking the limit as p tends to ∞ in the last inequality, we obtain that

$$\|Q_2(v) - Q'_2(v), m_1, \dots, m_{n-1}\|_\beta = 0, \quad (22)$$

for all $v \in E, m_1, \dots, m_{n-1} \in F$. By Lemma 6, we conclude that $Q_2(v) = Q'_2(v)$ for all $v \in E$. Hence, Q_2 is the unique quadratic mapping which satisfies (6).

Theorem 9. Let $\tau \in [0, \infty)$, $s \in (0, \infty)$ with $s\beta_1 < \beta$. Let χ

: $\underbrace{F \times F \times \dots \times F}_{n-1} \longrightarrow [0, \infty)$ be a function. Suppose that a mapping $\phi : E \longrightarrow F$ satisfies

$$\|\Delta\phi(v_1, v_2, \dots, v_l), m_1, \dots, m_{n-1}\|_\beta \leq \tau \sum_{1 \leq j \leq l} \|v_j\|_{\beta_1}^s \chi(m_1, \dots, m_{n-1}), \quad (23)$$

for all $v_1, v_2, \dots, v_l$ and $m_1, \dots, m_{n-1} \in F$. Then, there exists a unique quadratic mapping $Q_2 : E \longrightarrow F$ satisfies

$$\|\phi(v) - Q_2(v), m_1, \dots, m_{n-1}\|_\beta \leq 2\tau |2^{-s\beta_1}| \|v\|_{\beta_1}^s \chi(m_1, \dots, m_{n-1}), \quad (24)$$

for all $v \in E, m_1, \dots, m_{n-1} \in F$.

Proof. Switching $(v_1, v_2, \dots, v_l)$ by $(v, v, 0, \dots, 0)$ in (23), we have

$$\|\phi(2v) - 2^2\phi(v), m_1, \dots, m_{n-1}\|_\beta \leq 2\tau \|v\|_{\beta_1}^s \chi(m_1, \dots, m_{n-1}), \quad (25)$$

for all $v \in E, m_1, \dots, m_{n-1} \in F$. Interchanging v by $v/2$ in (25), we obtain

$$\left\| \phi(v) - 2^2\phi\left(\frac{v}{2}\right), m_1, \dots, m_{n-1} \right\|_\beta \leq 2\tau |2^{-s\beta_1}| \|v\|_{\beta_1}^s \chi(m_1, \dots, m_{n-1}), \quad (26)$$

for all $v \in E, m_1, \dots, m_{n-1} \in F$. Replacing v by $v/2^p$ in (26), we reach

$$\begin{aligned} & \left\| 2^{2p}\phi\left(\frac{v}{2^p}\right) - 2^{2(p+1)}\phi\left(\frac{v}{2^{p+1}}\right), m_1, \dots, m_{n-1} \right\|_\beta \\ &\leq 2 |2^{2p\beta}| \tau |2^{-(p+1)s\beta_1}| \|v\|_{\beta_1}^s \chi(m_1, \dots, m_{n-1}) \\ &\leq 2 |2^{-s\beta_1}| |2^{2\beta - s\beta_1}| p\tau \|v\|_{\beta_1}^s \chi(m_1, \dots, m_{n-1}), \end{aligned} \quad (27)$$

for all $v \in E$ and $m_1, \dots, m_{n-1} \in F$. As $s\beta_1 < \beta$ and $|2| \neq 1$, we have

$$\lim_{p \rightarrow \infty} \left\| 2^{2(p+1)}\phi\left(\frac{v}{2^{p+1}}\right) - 2^{2p}\phi\left(\frac{v}{2^p}\right), m_1, \dots, m_{n-1} \right\|_\beta = 0, \quad (28)$$

for all $v \in E$ and $m_1, \dots, m_{n-1} \in F$. From Remark 4, we conclude that the sequence $\{2^{2p}\phi(v/2^p)\}$ is a Cauchy sequence in F . As F is complete space. We can define $Q_2 : E \longrightarrow F$ by

$$Q_2(v) = \lim_{p \rightarrow \infty} 2^{2p}\phi\left(\frac{v}{2^p}\right), \quad (29)$$

for all $v \in E$. Next, our aim is to prove that the function Q_2 is

quadratic. From (23), (29) and Lemma 5 that

$$\begin{aligned} & \|\Delta Q_2(v_1, v_2, \dots, v_l), m_1, \dots, m_{n-1}\|_\beta \\ &= \lim_{p \rightarrow \infty} \left| 2^{2p\beta} \left\| \Delta \phi \left(\frac{v_1}{2^p}, \frac{v_2}{2^p}, \dots, \frac{v_l}{2^p} \right), m_1, \dots, m_{n-1} \right\|_\beta \right| \\ &\leq \lim_{p \rightarrow \infty} \left| 2^{2p\beta} \tau \sum_{1 \leq j \leq l} \left\| \frac{v_j}{2^p} \right\|_{\beta_1}^s \chi(m_1, \dots, m_{n-1}) \right| \\ &\leq \lim_{p \rightarrow \infty} \tau \left| 2^{2\beta-s\beta_1} \right|^p \sum_{1 \leq j \leq l} \|v_j\|_{\beta_1}^s \chi(m_1, \dots, m_{n-1}), \end{aligned} \quad (30)$$

for all $v_1, v_2, \dots, v_l \in E$ and $m_1, \dots, m_{n-1} \in F$. Since $s\beta_1 < \beta$ and $|2| \neq 1$, we have

$$\|\Delta Q_2(v_1, v_2, \dots, v_l), m_1, \dots, m_{n-1}\|_\beta = 0, \quad (31)$$

for all $v_1, v_2, \dots, v_l \in E, m_1, m_2, \dots, m_{n-1} \in F$. By Lemma 6, we obtain

$$\Delta Q_2(v_1, v_2, \dots, v_l) = 0, \quad (32)$$

for all $v_1, v_2, \dots, v_l \in E$. Therefore, the function Q_2 is quadratic. Switching v through $v/2$ in (25) and multiplying by $|2^{2\beta}|$, we reach

$$\left\| 2^4 \phi \left(\frac{v}{2^2} \right) - 2^2 \phi \left(\frac{v}{2} \right), m_1, \dots, m_{n-1} \right\|_\beta \leq 2\tau \left| 2^{2\beta} \right| \left\| \frac{v}{2^2} \right\|_{\beta_1}^s \chi(m_1, \dots, m_{n-1}). \quad (33)$$

Thus, by (25) and (33), we obtain

$$\begin{aligned} & \left\| \phi(v) - 2^4 \phi \left(\frac{v}{2^2} \right), m_1, \dots, m_{n-1} \right\|_\beta \\ &\leq \max \left\{ \left\| 2^2 \phi \left(\frac{v}{2} \right) - \phi(v), m_1, \dots, m_{n-1} \right\|_\beta, \left\| 2^4 \phi \left(\frac{v}{2^2} \right) - 2^2 \phi \left(\frac{v}{2} \right), m_1, \dots, m_{n-1} \right\|_\beta \right\} \\ &\leq \max \left\{ 2\tau \left\| \frac{v}{2} \right\|_{\beta_1}^s \chi(m_1, \dots, m_{n-1}), 2\tau \left| 2^{2\beta} \right| \left\| \frac{v}{2^2} \right\|_{\beta_1}^s \chi(m_1, \dots, m_{n-1}) \right\}. \end{aligned} \quad (34)$$

Since $s\beta_1 < \beta$ and $|2| \neq 1$, we attain

$$\left\| \phi(v) - 2^4 \phi \left(\frac{v}{2^2} \right), m_1, \dots, m_{n-1} \right\|_\beta \leq \left| 2^{-s\beta_1} \right| 2\tau \|v\|_{\beta_1}^s \chi(m_1, \dots, m_{n-1}), \quad (35)$$

for all $v \in E, m_1, \dots, m_{n-1} \in F$. By induction on p , we can conclude that

$$\left\| \phi(v) - 2^{2p} \phi \left(\frac{v}{2^p} \right), m_1, \dots, m_{n-1} \right\|_\beta \leq 2 \left| 2^{-s\beta_1} \right| \tau \|v\|_{\beta_1}^s \chi(m_1, \dots, m_{n-1}), \quad (36)$$

for all $p \in \mathbb{N}, v \in E$ and $m_1, \dots, m_{n-1} \in F$. Replacing v by $v/2$ in

(36) and multiplying both sides by $|2^{2\beta}|$, we get.

$$\begin{aligned} & \left\| 2^2 \phi \left(\frac{v}{2} \right) - 2^{2(p+1)} \phi \left(\frac{v}{2^{p+1}} \right), m_1, \dots, m_{n-1} \right\|_\beta \\ &\leq 2 \left| 2^{2\beta} \right| \left| 2^{-2s\beta_1} \right| \tau \|v\|_{\beta_1}^s \chi(m_1, \dots, m_{n-1}), \end{aligned} \quad (37)$$

for all $v \in E, m_1, \dots, m_{n-1} \in F$. It follows from (25) and (37) that

$$\left\| \phi(v) - 2^{2(p+1)} \phi \left(\frac{v}{2^{p+1}} \right), m_1, \dots, m_{n-1} \right\|_\beta \leq 2 \left| 2^{-s\beta_1} \right| \tau \|v\|_{\beta_1}^s \chi(m_1, \dots, m_{n-1}), \quad (38)$$

for all $v \in E, m_1, \dots, m_{n-1} \in F$, and $p \in \mathbb{N}$. Passing the limit as p tends to ∞ in (36), we can get (24). Finally, we want to prove that the function Q_2 is unique. Consider an another quadratic mapping Q'_2 satisfying (24),

$$\begin{aligned} & \|Q_2(v) - Q'_2(v), m_1, \dots, m_{n-1}\|_\beta \\ &= \left| 2^{2p\beta} \left\| Q_2 \left(\frac{v}{2^p} \right) - Q'_2 \left(\frac{v}{2^p} \right), m_1, \dots, m_{n-1} \right\|_\beta \right| \\ &\leq \left| 2^{2p\beta} \right| \max \left\{ \left\| Q_2 \left(\frac{v}{2^p} \right) - \phi \left(\frac{v}{2^p} \right), m_1, \dots, m_{n-1} \right\|_\beta, \left\| \phi \left(\frac{v}{2^p} \right) - Q'_2 \left(\frac{v}{2^p} \right), m_1, \dots, m_{n-1} \right\|_\beta \right\} \\ &\leq 2 \left| 2^{2p\beta} \right| \left| 2^{-s\beta_1} \right| \tau \left\| \frac{v}{2^p} \right\|_{\beta_1}^s \chi(m_1, \dots, m_{n-1}) \\ &\leq 2\tau \left| 2^{2\beta-s\beta_1} \right|^p \left| 2^{-s\beta_1} \right| \|v\|_{\beta_1}^s \chi(m_1, \dots, m_{n-1}), \end{aligned} \quad (39)$$

for all $v \in E$ and $m_1, \dots, m_{n-1} \in F$. Taking the limit as p tends to ∞ , we obtain that

$$\|Q_2(v) - Q'_2(v), m_1, \dots, m_{n-1}\|_\beta = 0, \quad (40)$$

for all $v \in E, m_1, \dots, m_{n-1} \in F$. By Lemma 6, we conclude that $Q_2(v) = Q'_2(v)$ for all $v \in E$. So that the function Q_2 is the unique quadratic function. Hence, the proof is completed.

We obtain the following results of theorem with a generalized control function when the domain E is a vector space and codomain F be a complete non-Archimedean (n, β) -normed space, where $n \geq 2$ and $0 < \beta \leq 1$.

Theorem 10. Let $\psi : E^l \rightarrow [0, \infty)$ be a function such that

$$\lim_{p \rightarrow \infty} \left| \frac{1}{2^{2p\beta}} \right| \psi(2^p v_1, 2^p v_2, \dots, 2^p v_l) = 0, \quad (41)$$

for all $v_1, v_2, \dots, v_l \in E$, and let $\chi : \underbrace{F \times F \times \dots \times F}_{n-1} \rightarrow [0, \infty)$

be a function. Suppose $\phi : E \rightarrow F$ be a mapping which satisfies

$$\|\Delta \phi(v_1, v_2, \dots, v_l), m_1, \dots, m_{n-1}\|_\beta \leq \psi(v_1, v_2, \dots, v_l) \chi(m_1, \dots, m_{n-1}), \quad (42)$$

for all $v_1, v_2, \dots, v_l \in E$ and $m_1, \dots, m_{n-1} \in F$. Then, there exists a quadratic mapping $Q_2 : E \longrightarrow F$ such that

$$\|\phi(v) - Q_2(v), m_1, \dots, m_{n-1}\|_\beta \leq \tilde{\psi}(v) \chi(m_1, \dots, m_{n-1}), \quad (43)$$

where

$$\tilde{\psi}(v) = \lim_{p \rightarrow \infty} \max \left\{ \left| 2^{-2i\beta} \right| \psi(2^{i-1}v, 2^{i-1}v, 0, \dots, 0) : 1 \leq i \leq p \right\}, \quad (44)$$

for all $v \in E, m_1, \dots, m_{n-1} \in F$. Moreover, if

$$\lim_{t \rightarrow \infty} \lim_{p \rightarrow \infty} \max \left\{ \left| 2^{-2i\beta} \right| \psi(2^{i-1}v, 2^{i-1}v, 0, \dots, 0) : 1+t \leq i \leq p+t \right\} = 0, \quad (45)$$

for all $v \in E$. Then, the unique mapping Q_2 is quadratic which satisfies the inequality (43).

Proof. Setting $(v_1, v_2, \dots, v_l)$ by $(v, v, 0, \dots, 0)$ in (42) and dividing both sides by $|2^{2\beta}|$, we have

$$\begin{aligned} & \left\| \frac{\phi(2v)}{2^2} - \phi(v), m_1, \dots, m_{n-1} \right\|_\beta \\ & \leq \left| 2^{-2\beta} \right| \psi(v, v, 0, \dots, 0) \chi(m_1, \dots, m_{n-1}), \end{aligned} \quad (46)$$

for all $v \in E$ and $m_1, \dots, m_{n-1} \in F$. Replacing v by $2^i v$ in (46) and dividing both sides by $|2^{2i\beta}|$, we obtain

$$\begin{aligned} & \left\| \frac{\phi(2^{i+1}v)}{2^{2(i+1)}} - \frac{\phi(2^i v)}{2^{2i}}, m_1, \dots, m_{n-1} \right\|_\beta \\ & \leq \left| 2^{-2i\beta} \right| \left| 2^{-2\beta} \right| \psi(2^i v, 2^i v, 0, \dots, 0) \chi(m_1, \dots, m_{n-1}), \end{aligned} \quad (47)$$

for all $v \in E, m_1, \dots, m_{n-1} \in F$, and $i \in \mathbb{N}$. Taking the limit as i tends to ∞ and considering (41), we attain

$$\lim_{i \rightarrow \infty} \left\| \frac{\phi(2^{i+1}v)}{2^{2(i+1)}} - \frac{\phi(2^i v)}{2^{2i}}, m_1, \dots, m_{n-1} \right\|_\beta = 0, \quad (48)$$

for all $v \in E$ and $m_1, \dots, m_{n-1} \in F$. Utilizing Remark 4, it is clear that the sequence $\{\phi(2^m v)/2^{2m}\}$ is a Cauchy sequence. As F is a complete space. We can define the mapping $Q_2 : E \longrightarrow F$ by

$$\lim_{i \rightarrow \infty} \left\| \frac{\phi(2^{i+1}v)}{2^{2(i+1)}} - \frac{\phi(2^i v)}{2^{2i}}, m_1, \dots, m_{n-1} \right\|_\beta = 0, \quad (49)$$

for all $v \in E, m_1, \dots, m_{n-1} \in F$. Next, we want to prove that the

function Q_2 is quadratic. So,

$$\begin{aligned} & \|\Delta Q_2(v_1, v_2, \dots, v_l), m_1, \dots, m_{n-1}\|_\beta \\ & \leq \left| 2^{-2p\beta} \right| \|\Delta Q_2(2^p v_1, 2^p v_2, \dots, 2^p v_l), m_1, \dots, m_{n-1}\|_\beta \\ & \leq \left| 2^{-2p\beta} \right| \psi(2^p v, 2^p v, 0, \dots, 0) \chi(m_1, \dots, m_{n-1}), \end{aligned} \quad (50)$$

for all $v_1, v_2, \dots, v_l \in E$ and $m_1, \dots, m_{n-1} \in F$. Taking the limit as $p \rightarrow \infty$ and considering (41), we arrive

$$\|\Delta Q_2(v_1, v_2, \dots, v_l), m_1, \dots, m_{n-1}\|_\beta = 0, \quad (51)$$

for all $v \in E, m_1, \dots, m_{n-1} \in F$. Using Lemma 6, we conclude that Q_2 is quadratic. Switching v through $2v$ in (46) and dividing by $|2^{2\beta}|$, we obtain

$$\begin{aligned} & \left\| \frac{\phi(2^2 v)}{2^4} - \frac{\phi(2v)}{2^2}, m_1, \dots, m_{n-1} \right\|_\beta \\ & \leq \left| 2^{-4\beta} \right| \psi(2v, 2v, 0, \dots, 0) \chi(m_1, \dots, m_{n-1}), \end{aligned} \quad (52)$$

for all $v \in E$ and $m_1, \dots, m_{n-1} \in F$. Considering (46), we receive

$$\begin{aligned} & \left\| \phi(v) - \frac{\phi(2^2 v)}{2^4}, m_1, \dots, m_{n-1} \right\|_\beta \\ & \leq \max \left\{ \left| 2^{-2\beta} \right| \psi(v, v, 0, \dots, 0), \left| 2^{-4\beta} \right| \psi(2v, 2v, 0, \dots, 0) \right\} \chi(m_1, \dots, m_{n-1}). \end{aligned} \quad (53)$$

By induction on p , we reach

$$\begin{aligned} & \left\| \phi(v) - \frac{\phi(2^p v)}{2^{2p}}, m_1, \dots, m_{n-1} \right\|_\beta \\ & \leq \max \left\{ \frac{\psi(2^{t-1}v, 2^{t-1}v, 0, \dots, 0)}{|2^{2t\beta}|} : 1 \leq t \leq p \right\} \chi(m_1, \dots, m_{n-1}), \end{aligned} \quad (54)$$

for all $v \in E$ and $m_1, \dots, m_{n-1} \in F$. Replacing v by $2v$ in (54) and dividing both sides by $|2^{2\beta}|$, we have

$$\begin{aligned} & \left\| \frac{\phi(2v)}{2^2} - \frac{\phi(2^{(m+1)}v)}{2^{2(m+1)}}, m_1, \dots, m_{n-1} \right\|_\beta \\ & \leq \max \left\{ \frac{\psi(2^t v, 2^t v, 0, \dots, 0)}{|2^{2(t+1)\beta}|} : 1 \leq t \leq p \right\} \chi(m_1, \dots, m_{n-1}), \end{aligned} \quad (55)$$

for all $v \in E, m_1, \dots, m_{n-1} \in F$, and $p \in \mathbb{N}$, which together with

(46) implies

$$\begin{aligned}
& \left\| \phi(v) - \frac{\phi(2^{p+1}v)}{2^{2(p+1)}}, m_1, \dots, m_{n-1} \right\|_\beta \\
& \leq \max \left\{ \frac{\psi(v, v, 0, \dots, 0)}{|2^{2\beta}|}, \frac{\psi(2^t v, 2^t v, 0, \dots, 0)}{|2^{2(t+1)\beta}|} : 1 \leq t \leq p \right\} \chi(m_1, \dots, m_{n-1}) \\
& \leq \max \left\{ \left| 2^{-2(t+1)\beta} \right| \psi(2^t v, 2^t v, 0, \dots, 0) : 0 \leq t \leq p \right\} \chi(m_1, \dots, m_{n-1}) \\
& \leq \max \left\{ \left| 2^{-2t\beta} \right| \psi(2^{t-1} v, 2^{t-1} v, 0, \dots, 0) : 1 \leq t \leq p+1 \right\} \chi(m_1, \dots, m_{n-1}),
\end{aligned} \quad (56)$$

for all $v \in E$, $m_1, \dots, m_{n-1} \in F$, and $p \in \mathbb{N}$. Passing the limit as p tends to ∞ in (54), we get (43). Finally, we want to show that the quadratic function Q_2 is unique. Consider another quadratic function Q'_2 satisfying (43). Hence,

$$\begin{aligned}
& \lim_{p \rightarrow \infty} \left| 2^{-2t\beta} \right| \tilde{\psi}(2^t v) \\
& = \lim_{t \rightarrow \infty} \left| 2^{-2t\beta} \right| \lim_{p \rightarrow \infty} \max \left\{ \left| 2^{-2i\beta} \right| \psi(2^{i+t-1} v, 2^{i+t-1} v, 0, \dots, 0) : 1 \leq i \leq p \right\} \\
& \leq \lim_{t \rightarrow \infty} \lim_{p \rightarrow \infty} \max \left\{ \left| 2^{-2i\beta} \right| \psi(2^{i-1} v, 2^{i-1} v, 0, \dots, 0) : 1+t \leq i \leq p+t \right\},
\end{aligned} \quad (57)$$

for all $v \in E$ and $m_1, \dots, m_{n-1} \in F$. It follows from (45) that

$$\begin{aligned}
& \left\| Q_2(v) - Q'_2(v), m_1, \dots, m_{n-1} \right\|_\beta \\
& \leq \lim_{t \rightarrow \infty} \left| 2^{-2t\beta} \right| \left\| Q_2(2^t v) - Q'_2(2^t v), m_1, \dots, m_{n-1} \right\|_\beta \\
& \leq \lim_{t \rightarrow \infty} \left| 2^{-2t\beta} \right| \max \left\{ \left\| Q_2(2^t v) - \phi(2^t v), m_1, \dots, m_{n-1} \right\|_\beta, \left\| \phi(2^t v) - Q'_2(2^t v), m_1, \dots, m_{n-1} \right\|_\beta \right\} \\
& \leq \lim_{t \rightarrow \infty} \left| 2^{-2t\beta} \right| \tilde{\psi}(2^t v) \chi(m_1, \dots, m_{n-1}) = 0.
\end{aligned} \quad (58)$$

From Lemma 6, we conclude that the quadratic function Q_2 is unique. This ends the proof of the theorem.

Theorem 11. Let $\psi : E^l \rightarrow [0, \infty)$ be a mapping such that

$$\lim_{p \rightarrow \infty} \left| 2^{2p\beta} \right| \psi\left(\frac{v_1}{2^p}, \frac{v_2}{2^p}, \dots, \frac{v_l}{2^p}\right) = 0, \quad (59)$$

for all $v_1, v_2, \dots, v_l \in E$, and let $\chi : \underbrace{F \times F \times \dots \times F}_{n-1} \rightarrow [0, \infty)$ be

a function. Suppose that $\phi : E \rightarrow F$ be a mapping satisfies

$$\left\| \Delta\phi(v_1, v_2, \dots, v_l), m_1, \dots, m_{n-1} \right\|_\beta \leq \psi(v_1, v_2, \dots, v_l) \chi(m_1, \dots, m_{n-1}), \quad (60)$$

for all $v_1, v_2, \dots, v_l \in E$ and $m_1, \dots, m_{n-1} \in F$. Then, there exists a quadratic mapping $Q_2 : E \rightarrow F$ such that

$$\left\| \phi(v) - Q_2(v), m_1, \dots, m_{n-1} \right\|_\beta \leq \tilde{\psi}(v) \chi(m_1, \dots, m_{n-1}), \quad (61)$$

where

$$\tilde{\psi}(v) = \lim_{p \rightarrow \infty} \max \left\{ \left| 2^{2(i-1)\beta} \right| \psi(2^{-i} v, 2^{-i} v, 0, \dots, 0) : 1 \leq i \leq p \right\}, \quad (62)$$

for all $v \in E$ and $m_1, \dots, m_{n-1} \in F$. Moreover, if

$$\lim_{t \rightarrow \infty} \lim_{p \rightarrow \infty} \max \left\{ \left| 2^{2(i-1)\beta} \right| \psi(2^{-i} v, 2^{-i} v, 0, \dots, 0) : 1+t \leq i \leq p+t \right\} = 0, \quad (63)$$

for all $v \in E$, then the quadratic mapping Q_2 is unique which satisfies (61).

Proof. Switching $(v_1, v_2, \dots, v_l)$ by $(v, v, 0, \dots, 0)$ in (60), we have

$$\left\| \phi(2v) - 2^2 \phi(v), m_1, \dots, m_{n-1} \right\|_\beta \leq \psi(v, v, 0, \dots, 0) \chi(m_1, \dots, m_{n-1}), \quad (64)$$

for all $v \in E$, $m_1, \dots, m_{n-1} \in F$. Interchanging v by $v/2$ in (64), we obtain

$$\left\| \phi(v) - 2^2 \phi\left(\frac{v}{2}\right), m_1, \dots, m_{n-1} \right\|_\beta \leq \psi\left(\frac{v}{2}, \frac{v}{2}, 0, \dots, 0\right) \chi(m_1, \dots, m_{n-1}), \quad (65)$$

for all $v \in E$, $m_1, \dots, m_{n-1} \in F$. Replacing v by $v/2^{i-1}$ in (65) and multiplying both sides by $|2^{2(i-1)\beta}|$, we obtain

$$\begin{aligned}
& \left\| 2^{2(i)} \phi\left(\frac{v}{2^i}\right) - 2^{2(i-1)} \phi\left(\frac{v}{2^{(i-1)}}\right), m_1, \dots, m_{n-1} \right\|_\beta \\
& \leq \left| 2^{2(i-1)\beta} \right| \psi\left(\frac{v}{2^i}, \frac{v}{2^i}, 0, \dots, 0\right) \chi(m_1, \dots, m_{n-1}),
\end{aligned} \quad (66)$$

for all $v \in E$, $m_1, \dots, m_{n-1} \in F$, and $i \in \mathbb{N}$. Taking the limit as i tends to ∞ and considering (59), we attain

$$\lim_{i \rightarrow \infty} \left\| 2^{2(i)} \phi\left(\frac{v}{2^i}\right) - 2^{2(i-1)} \phi\left(\frac{v}{2^{(i-1)}}\right), m_1, \dots, m_{n-1} \right\|_\beta = 0, \quad (67)$$

for all $v \in E$, $m_1, \dots, m_{n-1} \in F$. From Remark 4, we can conclude that the sequence $\{2^{2m} \phi(v/2^m)\}$ is a Cauchy sequence. We know that F is a complete space; we define a mapping $Q_2 : E \rightarrow F$ by

$$\lim_{i \rightarrow \infty} \left\| 2^{2(i)} \phi\left(\frac{v}{2^i}\right) - 2^{2(i-1)} \phi\left(\frac{v}{2^{(i-1)}}\right), m_1, \dots, m_{n-1} \right\|_\beta = 0, \quad (68)$$

for all $v \in E$, $m_1, \dots, m_{n-1} \in F$. Next, we prove that Q_2 is

quadratic. So,

$$\begin{aligned} & \|\Delta Q_2(v_1, v_2, \dots, v_l), m_1, \dots, m_{n-1}\|_\beta \\ & \leq |2^{2p\beta}| \left\| \Delta Q_2\left(\frac{v_1}{2^p}, \frac{v_2}{2^p}, \dots, \frac{v_l}{2^p}\right), m_1, \dots, m_{n-1} \right\|_\beta \quad (69) \\ & \leq |2^{2p\beta}| \psi\left(\frac{v}{2^p}, \frac{v}{2^p}, 0, \dots, 0\right) \chi(m_1, \dots, m_{n-1}), \end{aligned}$$

for all $v_1, v_2, \dots, v_l \in E$ and $m_1, \dots, m_{n-1} \in F$. Taking the limit as $p \rightarrow \infty$ and considering (59), we arrive

$$\|\Delta Q_2(v_1, v_2, \dots, v_l), m_1, \dots, m_{n-1}\|_\beta = 0, \quad (70)$$

for all $v \in E$ and $m_1, \dots, m_{n-1} \in F$. By Lemma 6, we know that Q_2 is quadratic. Replacing v by $v/2$ in (65) and multiplying both sides by $|2^{2\beta}|$, we obtain

$$\begin{aligned} & \left\| 2^4 \phi\left(\frac{v}{2^2}\right) - 2^2 \phi\left(\frac{v}{2}\right), m_1, \dots, m_{n-1} \right\|_\beta \\ & \leq |2^{2\beta}| \psi\left(\frac{v}{2^2}, \frac{v}{2^2}, 0, \dots, 0\right) \chi(m_1, \dots, m_{n-1}), \quad (71) \end{aligned}$$

for all $v \in E$ and $m_1, \dots, m_{n-1} \in F$. Considering (65), we receive

$$\begin{aligned} & \left\| \phi(v) - 2^4 \phi\left(\frac{v}{2^2}\right), m_1, \dots, m_{n-1} \right\|_\beta \\ & \leq \max \left\{ \psi\left(\frac{v}{2}, \frac{v}{2}, 0, \dots, 0\right), |2^{2\beta}| \psi\left(\frac{v}{2^2}, \frac{v}{2^2}, 0, \dots, 0\right) \right\} \chi(m_1, \dots, m_{n-1}). \quad (72) \end{aligned}$$

By induction on p , we reach

$$\begin{aligned} & \left\| \phi(v) - 2^{2p} \phi\left(\frac{v}{2^p}\right), m_1, \dots, m_{n-1} \right\|_\beta \\ & \leq \max \left\{ |2^{2(t-1)\beta}| \psi(2^{-t}v, 2^{-t}v, 0, \dots, 0): 1 \leq t \leq p \right\} \chi(m_1, \dots, m_{n-1}), \quad (73) \end{aligned}$$

for all $v \in E$ and $m_1, \dots, m_{n-1} \in F$. Replacing v by $v/2$ in (73) and multiplying both sides by $|2^{2\beta}|$, we have

$$\begin{aligned} & \left\| 2^2 \phi\left(\frac{v}{2}\right) - 2^{2(m+1)} \phi\left(\frac{v}{2^{(m+1)}}\right), m_1, \dots, m_{n-1} \right\|_\beta \\ & \leq \max \left\{ |2^{2(t-1)\beta}| \psi\left(\frac{v}{2^t}, \frac{v}{2^t}, 0, \dots, 0\right): 1 \leq t \leq p \right\} \chi(m_1, \dots, m_{n-1}), \quad (74) \end{aligned}$$

which together with (65) implies

$$\begin{aligned} & \left\| \phi(v) - 2^{2(p+1)} \phi\left(\frac{v}{2^{(p+1)}}\right), m_1, \dots, m_{n-1} \right\|_\beta \\ & \leq \max \left\{ \psi\left(\frac{v}{2}, \frac{v}{2}, 0, \dots, 0\right), |2^{2(t-1)\beta}| \psi\left(\frac{v}{2^t}, \frac{v}{2^t}, 0, \dots, 0\right): 1 \leq t \leq p \right\} \chi(m_1, \dots, m_{n-1}) \\ & \leq \max \left\{ |2^{2(t-1)\beta}| \psi\left(\frac{v}{2^t}, \frac{v}{2^t}, 0, \dots, 0\right): 1 \leq t \leq p+1 \right\} \chi(m_1, \dots, m_{n-1}), \quad (75) \end{aligned}$$

for all $v \in E$, $m_1, \dots, m_{n-1} \in F$, and $p \in \mathbb{N}$. Passing the limit as p tends to ∞ in (73), we get (61). At the end, we show that the

quadratic function Q_2 is unique. Consider an another quadratic mapping Q'_2 which satisfies (61). Therefore,

$$\begin{aligned} \lim_{p \rightarrow \infty} |2^{2t\beta}| \tilde{\psi}\left(\frac{v}{2^t}\right) &= \lim_{t \rightarrow \infty} |2^{2t\beta}| \lim_{p \rightarrow \infty} \max \left\{ |2^{2(i-1)\beta}| \psi\left(\frac{v}{2^{i+t}}, \frac{v}{2^{i+t}}, 0, \dots, 0\right): 1 \leq i \leq p \right\} \\ &\leq \lim_{t \rightarrow \infty} \lim_{p \rightarrow \infty} \max \left\{ |2^{2(i-1)\beta}| \psi\left(\frac{v}{2^i}, \frac{v}{2^i}, 0, \dots, 0\right): 1+t \leq i \leq p+t \right\}, \quad (76) \end{aligned}$$

for all $v \in E$. It follows from (63) that

$$\begin{aligned} & \|Q_2(v) - Q'_2(v), m_1, \dots, m_{n-1}\|_\beta \\ & \leq \lim_{t \rightarrow \infty} |2^{2t\beta}| \left\| Q_2\left(\frac{v}{2^t}\right) - Q'_2\left(\frac{v}{2^t}\right), m_1, \dots, m_{n-1} \right\|_\beta \\ & \leq \lim_{t \rightarrow \infty} |2^{2t\beta}| \max \left\{ \left\| Q_2\left(\frac{v}{2^t}\right) - \phi\left(\frac{v}{2^t}\right), m_1, \dots, m_{n-1} \right\|_\beta, \left\| \phi(2^t v) - Q'_2(2^t v), m_1, \dots, m_{n-1} \right\|_\beta \right\} \\ & \leq \lim_{t \rightarrow \infty} |2^{2(t-1)\beta}| \tilde{\psi}\left(\frac{v}{2^t}\right) \chi(m_1, \dots, m_{n-1}) = 0. \quad (77) \end{aligned}$$

By Lemma 6, we conclude that the quadratic function Q_2 is unique. This ends the proof of the theorem.

5. Conclusion

In this work, we introduced a new finite dimensional quadratic functional equation (1) and obtained its solution to show which satisfies the quadratic properties. Mainly, investigated Hyers-Ulam stability of considered domain E is a non-Archimedean β -normed space in Theorem 8 and Theorem 9 and examined by considered the domain E is a vector space in Theorem 10 and Theorem 11. The results obtained in this article may be further generalized to be in non-Archimedean quasi(n, β)-Banach spaces. This could be a potential future work.

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to this work. And all the authors have read and approved the final version manuscript.

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