Research Article

A New Result of Stability for Thermoelastic-Bresse System of Second Sound Related with Forcing, Delay, and Past History Terms

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We consider a one-dimensional linear thermoelastic Bresse system with delay term, forcing, and infinity history acting on the shear angle displacement. Under an appropriate assumption between the weight of the delay and the weight of the damping, we prove the well-posedness of the problem using the semigroup method, where an asymptotic stability result of global solution is obtained.

1. Introduction

In this work, we considered with the following problem:

$$\begin{cases} \rho_1 \varphi_{tt} - k(\varphi_x + l\psi)_x - k_0 l(w_x - l\varphi) + \mu_1 \varphi_t(x, t) + \mu_2 \varphi_t(x, t - \tau) = 0, \\ \rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + l\psi) + \int_0^\infty g(s)\psi_{xx}(x, t - s)ds + \gamma\theta_x + f(\psi) = 0, \\ \rho_1 w_{tt} - k_0(w_x - l\varphi)_x + kl(\varphi_x + l\psi) = 0, \\ \rho_3 \theta_t + \kappa q_x + \gamma\psi_{tx} = 0, \\ \alpha q_t + \beta q + \kappa\theta_x = 0, \end{cases} \quad (1)$$

$(x, t) \in (0, 1) \times (0, \infty)$, with initial-boundary conditions

$$\begin{aligned} \varphi(0, t) &= \varphi_x(1, t) = \psi_x(0, t) = \psi(1, t) = w_x(0, t) \\ &= w(1, t) = \theta(0, t) = q(1, t) = 0, t \geq 0, \end{aligned} \quad (2)$$

$$\begin{cases} \varphi(x, 0) = \varphi_0(x), \varphi_t(x, 0) = \varphi_1(x), x \in (0, 1), \\ \psi(x, 0) = \psi_0(x), \psi_t(x, 0) = \psi_1(x), x \in (0, 1), \\ w(x, 0) = w_0(x), w_t(x, 0) = w_1(x), x \in (0, 1), \\ \theta(x, 0) = \theta_0(x), q(x, 0) = q_0(x), \\ \varphi_t(x, t - \tau) = f_0(x, t - \tau), \end{cases} \quad (3)$$

with $\tau > 0$ is a time delay and μ_1 and μ_2 are positive real numbers. The function θ is the temperature difference, q is the heat flux, and $\rho_1, \rho_2, \rho_3, k, l, k_0, b, \gamma, \kappa, \alpha, \beta$ are positive constants. We use the energy method and assume that the relaxation function g satisfies the following hypotheses:

(G1) $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a C^1 function such that

$$g(0) > 0, b - \int_0^\infty g(s)ds = b - g_0 = L > 0. \quad (4)$$

(G2) Let ζ be a positive constant with

$$g'(t) \leq -\zeta g(t), \forall t \geq 0, \quad (5)$$

and we suppose that the forcing term $f(\psi(x, t))$ satisfies some hypotheses.

(A1) $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$|f(\psi^2) - f(\psi^1)| \leq k_0 \left(|\psi^1|^\theta - |\psi^2|^\theta \right) |\psi^1 - \psi^2| \quad (6)$$

for all $\psi^1, \psi^2 \in \mathbb{R}$,
where $k_0 > 0, \theta > 0$.
(A2)

$$0 \leq \widehat{f}(\psi) \leq f(\psi)\psi \text{ for all } \psi \in \mathbb{R}, \quad (7)$$

with

$$\widehat{f}(z) = \int_0^z f(s) ds. \quad (8)$$

Depending on some of the following parameters, we consider

$$\tilde{\eta} = \left(1 - \frac{\alpha k \rho_3}{\rho_1} \right) \left(\frac{\rho_1}{k} - \frac{\rho_2}{b} \right) - \frac{\gamma^2 \alpha}{b}, \quad (9)$$

$$k = k_0.$$

It is well known that, in the single wave equation, if $\mu_2 = 0$, that is, in the absence of a delay, the energy of system exponentially decays (see, e.g., [1–22]). On the contrary, if $\mu_1 = 0$, that is, there exists only the delay part in the interior, the system becomes unstable.

Bresse system is a mathematical model that describes the vibration of a planar, linear shearable curved beam. The model was first derived by Bresse [23], and it consists of three coupled wave equations given by

$$\begin{cases} \rho_1 \varphi_{tt} = Q_x + lN + F_1, \\ \rho_2 \psi_{tt} = M_x - Q + F_2, \\ \rho_1 w_{tt} = N_x - lQ + F_3, \end{cases} \quad (10)$$

where

$$\begin{aligned} N &= k_0(w_x - l\varphi), \\ Q &= k(\varphi_x + lw + \psi), \\ M &= b\psi_x. \end{aligned} \quad (11)$$

We use N, Q , and M to denote the axial force, the shear force, and the bending moment. By w, φ , and ψ , we are denoting the longitudinal, vertical, and shear angle displacements. Here, $\rho_1 = \rho A, \rho_2 = \rho I, b = EI, k_0 = EA, k = k' GA$, and $l = R^{-1}$ (see, e.g., [23]).

The Bresse system (10) is more general than the well-known Timoshenko system where the longitudinal displacement w is not considered $l = 0$. The reader may refer to, for example, [24–34].

System (10) is an undamped system, and its associated energy remains constant when the time t evolves. To stabilize

system (10), many damping terms have been considered by several authors (see, e.g., [1, 35–40]).

In the succeeding text, we will present some works, which studied the stability of the dissipative Bresse system. The paper [41] was concerned with asymptotic stability of a Bresse system with two frictional dissipations.

$$\begin{cases} \rho_1 \varphi_{tt} - k(\varphi_x + lw + \psi)_x - k_0 l(w_x - l\varphi) = -\gamma_1 \varphi_t, \\ \rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + lw + \psi) = -\gamma_2 \psi_t, \\ \rho_1 w_{tt} - k_0(w_x - l\varphi)_x + l k(\varphi_x + lw + \psi) = 0. \end{cases} \quad (12)$$

Under the condition of equal speeds of wave propagation, the authors proved that the system is exponentially stable. Otherwise, they show that Bresse system is not exponentially stable. Then, they proved that the solution decays polynomially to zero with optimal decay rate, depending on the regularity of initial data.

There are several works dedicated to the mathematical analysis of the Bresse system. They are mainly concerned with decay rates of solutions of the linear system. This is done by adding suitable damping effects that can be of thermal, viscous, or viscoelastic nature (see for instance [42–44]), among others.

Concerning thermoelastic Bresse system, [37] considered

$$\begin{cases} \rho_1 \varphi_{tt} - k(\varphi_x + lw + \psi)_x - k_0 l(w_x - l\varphi) + l\gamma \theta_1 = 0, \\ \rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + lw + \psi) + \gamma \theta_x = 0, \\ \rho_1 w_{tt} - k_0(w_x - l\varphi)_x + l k(\varphi_x + lw + \psi) + \gamma \theta_{1x} = 0, \\ \rho_3 \theta_t - \theta_{xx} + \gamma \psi_{tx} = 0, \\ \rho_3 \theta_{1x} - \theta_{1xx} + \gamma(w_{tx} - l\varphi_t) = 0, \end{cases} \quad (13)$$

together with initial and specific boundary conditions and proved an exponential and only polynomial-type decay stabilities results.

2. Preliminaries and Well-Posedness

Firstly, we assume the following hypothesis:

$$|\mu_2| < \mu_1. \quad (14)$$

Using semigroup theory, we will prove that systems (1)–(3) are well posed by introducing the following new variable [17].

$$z(x, \rho, t) = \varphi_t(x, t - \tau\rho), x \in (0, 1), \rho \in (0, 1), t > 0. \quad (15)$$

Then, we have

$$\tau z_t(x, \rho, t) + z_\rho(x, \rho, t) = 0 \text{ in } (0, 1) \times (0, 1) \times (0, \infty). \quad (16)$$

Further, let

$$\eta^t(x, s) = \psi(x, t) - \psi(x, t - s), s \geq 0. \quad (17)$$

For this reason, we observe that

$$\eta_t^t(x, s) + \eta_s^t(x, s) = \psi_t(x, t). \quad (18)$$

Therefore, problem (1) takes the form

$$\begin{cases} \rho_1 \varphi_{tt} - k(\varphi_x + lw + \psi)_x - lk_0(w_x - l\varphi) + \mu_1 \varphi_t(x, t) + \mu_2 z(x, 1, t) = 0, \\ \tau z_t(x, \rho, t) + z_\rho(x, \rho, t) = 0, \\ \rho_2 \psi_{tt} - L\psi_{xx} + k(\varphi_x + lw + \psi) + \int_0^\infty g(s) \eta_{xx}^t(x, s) ds + \gamma \theta_x + f(\psi(x, t)) = 0, \\ \rho_1 w_{tt} - k_0(w_x - l\varphi)_x + lk(\varphi_x + lw + \psi) = 0, \\ \rho_3 \theta_t + q_x + \gamma \psi_{tx} = 0, \\ \alpha q_t + \beta q + \theta_x = 0, \\ \eta_t^t(x, s) + \eta_s^t(x, s) = \psi_t(x, t). \end{cases} \quad (19)$$

The following are with the boundary conditions:

$$\begin{aligned} \varphi(0, t) = \varphi_x(1, t) = \psi_x(0, t) = \psi(1, t) = w_x(0, t) \\ = w(1, t) = \theta(0, t) = q(1, t) = 0, t \geq 0. \end{aligned} \quad (20)$$

The initial conditions are as follows:

$$\begin{cases} \varphi(x, 0) = \varphi_0(x), \varphi_t(x, 0) = \varphi_1(x), \\ \psi(x, 0) = \psi_0(x), x \in (0, 1), \\ \psi_t(x, 0) = \psi_1(x), w(x, 0) = w_0(x), \\ w_t(x, 0) = w_1(x), x \in (0, 1), \\ \theta(x, 0) = \theta_0(x), q(x, 0) = q_0(x), x \in (0, 1), \\ \varphi_t(x, -t) = f_0(x, t) \text{ in } (0, 1) \times (0, \tau), \\ z(x, 1, t) = f(x, t - \tau) \text{ in } (0, 1) \times (0, \tau), \\ \eta^t(x, 0) = 0, \forall t \geq 0, \\ \eta^t(0, s) = \eta^t(1, s) = 0 \forall s \geq 0, \\ \eta^0(x, s) = \eta_0(s) = 0 \forall s \geq 0. \end{cases} \quad (21)$$

Let ξ be positive constants such that

$$\tau|\mu_2| < \xi < \tau(2\mu_1 - |\mu_2|), \quad (22)$$

where τ is a real number with $0 < \tau$ and μ_1, μ_2 are a positive constants, and the initial data are $(\varphi_0, \varphi_1, \psi_0, \psi_1, w_0, w_1, f, \theta_0, q_0, \eta_0)$.

If we set

$$U = (\varphi, \varphi_t, z, \psi, \psi_t, w, w_t, \theta, q, \eta^t)^T, \quad (23)$$

then

$$U' = (\varphi_t, \varphi_{tt}, z_t, \psi_t, \psi_{tt}, w_t, w_{tt}, \theta_t, q_t, \eta_t^t)^T. \quad (24)$$

Therefore, problems (19)–(21) can be written as

$$\begin{cases} U'(t) = AU(t) + F, \\ U(0) = (\varphi_0, \varphi_1, f_1(\cdot, -\tau), \psi_0, \psi_1, w_0, w_1, \theta_0, q_0, \eta_0), \end{cases} \quad (25)$$

where the operator A is defined by

$$A \begin{pmatrix} \varphi \\ u \\ z \\ \psi \\ v \\ w \\ \bar{w} \\ \theta \\ q \\ \phi \end{pmatrix} = \begin{pmatrix} u \\ \frac{k}{\rho_1}(\varphi_x + lw + \psi)_x + \frac{k_0 l}{\rho_1}(w_x - l\varphi) - \frac{\mu_1}{\rho_1}u - \frac{\mu_2}{\rho_1}z(., 1) \\ -\left(\frac{1}{\tau}\right)z_\rho \\ v \\ \frac{L}{\rho_2}\psi_{xx} - \frac{k}{\rho_2}(\varphi_x + lw + \psi) + \frac{1}{\rho_2} \int_0^\infty g(s) \phi_{xx}(s) ds - \frac{\gamma}{\rho_2}\theta_x \\ \bar{w} \\ \frac{k_0}{\rho_1}(w_x - l\varphi)_x - \frac{kl}{\rho_1}(\varphi_x + lw + \psi) \\ -\frac{1}{\rho_3}q_x - \frac{\gamma}{\rho_3}v_x \\ -\frac{\beta}{\alpha}q - \frac{k}{\alpha}\theta_x \\ -\phi_s + v \end{pmatrix}, \quad (26)$$

$$F = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\frac{1}{\rho_2}f(\psi) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (27)$$

We consider the following spaces:

$$\begin{aligned} H_*^1(0, 1) &= \{h \in H^1(0, 1) : h(0) = 0\}, \\ \tilde{H}_*^1(0, 1) &= \{h \in H^1(0, 1) : h(1) = 0\}, \\ H_*^2(0, 1) &= H^2(0, 1) \cap H_*^1(0, 1), \\ \tilde{H}_*^2(0, 1) &= H^2(0, 1) \cap \tilde{H}_*^1(0, 1), \end{aligned} \quad (28)$$

$$\begin{aligned} \mathcal{H} &= H_*^1(0, 1) \times L^2(0, 1) \times L^2((0, 1), H_0^1(0, 1)) \\ &\times \tilde{H}_*^1(0, 1) \times L^2(0, 1) \times \tilde{H}_*^1(0, 1) \times L^2(0, 1) \\ &\times \tilde{H}_*^1(0, 1) \times L^2(0, 1) \times L^2(0, 1) \times L^2(0, 1) \\ &\times L_g^2(\mathbb{R}^+, H_0^1(0, 1)), \end{aligned}$$

where $L_g^2(\mathbb{R}^+, H_0^1(0, 1))$ denotes the Hilbert space of H_0^1 -valued functions on $\mathbb{R}^+$, endowed with the inner product

$$(V_1, V_2)_{L_g^2(\mathbb{R}^+, H_0^1(\Omega))} = \int_0^1 \int_0^1 g(s) V_{1x}(s) V_{2x}(s) ds dx. \quad (29)$$

We will show under the assumption (22) that A generates a C_0 semigroup on $\mathcal{H}$.

Now, we consider the vectors

$$\begin{aligned} U &= (\varphi, u, z, \psi, v, w, \bar{\omega}, \theta, q, \phi)^T, \\ \bar{U} &= (\bar{\varphi}, \bar{u}, \bar{z}, \bar{\psi}, \bar{v}, \bar{w}, \bar{\omega}, \bar{\theta}, \bar{q}, \bar{\phi})^T, \end{aligned} \quad (30)$$

and we define the inner product

$$\begin{aligned} \langle U, \bar{U} \rangle_{\mathcal{H}} &= k \int_0^1 (\varphi_x + \psi + lw)(\bar{\varphi}_x + \bar{\psi} + l\bar{w}) dx + \rho_2 \int_0^1 v \bar{v} dx \\ &\quad + \rho_1 \int_0^1 \bar{\omega} \bar{\omega} dx + k_0 \int_0^1 (w_x - l\varphi)(\bar{w}_x - l\bar{\varphi}) dx \\ &\quad + l \int_0^1 \psi_x \bar{\psi}_x dx + \rho_1 \int_0^1 u \bar{u} dx + \xi \int_0^1 \int_0^1 z \bar{z} dp dx \\ &\quad + \rho_3 \int_0^1 \theta \bar{\theta} dx + \alpha \int_0^1 q \bar{q} dx \\ &\quad + \int_0^1 \int_0^\infty g(s) \phi_x(s) \bar{\phi}_x(s) dx ds, \end{aligned} \quad (31)$$

where the domain of A is defined by

$$D(A) = \left\{ \begin{aligned} &U \in \mathcal{H} / \varphi \in H_*^2(0, 1); \psi, w \in \tilde{H}_*^2(0, 1), u, \theta \in H_*^1(0, 1); \\ &v, \bar{\omega}, q \in \tilde{H}_*^1(0, 1), u = z(., 0), z_\rho \in L^2((0, 1); L^2(0, 1)) \\ &\quad, \varphi_x(1) = 0, w_x(0) = \psi_x(0) = 0, \\ &\phi_s \in L_g^2(\mathbb{R}^+, H_0^1(0, 1)), \phi(x, 0) = 0, \end{aligned} \right\}. \quad (32)$$

Important properties of the above metrics are stated in the following lemmas. Although most of these results are followed straightforwardly from the known results, they are crucial for what follows. So for the convenience of the reader, we give their proofs here.

Lemma 1. *The operator A is dissipative and satisfies, for any $U \in D(A)$,*

$$\begin{aligned} \langle AU, U \rangle_{\mathcal{H}} &= -\beta \int_0^1 q^2 dx + \left(-\mu_1 + \frac{\mu_2}{2} + \frac{\xi}{2\tau} \right) \int_0^1 u^2 dx \\ &\quad + \left(\frac{\mu_2}{2} - \frac{\xi}{2\tau} \right) \int_0^1 z^2(x, 1) dx \\ &\quad + \frac{1}{2} \int_0^1 \int_0^\infty g'(s) |\phi_x(x, s)|^2 ds dx \leq 0. \end{aligned} \quad (33)$$

Proof. For any $U \in D(A)$, using the inner product,

$$\langle AU, U \rangle_{\mathcal{H}} = \left\langle \begin{pmatrix} u \\ \frac{k}{\rho_1}(\varphi_x + lw + \psi)_x + \frac{k_0 l}{\rho_1}(w_x - l\varphi) - \frac{\mu_1}{\rho_1}u - \frac{\mu_2}{\rho_1}z(., 1) \\ -\left(\frac{1}{\tau}\right)z_\rho \\ v \\ \frac{L}{\rho_2}\psi_{xx} - \frac{k}{\rho_2}(\varphi_x + lw + \psi) + \frac{1}{\rho_2} \int_0^\infty g(s) \phi_{xx}(s) ds - \frac{\gamma}{\rho_2} \theta_x \\ -\bar{\omega} \\ \frac{k_0}{\rho_1}(w_x - l\varphi)_x - \frac{kl}{\rho_1}(\varphi_x + lw + \psi) \\ -\frac{1}{\rho_3}q_x - \frac{\gamma}{\rho_3}v_x \\ -\frac{\beta}{\alpha}q - \frac{1}{\alpha}\theta_x \\ -\phi_s + v \end{pmatrix}, \begin{pmatrix} \varphi \\ u \\ z \\ \psi \\ v \\ w \\ \bar{\omega} \\ \theta \\ q \\ \phi \end{pmatrix} \right\rangle_{\mathcal{H}}. \quad (34)$$

Then,

$$\begin{aligned}
\langle AU, U \rangle_{\mathcal{H}} = & k \int_0^1 (u_x + v + l\bar{\omega})(\varphi_x + lw + \psi) dx + k_0 \int_0^1 (\bar{\omega}_x - l u) \\
& \cdot (w_x - l\varphi) dx + k \int_0^1 (\varphi_x + lw + \psi) u dx \\
& + k_0 l \int_0^1 (w_x - l\varphi) u dx - \mu_1 \int_0^1 u^2 dx \\
& - \mu_2 \int_0^1 z(x, 1) u dx + L \int_0^1 \psi_{xx} v dx \\
& - k \int_0^1 (\varphi_x + lw + \psi) v dx - \gamma \int_0^1 \theta_x v dx \\
& + k_0 \int_0^1 (w_x - l\varphi) \bar{\omega} dx - kl \int_0^1 (\varphi_x + lw + \psi) \bar{\omega} dx \\
& + L \int_0^1 v_x \psi_x dx + \int_0^1 \int_0^\infty g(s) \phi_x(s) (-\phi_s + v) dx ds \\
& - \int_0^1 q_x \theta dx - \gamma \int_0^1 u_x \theta dx - \beta \int_0^1 q^2 dx - \int_0^1 \theta_x q dx \\
& - \xi \int_0^1 \int_0^1 z z_\rho d\rho dx.
\end{aligned} \tag{35}$$

By the fact that

$$\begin{aligned}
& -\beta \int_0^1 q^2 dx - \mu_1 \int_0^1 u^2 dx - \mu_2 \int_0^1 z(x, 1) u dx + \int_0^1 \int_0^\infty g(s) \phi_x(s) \\
& \cdot (-\phi_s + v) dx ds - \frac{\xi}{\tau} \int_0^1 \int_0^1 z(x, \rho) z_\rho(x, \rho) d\rho dx \\
& = -\beta \int_0^1 q^2 dx - \mu_1 \int_0^1 u^2 dx - \mu_2 \int_0^1 z(x, 1) u dx + \int_0^1 \int_0^\infty g(s) \phi_x(s) \\
& \cdot (-\phi_s + v) dx ds - \frac{\xi}{2\tau} \int_0^1 \int_0^1 \frac{\partial}{\partial \rho} z^2(x, \rho) d\rho dx \\
& = -\beta \int_0^1 q^2 dx - \mu_1 \int_0^1 u^2 dx - \mu_2 \int_0^1 z(x, 1) u dx + \int_0^1 \int_0^\infty g(s) \phi_x(s) \\
& \cdot (-\phi_s + v) dx ds - \frac{\xi}{2\tau} \int_0^1 \{z^2(x, 1) - z^2(x, 0)\} dx \\
& = -\beta \int_0^1 q^2 dx - \mu_1 \int_0^1 u^2 dx - \mu_2 \int_0^1 z(x, 1) u dx + \int_0^1 \int_0^\infty g(s) \phi_x(s) \\
& \cdot (-\phi_s + v) dx ds - \frac{\xi}{2\tau} \int_0^1 z^2(x, 1) dx + \frac{\xi}{2\tau} \int_0^1 u^2 dx,
\end{aligned} \tag{36}$$

and using Young's inequality, we find

$$\begin{aligned}
\langle AU, U \rangle_H \leq & -\beta \int_0^1 q^2 dx + \left(-\mu_1 + \frac{\mu_2}{2} + \frac{\xi}{2\tau} \right) \int_0^1 u^2 dx \\
& + \left(\frac{\mu_2}{2} - \frac{\xi}{2\tau} \right) \int_0^1 z^2(x, 1) dx \\
& + \frac{1}{2} \int_0^1 \int_0^\infty g'(s) |\phi_x(x, s)|^2 ds dx.
\end{aligned} \tag{37}$$

Keeping in mind condition (22), the desired result yields.

Lemma 2. *The operator $I - A$ is surjective.*

Proof. We need to show that for all $\mathcal{F} = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10})^T \in \mathcal{H}$, there exists $U \in D(A)$ such that

$$U - AU = \mathcal{F}, \tag{38}$$

that is,

$$\begin{cases} -u + \varphi = f_1 \in H^1_*(0, 1), \\ -k(\varphi_x + lw + \psi)_x - k_0 l(w_x - l\varphi) + \rho_1 u + \mu_1 u + \mu_2 z(\cdot, 1) = \rho_1 f_2 \in L^2(0, 1), \\ z + \tau^{-1} z_\rho = \tau f_3 \in L^2((0, 1), H^1(0, 1)), \\ -v + \psi = f_4 \in \tilde{H}^1_*(0, 1), \\ -L\psi_{xx} + k(\varphi_x + lw + \psi) + \rho_2 v - \int_0^\infty g(s) \phi_{xx}(s) ds + \gamma \theta_x = \rho_2 f_5 \in L^2(0, 1), \\ -\bar{\omega} + w = f_6 \in \tilde{H}^1_*(0, 1), \\ -k_0(w_x - l\varphi)_x + kl(\varphi_x + lw + \psi) + \rho_1 \bar{\omega} = \rho_1 f_7 \in L^2(0, 1), \\ q_x + \gamma v_x + \rho_3 \theta = \rho_3 f_8 \in L^2(0, 1), \\ (\beta + \alpha)q + \theta_x = \alpha f_9 \in L^2(0, 1), \\ \phi + \phi_s - v = f_{10} \in L^2(0, 1). \end{cases} \tag{39}$$

From (39), we define

$$\theta = \frac{\alpha}{k} \int_0^x f_9(y) dy - \frac{\alpha}{k} (\beta + \alpha) \int_0^x q(y) dy, \tag{40}$$

so $\theta(0, t) = 0$.

Inserting $u = \varphi - f_1$, $v = \psi - f_4$, $\bar{\omega} = w - f_6$ and (39) into (40), we get

$$\begin{cases} -k(\varphi_x + lw + \psi)_x - k_0 l(w_x - l\varphi) + \rho_1 \varphi + \mu_1 u + \mu_2 z(\cdot, 1) = h_1 \in L^2(0, 1), \\ -L\psi_{xx} + k(\varphi_x + lw + \psi) + \rho_2 \psi - \int_0^\infty g(s) \phi_{xx}(s) ds - \gamma(\beta + \alpha)q = h_2 \in L^2(0, 1), \\ -k_0(w_x - l\varphi)_x + kl(\varphi_x + lw + \psi) + \rho_1 w = h_3 \in L^2(0, 1), \\ q_x + (\beta + \alpha) \int_0^x q(y) dy - \gamma \psi_x = h_4 \in L^2(0, 1), \\ z + \tau^{-1} z_\rho = h_5 \in L^2(0, 1), \\ \phi + \phi_s - v = h_6 \in L^2(0, 1), \end{cases} \tag{41}$$

where

$$\begin{cases} h_1 = \rho_1(f_1 + f_2), \\ h_2 = \rho_2(f_4 + f_5) - \frac{\alpha}{k} \gamma f_9, \\ h_3 = \rho_1(f_6 + f_7), \\ h_4 = \gamma f_{4x} + \rho_3 \left(f_8 - \frac{\alpha}{k} \int_0^x f_9(y) dy \right), \\ h_5 = z + \tau^{-1} z_\rho, \\ h_6 = \phi + \phi_s - v. \end{cases} \tag{42}$$

Furthermore, by (39), we can find as $z(x, 0) = u(x)$ for $x \in (0, 1)$. Following the same last approach, we obtain by using equation for z in (39)

$$z(x, \rho) = u(x)e^{-\tau\rho} + \tau e^{-\tau\rho} \int_0^\rho f_3(x, s)e^{\tau\rho s} ds. \quad (43)$$

From (39), we obtain

$$z(x, \rho) = \varphi(x)e^{-\tau\rho} - f_1 e^{-\tau\rho} + \tau e^{-\tau\rho} \int_0^\rho f_3(x, s)e^{\tau\rho s} ds. \quad (44)$$

Then,

$$z(x, 1) = \varphi(x)e^{-\tau} + z_0(x), \quad (45)$$

such that

$$z_0(x) = -f_1 e^{-\tau} + \tau e^{-\tau} \int_0^\rho f_3(x, s)e^{\tau s} ds. \quad (46)$$

We note that the last equation in (41) with $\phi(x, 0) = 0$ has a unique solution

$$\begin{aligned} \phi(x, s) &= \left(\int_0^x e^y (f_{10}(x, y) + v(x)) dy \right) e^{-s} \\ &= \left(\int_0^x e^y (f_{10}(x, y) + \psi(x) - f_4(x)) dy \right) e^{-s}. \end{aligned} \quad (47)$$

In order to solve (42), we consider

$$a((\varphi, \psi, w, q), (\tilde{\varphi}, \tilde{\psi}, \tilde{w}, \tilde{q})) = L(\tilde{\varphi}, \tilde{\psi}, \tilde{w}, \tilde{q}), \quad (48)$$

where

$$a : \left[H_*^1(0, 1) \times \tilde{H}_*^1(0, 1) \times \tilde{H}_*^1(0, 1) \times L^2(0, 1) \right]^2 \longrightarrow \mathbb{R} \quad (49)$$

is the bilinear form given by

$$\begin{aligned} a((\varphi, \psi, w, q), (\tilde{\varphi}, \tilde{\psi}, \tilde{w}, \tilde{q})) &= k \int_0^1 (\varphi_x + l w + \psi)(\tilde{\varphi}_x + l \tilde{w} + \tilde{\psi}) dx + (\beta + \alpha) \int_0^1 q \tilde{q} dx \\ &+ b \int_0^1 \psi_x \tilde{\psi}_x dx + \rho_2 \int_0^1 \psi \tilde{\psi} dx - \gamma(\beta + \alpha) \int_0^1 q \tilde{\psi} dx \\ &+ \rho_1 \int_0^1 \psi \tilde{\psi} dx + \gamma(\beta + \alpha) \int_0^1 \psi \tilde{q} dx + \rho_1 \int_0^1 w \tilde{w} dx \\ &+ k_0 \int_0^1 (w_x - l \varphi)(\tilde{w}_x - l \tilde{\varphi}) dx + \int_0^1 \varphi \tilde{\varphi} (\mu_1 + \mu_2 e^{-\tau}) dx \\ &+ \rho_3(\beta + \alpha) \int_0^1 \left(\int_0^x q(y) dy \int_0^x \tilde{q}(y) dy \right) dx. \end{aligned}$$

$$L : \left[H_*^1(0, 1) \times \tilde{H}_*^1(0, 1) \times \tilde{H}_*^1(0, 1) \times L^2(0, 1) \right] \longrightarrow \mathbb{R} \quad (50)$$

is the linear form defined by

$$\begin{aligned} L(\tilde{\varphi}, \tilde{\psi}, \tilde{w}, \tilde{q}) &= \int_0^1 h_1 \tilde{\varphi} dx + \int_0^1 h_2 \tilde{\psi} dx + \int_0^1 h_3 \tilde{w} dx \\ &+ (\alpha + \beta) \int_0^1 h_4 \int_0^x \tilde{q}(y) dy dx + \int_0^1 (\mu_1 f_1 \mu_2 z_0) \tilde{\varphi} dx. \end{aligned} \quad (51)$$

It is easy to verify that a is continuous and coercive, and L is continuous. So applying the Lax-Milgram theorem, we deduce that for all $(\tilde{\varphi}, \tilde{\psi}, \tilde{w}, \tilde{q}) \in H_*^1(0, 1) \times \tilde{H}_*^1(0, 1) \times \tilde{H}_*^1(0, 1) \times L^2(0, 1)$, problem (48) admits a unique solution $(\varphi, \psi, w, q) \in H_*^1(0, 1) \times \tilde{H}_*^1(0, 1) \times \tilde{H}_*^1(0, 1) \times L^2(0, 1)$. Since $D(A)$ is dense in $\mathcal{H}$ consequently, using Lemmas 1 and 2, we conclude that A is a maximal monotone operator. Hence, by Hille-Yosida theorem (see [45]), we have the following well-posedness result such that (25) is satisfied.

Theorem 3. *Let $U_0 \in \mathcal{H}$, then there exists a unique weak solution $U \in C(\mathbb{R}^+, \mathcal{H})$ of problems (1)–(3). Moreover, if $U_0 \in D(A)$, then $U \in C(\mathbb{R}^+, D(A)) \cap C^1(\mathbb{R}^+, \mathcal{H})$.*

Lemma 4. *The operator F defined in (26) is locally Lipschitz in $\mathcal{H}$.*

Proof. Let $U = (\varphi, u, z, \psi, v, w, \bar{w}, \theta, q, \phi)^T$, $\bar{U} = (\bar{\varphi}, \bar{u}, \bar{z}, \bar{\psi}, \bar{v}, \bar{w}, \bar{\theta}, \bar{q}, \bar{\phi})^T$, then we have

$$\|F(U) - F(\bar{U})\|_{\mathcal{H}} \leq \|f(\psi) - f(\bar{\psi})\|_{L^2}. \quad (52)$$

By using (6), Holder's and Poincaré's inequalities, we can obtain

$$\|f(\psi) - f(\bar{\psi})\|_{L^2} \leq \left(\|\psi\|_{2\theta}^\theta + \|\bar{\psi}\|_{2\theta}^\theta \right) \|\psi - \bar{\psi}\| \leq c_1 \|\psi - \bar{\psi}\|, \quad (53)$$

which gives us

$$\|F(U) - F(\bar{U})\|_{\mathcal{H}} \leq c_1 \|\psi - \bar{\psi}\|_{\mathcal{H}}. \quad (54)$$

Then, the operator F is locally Lipschitz in $\mathcal{H}$. The proof is hence complete.

3. Exponential Stability

Here, we present our stability result for the energy of the solution of systems (1)–(3), by using the multiplier technique. So we define the energy of our system by

$$\begin{aligned}
E(t) = & \frac{1}{2} \int_0^1 [\rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + \rho_1 w_t^2 + b \psi_x^2 + \rho_3 \theta^2 + \alpha q^2 \\
& + k(\varphi_x + \psi + lw)^2 + k_0(w_x - l\varphi)^2] dx \\
& + \frac{\xi}{2} \int_0^1 \int_0^1 z^2(x, \rho, t) d\rho dx + \frac{1}{2} \int_0^1 \int_0^\infty g(s) |\eta_x^t(x, s)|^2 ds dx \\
& + \int_0^1 \hat{f}(\psi(t)) dx.
\end{aligned} \tag{55}$$

The proof of the stability for our system is based on the following lemmas:

Lemma 5. Let $(\varphi, \psi, w, \theta, q, z, \eta^t)$ be the solution of (19)–(21). Then, the energy functional, defined by (55), satisfies

$$\begin{aligned}
E'(t) \leq & -\beta \int_0^1 q^2 dx - C \int_0^1 \psi_t^2 dx - \left(\mu_1 - \frac{\xi}{2\tau} - \frac{|\mu_2|}{2} \right) \\
& \cdot \|\varphi_t\|_2^2 - \left(\frac{\xi}{2\tau} - \frac{|\mu_2|}{2} \right) \|z(x, 1, t)\|_2^2 \\
& + \frac{1}{2} \int_0^1 \int_0^\infty g'(s) |\eta_x^t(x, s)|^2 ds dx,
\end{aligned} \tag{56}$$

such that $C > 0$.

Proof. Multiplying (1.1)₁, (1.1)₂, (1.1)₃, (1.1)₄, and (1.1)₅ by $\varphi_t, \psi_t, w_t, \theta$, and q , respectively, and after simplification, we have (56).

With the fact

$$\frac{d}{dt} \hat{f}(\psi) = f(\psi) \psi, \tag{57}$$

it gives us (56).

Lemma 6. Let $(\varphi, \psi, w, \theta, q, z, \eta^t)$ be the solution of (19)–(21). We have

$$F_1(t) := \alpha \rho_3 \int_0^1 \theta \int_0^x q(y) dy dx \tag{58}$$

satisfies, for any $\varepsilon_1 > 0$, the estimate

$$F_1'(t) \leq -\frac{\rho_3}{2} \int_0^1 \theta^2 dx + \varepsilon_1 \int_0^1 \psi_t^2 dx + c \left(1 + \frac{1}{\varepsilon_1} \right) \int_0^1 q^2 dx. \tag{59}$$

Proof. Taking the derivative of F_1 , using the fourth and fifth equations in (1) and performing integration by parts, we get

$$\begin{aligned}
F_1'(t) = & -\rho_3 k \int_0^1 \theta^2 dx - \alpha k \int_0^1 q^2 dx - \alpha \gamma \int_0^1 \psi_{tx} \int_0^1 q(y) dy dx \\
& - \beta \rho_3 \int_0^1 \theta \int_0^x q(y) dy dx.
\end{aligned} \tag{60}$$

According to Cauchy–Schwarz and Young’s inequalities with $\varepsilon_1 > 0$, we get (59).

Lemma 7. Let $(\varphi, \psi, w, \theta, q, z, \eta^t)$ be the solution of (19)–(21). We have

$$F_2(t) := -\frac{\rho_2 \rho_3}{\gamma} \int_0^1 \theta \int_0^x \psi_t(y) dy dx \tag{61}$$

satisfies, for any $\varepsilon_1, \varepsilon_2, \delta_1 > 0$, the estimate

$$\begin{aligned}
F_2'(t) \leq & -\frac{\rho_2}{\gamma} \int_0^1 \psi_t^2 dx + \varepsilon_2 \int_0^1 (\varphi_x + \psi + lw)^2 dx \\
& + \left(\varepsilon_3 + \frac{\rho_3}{\gamma} \left(\frac{\varepsilon_2}{b^2 \lambda_2} + \frac{b^2}{2\varepsilon_2 \lambda_2} \right) \right) \int_0^1 \psi_x^2 dx \\
& + c \left(1 + \frac{1}{\varepsilon_2} + \frac{1}{\varepsilon_3} \right) \int_0^1 \theta^2 dx + c \int_0^1 q^2 dx \\
& + \frac{g_0}{4\delta_1} \int_0^1 \int_0^\infty g(s) |\eta_x^t(x, s)|^2 ds dx \\
& + (\delta_1 + C_1) \int_0^1 \theta_x^2 dx.
\end{aligned} \tag{62}$$

Proof. For differentiation of F_2 , using equations in (1) and integration by parts, we obtain

$$\begin{aligned}
F_2'(t) = & -\rho_2 \int_0^1 \psi_t^2 dx - \frac{\rho_2 k}{\gamma} \int_0^1 q \psi_t dx + \rho_3 \int_0^1 \theta^2 dx \\
& - \frac{b \rho_3}{\gamma} \int_0^1 \theta \psi_x dx + \frac{k \rho_3}{\gamma} \int_0^1 (\varphi_x + \psi + lw) \int_0^x \theta(y) dy dx \\
& + \frac{\rho_3}{\gamma} \int_0^1 \int_0^\infty g(s) \eta_x^t(x, s) ds \int_0^x \theta_x(y) dy dx \\
& + \frac{\rho_3}{\gamma} \int_0^1 \theta \int_0^x f(\psi) dy dx.
\end{aligned} \tag{63}$$

Estimate (62) follows by using Cauchy–Schwarz, Young’s, and Poincaré’s inequalities that

$$\begin{aligned}
\int_0^1 |f(\psi) \theta| dx & \leq \int_0^1 |\psi|^\theta |\psi| |\theta| dx \leq \|\psi\|_{2(\theta+1)}^\theta \|\psi\|_{2(\theta+1)} \|\theta\| \\
& \leq C_1 \int_0^1 \theta^2 dx.
\end{aligned} \tag{64}$$

Lemma 8. Let $(\varphi, \psi, w, \theta, q, z, \eta^t)$ be the solution of (19)–(21). Then, the energy functional

$$F_3(t) := -\rho_1 \int_0^1 (\varphi \varphi_t + w w_t) dx \tag{65}$$

satisfies the estimate

$$\begin{aligned} F'_3(t) \leq & -\left(\rho_1 - \frac{1}{4\varepsilon_4}\right) \int_0^1 \varphi_t^2 dx + c \int_0^1 \psi_x^2 dx + k_0 \int_0^1 (w_x - l\varphi)^2 dx \\ & + c \int_0^1 (\varphi_x + \psi + lw)^2 dx - \rho_1 \int_0^1 w_t^2 dx \\ & + (\varepsilon_5 \mu_2 + \mu_1 \varepsilon_4) \int_0^1 \varphi^2 dx + \frac{\mu_2}{4\varepsilon_5} \int_0^1 z^2(x, 1, t) dx. \end{aligned} \quad (66)$$

Proof. Using (1)–(3) gives

$$\begin{aligned} F'_3(t) = & -\rho_1 \int_0^1 \varphi_t^2 dx + k \int_0^1 (\varphi_x + \psi + lw)^2 dx \\ & - k \int_0^1 (\varphi_x + \psi + lw) \psi dx - \rho_1 \int_0^1 w_t^2 dx \\ & + k_0 \int_0^1 (w_x - l\varphi)^2 dx + \mu_1 \int_0^1 \varphi \varphi_t dx \\ & + \mu_2 \int_0^1 \varphi z(x, 1, t) dx. \end{aligned} \quad (67)$$

Using Young's and Poincaré's inequalities, estimate (66) is established.

Lemma 9. Let $(\varphi, \psi, w, \theta, q, z, \eta^t)$ be the solution of (19)–(21). Then, the energy functional

$$F_4(t) := \rho_2 \int_0^1 \psi \psi_t dx \quad (68)$$

satisfies for any $\delta_2 > 0$ the estimate

$$\begin{aligned} F'_4(t) \leq & \left(\frac{b}{2} + \delta_2 + C_2\right) \int_0^1 \psi_x^2 dx + \rho_2 \int_0^1 \psi_t^2 dx \\ & + \frac{k^2}{b} \int_0^1 (\varphi_x + \psi + lw)^2 dx + c \int_0^1 \theta^2 dx \\ & + \frac{g_0}{4\delta_2} \int_0^1 \int_0^\infty g(s) |\eta_x^t(x, s)|^2 ds dx. \end{aligned} \quad (69)$$

Proof. Taking the derivative of F_4 and using the second equation in (1), it follows that

$$\begin{aligned} F'_4(t) = & -b \int_0^1 \psi_x^2 dx + \rho_2 \int_0^1 \psi_t^2 dx + \gamma \int_0^1 \psi_x \theta dx \\ & - k \int_0^1 (\varphi_x + \psi + lw) dx \\ & + \int_0^1 \psi_x(x) \int_0^\infty g(s) \eta_x^t(x, s) ds dx - \int_0^1 \psi f(\psi) dx, \end{aligned} \quad (70)$$

$$\begin{aligned} \int_0^1 |f(\psi) \psi| dx & \leq \int_0^1 |\psi|^\theta |\psi| |\psi| dx \leq \|\psi\|_{2(\theta+1)}^\theta \|\psi\|_{2(\theta+1)} \|\psi\| \\ & \leq C_2 \int_0^1 \psi_x^2 dx. \end{aligned} \quad (71)$$

Young's and Poincaré's inequalities for (70) yield (69).

Lemma 10. Let $(\varphi, \psi, w, \theta, q, z, \eta^t)$ be the solution of (19)–(21). Then, the energy functional

$$F_5(t) := -\rho_1 \int_0^1 \varphi_t (w_x - l\varphi) dx - \rho_1 \int_0^1 w_t (\varphi_x + \psi + lw) dx \quad (72)$$

satisfies the estimate

$$\begin{aligned} F'_5(t) \leq & -\left(lk_0 - \frac{\mu_1}{4\varepsilon_6} - \frac{\mu_2}{4\varepsilon_7}\right) \int_0^1 (w_x - l\varphi)^2 dx - \frac{l\rho_1}{2} \int_0^1 w_t^2 dx \\ & + (l\rho_1 + \varepsilon_6 \mu_1) \int_0^1 \varphi_t^2 dx + c \int_0^1 \psi_t^2 dx \\ & + lk \int_0^1 (\varphi_x + \psi + lw)^2 dx + \varepsilon_7 \mu_2 \int_0^1 z^2(x, 1, t) dx. \end{aligned} \quad (73)$$

Proof. For differentiation of F_5 , using (1.1)₁ and (1.1)₃, we arrive at

$$\begin{aligned} F'_5(t) = & -lk_0 \int_0^1 (w_x - l\varphi)^2 dx - l\rho_1 \int_0^1 w_t^2 dx + l\rho_1 \int_0^1 \varphi_t^2 dx \\ & + lk \int_0^1 (\varphi_x + \psi + lw)^2 dx - \rho_1 \int_0^1 \psi_t w_t dx \\ & + \mu_1 \int_0^1 \varphi_t (w_x - l\varphi) dx + \mu_2 \int_0^1 z(x, 1, t) (w_x - l\varphi) dx. \end{aligned} \quad (74)$$

Young's inequality for (74) yields (73).

Lemma 11. Let $(\varphi, \psi, w, \theta, q, z, \eta^t)$ be the solution of (19)–(21) and let $k = k_0$. Then, the functional

$$\begin{aligned} F_6(t) := & -\rho_1 \int_0^1 (w_x - l\varphi) \int_0^x w_t(y) dy dx \\ & - \rho_1 \int_0^1 \varphi_t \int_0^x (\varphi_x + \psi + lw) dy dx \end{aligned} \quad (75)$$

satisfies the estimate

$$\begin{aligned} F'_6(t) \leq & -\frac{\rho_1}{2} \int_0^1 \varphi_t^2 dx - k_0 \int_0^1 (w_x - l\varphi)^2 dx + \rho_1 \int_0^1 w_t^2 dx \\ & + k \int_0^1 (\varphi_x + \psi + lw)^2 dx + \frac{\rho_1}{2} \int_0^1 \psi_t^2 dx. \end{aligned} \quad (76)$$

Proof. A simple differentiation of F_6 , using the first and third equations in (1), leads to

$$\begin{aligned} F'_6(t) = & -\rho_1 \int_0^1 \varphi_t^2 dx - k_0 \int_0^1 (w_x - l\varphi)^2 dx + \rho_1 \int_0^1 w_t^2 dx \\ & - \rho_1 \int_0^1 \varphi_t \int_0^x \psi_t(y) dy + k \int_0^1 (\varphi_x + \psi + lw)^2 dx \\ & + l(k - k_0) \int_0^1 (w_x - l\varphi) \int_0^x (\varphi_x + \psi + lw) dy dx, \end{aligned} \quad (77)$$

and using Young's and Cauchy-Schwarz inequalities, with the fact that $k = k_0$, gives (76).

Lemma 12. Let $(\varphi, \psi, w, \theta, q, z, \eta^t)$ be the solution of (19)–(21) and let (9) holds, and we have

$$\begin{aligned} F_7(t) := & \rho_2 \int_0^1 \psi_t(\varphi_x + \psi + lw) dx + \frac{b\rho_1}{k} \int_0^1 \varphi_t \psi_x dx \\ & + \frac{b\rho_3}{\gamma} \left(\frac{\rho_1}{k} - \frac{\rho_2}{b} \right) \int_0^1 \theta \varphi_t dx \\ & - \frac{b}{\gamma} \left(\frac{\rho_1}{k} - \frac{\rho_2}{b} \right) \int_0^1 q(\varphi_x + \psi + lw) dx \\ & - \frac{bl^2\rho_2}{k_0} \int_0^1 \psi \psi_t dx + \frac{bl\rho_1}{k_0} \int_0^1 \psi w_t dx \end{aligned} \quad (78)$$

satisfies, for any $\varepsilon_4, \varepsilon_5, \delta_3 > 0$, the estimate

$$\begin{aligned} F'_7(t) \leq & -\left(\frac{k}{2} - \frac{b\eta}{\gamma\alpha\varepsilon_{10}} + \frac{\gamma}{4\varepsilon_1} + \frac{kb\rho_3}{\gamma 4\varepsilon_2\rho_1} \left(\frac{\rho_1}{k} - \frac{\rho_2}{b} \right) \right. \\ & \left. + \frac{b}{4\varepsilon_3} \right) \int_0^1 (\varphi_x + \psi + lw)^2 dx + \varepsilon_8 \int_0^1 w_t^2 dx \\ & + \left(\frac{b^2l^2}{k} + \frac{bl^2\rho_2\delta_3}{k_0} + b\varepsilon_3 + \frac{bl^2}{k_0} c_1 \right. \\ & \left. + 2 \left(\frac{\varepsilon}{b^2\lambda_1} + \frac{b^2}{2\varepsilon\lambda_1} \right) + c_2 \right) \int_0^1 \psi_x^2 dx \\ & + \varepsilon_9 \int_0^1 (w_x - l\varphi)^2 dx + c \left(1 + \frac{1}{\varepsilon_8} + \frac{b\rho_1\varepsilon_4}{k} \right) \\ & \cdot \int_0^1 \psi_t^2 dx + c \left(1 + \frac{1}{\varepsilon_8} \right) \int_0^1 q^2 dx + c \left(1 + \frac{1}{\varepsilon_9} \right. \\ & \left. + \frac{b\rho_3\mu_1}{\gamma\rho_1} \varepsilon_5 \left(\frac{\rho_1}{k} - \frac{\rho_2}{b} \right) \right) \int_0^1 \theta^2 dx + \left(\frac{b\eta}{\gamma\alpha} \varepsilon_{10} \right. \\ & \left. + \gamma\varepsilon_1 + \frac{kb\rho_3}{\gamma\rho_1} \varepsilon_2 \left(\frac{\rho_1}{k} - \frac{\rho_2}{b} \right) \right) \int_0^1 \theta_x^2 dx \\ & + \left(\frac{b\rho_1}{k 4\varepsilon_4} + \frac{b\rho_3\mu_1}{4\varepsilon_5\gamma\rho_1} \left(\frac{\rho_1}{k} - \frac{\rho_2}{b} \right) \right) \int_0^1 \varphi_t^2 dx \\ & + \frac{l\varepsilon_1}{b^2} \int_0^1 (w + \psi)^2 dx + \frac{\varepsilon}{2} \int_0^1 (\varphi_x + \psi)^2 \\ & + \frac{g_0 bl^2\rho_2}{k_0 4\delta_3} \int_0^1 \int_0^\infty g(s) |\eta_x^t(x, s)|^2 ds dx. \end{aligned} \quad (79)$$

Proof. Taking the deviate of F_7 , we obtain

$$\begin{aligned} F'_7(t) = & \rho_2 \int_0^1 \psi_{tt}(\varphi_x + \psi + lw) dx + \rho_2 \int_0^1 \psi_t(\varphi_x + \psi + lw)_t dx \\ & + \frac{b\rho_1}{k} \int_0^1 \varphi_{tt} \psi_x dx - \frac{b\rho_1}{k} \int_0^1 \varphi_t \psi_{xt} dx \\ & + \frac{b\rho_3}{\gamma} \left(\frac{\rho_1}{k} - \frac{\rho_2}{b} \right) \int_0^1 \theta_t \varphi_t dx \\ & + \frac{b\rho_3}{\gamma} \left(\frac{\rho_1}{k} - \frac{\rho_2}{b} \right) \int_0^1 \theta \varphi_{tt} dx \\ & - \frac{b}{\gamma} \left(\frac{\rho_1}{k} - \frac{\rho_2}{b} \right) \int_0^1 q_t(\varphi_x + \psi + lw) dx \\ & - \frac{b}{\gamma} \left(\frac{\rho_1}{k} - \frac{\rho_2}{b} \right) \int_0^1 q(\varphi_x + \psi + lw)_t dx \\ & - \frac{bl^2\rho_2}{k_0} \int_0^1 \psi_t^2 dx - \frac{bl^2\rho_2}{k_0} \int_0^1 \psi_{tt} \psi dx \\ & + \frac{bl\rho_1}{k_0} \int_0^1 w_{tt} \psi dx + \frac{bl\rho_1}{k_0} \int_0^1 w_t \psi_t dx. \end{aligned} \quad (80)$$

From the RHS of (80) and the relations in (1)–(3), we arrive at

$$\begin{aligned} & \rho_2 \int_0^1 \psi_{tt}(\varphi_x + \psi + lw) dx \\ & = -k \int_0^1 (\varphi_x + \psi + lw)^2 dx - \gamma \int_0^1 \theta_x(\varphi_x + \psi + lw) dx \\ & \quad - b \int_0^1 \psi_x(\varphi_x + \psi + lw)_x dx \\ & \quad - \int_0^1 \int_0^\infty g(s) \psi_{xx}(x, t-s) ds (\varphi_x + \psi + lw) dx \\ & \quad - \int_0^1 f(\psi)(\varphi_x + \psi + lw) dx, \end{aligned} \quad (81)$$

$$\begin{aligned} \rho_1 \int_0^1 \varphi_{tt} \psi_x dx = & k \int_0^1 \psi_x(\varphi_x + \psi + lw)_x dx \\ & + k_0 l \int_0^1 (w_x - l\varphi) dx - \mu_1 \int_0^1 \varphi_t \psi_x dx, \end{aligned} \quad (82)$$

$$\rho_3 \int_0^1 \theta_t \varphi_t dx = k \int_0^1 q \varphi_{xt} dx + \gamma \int_0^1 \psi_t \varphi_{xt} dx, \quad (83)$$

$$\begin{aligned} \int_0^1 \theta \varphi_{tt} dx = & -\frac{k}{\rho_1} \int_0^1 \theta_x(\varphi_x + \psi + lw) dx \\ & + \frac{lk_0}{\rho_1} \int_0^1 \theta(w_x - l\varphi) dx - \frac{\mu_1}{\rho_1} \int_0^1 \theta \varphi_t dx, \end{aligned} \quad (84)$$

$$\begin{aligned} -\int_0^1 q_t(\varphi_x + \psi + lw) dx = & \frac{\beta}{\alpha} \int_0^1 q(\varphi_x + \psi + lw) dx \\ & + \frac{k}{\alpha} \int_0^1 \theta_x(\varphi_x + \psi + lw) dx, \end{aligned} \quad (85)$$

$$\begin{aligned}
-\rho_2 \int_0^1 \psi_{tt} \psi dx &= b \int_0^1 \psi_x^2 dx + k \int_0^1 \psi(\varphi_x + \psi + lw) dx \\
&\quad - \gamma \int_0^1 \theta \psi_x dx + \int_0^1 \int_0^\infty g(s) \psi_{xx} ds \psi dx \quad (86) \\
&\quad + \int_0^1 f(\psi) \psi dx,
\end{aligned}$$

$$\begin{aligned}
\rho_1 \int_0^1 w_{tt} \psi dx &= -k_0 \int_0^1 \psi_x (w_x - l\varphi) dx \\
&\quad - kl \int_0^1 \psi(\varphi_x + \psi + lw) dx. \quad (87)
\end{aligned}$$

Invoking to (81)–(87) into (80), we arrive at

$$\begin{aligned}
F'_7(t) &= -k \int_0^1 (\varphi_x + \psi + lw)^2 dx + \left(\rho_2 - \frac{bl^2 \rho_2}{k_0} \right) \int_0^1 \psi_t^2 dx \\
&\quad + \left(l\rho_2 + \frac{bl\rho_1}{k_0} \right) \int_0^1 \psi_t w_t dx + \frac{b\eta}{\alpha\gamma} \int_0^1 \theta_x (\varphi_x + \psi + lw) dx \\
&\quad - \frac{b}{\gamma} \left(\frac{\rho_1}{k} - \frac{\rho_2}{b} \right) \int_0^1 q \psi_t dx - \frac{bl}{\gamma} \left(\frac{\rho_1}{k} - \frac{\rho_2}{b} \right) \int_0^1 q w_t dx \\
&\quad + \frac{blk_0 \rho_3}{\gamma \rho_1} \left(\frac{\rho_1}{k} - \frac{\rho_2}{b} \right) \int_0^1 \theta (w_x - l\varphi) dx - \frac{\gamma bl^2}{k_0} \int_0^1 \theta \psi_x dx \\
&\quad + \frac{b\beta}{\alpha\gamma} \left(\frac{\rho_1}{k} - \frac{\rho_2}{b} \right) \int_0^1 q (\varphi_x + \psi + lw) dx + \frac{b^2 l^2}{k_0} \int_0^1 \psi_x^2 dx \\
&\quad - bl \int_0^1 \psi_x (w_x - l\varphi) dx - \gamma \int_0^1 \theta_x (\varphi_x + \psi + lw) dx \\
&\quad - \frac{kb\rho_3}{\gamma \rho_1} \left(\frac{\rho_1}{k} - \frac{\rho_2}{b} \right) \int_0^1 \theta_x (\varphi_x + \psi + lw) dx \\
&\quad - b \int_0^1 \psi_x (\varphi_x + \psi + lw)_x dx + \frac{b\rho_1}{k} \int_0^1 \psi_t \varphi_{xt} dx \\
&\quad - \frac{b\rho_3 \mu_1}{\gamma \rho_1} \left(\frac{\rho_1}{k} - \frac{\rho_2}{b} \right) \int_0^1 \theta \varphi_t dx - \int_0^1 f(\psi) (\varphi_x + \psi + lw) dx \\
&\quad + \frac{bl^2}{k_0} \int_0^1 f(\psi) \psi dx + \frac{bl^2 \rho_2}{k_0} \int_0^1 \psi_x(x) \int_0^\infty g(s) \eta_x^t(x, s) ds dx. \quad (88)
\end{aligned}$$

We thus have

$$\begin{aligned}
\int_0^1 |\varphi_x f(\psi)| dx &\leq \|\varphi_x\| \|\psi\|_{2(\theta+1)}^\theta \|\psi\|_{2(\theta+1)} \\
&\leq \frac{\varepsilon}{2b^2} \int_0^1 \varphi_x^2 dx + \frac{b^2}{2\varepsilon \lambda_1} \int_0^1 \psi_x^2 dx \\
&\leq \frac{\varepsilon}{b^2} \int_0^1 (\varphi_x + \psi)^2 dx + \frac{\varepsilon}{b^2} \int_0^1 \psi^2 dx + \frac{b^2}{2\varepsilon \lambda_1} \int_0^1 \psi_x^2 dx \\
&\leq \frac{\varepsilon}{b^2} \int_0^1 (\varphi_x + \psi)^2 dx + \left(\frac{\varepsilon}{b^2 \lambda_1} + \frac{b^2}{2\varepsilon \lambda_1} \right) \int_0^1 \psi_x^2 dx. \quad (89)
\end{aligned}$$

Estimate (79) follows thanks to Young's inequality and the fact that $k = k_0$.

Lemma 13. Let $(\varphi, \psi, w, \theta, q, z, \eta^t)$ be the solution of (19)–(21). Then, the energy functional

$$F_8(t) := \int_0^1 \rho_1 \varphi_t \varphi dx + \frac{\mu_1}{2} \int_0^1 \varphi^2 dx. \quad (90)$$

Then, we have the following estimate, for any $\varepsilon_{11} > 0$,

$$\begin{aligned}
F'_8(t) &\leq \left(-K + \varepsilon_{11} \left(\frac{K}{2} + \frac{\mu_2 c}{2} \right) \right) \int_0^1 \varphi_x^2 dx + \frac{K}{2\varepsilon_{11}} \int_0^1 \psi_x^2 dx \\
&\quad + \frac{\mu_2}{2\varepsilon_{11}} \int_0^1 z^2(x, 1, t) dx + \rho_1 \int_0^1 \varphi_t^2 dx, \quad (91)
\end{aligned}$$

where $c = 1/\pi^2$ is the Poincaré constant.

Proof. Taking the derivative of (90) with respect to t , we have

$$F'_8(t) = \rho_1 \int_0^1 \varphi_{tt} \varphi dx + \rho_1 \int_0^1 \varphi_t^2 dx + \mu_1 \int_0^1 \varphi_t \varphi dx. \quad (92)$$

Then, by using the first equation in (1), we find

$$F'_8(t) = k \int_0^1 (\varphi_x + \psi + lw)_x \varphi dx - \mu_2 \int_0^1 \varphi z(x, 1, t) dx + \rho_1 \int_0^1 \varphi_t^2 dx. \quad (93)$$

Consequently, we arrive at

$$F'_8(t) = -k \int_0^1 (\varphi_x + \psi + lw) \varphi_x dx - \mu_2 \int_0^1 \varphi z(x, 1, t) dx + \rho_1 \int_0^1 \varphi_t^2 dx. \quad (94)$$

Applying Young's inequality and Poincaré's inequality, we find (90).

Lemma 14. Let $(\varphi, \psi, w, \theta, q, z, \eta^t)$ be the solution of (19)–(21). Then, we define the functional

$$F_9(t) := \int_0^1 \int_0^1 e^{-2\tau\rho} z^2(x, \rho, t) d\rho dx. \quad (95)$$

Then, the following result holds.

$$F'_9(t) \leq -F_9(t) - \frac{c_1}{2\tau} \int_0^1 z^2(x, 1, t) dx + \frac{1}{2\tau} \int_0^1 \psi_t^2(x, t) dx, \quad (96)$$

where c is a positive constant.

Proof. Taking the deviate of (95) with respect to t and using the equation (16), we get

$$\begin{aligned} & \frac{d}{dt} \left(\int_0^1 \int_0^1 e^{-2\tau\rho} z^2(x, \rho, t) d\rho dx \right) \\ &= -\frac{1}{\tau} \int_0^1 \int_0^1 e^{-2\tau\rho} z z_\rho(x, \rho, t) d\rho dx \\ &= -\int_0^1 \int_0^1 e^{-2\tau\rho} z^2(x, \rho, t) d\rho dx \\ &\quad - \frac{1}{2\tau} \int_0^1 \int_0^1 \frac{\partial}{\partial \rho} (e^{-2\tau\rho} z^2(x, \rho, t)) d\rho dx. \end{aligned} \quad (97)$$

Making use of the estimate above, implies that there exists a positive constant c_1 such that (96) holds.

Theorem 15. Assume that $\eta = 0$ and $k = k_0$. Then, $(\varphi, \psi, w, \theta, q, z, \eta^t)$ the solution of (19)–(21) satisfies

$$E(t) \leq c_0 e^{-c_1 t}, \quad t \geq 0, \quad (98)$$

where the positive constant c_0 is directly depending on initial data and the uniform constant c_1 is depending only on the coefficients of the system. For $N, N_i > 0$,

$$\mathcal{L}(t) := NE(t) + \sum_{i=1}^{i=9} N_i F_i(t), \quad (99)$$

Then, from (56), (59), (62), (66), (69), (73), (76), (79), (91), and (96), we have

$$\begin{aligned} \mathcal{L}'(t) \leq & \left[-\beta N + c_1 \left(1 + \frac{1}{\varepsilon_1} \right) + cN_2 + c \left(1 + \frac{1}{\varepsilon_8} \right) N_7 \right] \int_0^l q^2 dx \\ & - N \left[\mu_1 - \frac{\xi}{2} - \frac{|\mu_2|}{2} \right] \|\varphi_t\|_2^2 - N \left[\frac{\xi}{2} - \frac{|\mu_2|}{2} \right] \|z(x, l, t)\|_2^2 \\ & - \left[\frac{N_1 \rho_3}{2} - N_2 \left(C_1 + 1 + \frac{1}{\varepsilon_2} + \frac{1}{\varepsilon_3} \right) - cN_4 \right. \\ & \left. - c \left(1 + \frac{1}{\varepsilon_9} + \frac{b\rho_3\mu_1}{\gamma\rho_1 c} \varepsilon_5 \left(\frac{\rho_1}{k} - \frac{\rho_2}{b} \right) \right) N_7 \right] \int_0^l \theta^2 dx \\ & + \left[\varepsilon_1 N_1 - cN - N_2 \frac{\rho_2}{\gamma} + \rho_2 N_4 + cN_5 + \frac{\rho_1}{2} N_6 \right. \\ & \left. + c \left(1 + \frac{1}{\varepsilon_8} + \frac{b\rho_1\varepsilon_4}{k} \right) N_7 + \rho_1 N_8 + \frac{1}{2\tau} N_9 \right] \int_0^l \psi_t^2 dx \\ & + \left[\varepsilon_2 N_2 + cN_3 + \frac{k^2}{b} N_4 + lkN_5 + kN_6 \right. \\ & \left. - \left(\frac{k}{2} - \frac{b\tilde{\eta}}{\alpha\gamma\varepsilon_{10}} + \frac{\gamma}{4\varepsilon_1} + \frac{kb\rho_3}{\gamma 4\varepsilon_2\rho_1} \left(\frac{\rho_1}{k} - \frac{\rho_2}{b} \right) \right) \right. \\ & \left. + \frac{b}{4\varepsilon_3} \right] N_7 \int_0^l (\varphi_x + lw + \psi)^2 dx \end{aligned}$$

$$\begin{aligned} & + \left[\left(\varepsilon_3 + \frac{\rho_3}{\gamma} \left(\frac{\varepsilon_2}{b^2\lambda_2} + \frac{b^2}{2\varepsilon_2\lambda_2} \right) \right) N_2 + cN_3 \right. \\ & + \left(\delta_2 + \frac{b}{2} + C_2 \right) N_4 + \left(\frac{bl^2\rho_2\delta_3}{k_0} + \frac{b^2l^2}{k} + b\varepsilon_3 \right. \\ & + \frac{bl^2}{k_0} c_1 + 2 \left(\frac{\varepsilon}{b^2\lambda_1} + \frac{b^2}{2\varepsilon\lambda_1} \right) + c_2 \left. \right) N_7 \\ & + \left(\frac{k}{2\varepsilon_{11}} - k + \varepsilon_{11} \frac{k}{2} + \varepsilon_{11} \frac{\mu_2 c}{2} \right) N_8 \left. \right] \int_0^l \psi_x^2 dx \\ & + \left[-\rho_1 N_3 - \frac{l\rho_1}{2} N_5 + \rho_1 N_6 + \varepsilon_8 N_7 \right] \int_0^l w_t^2 dx \\ & + \left[k_0 N_3 - \left(lk_0 - \frac{\mu_1}{4\varepsilon_6} - \frac{\mu_2}{4\varepsilon_7} \right) N_5 - k_0 N_6 + \varepsilon_9 N_7 \right] \\ & \cdot \int_0^l (w_x - l\varphi)^2 dx + \left[\frac{\mu_2}{4\varepsilon_5} N_3 + \varepsilon_7 \mu_2 N_5 + \frac{\mu_2}{2\varepsilon_{11}} N_8 - \frac{c_1}{2\tau} N_9 \right] \\ & \cdot \int_0^l z^2(x, l, t) dx + [\varepsilon_5 \mu_2 + \varepsilon_4 \mu_1] N_3 \int_0^l \varphi^2 dx \\ & + \left[- \left(1 - \frac{1}{4\varepsilon_4} \right) N_3 + (l\rho_1 + \varepsilon_6 \mu_1) N_5 - \frac{\rho_1}{2} N_6 \right. \\ & + \rho_1 N_8 + \left(\frac{b\rho_1}{k4\varepsilon_4} + \frac{b\rho_3\mu_1}{4\varepsilon_5\gamma\rho_1} \left(\frac{\rho_1}{k} - \frac{\rho_2}{b} \right) \right) N_7 \left. \right] \int_0^l \varphi_t^2 dx \\ & + \left[\left(-k + \varepsilon_{11} \left(\frac{k}{2} + \frac{\mu_2 c}{2} \right) \right) N_8 \right] \int_0^l \varphi_x^2 dx \\ & + \left[\left(\frac{b\tilde{\eta}\varepsilon_{10}}{\alpha\gamma} + \gamma\varepsilon_1 + \frac{kb\rho_3}{\gamma\rho_1} \varepsilon_2 \left(\frac{\rho_1}{k} - \frac{\rho_2}{b} \right) \right) N_7 \right. \\ & + N_2 \delta_1 \left. \right] \int_0^l \theta_x^2 dx - N_9 F_9(t) + \left(\frac{N_2 g_0}{4\delta_1} + \frac{N_4 g_0}{4\delta_2} \right. \\ & + \frac{N_7 g_0 b l^2 \rho_2}{k_0 4\delta_3} - N \frac{\zeta}{2} \left. \right) \int_0^l \int_0^\infty g(s) |\eta_x^t(x, s)|^2 ds dx. \end{aligned} \quad (100)$$

At this point, we have to choose our constants very carefully. First, choosing $\varepsilon_i, i = 1, \dots, 10$ small enough such that

$$\varepsilon_1 \leq \frac{N_2(\rho_2/\gamma) + \rho_2 N_4 + cN_5 + (\rho_1/2)N_6}{N_1}. \quad (101)$$

Moreover, we pick N_9 large enough so that

$$\begin{aligned} & \frac{\mu_2}{4\varepsilon_5} N_3 + \varepsilon_7 \mu_2 N_5 + \frac{\mu_2}{2\varepsilon_{11}} N_8 - \frac{c_1}{2\tau} N_9 \leq 0, \\ & N_9 \geq \frac{(\mu_2/4\varepsilon_5)N_3 + \varepsilon_7 \mu_2 N_5 + (\mu_2/2\varepsilon_{11})N_8}{c_1/2\tau}, \end{aligned} \quad (102)$$

and we take ε_{11} small enough such that

$$\varepsilon_{11} \leq \frac{k}{(k/2 + \mu_2 c/2)N_8}. \quad (103)$$

Next, choosing N_5 large enough such that

$$\frac{N_5 \rho_3 \kappa}{4} \geq N_4 \left(\gamma \rho_3 + \frac{\rho_3}{2\varepsilon_4} (b + 2\kappa) \right). \quad (104)$$

After that, we can choose N large enough such that

$$N \geq \frac{c_1(1 + 1/\varepsilon_1) + cN_2 + c(1 + 1/\varepsilon_8)N_7}{\beta}, \quad (105)$$

$$\frac{N_2 g_0}{4\delta_1} + \frac{N_4 g_0}{4\delta_2} + \frac{g_0 b l^2 \rho_2}{k_0 4\delta_3} - N \frac{\zeta}{2} \leq 0.$$

Thus, the relation (100) becomes

$$\begin{aligned} \frac{d}{dt} \mathcal{L}(t) &\leq -\eta_1 \int_0^1 (\psi_t^2 + \psi_x^2 + \varphi_t^2 + (\varphi_x + lw + \psi)^2 \\ &\quad + \theta^2 + q^2) dx - \eta_1 \int_0^1 \int_0^1 z^2(x, \rho, t) d\rho dx, \end{aligned} \quad (106)$$

which leads by (55) that there exists also η_2 , such that

$$\frac{d}{dt} \mathcal{L}(t) \leq -\eta_2 E(t), \forall t \geq 0. \quad (107)$$

Lemma 16. For N large enough, there exist two positive constants β_1 and β_2 depending on $N_i, i = 1, \dots, 9$ and $\varepsilon_i, i = 1, \dots, 11$ such that

$$\beta_1 E(t) \leq \mathcal{L}(t) \leq \beta_2 E(t), \forall t \geq 0. \quad (108)$$

Proof. We consider the functional

$$H(t) = \sum_{i=1}^{i=9} N_i F_i(t) \quad (109)$$

and show that

$$|H(t)| \leq CE(t), C > 0. \quad (110)$$

From (58), (61), (65), (68), (72), (75), (78), (90), and (95), we obtain

$$\begin{aligned} |H(t)| &\leq N_1 \left| \alpha \rho_3 \int_0^1 \theta \int_0^x q(y) dy dx \right| \\ &\quad + N_2 \left| -\frac{\rho_2 \rho_3}{\gamma} \int_0^1 \theta dx \int_0^x \psi_t(y) dy dx \right| \\ &\quad + N_3 \left| \rho_1 \int_0^1 (\varphi \varphi_t + w w_t) dx \right| \end{aligned}$$

$$\begin{aligned} &+ N_4 \left| \rho_2 \int_0^1 \int_0^x \psi \psi_t(t, x) dx \right| \\ &+ N_5 \left| -\rho_1 \int_0^1 \varphi_t(w_x - l\varphi) dx - \rho_1 \int_0^1 w_t(\varphi_x + \psi + lw) dx \right| \\ &+ N_6 \left| -\rho_1 \int_0^1 (w_x - l\varphi) \int_0^x w_t(y) dy dx \right. \\ &\quad \left. - \rho_1 \int_0^1 \varphi_t \int_0^x (\varphi_x + \psi + lw) dy dx \right| \\ &+ N_7 \left| \rho_2 \int_0^1 (\varphi_x + \psi + lw) dx + \frac{b\rho_1}{k} \int_0^1 \varphi_t \psi_x dx \right| \\ &+ N_8 \left| \int_0^1 \rho_1 \varphi \varphi_t dx + \frac{\mu_1}{2} \int_0^1 \varphi^2 dx \right| \\ &+ N_9 \int_0^1 \int_0^1 e^{-2\tau\rho} z^2(x, \rho, t) d\rho dx. \end{aligned} \quad (111)$$

By using, the trivial relation

$$\int_0^1 (\varphi + lw)^2 dx \leq 2c \int_0^1 (\varphi_x + lw + \psi)^2 dx + 2c \int_0^1 \psi_x^2 dx, \quad (112)$$

Young's and Poincaré's inequalities, we get

$$\begin{aligned} |H(t)| &\leq \alpha_1 \int_0^1 \varphi_t^2 dx + \alpha_2 \int_0^1 \psi_t^2 dx + \alpha_3 \int_0^1 w_t^2 \\ &\quad + \alpha_4 \int_0^1 \psi_x^2 + \alpha_5 \int_0^1 \theta^2 dx + \alpha_6 \int_0^1 q^2 dx \\ &\quad + \alpha_7 \int_0^1 ((\varphi_x + lw + \psi)^2 + (w_x - l\varphi)^2) dx \\ &\quad + \int_0^1 \int_0^1 z^2(x, \rho, t) d\rho dx, \end{aligned} \quad (113)$$

where $\alpha_1, \dots, \alpha_6$ are the positive constants as follows:

$$\left\{ \begin{aligned} \alpha_1 &:= \frac{1}{2} (N_3 \rho_1 + N_8 \rho_1), \\ \alpha_2 &:= \frac{1}{2} \left(N_4 \rho_2 + N_2 \frac{\rho_2 \rho_3}{\gamma} \right), \\ \alpha_3 &:= \frac{1}{2} (N_3 \rho_1 + N_6 \rho_1), \\ \alpha_4 &:= \frac{b\rho_1}{2k}, \\ \alpha_5 &:= \frac{1}{2} \left(N_1 \rho_3 + \frac{\rho_2 \rho_3}{\gamma} \right), \\ \alpha_6 &:= \frac{1}{2} (N_1 \rho_3 + N_5 \tau_0 \rho_3), \\ \alpha_7 &:= \frac{1}{2} (N_7 \rho_2 + 3\rho_1). \end{aligned} \right. \quad (114)$$

From (113), we have

$$|H(t)| \leq \widehat{C}E(t), \quad (115)$$

for

$$\widehat{C} = \frac{\max \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}}{\min \{\rho_1, \rho_2, \rho_3, k, b, \kappa, \gamma, \delta, \tau_0\}}. \quad (116)$$

Therefore, we get

$$|\mathcal{L}(t) - NE(t)| \leq \widehat{C}E(t). \quad (117)$$

Then, we can choose N large enough so that $\beta_1 = N - \widehat{C} > 0$. Then, (108) holds true for $\beta_2 = N + \widehat{C} > 0$, and this concludes the proof of the Lemma.

Combining now (107) and (108), we conclude that there exists some $\Lambda > 0$ such that

$$\frac{d}{dt} \mathcal{L}(t) \leq -\Lambda \mathcal{L}(t), \forall t \geq 0. \quad (118)$$

Integration of (118) yields

$$\mathcal{L}(t) \leq \mathcal{L}(0)e^{-\Lambda t}, \forall t \geq 0. \quad (119)$$

Finally, using (108) and (119), so (98) is satisfied, we thus immediately reach to Theorem 15.

4. Conclusion and Perspective

In this current study, a one-dimensional linear thermoelastic Bresse system with delay term, forcing, and infinity history acting on the shear angle displacement is considered. According to an appropriate assumption between the weight of the delay and the weight of the damping, the well-posedness of the problem using the semigroup method is proved, where an asymptotic stability result of global solution is obtained. In next article, we will generalize this result to convex bounded domain with a holomorphic map, and let x and y be two distinct fixed points for our problem. We will suppose there is at least one complex geodesics passing through two distinct variables. We will see that this method of proof cannot be generalized to the case of a bounded domain of a complex Banach space. Also, in the last part of the next article, we will study the fixed points of the analytical automorphisms of the open unit-ball B of a complex Banach space. More precisely, we will assume that B is homogeneous and we will show that, if the right hand side is an analytical automorphism of B , there exists a complex geodesic which we will specify formed of fixed points of the right hand. We will see that the set of fixed points of the right hand can be much larger by using the studied algorithm in ([46–51]).

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no competing interests.

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