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Research Article

Analysis of a Predator-Prey Model with Distributed Delay

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In this paper, we consider a predator-prey model, where we assumed that the model to be an infected predator-free equilibrium one. The model includes a distributed delay to describe the time between the predator's capture of the prey and its conversion to biomass for predators. When the delay is absent, the model exhibits asymptotic convergence to an equilibrium. Therefore, any nonequilibrium dynamics in the model when the delay is included can be attributed to the delay's inclusion. We assume that the delay is distributed and model the delay using integrodifferential equations. We established the well-posedness and basic properties of solutions of the model with nonspecified delay. Then, we analyzed the local and global dynamics as the mean delay varies.

1. Introduction

In applied engineering and complex system sciences, mathematical models that display deterministic chaotic dynamical behaviour are of interest. The majority of encounters in nature are admittedly delayed or isolated, as both predator and prey function stochastically in absorbing available resources. This can be used to share bandwidth and resources among network users at a bottleneck node or a leaky bucket used to track flows, for example. If we assume that network users' behaviour is stochastic and that the accommodating segment has limited buffering space, then forwarding generated data packets can be compared to a predator-prey style interaction with limited resources characteristics during rush hours, when users interact intensively. One approach to examining a heterogeneous network susceptible to attack is modeling cyberspace as a predator-prey landscape. The predator-prey model of Gauss type is a well-known simple mathematical model describing the interaction between species. Its variations and extensions are studied in modern day population dynamics theory (see, for example, [1–14]). This model is based on the assumption that in real-world ecosystems prey populations do not grow exponentially in the absence of a predator, but rather their size is eventually limited by the absence of resources. Fan and Wolkowicz studied the effects of incorporating discrete delay in [15]. The delay corresponds to the time lag between predator capturing the prey and its conversion to biomass for predators. Their research focused on switches of stability of the coexistence equilibrium, the occurrence of periodic solutions, and subsequent bifurcation dynamics as the length of the delay increased. Li et al.[16] analyzed a Gause-type predator-prey model in which adult and juvenile death rates were taken to be different. In their work, the delay denoted the maturation period of the predator. They studied the dynamical behaviour of the system for the functional responses of Holling type I and Holling type II. They established the existence of stability switches due to Hopf bifurcations. These bifurcations occur in pairs that are connected and are nested. They have also shown that there is a range of parameters for which there exist two or more stable periodic solutions.

In nature, for each case, the processing delay rarely has the same duration, and instead follows a distribution of some mean value. Recently, Chaudhuri et al. [17] studied the following epidemic model consisting of four species, namely, sound prey, infected prey, sound predator, and infected predators.

$$\begin{aligned}
\frac{dX_1(\tau)}{d\tau} &= X_1 r_1 - X_1\left(\frac{X_1+X_2}{\tilde{k}}\right) - \gamma X_1 X_2 - \beta X_4 X_1 - b_1 X_1 X_3 - b_3 X_1 X_4 \\
\frac{dX_2(\tau)}{d\tau} &= X_2 r_2 - X_2 \nu + \gamma X_1 X_2 - X_2\left(\frac{X_1+X_2}{\tilde{k}}\right) - b_2 X_2 X_3 - b_4 X_2 X_4 + \beta X_1 X_4 \\
\frac{dX_3(\tau)}{d\tau} &= X_3\left(-m - \alpha X_2 - \eta X_4 + d\left(b_1 X_1 + b_2 X_2\right)\right) \\
\frac{dX_4(\tau)}{d\tau} &= X_4\left(-m - \mu + \eta X_3 + d\left(b_3 X_1 + b_4 X_2\right) + \alpha X_2 X_3\right).
\end{aligned} \tag{1}$$

In [4], we have modified the system (1) with discrete delay.

$$\begin{aligned}
\frac{dX_1(\tau)}{d\tau} &= X_1 r_1 - X_1\left(\frac{X_1+X_2}{\tilde{k}}\right) - \gamma X_1 X_2(t-\tau_1) - \beta X_4 X_1 - b_1 X_1 X_3 - b_3 X_1 X_4 \\
\frac{dX_2(\tau)}{d\tau} &= X_2 r_2 - X_2 \nu + \gamma X_1 X_2(t-\tau_1) - X_2\left(\frac{X_1+X_2}{\tilde{k}}\right) - b_2 X_2 X_3 - b_4 X_2 X_4 + \beta X_1 X_4 \\
\frac{dX_3(\tau)}{d\tau} &= X_3\left(-m - \alpha X_2 - \eta X_4 + d\left(b_1 X_1 + b_2 X_2\right)\right) \\
\frac{dX_4(\tau)}{d\tau} &= X_4\left(-m - \mu + \eta X_3 + d\left(b_3 X_1 + b_4 X_2\right) + \alpha X_2 X_3\right).
\end{aligned} \tag{2}$$

They investigated the stability properties and the existence of Hopf bifurcation. In this paper, we study the effects of incorporating distributed delay in the system (1) for infected predator-free equilibrium.

In the next section, an analysis of infected predator-free equilibrium of (1) is presented. In Section 3, we established the well posedness and basic properties of the model. We investigated the stability properties for different equilibriums in Section 4. Section 5 with conclusions completes the paper.

2. Infected Predator-Free Equilibrium

Consider (1)

$$\begin{aligned}
\frac{dX_1(\tau)}{d\tau} &= X_1 r_1 - X_1\left(\frac{X_1+X_2}{\tilde{k}}\right) - \gamma X_1 X_2 - \beta X_4 X_1 - b_1 X_1 X_3 - b_3 X_1 X_4 \\
\frac{dX_2(\tau)}{d\tau} &= X_2 r_2 - X_2 \nu + \gamma X_1 X_2 - X_2\left(\frac{X_1+X_2}{\tilde{k}}\right) - b_2 X_2 X_3 - b_4 X_2 X_4 + \beta X_1 X_4 \\
\frac{dX_3(\tau)}{d\tau} &= X_3\left(-m - \alpha X_2 - \eta X_4 + e\left(b_1 X_1 + b_2 X_2\right)\right) \\
\frac{dX_4(\tau)}{d\tau} &= X_4\left(-m - \mu + \eta X_3 + e\left(b_3 X_1 + b_4 X_2\right) + \alpha X_2 X_3\right).
\end{aligned} \tag{3}$$

By introducing scaling variables $x_1(t) = \theta X_1(\wp)$, $x_2(t) = \psi X_2(\wp)$, $x_3(t) = \psi X_3(\wp)$, $x_4(t) = \omega X_4(\wp)$, $t = \wp\sigma$ where $\sigma = m$, $\phi = \gamma/m$, $\omega = \beta/m$, $\theta = (eb_1)/m$, $\psi = b_1/m$.

Let $A = \gamma/eb_1$, $B = 1/\tilde{K}\gamma$, $C = b_3/\beta$, $C = b_4/\beta$, $E = \alpha/\gamma$, $F = \eta/\beta$, $G = b_3/b_1$, $X = eb_2/\gamma$, $R_1 = r_1/m$, $R_2 = r_2 - \nu/m$, $M = \mu/m$. We obtain

$$\begin{aligned}
\dot{x}_1 &= x_1 R_1 - ABx_1^2 - Bx_2 x_1 - x_2 x_1 - x_1 x_4 - x_1 x_3 - Cx_1 x_4 \\
\dot{x}_2 &= x_2 R_2 - x_2 ABx_1 - Bx_2^2 + Ax_2 x_1 - AXx_3 x_2 - Dx_2 x_4 + Ax_1 x_4 \\
\dot{x}_3 &= -x_3 + x_1 x_3 + Xx_2 x_3 - Ex_2 x_3 - Fx_3 x_4 \\
\dot{x}_4 &= -x_4 - Mx_4 + Gx_1 x_4 + \frac{GD}{AC} x_2 x_4 + \frac{GFx_3 x_4}{C} + \frac{EGx_2 x_3}{C}.
\end{aligned} \tag{4}$$

Now assume that the predator becomes disease free and for simplicity let us consider $X = E = 1$. Then, (4) becomes

$$\begin{aligned}
\dot{x}_1 &= x_1 R_1 - ABx_1^2 - Bx_2 x_1 - x_2 x_1 - x_1 x_3 \\
\dot{x}_2 &= x_2 R_2 - x_2 ABx_1 - Bx_2^2 + Ax_2 x_1 - Ax_2 x_3 \\
\dot{x}_3 &= -x_3 + x_1 x_3.
\end{aligned} \tag{5}$$

Now, we introduce distributed delay to (5)

$$\begin{aligned}
\dot{x}_1 &= x_1\left(R_1 - ABx_1\right) - (B+1) x_2 x_1 - x_1 x_3 \\
\dot{x}_2 &= x_2\left(R_2 - Bx_2\right) - x_2 x_1 (AB - A) - Ax_2 x_3 \\
\dot{x}_3 &= -x_3 + \int_0^{\infty} x_1(t-u) x_3(t-u) e^{-u} h(u)\, du.
\end{aligned} \tag{6}$$

Here, the function $h(u)$ is the kernel of the distributed delay with the following properties

$$\begin{aligned}
&\int_0^{\infty} h(u)\, du = 1 \\
&\int_0^{\infty} uh(u)\, du = \wp,
\end{aligned} \tag{7}$$

where $\wp$ is the mean delay between the capture of the prey to the conversion into the biomass of the predator.

Denote by $\wr^3$, the Banach space of bounded continuous functions mapping from $(-\infty, 0]$ into R^3 fitted with the uniform norm. We consider initial data $\Phi = (\Phi_1, \Phi_2, \Phi_3) \in \wr_+^3 = (\Phi \in \wr^3 : \Phi_i(\Theta) \geq 0, i = 1, 2, 3, \Theta \leq 0)$. Define int $\wr_+^3 = (\Phi \in \wr^3 : \phi_i(\Theta) > 0, i = 1, 2, 3\, \Theta \leq 0)$. Denote the solutions of (6) with initial data $\Phi \in \wr_+^3$ at time t by $\Pi(\Phi, t)$ when they exist. Hence, for mentioning the positive solutions, we are referring to the solutions $\Pi(\Phi, t)$ with $\Phi \in$ int $\wr_+^3$. Later, we show that each component is positive for all $t > 0$ in this case.

3. Well Posedness and Basic Properties of the Model

Define $L > 0$ and assume that $g(s) = 0$ for all $s \in (L, \infty)$. We allow $L = \infty$.

Theorem 1. *Solutions of (6) exist, with initial data in $\wr_+^3$, and for all $t > 0$, they are unique and remain in $\wr_+^3$.*

Proof. For each bounded functions $\Phi \in \wr_+^3$, there exists a unique solution of (6), $\Pi(\Phi, t)$ such that $\Pi(\Phi,.)|_{(-\infty,0]} = \Phi(t)$,

For all $t \geq 0$, $x_1(t) = x_2(t) = 0$ if $x_1(0) = x_2(0) = 0$. If $x_1(0) > 0$, then $x_1(t)$ will remain positive. Similarly, if $x_2(0) > 0$, $x_2(t)$ will remain positive. Hence, for all $t > 0$, there exists a unique solution.

Finally, as $x_3(t) \geq 0$ on $[-L, 0]$, $\dot{x}_3(t) \geq -x_3(t)$ for all $0 \leq t \leq L$. Hence, $x_3(t) \geq 0$ for all $-L \leq t \leq L$. By induction, $x_3(t) \geq 0$, for an $t \in [L(n-1), Ln]$ and $n \in N, n > 0$. Hence, for all $t > 0$, $x_3(t) \geq 0$.

Proposition 2. *Solutions of (6) with positive initial conditions remain positive for all $t > 0$.*

Proof. By the previous theorem, $x_1(0) > 0$, then $x_1(t) > 0$ for all $t > 0$ and $x_2(0) > 0$, then $x_2(t) > 0$ for all $t > 0$. Assume that $x_3(t) = 0$ at $\hat{t}$. This implies that $\dot{x}_3(\hat{t}) \leq 0$. $d x_3(\hat{t})/dt = \int_0^\infty x_1(\hat{t} - u)x_3(\hat{t} - u)e^{-u}h(u)du$ which is positive, a contradiction.

Lemma 3. *Solutions of (6) are bounded and $\limsup_{t\longrightarrow\infty} x_1(t) \leq R_1/AB$, $\limsup_{t\longrightarrow\infty} x_2(t) \leq R_2/B$ and $x_3(t) \leq (1 + R_1)^2/4R_1 \int_0^\infty e^{-u}h(u)du$.*

Proof. Note that $\dot{x}_1(t) \leq x_1(t)(R_1 - ABx_1(t))$. Also, $x_1(0) \geq 0$, given $\varepsilon > 0\exists$ a $T > 0 \ni x_1(t) < (R_1/AB) + \varepsilon$ for all $t \geq T$. Therefore, $\limsup_{t\longrightarrow\infty} x_1(t) \leq R_1/AB$. Similarly, $\limsup_{t\longrightarrow\infty} x_2(t) \leq R_2/B$.

Consider

$$z(t) = x_3(t) + \int_0^\infty x_1(t-u)e^{-u}h(u)du. \tag{8}$$

The derivative of $z(t)$ with respect to t,

$$\dot{z}(t) = \dot{x}_3(t) + \int_0^\infty \dot{x}_1(t-u)e^{-u}h(u)du. \tag{9}$$

Now,

$$\begin{aligned}\dot{z}(t) = {}& -x_3(t) + \int_0^\infty x_1(t-u)x_3(t-u)e^{-u}h(u)du \\ &+ e^{-u}h(u)du\left(\int_0^\infty x_1(t-u)(R_1 - ABx_1(t-u)\right. \\ &\left. - x_1(t-u)x_2(t-u)(B+1) - x_1(t-u)x_3(t-u)\right), \\ \dot{z}(t) = {}& -z(t) + \int_0^\infty x_1(t-u)e^{-u}h(u)du(1 + R_1 - ABx_1(t-u) - (B+1)x_2(t-u)).\end{aligned} \tag{10}$$

Note that $(1 + R_1/2 - (ABx_1(t-u) + (B+1)x_2(t-u)))^2 \geq 0$. Therefore,

$$\dot{z}(t) \leq -z(t) + \frac{(1+R_1)^2}{4}\int_0^\infty e^{-u}h(u)du. \tag{11}$$

Also,

$$\dot{w}(t) = -w(t) + \frac{(1+R_1)^2}{4}\int_0^\infty e^{-u}h(u)du, \tag{12}$$

has a solution

$$w(t) = w(0)e^{-t} + \left(1 - e^{-t}\right)\frac{(1+R_1)^2}{4R_1}\int_0^\infty e^{-u}h(u)du. \tag{13}$$

For each $w(t)$ with $w(0) \geq 0$ and for every $\varepsilon > 0\exists T > 0 \ni w(t) \leq (1 + R_1)^2/4R_1 \int_0^\infty e^{-u}h(u)du + \varepsilon$, for all $t > T$. Provided $w(0) = z(0)$, by the comparison theorem, we conclude that $\limsup_{t\longrightarrow\infty} z(t)$ is also bounded by $(1 + R_1)^2/4R_1 \int_0^\infty e^{-u}h(u)du$, and therefore, so is $\limsup_{t\longrightarrow\infty} x_3(t)$.

Set

$$\begin{gathered} X^0 \equiv \left\{\wr_+^3 : \Phi_2(0) > 0 \text{ and} \exists \Theta \in [-L, 0] \text{such that} \, \Phi_2(\Theta)\Phi_3(\Theta) > 0\right\} \\ X_1 \equiv \left\{\wr_+^2 : \Phi_2(0) = 0\right\} \\ X_2 \equiv \left\{\wr_+^2 : \Phi_2(0) > 0 \text{ and } \Phi_2(\Theta)\Phi_3(\Theta) = 0, \forall\Theta \in [-L, 0]\right\} \\ X = X^0 \cup X_1 \cup X_2. \end{gathered} \tag{14}$$

Now,

(1) If $\Phi \in X^0$, since it is continuous and $\Phi_2(0) > 0$, then there exists $a > 0$, such that for all $\Theta \in (-a, 0]\Phi_2(\Theta) > 0$

(2) If $\Phi_2(\Theta)\Phi_3(\Theta) = 0$ for all $\Theta \in [-L, 0)$, then $\int_0^\infty \Phi_2(-u)\Phi_3(-u)e^{-u}h(u)du = 0$

(3) If $\Phi \in X_2$, since Φ is continuous, there exists $a > 0$ such that for all $\Theta \in (-a, 0]$, $\Phi_2(\Theta) > 0$ and $\Phi_3(\Theta) = 0$

Denote $(x_1(t), x_2(t), x_3(t)) = (x_1(t, \Phi(t)), x_2(t, \Phi(t)), x_3(t, \Phi(t)))$ to be the solution of (6) with the initial data $\Phi(t)$ in X^0. Set $T(t)(\Phi)(\Theta) = (x_1(t+\Theta), x_2(t+\Theta), x_3(t+\Theta)), \Theta \in (-\infty,0]$.

Lemma 4. *If the solutions of (6) have initial conditions in X^0, then $\forall t > 0\exists\Theta \in [t - L, t] \ni x_1(\Theta)x_3(\Theta) > 0$. Also, $\exists M > 0 \ni x_1(t) > 0, x_3(t) > 0$ and $\int_0^\infty x_1(t-u)x_3(t-u)e^{-u}h(u)du > 0 \forall t > M$.*

Proof. As $\Phi_1(0) > 0$ and $\Phi_2(0) > 0$, $x_1(t) > 0, x_2(t) > 0$ for all $t > 0$. Consider the case $\Phi_3(0) > 0$. Then, by Theorem 1, $x_3(t) \geq 0 \forall t$, and hence, $\dot{x}_3(t) \geq -x_3(t)$, and $x_3(t) > 0 \forall t \geq 0$.

Consider the case $\Phi_3(0) = 0$. Since $\Phi_2(t)$ and $\Phi_3(t)$ are continuous functions, then there exists $t_1, t_2 \in R \ni -L < t_1 < \Theta_0 < t_2 \leq 0$ and $\Phi_2(t)\Phi_3(t) > 0 \forall t \in [t_1, t_2]$. Therefore, there exists time $T \leq L + t_1$ such that $\dot{x}_3(T) > 0$ and $\exists$

$\varepsilon > 0$ such that $x_3(t) > 0 \forall t \in (T, T+\varepsilon)$. Then, as $\dot{x}_3(t) \geq -x_3(t)$, $x_3(t) > 0 \forall t > T$, it follows that $\theta_0 \in [t_1, t_2] \subset [t_1, T]$.

Lemma 5. *Sets X^0, X_1, and X_2 are positively invariant under $T(t)$.*

Proof. For X^0, the result is true under $T(t)$ by Lemma 4. If the solution has initial conditions in X_1, then, by (6), $x_2(t) = 0 \forall t > 0$. By Theorem 1, $x_3(t) \geq 0 \forall t > 0$. Now, the solutions with initial conditions in X_2 are taken into consideration. aS $x_3(0) = 0$ and since $x_3(t) = 0$ is a solution of (6) with $x_3(0) = 0$ and $\Phi_2(\Theta)\Phi_3(\Theta) = 0 \forall \Theta \in [-L, 0]$, then by the uniqueness of solutions, $x_3(t) = 0 \forall t > 0$. Hence, $\int_0^\infty x_1(t-u)x_3(t-u)e^{-u} h(u)du \forall t > 0$. Now, from Theorem 1, $x_1(t) \geq 0$ and $x_2(t) > 0$ $\forall t > 0$.

4. Stability Results with General Delay

Consider three equilibria of (6), $E_0 = (0,0,0)$, $E_1 = ((R_1/AB), 00)$ and $E_+ = ((\int_0^\infty e^{-u}h(u)du)^{-1}, (R_2 - AR_1 - AB - A - A^2 B\int_0^\infty e^{-u}h(u)du/B - AB - A), (R_2 - (AB - A)\int_0^\infty e^{-u}h(u)du)^{-1} - Bx_2^*/A))$.

The linearization of the system (6) around an equilibrium $E_+ = (x_1^*, x_2^*, x_3^*)$ is given by

$$\dot{X}(t) = AX(t) + B\int_0^\infty e^{-u}h(u)X(t-u)du. \tag{15}$$

Here,

$$A = \begin{vmatrix} R_1 - 2ABx_1^* - x_2^*(B+1) - x_3^* & -x_1^*(B+1) & -x_1^* \\ A(1-B)x_2^* & R_2 - 2Bx_2^* + x_1^*(A-AB) - Ax_3^* & -Ax_2^* \\ 0 & 0 & -1 \end{vmatrix},$$
$$B = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x_3^* & 0 & x_1^* \end{vmatrix}. \tag{16}$$

4.1. Stability at E_0. At E_0 (15) becomes

$$\begin{vmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{vmatrix} = \begin{vmatrix} R_1 & 0 & 0 \\ 0 & R_2 & 0 \\ 0 & 0 & -1 \end{vmatrix} \begin{vmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{vmatrix}. \tag{17}$$

Since two of the eigen values are positive E_0 is an unstable saddle point.

Lemma 6. *E_0 is globally asymptotically stable with initial data in X_1.*

Proof. We know that $x_1(t)$ and $x_2(t)$ is equal to 0 for all $t > 0$. Now, consider $L = \infty$. If $\Phi_2(\Theta)\Phi_3(\Theta) = 0 \forall \Theta \in [-\infty, 0]$ and therefore $\int_0^\infty \Phi_2(-u)\Phi_3(-u)e^{-u}h(u)du$, then since $x_1(t) = 0$ $\forall t > 0$, $\int_0^\infty x_1(t-u)x_3(t-u)e^{-u}h(u)du = 0 \forall t > 0$. Hence, $\dot{x}_3(t) = -(x_3 t) \forall t > 0$. Hence, $\lim_{t\to\infty} x_3(t) = 0$.

If $\exists \Theta \in [-\infty, 0] \ni t$ $\Phi_2(\Theta)\Phi_3(\Theta) > 0$, then $\exists T > 0$, $\ni$ $\int_0^\infty x_1(t-u)x_3(t-u)e^{-u}h(u)du > 0 \forall t > T$. Also, as $x_1(t) = 0$ $\forall t > 0$, and hence,

$$\begin{aligned} \dot{x}_3(t) &= -x_3(t) + \int_0^t x_1(t-u)x_3(t-u)e^{-u}h(u)du \\ &\quad + \int_t^\infty x_1(t-u)x_3(t-u)e^{-u}h(u)du \\ &= -x_3(t) + \int_t^\infty x_1(t-u)x_3(t-u)e^{-u}h(u)du. \end{aligned} \tag{18}$$

The limit of $\int_t^\infty x_1(t-u)x_3(t-u)e^{-u}h(u)du$ as $t \longrightarrow \infty$ is 0. Therefore, $\lim_{t\longrightarrow\infty} x_3(t) = 0$.When $L < \infty$, as $x_1(t) = 0$ for $t > 0$, then $\int_t^\infty x_1(t-u)x_3(t-u)e^{-u}h(u)du = 0 \forall t > L$. Then, $\dot{x}_3(t) = -x_3(t) \forall t > L$.

4.2. Stability at E_1. The linearization around E_1 takes the form

$$\begin{vmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{vmatrix} = \begin{vmatrix} -R_1 & -\frac{R_1}{AB}(B+1) & -\frac{R_1}{AB} \\ 0 & R_2 + \frac{R_1}{AB}(A-AB) & 0 \\ 0 & 0 & -1 \end{vmatrix} \begin{vmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{vmatrix} + \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{R_1}{AB} \end{vmatrix} \begin{vmatrix} \int_0^\infty x_1(t-u)e^{-u}h(u)du \\ \int_0^\infty x_2(t-u)e^{-u}h(u)du \\ \int_0^\infty x_3(t-u)e^{-u}h(u)du \end{vmatrix}. \tag{19}$$

The characteristic equation takes the form

$$\begin{aligned} &\left(\lambda + 1 - \int_0^\infty e^{(-1+\lambda)u}h(u)du\right)\left[\lambda^2 + \lambda\left(R_2 + \frac{R_1}{AB}(A-AB) - R_1\right)\right. \\ &\left.\quad + (-R_1)\left(R_2 + \frac{R_1}{AB}(A-AB)\right)\right] = 0. \end{aligned} \tag{20}$$

Theorem 7. *E_1 is locally asymptotically stable if $\int_0^\infty e^{-u}h(u)du < 1$ and unstable if either inequality is reversed.*

Proof. The term in the square brackets has roots $-(R_2 + R_1/AB(A-AB))$, $-R_1$ which are both negative iff $R_2 > R_1/AB(A-AB)$. The stability of E_1 is determined by the sign of the real parts of the roots of $m(\lambda) = (\lambda + 1 - \int_0^\infty e^{(-1+\lambda)u}h(u)du)$.

Substituting $\lambda = \beta + i\gamma$, $\gamma \geq 0$ in $m(\lambda)$ and separating real and imaginary parts, we obtain

$$L(\beta) = (\beta + 1) = \int_0^\infty e^{(-1+\lambda)u}\cos(\gamma u)h(u)du = R(\beta), \tag{21}$$

$$\gamma = -\int_0^\infty e^{(-1+\lambda)u}\sin(\gamma u)h(u)du = 0. \tag{22}$$

First, we show that if $\int_0^\infty e^{-u}h(u)du > 1$, then $m(\lambda)$ has a positive real root. Note that if $\gamma = 0$, then (22) is satisfied. In this case in (21), $L(0) < R(0)$, $R(\beta)$ is a decreasing function of β and $L(\beta)$ is an increasing function of β and $\lim_{\beta\longrightarrow\infty} L(\beta) = +\infty$. Therefore, $m(\lambda)$ has a real root which is positive and E_1 is unstable. Also, if $\int_0^\infty e^{-u}h(u)du < 1$, $L(0) > R(0)$, $R(\beta)$ is decreasing and $L(\beta)$ is increasing, and (21) can never be satisfied for $\beta > 0$. Hence, E_1 is locally asymptotically stable.

Lemma 8. *E_1 is globally asymptotically stable with initial data in X_2.*

Proof. We know that $x_3(t) = 0 \forall t > 0$. Then, (6) becomes an ODE model. By Lemma 6 in [9], this lemma is true.

4.3. Properties of the Model when $\mathbf{E}_+$ Exists. The characteristic equation around $E_+ = (x_1^*, x_2^*, x_3^*)$ is

$$\lambda^3 + A\lambda^2 + B\lambda + C + \int_0^\infty e^{(-1+\lambda)u} h(u) du \left(D\lambda^2 + \lambda E + F\right) = 0, \tag{23}$$

where $A = 1 - x_1^* A + x_2^* + 3x_1^* AB + 3x_2^* B + x_3^* + Ax_3^* - R_1 - R_2$,

$$\begin{aligned} B = & -x_1^* A + x_2^* + 3x_1^* AB - 2x_1^{*2} A^2 B + 3x_2^* B + 2x_2^{*2} B + 2x_1^{*2} A^2 B^2 \\ & + 4x_1^* A x_2^* B^2 + 2x_2^{*2} B^2 + x_3^* + AX_3^* - x_1^* x_3^* A + A x_2^* x_3^* \\ & + x_1^* x_3^* AB + 2x_1^* x_3^* A^2 B + 2x_2^* x_3^* B + ABx_2^* x_3^* + Ax_3^2 - R_1 \\ & + x_1^* AR_1 - x_1^* ABR_1 - 2x_2^* BR_1 - Ax_3^* R_1 - R_2 - x_2^* R_2 \\ & - 2x_1^* ABR_2 - x_2^* BR_2 - x_3^* R_2 + R_1 R_2), \\ C = & -2x_1^{*2} A^2 B + 2x_2^{*2} B + 2x_1^{*2} A^2 B^2 + 4x_1^* x_2^* AB^2 + 2x_2^{*2} B^2 \\ & - x_1^* x_3^* A + x_2^* x_3^* A + x_2^* x_3^* AB + 2x_1^* A^2 B x_3^* + 2Bx_2^* x_3^* \\ & + ABx_2^* x_3^* + Ax_3^2 + x_1^* AR_1 - x_1^* ABR_1 - 2x_2^* BR_1 - Ax_3^* R_1 \\ & - x_2^* R_2 - 2x_1^* ABR_2 - x_2^* BR_2 - x_3^* R_2 + R_1 R_2 \\ \mathrm{D} = & -x_1^*, \mathrm{E} = x_1^{*2} A - x_2^* x_1^* - 3x_1^{*2} AB - 3x_1^* x_2^* B - Ax_1^* x_3^* \\ & + R_1 x_1^* + R_2 x_1^*, \\ \mathrm{F} = & 2x_1^{*3} A^2 B - 2x_2^{*3} B - 2x_1^{*3} A^2 B^2 - 4x_1^{*2} x_2^* AB^2 - 2x_1^* x_2^{*2} B^2 \\ & - 2Ax_1^* x_2^* x_3^* - 2x_1^{*2} x_3^* A^2 B - 2ABx_1^* x_2^* x_3^* - x_1^{*2} AR_1 \\ & + x_1^{*2} ABR_1 + 2x_1^* x_2^* BR_1 + Ax_1^* x_3^* R_1 + x_1^* x_2^* R_2 + 2x_1^{*2} ABR_1 \\ & + x_1^* x_2^* BR_2 - x_1^* R_1 R_2. \end{aligned} \tag{24}$$

Lemma 9. *If $\wp = 0$, E_+ is locally asymptotically stable.*

Proof. Since $\wp = 0$, then $h(u) = \delta_0(u)$. $\int_0^\infty e^{-\alpha u} h(u) du = 1$ and E_+ becomes

$(1, (R_2 - AR_1 - AB - A - A^2 B/B - AB - A), (R_2 - (AB - A) - Bx_2^*/A))$. Then, (23) becomes

$$\lambda^3 + A\lambda^2 + B\lambda + C + \left(D\lambda^2 + \lambda E + F\right) = 0. \tag{25}$$

Simplify (25) to the following equation

$$\lambda^3 + (A + D)\lambda^2 + (B + E)\lambda + (C + F) = 0. \tag{26}$$

By Routh hurwitz criterion if $(A + D)(B + E) - (C + F) > 0$, E_+ is locally asymptotically stable.

Theorem 10. *As $\wp$ increases from zero, if a root appears on or crosses the imaginary axis as $\wp$ increases from 0, then the number of roots of (23) with a positive real part can change.*

Proof. For $g(\lambda) = A\lambda^2 + B\lambda + C + \int_0^\infty e^{(-1+\lambda)u} h(u) du (D\lambda^2 + \lambda E + F)$, it is easy that $\limsup_{\lambda \longrightarrow \infty} |\lambda^{-3} g(\lambda)| = 0 < 1$.

Hence, none of the root of (23) with positive real part can enter from ∞ as $\wp$ bifurcates from 0. As Lemma 6 holds, the result follows.

Also, if $\int_0^\infty e^{-u} h(u) du > 1$, then E_+ is locally asymptotically stable and if $\int_0^\infty e^{-u} h(u) du > 1$, then (23) has no positive real roots.

4.3.1. Global Dynamics

Lemma 11. *If $\int_0^\infty e^{-u} h(u) du > 1$ and $\wp = 0$, then E_+ is globally asymptotically stable with respect to the solutions of (6) with $x_1(0) > 0, x_2(0) > 0$ and $x_3(0) > 0$.*

Proof. Since $\wp = 0$, then $h(u) = \delta_0(u)$, and therefore, system (6) reduces to its ODE prototype

$$\begin{aligned} \dot{x}_1(t) &= x_1(R_1 - ABx_1 - x_2(B + 1) - x_3) \\ \dot{x}_2(t) &= x_2(R_2 - Bx_2 - x_1(AB - A) - Ax_3) \\ \dot{x}_2(t) &= -x_3 + x_1 x_3. \end{aligned} \tag{27}$$

Solutions with positive initial conditions will remain positive for all $t > 0$. Using the Dulac criterion, we observe that there are no periodic solutions lying in int R_+^2. Observe that only the solutions with initial conditions on the y-axis converge to E_0, while solutions on the x-axis, not including the origin, converge to E_1. On the other hand, E_1 repels the solutions with initial data not on the x-axis. Using straightforward phase plane argument, one can see that neither E_0, nor E_1 is in the ω-limit set of solutions with initial data in int R_+^2. Then, by the Poincare-Bendixson theorem [10], E_+ is globally asymptotically stable.

Lemma 12. *Consider the solutions of (6) with initial data in $X^0 \cup X^2$. There is no positive monotonically increasing sequence $\{t_n\}$, with $t_n \longrightarrow \infty$ as $n \longrightarrow \infty$ such that $(x_1(t_n), x_2(t_n), x_3(t_n))$ converges to E_0.*

Here, for every solution with initial data in $X^0 \cup X_2$, $x(t) > 0 \forall t \geq 0$. We prove this theorem by contradiction.

Assume that $\exists$ a monotonically increasing sequence $\{t_n\}$ which is positive, with $t_n \longrightarrow \infty$ such that $\dot{x}(t_n) \leq 0$ and $(x_1(t_n), x_2(t_n), x_3(t_n))$ converges to E_0 as $n \longrightarrow \infty$. For every $\varepsilon > 0$, $\exists T > 0 \ni x_1(t_n) < \varepsilon$, $x_2(t_n) < \varepsilon$ and $x_3(t_n) < \varepsilon$, for all $t_n >$

T. Set $\varepsilon < 1/2$. Then, $\dot{x}(t_n) > x(t_n)(1 - 2l\varepsilon) > 0$, for sufficiently large n, which is a contradiction. Also, if $\int_0^\infty e^{-u} h(u) du > 1$, then no solution of (6) with initial data in X^0 converges to E_1.

5. Conclusion

Through evolution, nature has developed natural propensities in complex systems (including animalia and plants) that enable survival through adaptation. Malicious agents, such as viruses, worms, and denial-of-service attacks, plague the Internet and the vast array of networks and applications that link to it. For example, using the Internet as an environment, the malicious attacks described above (viruses) can be viewed as predators, with their interactions with the ecosystem (servers) resembling a predator-prey relationship. A predator-prey model with distributed delay is considered in this paper. For infected predator-free equilibrium, we established properties of the system such as positivity and boundedness and conditions for global asymptotic stability of some equilibria for the general delay. We were particularly interested in the dynamics when E_+ exists. We showed that solutions with positive initial data remain positive for all time. Moreover, we determined the set of initial data such that the solutions eventually become positive.

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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